Exceptional points and ground-state entanglement spectrum of a fermionic extension of the Swanson oscillator

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Motivated by the structure of the Swanson oscillator, which is a well-known example of a non-hermitian quantum system consisting of a general representation of a quadratic Hamiltonian, we propose a fermionic extension of such a scheme which incorporates two fermionic oscillators, together with bilinear-coupling terms that do not conserve particle number. We determine the eigenvalues and eigenvectors, and expose the appearance of exceptional points where two of the eigenstates coalesce with the corresponding eigenvectors exhibiting the self-orthogonality relation. We compute the entanglement spectrum and entanglement entropy of the ground state in two different ways, with one of them being via the Gelfand-Naimark-Segal construction. In addition to the approach involving the usual bi-normalization of the eigenvectors of the non-hermitian Hamiltonian, we also discuss the case where the eigenvectors are normalized with respect to the Dirac norms. It is found that the model exhibits a quantum phase transition due to the presence of a ground-state crossing.

I. INTRODUCTION

In recent times, the study of non-hermitian systems in quantum mechanics has evinced a lot of interest due to its relevance in open quantum systems [1–6]. Parity-time-symmetric Hamiltonians, where the parity operator $\mathcal{P}$ is defined by the operations $(i, x, p) \rightarrow (i, -x, -p)$ and the time-reversal operator $\mathcal{T}$ by the ones $(i, x, p) \rightarrow (-i, x, -p)$, form a distinct subclass of a wider branch of non-hermitian Hamiltonians. Such Hamiltonians have drawn considerable attention because a system featuring unbroken $\mathcal{PT}-$symmetry generally preserves the reality of the corresponding bound-state eigenvalues, unless $\mathcal{PT}$ be broken when the eigenvectors cease to be simultaneous eigenfunctions of the joint $\mathcal{PT}$ operator, and as a result, complex eigenvalues spontaneously appear in conjugate pairs [7–10]. The last two decades have witnessed the relevance of $\mathcal{PT}-$symmetry in optics [11], including non-hermitian photonics [12, 13], wherein balancing gain and loss provides a powerful toolbox towards the exploration of new types of light-matter interaction [14].

A remarkable feature associated with many non-hermitian systems is the unique presence of exceptional points, which are singular points in the parameter space at which two or more eigenstates (eigenvalues and eigenstates) coalesce [15–20]. Such points, including the existence of their higher orders [21], are of great interest especially in the context of optics [22–25], as well as while going for the experimental observations in thermal atomic ensembles [26]. It is worthwhile noting that a non-hermitian operator (even with real eigenvalues) admits distinct left and right eigenvectors; at the exceptional point, the coalescing eigenvectors become orthogonal to each other, i.e., they exhibit the so-called self-orthogonality condition in which the inner product between the corresponding left and right eigenvectors becomes zero [15]. This result has found interesting physical implications such as stopping of light in $\mathcal{PT}$-symmetric optical waveguides, as reported in Ref. [27].

A particularly simple yet interesting example of a non-hermitian system is the Swanson oscillator [28–31], being described by the Hamiltonian (we take $\hbar = k_B = 1$)

$$H = \omega a^\dagger a + \alpha(a^\dagger)^2 + \beta a^2, \tag{1}$$

where $\omega, \alpha, \beta \in \mathbb{R}$, with $\omega > 0$ and $\alpha \neq \beta$; the latter condition ensures that the Hamiltonian is non-hermitian. The Hamiltonian is $\mathcal{PT}-$symmetric and also pseudo-hermitian [32–34], thereby holding a real and positive spectrum for a certain range of the parameter values. The remarkable feature of the Swanson model is the existence of the terms $(a^\dagger)^2$ and a^2, which are not 'number conserving', respectively leading to the transitions $|n\rangle \mapsto |n+2\rangle$ and $|n\rangle \mapsto |n-2\rangle$. Exceptional points arising from a situation involving coupled oscillators where each mode is described by a Swanson-like Hamiltonian have been reported recently in Ref. [35].

In this paper, we present a formalism that addresses a fermionic extension of the Swanson oscillator [Sec. (II)]. In particular, we demonstrate the existence of exceptional points in the parameter space describing the system; at such points, two of the eigenstates coalesce with the eigenvectors conforming to the self-orthogonality condition [Sec. (III)]. Employing the so-called bi-orthogonal approach to non-hermitian quantum mechanics (details to be explained later), we compute the ground-state entanglement entropy of the model [Sec. (IV)], placing particular emphasis on the algebraic approach to quantum

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mechanics based on the Gelfand-Naimark-Segal (GNS) construction [Sec. (IV B)]. This is followed by the computation of the entanglement spectrum based on the prescription of Dirac normalization of the eigenvectors of the non-hermitian Hamiltonian [Sec. (V)]. We conclude the paper in Sec. (VI).

II. THE FERMIONIC EXTENSION: HAMILTONIAN AND HILBERT SPACE

Towards this end we consider a quadratic (oscillator) Hamiltonian, but incorporate additional terms that do not lead to conservation of particle number. Since for fermionic operators, say c and $c^\dagger$, the properties $c^2 = (c^\dagger)^2 = 0$ need to be satisfied, a straightforward generalization of Eq. (1) would be quite unfeasible. One could however, resort to a situation with two fermionic sets of operators $(c_1, c_1^\dagger)$ and $(c_2, c_2^\dagger)$, defining a Hamiltonian that goes as

$$H = \omega_1 c_1^\dagger c_1 + \omega_2 c_2^\dagger c_2 + \alpha c_1^\dagger c_2^\dagger + \beta c_2 c_1, \quad \alpha \neq \beta, \tag{2}$$

where $\omega_{1,2}, \alpha, \beta \in \mathbb{R}$, with $\omega_{1,2} > 0$. One has the usual anti-commutation relations, i.e.,

$$\{c_j, c_k^\dagger\} = \delta_{j,k}, \quad \{c_j, c_k\} = 0 = \{c_j^\dagger, c_k^\dagger\}, \quad j,k = 1,2. \tag{3}$$

It may be speculated that such a non-hermitian system may emerge from the interaction of two uncoupled fermionic oscillators with some external agent whose effect is to ensure that a transition from the zero-particle state to the two-particle state happens with a different weight as compared to the reverse transition, i.e., one of these transitions is favored over the other. A schematic diagram is shown in Fig. (1) wherein one has a pair of single-occupancy quantum dots with external biases (denoted with arrows) corresponding to the terms in the Hamiltonian with coefficients α and β. With Eq. (2) as the candidate for the fermionic extension of the Swanson oscillator, we now proceed to investigate the associated exceptional points, which are basically the fingerprints signifying the character of a non-hermitian system. A similar quadratic and non-hermitian model with number-conserving interactions may also be studied as presented in Appendix (A). It may be pointed out that the indices '1' and '2' can be looked upon as serving internal indices (like spin, color, etc.), in which case the above Hamiltonian describes a system with a single site which can accommodate two different types of fermions, labeled by '1' and '2'. Arguably, this interpretation bears a closer resemblance to the bosonic Swanson oscillator. However, as far as our analysis is concerned, such interpretations do not play a role; thus we continue to treat the indices '1' and '2' as different position labels throughout the rest of the work.

For the fermionic system at hand, the complete Hilbert space can be decomposed as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{4}$$

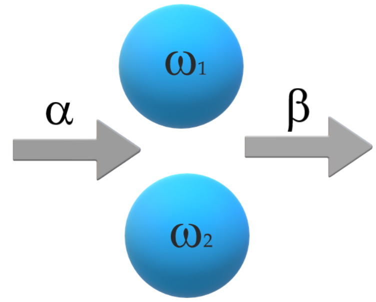

FIG. 1: Schematic setup showing two single-occupancy quantum dots with external biases (denoted with arrows) corresponding to the non-number-conserving interactions with coefficients α and β.

where $\mathcal{H}_0$ and $\mathcal{H}_2$ are one-dimensional (each) and are spanned by the vectors $|\Omega\rangle$ and $c_1^\dagger c_2^\dagger|\Omega\rangle$, respectively; $\mathcal{H}_1$ is two-dimensional and is spanned by $c_1^\dagger|\Omega\rangle$ and $c_2^\dagger|\Omega\rangle$. Here $|\Omega\rangle$ is the zero-particle (vacuum) state. We relabel the basis vectors as $|1\rangle := |\Omega\rangle$, $|2\rangle := c_1^\dagger|\Omega\rangle$, $|3\rangle := c_2^\dagger|\Omega\rangle$, and $|4\rangle := c_1^\dagger c_2^\dagger|\Omega\rangle$. In this (natural) basis, the Hamiltonian is expressible as a 4×4 matrix which reads (we pick $\omega_1 = \omega$ and $\omega_2 = 1 - \omega$, with $\omega \in (0, 1)$)

$$H = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & \omega & 0 & 0 \\ 0 & 0 & (1-\omega) & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

It should be remarked that just as the (bosonic) Swanson oscillator, the fermionic extension is pseudo-hermitian for a certain range of the parameter values, i.e., one can find some matrix η, such that $h = \eta^{-1} H \eta$ is hermitian. Explicitly, for $\alpha\beta > 0$, we have

$$\eta = \begin{pmatrix} 1 & 0 & 0 & (\alpha - \sqrt{\alpha\beta}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (\sqrt{\alpha\beta} - \beta) & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

giving

$$h = \begin{pmatrix} 0 & 0 & 0 & \sqrt{\alpha\beta} \\ 0 & \omega & 0 & 0 \\ 0 & 0 & (1-\omega) & 0 \\ \sqrt{\alpha\beta} & 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

which is hermitian. It may be remarked that the system is pseudo-hermitian for $4\alpha\beta + 1 \geq 0$ [Sec. (III B)], which ensures the reality of the spectrum [32], although the explicit forms of the matrices η and h given above for the purpose of illustration are specific to the region in the parameter space for which $\alpha\beta > 0$ (h ceases to be hermitian for $\alpha\beta < 0$). In what follows, we explore the existence of exceptional points associated with H.

III. EIGENSTATES, PARAMETER SPACE, AND EXCEPTIONAL POINTS

For ease of demonstration, we go for the choice $\omega_1 = \omega$ and $\omega_2 = 1-\omega$, with $\omega \in (0,1)$. The four-dimensional problem admits four eigenstates. Two of the right eigenvectors in the $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ basis are

$$|\psi_R^{\mathrm{I}}\rangle = \begin{pmatrix} -\frac{\sqrt{4\alpha\beta+1}+1}{2\alpha} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\psi_R^{\mathrm{II}}\rangle = \begin{pmatrix} -\frac{1-\sqrt{4\alpha\beta+1}}{2\alpha} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{8}$$

with respective eigenvalues $E^{\mathrm{I,II}} = \frac{1}{2}\left(1 \mp \sqrt{4\alpha\beta+1}\right)$. The other two eigenvectors are

$$|\psi_R^{\mathrm{III}}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_R^{\mathrm{IV}}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{9}$$

with respective eigenvalues $E^{\mathrm{III,IV}} = \omega, 1-\omega$.

A. Exceptional points

The states described by $|\psi_R^{\mathrm{III,IV}}\rangle$ are independent of the 'non-Hermiticity' parameters α and β, and therefore cannot be made to coalesce by tuning the parameters α and β. On the other hand, it is clear that the states described by $|\psi_R^{\mathrm{I,II}}\rangle$ depend upon α and β rather strongly. The corresponding left eigenvectors read

$$\langle\psi_L^{\mathrm{I}}| = \begin{pmatrix} -\frac{\sqrt{4\alpha\beta+1}+1}{2\beta} \\ 0 \\ 0 \\ 1 \end{pmatrix}^T, \quad \langle\psi_L^{\mathrm{II}}| = \begin{pmatrix} -\frac{1-\sqrt{4\alpha\beta+1}}{2\beta} \\ 0 \\ 0 \\ 1 \end{pmatrix}^T. \tag{10}$$

At an exceptional point, it is expected that both the eigenvalues and the eigenvectors coalesce. For the present case, it is found to happen for $1+4\alpha\beta = 0$, for which $E^{\mathrm{I}} = E^{\mathrm{II}} = \frac{1}{2}$ and $|\psi_R^{\mathrm{I}}\rangle = |\psi_R^{\mathrm{II}}\rangle$; quite naturally then, one also has $\langle\psi_L^{\mathrm{I}}| = \langle\psi_L^{\mathrm{II}}|$, which gives

$$\langle\psi_L^{\mathrm{I,II}}|\psi_R^{\mathrm{I,II}}\rangle = \frac{1}{4\alpha\beta} + 1 = 0, \tag{11}$$

confirming the self-orthogonality condition [15]. On the $\alpha\beta$-parameter space, the rectangular hyperbola $4\alpha\beta + 1 = 0$ describes the set of points (infinitely many) for which the eigenvalues and eigenvectors coalesce. Thus, the condition $4\alpha\beta+1 = 0$ may be interpreted as pointing to the 'exceptional curve'.

B. $\alpha\beta$-parameter space

Let us comment on the parameter space which is induced by the parameters α and β (assuming $\alpha, \beta \neq 0$).

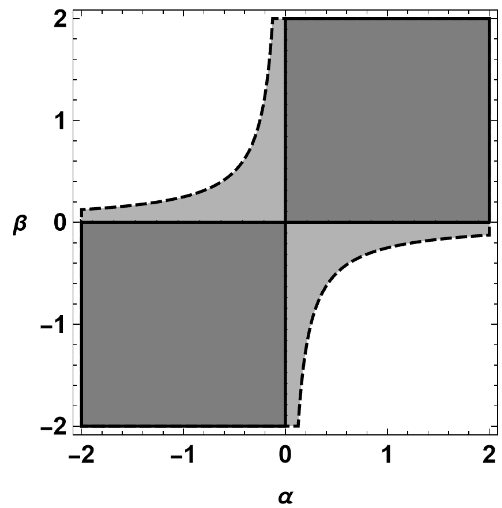

FIG. 2: Region in the $\alpha\beta$-parameter space conforming to $4\alpha\beta + 1 > 0$ (light and dark gray) and $\alpha\beta > 0$ (dark gray). The black-dashed curve is $4\alpha\beta + 1 = 0$.

Since we are looking for real eigenvalues, we restrict our attention to the points for which $4\alpha\beta + 1 \geq 0$. We note that the norm of the states 'I' and 'II' can be determined to be

$$\langle\psi_L^{\mathrm{I}}|\psi_R^{\mathrm{I}}\rangle = 1 + \frac{(1+\sqrt{4\alpha\beta+1})^2}{4\alpha\beta}, \tag{12}$$

$$\langle\psi_L^{\mathrm{II}}|\psi_R^{\mathrm{II}}\rangle = 1 + \frac{(1-\sqrt{4\alpha\beta+1})^2}{4\alpha\beta}. \tag{13}$$

Although the norms coalesce and vanish at exceptional points for which $4\alpha\beta + 1 = 0$, demanding that they are to be positive furnishes the additional condition $\alpha\beta > 0$. In Fig. (2), the region shaded in dark gray (the first and third quadrants excluding the lines $\alpha = 0$ and $\beta = 0$) are where the following two conditions hold: (a) spectrum is real, (b) norms are positive. The region shaded in light gray contains those points for which the norms $\langle\psi_L^{\mathrm{I}}|\psi_R^{\mathrm{I}}\rangle$ and $\langle\psi_L^{\mathrm{II}}|\psi_R^{\mathrm{II}}\rangle$ are not positive definite, although the spectrum is still real. The exceptional curve $4\alpha\beta + 1 = 0$ is shown as the dashed curve, on which the norms vanish.

IV. GROUND-STATE ENTANGLEMENT SPECTRUM AND ENTANGLEMENT ENTROPY

Let us now evaluate the entanglement spectrum of the ground state. We adopt two distinct ways to approach the problem; the first one is based on the bi-orthogonal interpretation of non-hermitian quantum mechanics [36], while the second one uses a Dirac-normalization scheme (see for instance, Ref. [37]) to produce right and left (reduced) density matrices [38]. Below, we briefly digress upon the two above-mentioned schemes.

In the bi-orthogonal scheme, for a generic eigenstate with right and left eigenvectors $|\psi_R\rangle$ and $\langle\psi_L|$, respectively, one forms the norm as $||\psi|| = \sqrt{\langle\psi_L|\psi_R\rangle}$, while the expectation value of some operator $\mathcal{O}$ reads as $\langle\psi_L|\mathcal{O}|\psi_R\rangle$. The states can be bi-normalized by redefining $|\psi_R^N\rangle = \frac{|\psi_R\rangle}{\sqrt{\langle\psi_L|\psi_R\rangle}}$ and $\langle\psi_L^N| = \frac{\langle\psi_L|}{\sqrt{\langle\psi_L|\psi_R\rangle}}$, such that $\langle\psi_L^N|\psi_R^N\rangle = 1$. This produces reliable results, especially for the reduced density matrix that is described subsequently, provided that $\langle\psi_L|\psi_R\rangle > 0$ for non-trivial eigenvectors. This however, cannot be guaranteed in general; for instance, the norms given in Eqs. (12) and (13) are positive definite only for parameter choices satisfying $\alpha\beta > 0$. For the other cases where the norms are not positive definite, one can resort to the so-called Dirac norms $\langle\psi_L|\psi_L\rangle$ and $\langle\psi_R|\psi_R\rangle$, which are guaranteed to be positive definite for non-trivial eigenvectors. One then has either left or right normalization, which can produce positive-semidefinite (reduced) density matrices. In this section, we compute the ground-state entanglement spectrum and entanglement entropy via the normalization scheme based on the bi-orthogonal interpretation of non-hermitian quantum mechanics (we call it bi-normalization) and deal with Dirac normalization later in Sec. (V).

A. Density matrix from bi-normalization

We consider the situation where Eqs. (12) and (13) describe positive-definite norms, i.e., we consider parameter values from the region shaded in dark gray in Fig. (2). It is equivalent to the condition $\alpha\beta > 0$, for which in the $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ basis, the ground state is described by

$$|G_R\rangle = \begin{pmatrix} -\frac{\sqrt{4\alpha\beta+1}+1}{2\alpha} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \langle G_L| = \begin{pmatrix} -\frac{\sqrt{4\alpha\beta+1}+1}{2\beta} \\ 0 \\ 0 \\ 1 \end{pmatrix}^T. \tag{14}$$

For constructing the (reduced) density matrix, we bi-normalize them as

$$|G_R^N\rangle := \frac{|G_R\rangle}{\sqrt{\langle G_L|G_R\rangle}}, \quad \langle G_L^N| := \frac{\langle G_L|}{\sqrt{\langle G_L|G_R\rangle}}, \tag{15}$$

where $\langle G_L|G_R\rangle = 1 + \frac{(1+\sqrt{4\alpha\beta+1})^2}{4\alpha\beta}$. The ground state can equivalently be described by the global density matrix that is defined as

$$\varrho = |G_R^N\rangle\langle G_L^N|. \tag{16}$$

To compute the entanglement entropy of, say, the first fermion (index '1'), we now need to find the reduced density matrix for the first particle (call it ρ_1), from where we can compute the von Neumann entropy as

$$S(\rho_1) = -\mathrm{Tr}_1[\rho_1 \ln\rho_1]. \tag{17}$$

The usual way to obtain the reduced density matrix describing a particular subsystem demands performing a partial trace on the global density matrix over the rest of the system. Although this a sensible operation to perform on a system with non-identical particles, the case of identical and indistinguishable particles requires more careful treatment; this is because when one works with a system of indistinguishable particles, the Hilbert space usually has a richer structure as compared to that of a system with non-identical constituents. For example, a system of two non-identical particles is described by the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, where '1' and '2' label the two particles. However, if the particles obey a specific statistics, like Bose or Fermi statistics, the corresponding Hilbert space is not the whole of $\mathcal{H}_1 \otimes \mathcal{H}_2$, but rather the symmetric and antisymmetric subspaces of $\mathcal{H}_1 \otimes \mathcal{H}_2$, respectively. This induces intrinsic correlations between different subsystems, which arise purely due to the indistinguishability and the statistics of the particles. Taking this important issue into account, Balachandran *et al.* proposed the algebraic framework [39, 40] for computing the entanglement entropy, which we intend to follow in this work. However, before that, we demonstrate a relatively-easier method to obtain the entanglement entropy for our present setting.

1. *Reduced density matrix for fermion '1'*

The strategy towards finding the reduced density matrix for one of the fermions, say, fermion '1' is as follows (see [41] for some related discussions). The local Hilbert space is spanned by $|1\rangle := |\Omega\rangle$ and $|2\rangle := c_1^\dagger|\Omega\rangle$. Thus any local operator is expressible as

$$\mathcal{O}_{a_0,a_1,a_2,a_3} = a_0\mathbb{I} + a_1 c_1 + a_2 c_1^\dagger + a_3 c_1^\dagger c_1. \tag{18}$$

However, the parity-superselection rule [42] insists that the density matrix must have the following form [39]:

$$\rho_1 = a\mathbb{I} + b c_1^\dagger c_1, \tag{19}$$

for some coefficients $\{a, b\}$. Now, for a generic local observable $\mathcal{O}$, one must have

$$\mathrm{Tr}_1[\rho_1\mathcal{O}] = \langle G_L^N|\mathcal{O}|G_R^N\rangle, \tag{20}$$

where $\mathrm{Tr}_1[\cdot]$ is evaluated on the basis $|\Omega\rangle$ and $c_1^\dagger|\Omega\rangle$. This serves as a consistency condition allowing one to determine the constants $\{a, b\}$. A straightforward computation leads to

$$\rho_1 = \begin{pmatrix} \frac{\chi}{1+\chi} & 0 \\ 0 & \frac{1}{1+\chi} \end{pmatrix}, \tag{21}$$

where $\chi = \frac{\left(\sqrt{4\alpha\beta+1}+1\right)^2}{4\alpha\beta}$. Notice that $\mathrm{Tr}_1[\rho] = 1$, as anticipated. Moreover, for consistency, one requires $\chi \geq 0$, which is equivalent to the restriction $\alpha\beta > 0$. In Fig.

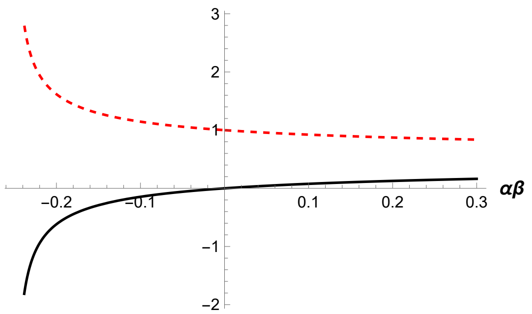

FIG. 3: $(\rho_1)_{11}$ (red) and $(\rho_1)_{22}$ (black) as functions of $\alpha\beta$, justifying the choice of parameters for which $\alpha\beta > 0$.

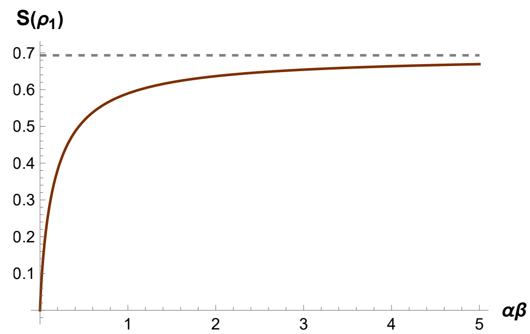

FIG. 4: Ground-state entanglement entropy as a function of $\alpha\beta$. The dashed line corresponds to $\ln 2$.

(3), we have plotted the ground-state entanglement spectrum, i.e., the elements $(\rho_1)_{11}$ and $(\rho_1)_{22}$ as functions of $\alpha\beta$, from which one clearly sees that $(\rho_1)_{22}$ becomes negative for $\alpha\beta < 0$, although one still has $(\rho_1)_{11}+(\rho_1)_{22} = 1$. We may now easily compute the ground-state entanglement entropy from the standard representation:

$$S(\rho_1) = -\mathrm{Tr}_1[\rho_1 \ln \rho_1], \tag{22}$$

which is plotted in Fig. (4) as a function of $\alpha\beta$. It is found that it is nearly zero for $\alpha\beta \approx 0$ and increases as $\alpha\beta$ is increased, approaching finally towards $\ln 2$, which is the maximum entropy of a two-state system.

B. Algebraic approach to ground-state entanglement entropy

We now demonstrate the computation of the ground-state entanglement entropy of our model based on the algebraic approach to quantum mechanics *à la* the Gelfand-Naimark-Segal or GNS construction. We have already emphasized how important it is to use this specific approach. But there is more good reason to choose this course of action. The traditional approach towards entanglement relies rather heavily on the notion of separability of the concerned state. Although here we work with two fermionic degrees of freedom, the separability of the ground state itself is not so transparent; as one can see from Eq. (14), the ground state is a linear superposition of the vacuum state and a two-particle state. This renders a conceptual difficulty for the analysis of entanglement. Nevertheless, one can tacitly avoid the debate over whether the given state is separable or not and still get an answer for the entanglement entropy. Here, adopting the prescriptions provided by the algebraic approach quantum mechanics based on the GNS construction [43, 44], we demonstrate that indeed the aforementioned state is entangled. In this framework, the entanglement of a subsystem depends on how the Hilbert space of the subsystem, which results from the restriction of the quantum state to the algebra generated by the observables particular to the subsystem, decomposes into different irreducible spaces [39, 40].

1. *Basic framework*

Let us briefly review the fundamental ideas behind this approach [43, 44]. The observables of a quantum system generate a non-abelian C^*-algebra (call it $\mathcal{A}$); a state Ω of the system is a positive linear functional on it, i.e., $\forall \alpha, \beta \in \mathcal{A}$ (not to be confused with the α and β parameters of the Swanson Hamiltonian), one has

$$\Omega(\alpha) \in \mathbb{C}, \quad \Omega(\alpha+\beta) = \Omega(\alpha) + \Omega(\beta), \quad \Omega(\alpha^*\alpha) \geq 0. \tag{23}$$

One then considers the vector space $\hat{\mathcal{A}}$ with the elements $\alpha \in \mathcal{A}$ and label them as $|\alpha\rangle$, such that the observables act on them as $\beta|\alpha\rangle = |\beta\alpha\rangle$. Introducing the inner product as $\langle\alpha|\beta\rangle = \Omega(\alpha^*\beta)$, one further constructs the null space as $\hat{\mathcal{N}}_\Omega = \{\alpha \in \hat{\mathcal{A}} \big| \langle\alpha|\alpha\rangle = 0\}$. From Schwarz inequality it follows then that $\langle a|\alpha\rangle = 0$, $\forall a \in \mathcal{A}$, $\alpha \in \mathcal{N}_\Omega$. The GNS Hilbert space $\mathcal{H}_\Omega$ is then identified with the space $\hat{\mathcal{A}}/\hat{\mathcal{N}}_\Omega$, whose elements $|[\alpha]\rangle$ are the equivalence classes $|[\alpha]\rangle = |\alpha + \hat{\mathcal{N}}_\Omega\rangle$, completed with respect to the inner product. This also induces the representation π_Ω of $\mathcal{A}$ on $\mathcal{H}_\Omega$ as $\pi_\Omega(\alpha)|[\beta]\rangle = |[\alpha\beta]\rangle$. An important feature of this representation is that it is irreducible if and only if Ω is pure.

2. *Computing ground-state entanglement entropy*

For our purposes, let us take the state Ω to be pure. Now, if one considers a subsystem whose observables form a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, the restriction of the state Ω on $\mathcal{A}$ to $\mathcal{A}_0$, dubbed as Ω_0, may or may not be pure. In general, it can be expressed as a density matrix ρ_0, satisfying $\mathrm{Tr}_{\mathcal{H}_{\Omega_0}}(\rho_0 \pi_{\Omega_0}(\alpha_0)) = \Omega(\alpha_0)$, with $\alpha_0 \in \mathcal{A}_0$. Here $\mathcal{H}_{\Omega_0}$ is the Hilbert space of the subsystem, obtained by applying the GNS construction to $(\mathcal{A}_0, \Omega_0)$ and π_{Ω_0} is the induced representation on it. If Ω_0 is not pure, $\mathcal{H}_{\Omega_0}$ decomposes as $\mathcal{H}_{\Omega_0} = \oplus_i \mathcal{H}_i$. Following [39, 40], let us denote the orthogonal projectors $P_i : \mathcal{H}_{\Omega_0} \to \mathcal{H}_i$. This finally leads to the entanglement entropy as

$$S_{\Omega_0} = -\sum_i \sigma_i \ln \sigma_i, \tag{24}$$

where $\sigma_i = \big|\big|P_i|[\mathbb{I}_{\mathcal{A}}]\rangle\big|\big|^2$, with $\mathbb{I}_{\mathcal{A}}$ being the identity element of the algebra $\mathcal{A}$ and as a consequence, of $\mathcal{A}_0$ too.

Let us take our subsystem to be the first fermion for which the subalgebra $\mathcal{A}_0$ is generated by $\mathbb{I}$ and $c_1^\dagger c_1$. Then, Ω is given by

$$\Omega(\alpha) = \langle G_N^L|\alpha|G_N^R\rangle. \tag{25}$$

A general element is expressible as $x\mathbb{I} + yc_1^\dagger c_1$. If an element (x, y) qualifies to be a null vector, it should satisfy $\Omega\left((x\mathbb{I} + yc_1^\dagger c_1)^\dagger(x\mathbb{I} + yc_1^\dagger c_1)\right) = 0$, which implies

$$\frac{1}{1+\chi}\left(|x|^2\chi + |x+y|^2\right) = 0, \tag{26}$$

where χ is the parameter defined below Eq. (21). For $\chi > 0$, there is no null state and the GNS Hilbert space $\mathcal{H}_{\Omega_0}$ is spanned by $|[\mathbb{I}]\rangle$ and $|[c_1^\dagger c_1]\rangle$. Further, one can also show that $\mathcal{H}_{\Omega_0} = \mathcal{H}_1 \oplus \mathcal{H}_2$, with $\mathcal{H}_1$ spanned by $|[\mathbb{I} - c_1^\dagger c_1]\rangle$ and $\mathcal{H}_2$ spanned by $|[c_1^\dagger c_1]\rangle$. The identity element can be decomposed as $|[\mathbb{I}]\rangle = |[\mathbb{I} - c_1^\dagger c_1]\rangle + |[c_1^\dagger c_1]\rangle$, leading to the entanglement entropy which reads

$$S_{\Omega_0} = -\frac{1}{1+\chi}\ln\left[\frac{1}{1+\chi}\right] - \frac{\chi}{1+\chi}\ln\left[\frac{\chi}{1+\chi}\right], \tag{27}$$

and which exactly agrees with the result of Sec. (IV A). It may be remarked that although for illustrative purpose, we focused on defining states via the bi-normalization scheme as discussed in Sec. (IV A), it is straightforward to perform analogous computations using the scheme of Dirac normalization which will be presented in Sec. (V).

C. An observation

To this end, we obtained the reduced matrix and computed the entanglement entropy, opting the bi-normalization prescription. Let us end this discussion by discussing an intriguing observation. Consider a modified form of the same Hamiltonian

$$H = \omega c_1^\dagger c_1 + (1-\omega)c_2^\dagger c_2 + \alpha c_1^\dagger c_2^\dagger + \frac{\Lambda}{\alpha}c_2 c_1, \tag{28}$$

with $\Lambda = \alpha\beta$. Let us denote the Hamiltonian by $H(\omega, \Lambda, \alpha)$. The spectrum is found to be independent of α and so is the reduced density matrix for a single particle. However, the parent density matrix $\varrho = |G_N^R\rangle\langle G_N^L|$, which describes the state of the full system, explicitly depends on the parameter α. This has to do with the fact that when one considers only a single fermion, the corresponding subalgebra has no non-hermitian element that has a non-zero expectation value. However, when the full system is considered, there exist certain non-hermitian operators like $c_1^\dagger c_2^\dagger$ and $c_2 c_1$, which have non-vanishing expectation values in this state.

V. GROUND-STATE ENTANGLEMENT ENTROPY FROM DIRAC NORMALIZATION

Notice that in the preceding discussion, we considered $\alpha\beta > 0$, thereby excluding parameters from the region shaded in light gray in Fig. (2), for which the spectrum is real but Eqs. (12) and (13) are not positive definite and coalesce to zero for $4\alpha\beta + 1 = 0$. Thus, we can no longer rely on the bi-normalization procedure as given in Eq. (15) to produce a reduced density matrix that is positive semidefinite. Instead, we may normalize using the Dirac norms [37] $\langle G_R|G_R\rangle$ and $\langle G_L|G_L\rangle$, which lead to right and left (reduced) density matrices, respectively (see [38] and references therein). Below, we focus on the right density matrix.

Let us analyze the special case for which $\omega \leq \frac{1}{2}$, and then ground-state energy reads

$$\begin{aligned} E_G &= \omega, & -\frac{1}{4} < \alpha\beta < \omega^2 - \omega, \\ &= \frac{1}{2}\left(1 - \sqrt{4\alpha\beta + 1}\right), & \alpha\beta > \omega^2 - \omega. \end{aligned} \tag{29}$$

For the purpose of illustration, we have plotted all the four eigenvalues as a function of $\alpha\beta$ in Fig. (5) for the choice $\omega = 1/4$, and one can observe a ground-state crossing. The ground state reads as

$$|G_R\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{30}$$

for $-\frac{1}{4} < \alpha\beta < \omega^2 - \omega$, and

$$|G_R\rangle = \begin{pmatrix} -\frac{\sqrt{4\alpha\beta+1}+1}{2\alpha} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{31}$$

for $\alpha\beta > \omega^2 - \omega$. We denote the corresponding reduced density matrix for the fermion '1' as ρ_1, such that

$$\mathrm{Tr}_1[\rho_1\mathcal{O}] = \frac{\langle G_R|\mathcal{O}|G_R\rangle}{\langle G_R|G_R\rangle}. \tag{32}$$

It then simply follows that

$$\rho_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{33}$$

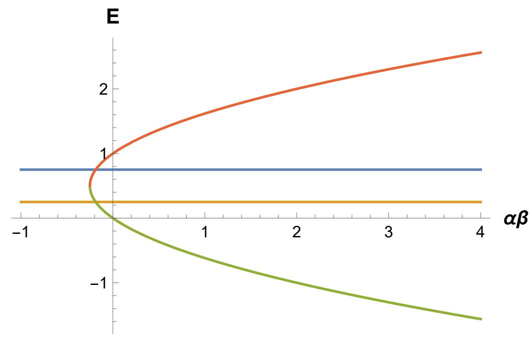

FIG. 5: Energy eigenvalues E^{I} (green), E^{II} (orange), E^{III} (yellow), and E^{IV} (blue), as a function of $\alpha\beta$. We have chosen $\omega = 1/4$.

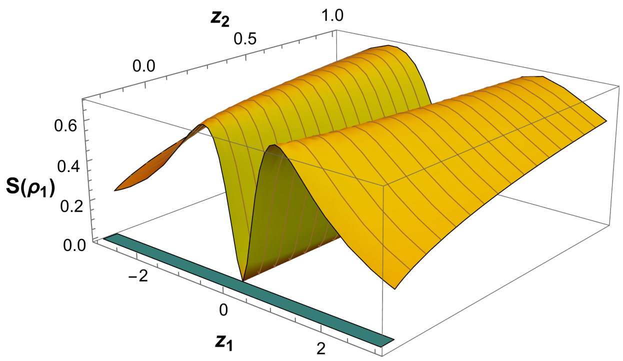

FIG. 6: Ground-state entanglement entropy as a function of z_1 and z_2. The discontinuity between the yellow and blue-green regions indicates the phase transition.

for $-\frac{1}{4} < \alpha\beta < \omega^2 - \omega$, while

$$\rho_1 = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}, \tag{34}$$

for $\alpha\beta > \omega^2 - \omega$. Here, $\lambda_{22} = \left[1 + \left(\frac{1+\sqrt{1+4z_2}}{2z_1}\right)^2\right]^{-1}$ and $\lambda_{11} = 1 - \lambda_{22}$, with $\alpha = z_1, \beta = \frac{z_2}{z_1}$; for $\alpha \neq 0$ the transformation $(\alpha, \beta) \to (z_1, z_2)$ is well defined and is invertible. We have plotted the ground-state entanglement entropy in Fig. (6) which shows a discontinuous jump between the two regimes $-\frac{1}{4} < \alpha\beta < \omega^2 - \omega$ and $\alpha\beta > \omega^2 - \omega$. This seems to indicate a quantum phase transition, being characterized by the discontinuous jump in the entanglement entropy due to the ground-state crossing [45].

It should be emphasized that we have normalized the ground state here with respect to the 'right' Dirac norm $\langle G_R|G_R\rangle$, such that the reduced density matrix so obtained turns out to be positive semidefinite. One could have alternatively employed the 'left' Dirac norm $\langle G_L|G_L\rangle$ and then, the result for the reduced density matrix and entanglement entropy would have the same form as obtained above under the interchange $\alpha \leftrightarrow \beta$. The same phase transition can be observed for both the cases, and since the ground-state crossing happens for parameter values for which $\alpha\beta < 0$, no such phase transition was observed in Sec. (IV A), in which we specifically restricted ourselves to the cases with $\alpha\beta > 0$.

VI. CONCLUDING REMARKS

We have proposed a fermionic extension of the Swanson oscillator, which admits a quadratic but non-hermitian Hamiltonian by including terms which do not conserve particle number. We have shown that our proposed model admits of an infinite number of exceptional points, being given by the points residing on the exceptional curve $4\alpha\beta + 1 = 0$. Restricting to parameter values which produce a positive norm (in the sense of bi-normalization of left and right eigenvectors) and a real spectrum, we have computed the entanglement spectrum and entanglement entropy of the ground state in Sec. (IV), wherein we also made use of the approach based on the GNS construction [39, 40]. In Sec. (V), upon adopting a different scheme which relies on Dirac normalization rather than bi-normalization, we were able to compute the entanglement spectrum even in the region of the parameter space for which the norms given in Eqs. (12) and (13) turn out to be complex, coalescing to zero at the exceptional points. A quantum phase transition was observed, which corresponds to the ground-state crossing. The analogous number-preserving case is presented in Appendix (A).

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Appendix A: Two-fermion model with non-hermitian and number-conserving interactions

In this appendix, we consider the situation with bilinear-coupling terms between the two fermions that conserve particle number, i.e., we have terms that go as $c_1^\dagger c_2$ and $c_2^\dagger c_1$. Then the Hamiltonian reads

$$H = \omega_1 c_1^\dagger c_1 + \omega_2 c_2^\dagger c_2 + \gamma c_1^\dagger c_2 + \delta c_2^\dagger c_1, \tag{A1}$$

where $\omega_{1,2}, \gamma, \delta \in \mathbb{R}$, with $\omega_{1,2} > 0$ and $\gamma \neq \delta$. A schematic diagram is shown in Fig. (7).

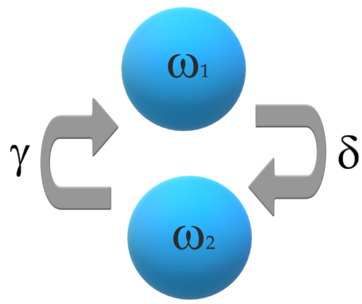

FIG. 7: Schematic setup showing two single-occupancy quantum dots with number-conserving interactions (denoted with arrows) characterized by the coefficients γ and δ.

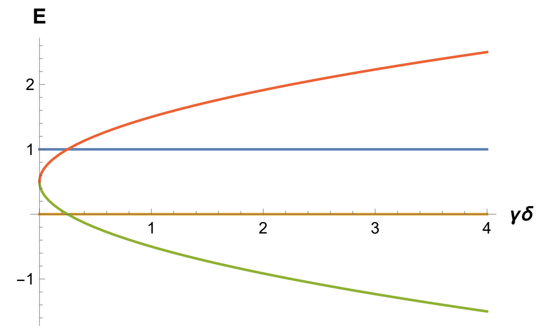

FIG. 8: Energy eigenvalues E^{I} (yellow), E^{II} (blue), E^{III} (green), and E^{IV} (orange), as a function of $\gamma\delta$. We have chosen $\omega = 1/2$.

As before, the Hilbert space is four-dimensional and may be decomposed as $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. In terms of the basis vectors $|1\rangle := |\Omega\rangle$, $|2\rangle := c_1^\dagger|\Omega\rangle$, $|3\rangle := c_2^\dagger|\Omega\rangle$, and $|4\rangle := c_1^\dagger c_2^\dagger|\Omega\rangle$, H has the following matrix representation:

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \omega_1 & \gamma & 0 \\ 0 & \delta & \omega_2 & 0 \\ 0 & 0 & 0 & (\omega_1+\omega_2) \end{pmatrix}. \tag{A2}$$

Resorting to the choice $\omega_1 = \omega$ and $\omega_2 = 1-\omega$, for $\omega \in (0,1)$, two of the right eigenvectors read

$$|\psi_R^{\mathrm{I}}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_R^{\mathrm{II}}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{A3}$$

with eigenvalues $E^{\mathrm{I,II}} = 0, 1$. The remaining two right eigenvectors are

$$|\psi_R^{\mathrm{III}}\rangle = \begin{pmatrix} 0 \\ -\frac{\sqrt{4\gamma\delta+4\omega^2-4\omega+1}-2\omega+1}{2\delta} \\ 1 \\ 0 \end{pmatrix}, \tag{A4}$$

$$|\psi_R^{\mathrm{IV}}\rangle = \begin{pmatrix} 0 \\ -\frac{-\sqrt{4\gamma\delta+4\omega^2-4\omega+1}-2\omega+1}{2\delta} \\ 1 \\ 0 \end{pmatrix}, \tag{A5}$$

with the corresponding eigenvalues $E^{\mathrm{III,IV}} = \frac{1}{2}\left(1 \mp \sqrt{4\gamma\delta+4\omega^2-4\omega+1}\right)$. The reality of the eigenvalues requires $4\gamma\delta + 4\omega^2 - 4\omega + 1 \geq 0$, a condition that is dependent on the choice of ω, unlike in the previously-studied case.

1. Exceptional points

Notice that the eigenvectors $|\psi_R^{\mathrm{I,II}}\rangle$ are insensitive to the choice of the parameters γ and δ, and therefore are not involved in coalescence at any point by tuning γ and δ. However, the eigenvectors $|\psi_R^{\mathrm{III,IV}}\rangle$ and the corresponding eigenvalues $E^{\mathrm{III,IV}}$ may coalesce for particular values of γ and δ. Such a coalescence occurs for points on the rectangular hyperbola $4\gamma\delta + 4\omega^2 - 4\omega + 1 = 0$, on the $\gamma\delta$-parameter space (fixing ω). In a sense, therefore, one has an exceptional curve (rather than isolated points) in the parameter space. The left eigenvectors $\langle\psi_L^{\mathrm{III,IV}}|$ are computed straightforwardly as

$$\langle\psi_L^{\mathrm{III}}| = \begin{pmatrix} 0 \\ -\frac{\sqrt{4\gamma\delta+4\omega^2-4\omega+1}-2\omega+1}{2\gamma} \\ 1 \\ 0 \end{pmatrix}^T, \tag{A6}$$

$$\langle\psi_L^{\mathrm{IV}}| = \begin{pmatrix} 0 \\ -\frac{-\sqrt{4\gamma\delta+4\omega^2-4\omega+1}-2\omega+1}{2\gamma} \\ 1 \\ 0 \end{pmatrix}^T. \tag{A7}$$

At exceptional points, i.e., for $4\gamma\delta + 4\omega^2 - 4\omega + 1 = 0$, one finds that $|\psi_R^{\mathrm{III}}\rangle = |\psi_R^{\mathrm{IV}}\rangle$ (and also $\langle\psi_L^{\mathrm{III}}| = \langle\psi_L^{\mathrm{IV}}|$), along with $E^{\mathrm{III}} = E^{\mathrm{IV}} = 1/2$. Thus, it is straightforward to verify that $\langle\psi_L^{\mathrm{III,IV}}|\psi_R^{\mathrm{III,IV}}\rangle = 0$, reproducing the self-orthogonality relation.

2. Ground-state entanglement

Let us now describe entanglement properties of the ground state in this model. Some intriguing features can be exposed, for which we pick $\omega = 1/2$. In this case, we must restrict ourselves to parameter choices leading to $\gamma\delta > 0$ to ensure that the spectrum is real. The eigenvalues corresponding to the four eigenstates are plotted in Fig. (8) and for $\gamma\delta = 1/4$, one observes a ground-state crossing. Thus, $|\psi^{\mathrm{I}}\rangle$ describes the ground state for $\gamma\delta < 1/4$, while $|\psi^{\mathrm{III}}\rangle$ describes the ground state

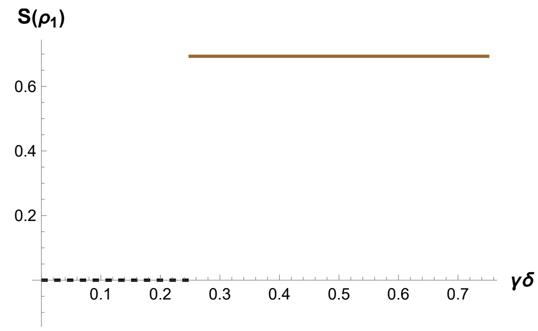

FIG. 9: Entanglement entropy of the ground state showing a discontinuous jump at $\gamma\delta = 1/4$, indicative of a phase transition.

for $\gamma\delta > 1/4$. The corresponding reduced density matrix for the fermion '1' can be obtained by the procedure described in the previous section. It reads

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \gamma\delta < \frac{1}{4}, \tag{A8}$$

and

$$\rho_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad \gamma\delta > \frac{1}{4}. \tag{A9}$$

The entanglement entropy, i.e., $S(\rho_1) = -\mathrm{Tr}_1[\rho_1 \ln \rho_1]$, is 0 when $\gamma\delta < \frac{1}{4}$ and jumps to $\ln 2$ when $\gamma\delta > \frac{1}{4}$. This indicates a quantum phase transition, being characterized by the discontinuous jump in the entanglement entropy at $\gamma\delta = 1/4$, as shown in Fig. (9). It should be remarked that although we have picked $\omega = 1/2$ to simplify our calculations, similar discontinuous jumps can be observed for other values of $\omega \in (0, 1)$. One could similarly define reduced density matrices by considering the Dirac-normalization scheme as in Sec. (V). In that case, one still observes the phase transition due to the ground-state crossing. We do not pursue this further as the calculations can be easily performed in the same spirit as those presented in Sec. (V).

[1] I. Rotter, *A non-hermitian Hamilton operator and the physics of open quantum systems*, J. Phys. A: Math. Theor. 42, 153001 (2009).

[2] Y. Michishita and R. Peters, *Equivalence of Effective Non-Hermitian Hamiltonians in the Context of Open Quantum Systems and Strongly Correlated Electron Systems*, Phys. Rev. Lett. 124, 196401 (2020).

[3] K. Holmes, W. Rehman, S. Malzard, and E.-M. Graefe, *Husimi Dynamics Generated by non-Hermitian Hamiltonians*, Phys. Rev. Lett. 130, 157202 (2023).

[4] E.-M. Graefe, M. Höning, and H. J. Korsch, *Classical limit of non-hermitian quantum dynamics – a generalized canonical structure*, J. Phys. A: Math. Theor. 43, 075306 (2010).

[5] Á. Gómez-León, T. Ramos, A. González-Tudela, and D. Porras, *Bridging the gap between topological non-Hermitian physics and open quantum systems*, Phys. Rev. A 106, L011501 (2022).

[6] X. Niu, J. Li, S. L. Wu, and X. X. Yi, *Effect of quantum jumps on non-Hermitian systems*, Phys. Rev. A 108, 032214 (2023).

[7] C. M. Bender and S. Boettcher, *Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry*, Phys. Rev. Lett. 80, 5243 (1998).

[8] C. M. Bender, S. Boettcher, and P. N. Meisinger, *$\mathcal{PT}$-symmetric quantum mechanics*, J. Math. Phys. 40, 2201 (1999).

[9] C. M. Bender and D. W. Hook, *$\mathcal{PT}$-symmetric quantum mechanics*, arXiv: 2312.17386 [quant-ph].

[10] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, *Non-Hermitian physics and PT symmetry*, Nat. Phys. 14, 11 (2018).

[11] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, *Optical Solitons in $\mathcal{PT}$ Periodic Potentials*, Phys. Rev. Lett. 100, 030402 (2008).

[12] L. Feng, R. El-Ganainy, and L. Ge, *Non-Hermitian photonics based on parity–time symmetry*, Nat. Photonics 11, 752 (2017).

[13] R. El-Ganainy, M. Khajavikhan, D. N. Christodoulides, and S. K. Ozdemir, *The dawn of non-Hermitian optics*, Commun. Phys. 2, 37 (2019).

[14] C. Wang, Z. Fu, W. Mao, J. Qie, A. D. Stone, and L. Yang, *Non-Hermitian optics and photonics: from classical to quantum*, Adv. Opt. Photonics 15, 442 (2023).

[15] W. D. Heiss, *The physics of exceptional points*, J. Phys. A: Math. Theor. 45, 444016 (2012).

[16] M. Znojil, *Exceptional points and domains of unitarity for a class of strongly non-Hermitian real-matrix Hamiltonians*, J. Math. Phys. 62, 052103 (2021).

[17] T. Kato, *Perturbation Theory for Linear Operators: Classics in Mathematics*, Springer (1995).

[18] F. Correa and M. L. Plyushchay, *Spectral singularities in $\mathcal{PT}-$symmetric periodic finite-gap systems*, Phys. Rev. D 86, 085028 (2012).

[19] K. Zelaya and O. Rosas-Ortiz, *Exact Solutions for Time-Dependent Non-Hermitian Oscillators: Classical and Quantum Pictures*, Quantum Rep. 3, 458 (2021).

[20] V. Fernández, R. Ramírez, and M. Reboiro, *Swanson Hamiltonian: non-PT-symmetry phase*, J. Phys. A: Math. Theor. 55, 015303 (2022).

[21] I. Mandal and E. J. Bergholtz, *Symmetry and Higher-Order Exceptional Points*, Phys. Rev. Lett. 127, 186601 (2021).

[22] M.-A. Miri and A. Alú, *Exceptional points in optics and photonics*, Science 363, eaar7709 (2019).

[23] Y. Zhiyenbayev, Y. Kominis, C. Valagiannopoulos, V. Kovanis, and A. Bountis, *Enhanced stability, bistability, and exceptional points in saturable active photonic couplers*, Phys. Rev. A 100, 043834 (2019).

[24] J. Wiersig, *Review of exceptional point-based sensors*, Photonics Res. 8, 1457 (2020).

[25] A. Li, H. Wei, M. Cotrufo, W. Chen, S. Mann, X. Ni, B. Xu, J. Chen, J. Wang, S. Fan, C.-W. Qiu, A. Alú, and L. Chen, *Exceptional points and non-hermitian photonics at the nanoscale*, Nat. Nanotechnol. 18, 706 (2023).

[26] C. Liang, Y. Tang, A.-N. Xu, and Y.-C. Liu, *Observation of Exceptional Points in Thermal Atomic Ensembles*, Phys. Rev. Lett. 130, 263601 (2023).

[27] T. Goldzak, A. A. Mailybaev, and N. Moiseyev, *Light Stops at Exceptional Points*, Phys. Rev. Lett. 120, 013901 (2018).

[28] M. S. Swanson, *Transition elements for a non-hermitian quadratic Hamiltonian*, J. Math. Phys. 45, 585 (2004).

[29] E.-M. Graefe, H. J. Korsch, A. Rush, and R. Schubert, *Classical and quantum dynamics in the (non-hermitian) Swanson oscillator*, J. Phys. A: Math. Theor. 48, 055301 (2015).

[30] B. Bagchi and I. Marquette, *New 1-step extension of the Swanson oscillator and superintegrability of its two-dimensional generalization*, Phys. Lett. A 379, 1584 (2015).

[31] A. Fring and M. H. Y. Moussa, *Non-Hermitian Swanson model with a time-dependent metric*, Phys. Rev. A 94, 042128 (2016).

[32] A. Mostafazadeh, *Pseudo-Hermiticity versus* PT *symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian*, J. Math. Phys. 43, 205 (2002).

[33] H. F. Jones, *On pseudo-Hermitian Hamiltonians and their Hermitian counterparts*, J. Phys. A: Math. Gen. 38, 1741 (2005).

[34] B. Bagchi, C. Quesne, and R. Roychoudhury, *Pseudo-Hermiticity and some consequences of a generalized quantum condition*, J. Phys. A: Math. Gen. 38, L647 (2005).

[35] B. Bagchi, R. Ghosh, and S. Sen, *Exceptional point in a coupled Swanson system*, EPL 137, 50004 (2022).

[36] D. C. Brody, *Biorthogonal quantum mechanics*, J. Phys. A: Math. Theor. 47, 035305 (2014).

[37] F. Ruzicka, K. S. Agarwal, and Y. N. Joglekar, *Conserved quantities, exceptional points, and antilinear symmetries in non-hermitian systems*, J. Phys.: Conf. Ser. 2038, 012021 (2021).

[38] L. Herviou, N. Regnault, and J. H. Bardarson, *Entanglement spectrum and symmetries in non-Hermitian fermionic non-interacting models*, SciPost Phys. 7, 069 (2019).

[39] A. P. Balachandran, T. R. Govindarajan, A. R. de Queiroz, and A. F. Reyes-Lega, *Entanglement and Particle Identity: A Unifying Approach*, Phys. Rev. Lett. 110, 080503 (2013).

[40] A. P. Balachandran, T. R. Govindarajan, A. R. de Queiroz, and A. F. Reyes-Lega, *Algebraic approach to entanglement and entropy*, Phys. Rev. A 88, 022301 (2013).

[41] A. F. Reyes-Lega, *Some aspects of operator algebras in quantum physics*, Geometric, Algebraic, and Topological Methods for Quantum Field Theory: Proceedings of the 2013 Villa de Leyva Summer School, Villa de Leyva, Colombia, 15–27 July 2013, pp. 1-74 (2017).

[42] G. C. Wick, A. S. Wightman, and E. P. Wigner, *The Intrinsic Parity of Elementary Particles*, Phys. Rev. 88, 101 (1952).

[43] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, 2nd. ed., Springer (1996).

[44] F. Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*, 2nd. ed., World Scientific (2008).

[45] S. Sachdev, *Quantum Phase Transitions*, 2nd. ed., Cambridge University Press (2011).