# Unitary Owen points in cooperative lot-sizing models with backlogging

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#### Abstract

Cooperative lot-sizing models with backlogging and heterogeneous costs are studied in Guardiola et al. (2020). In this model several firms participate in a consortium aiming at satisfying their demand over the planing horizon with minimal operation cost. Each firm uses the best ordering channel and holding technology provided by the participants in the consortium. The authors show that there are always fair allocations of the overall operation cost among the firms so that no group of agents profit from leaving the consortium. This paper revisits those cooperative lot-sizing models and presents a new family of cost allocations, the unitary Owen points. This family is an extension of the Owen set which enjoys very good properties in production-inventory proble, introduced by Guardiola et al. (2008). Necessary and sufficient conditions are provided for the unitary Owen points to be fair allocations. In addition, we provide empirical evidence, throughout simulation, showing that the above condition is fulfilled in most cases. Additionally, a relationship between lotsizing games and a certain family of production-inventory games, through Owen's points of the latter, is described. This interesting relationship enables to easily construct a variety of fair allocations for cooperative lot-sizing models.

Key words: lot-sizing models, cooperative game theory, fair division of costs

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## 1 Introduction

Lot-sizing problems with backlogging have been widely studied, during the last 40 years, proving different reformulations that have made it possible to find more efficiently solutions to the problem. Pochet and Wolsey (1988) examine mixed integer programming reformulations of the uncapacitated lot-sizing problem with backlogging which in an extended space of variables give strong reformulations using linear programming.

More recently, a series of papers (Van Den Heuvel et al., 2007; Guardiola et al. 2008, 2009; Xu and Yang, 2009; Dreschel, 2010; Gopaladesikan and Uhan, 2011; Zeng et al. 2011; Tsao et al. 2013; Chen and Zhang, 2016) have addressed the lot sizing problem under the perspective of cooperation considering cost sharing aspects of these models. Specifically, Van Den Heuvel et al. (2007) focus on the cooperation in economic lot-sizing situations with homogeneous costs but without backlogging (henceforth ELS-games). Subsequently, Guardiola et al. (2008, 2009) present the class of production-inventory games (henceforth, PI-games). Those papers provide a cooperative approach to analyze the production and storage of indivisible items being their characteristic function given as the optimal objective function of a linear optimization problem. PI-games may be considered ELS games without setup costs but with backlogging and heterogeneous costs. Guardiola et al. (2009) prove that the Owen set, the set of allocations which are achievable through dual solutions reduces to a singleton in the class of PI-games and these authors coined the term *Owen point* to refer to that solution. The authors prove that the Owen point is always a consistent core allocation. More recently, Chen and Zhang (2016) consider the ELS-game with general concave ordering cost and they show that a core allocation is computed in polynomial time under the assumption that all retailers have the same cost parameters (again homogeneous costs). Their approach is based on linear programming (LP) duality. Specifically, they prove that there is an optimal dual solution that defines an allocation in the core and point out that it is not necessarily true that every core allocation can be obtained by means of dual solutions.

Finally, setup-inventory games (henceforth, SI-games) are introduced in Guardiola et. al. (2020) as a new class of combinatorial optimization games that arises from cooperation in lotsizing problems with backlogging and heterogeneous costs. Each firm faces demand for a single product in each period and coalitions can pool orders. Firms cooperate by using the best ordering channel and holding technology provided by the participants in the consortium, e.g. they produce, hold inventory, pay backlogged demand and make orders at the minimum cost among the members of the coalition. Thus, firms aim at satisfying their demand over the planing horizon with minimal operation cost. The authors show that there are always fair allocations of the overall operation cost among the firms so that no group of agents profit from leaving the consortium. That paper proposes a parametric family of cost allocations and provides sufficient conditions for this to be a stable family against coalitional defections of firms.

The Owen point works very well as long as one has a strong formulation for the underlying optimization problem, such as PI-problems, because the dual variables (shadow prices) are used to construct the core allocations. However, this does not work for SI-problems, because in the original space of variables the corresponding optimization problem has integer variables and strong duality does not apply. In this paper we extend further the idea of dual prices and we construct an *ad hoc* kind of prices as the sum of the production, inventory and backlogging costs plus a proportion of the fixed order cost which depends on the total demand satisfied in that period. They are called unitary prices. These unit prices enable to replicate the construction of the Owen point by multiplying such unit prices by the demands and adding in all the periods. These allocations "a la Owen" are called unitary Owen points. Unfortunately one cannot always guarantee that unitary Owen points are core allocations. Nevertheless, we provide necessary and sufficient conditions for this situation to hold, i.e. for unitary Owen points to be core allocations and also show by simulation empirical evidence that this condition is satisfied in most cases. Furthermore, we consider whether it is possible to relate general SI-situations to simpler situations where the core is well-known and characterized as in PI-games. In this regard, we prove that the answer to this question is yes: one can use the Owen point of the surplus game a PI-game which measures the excess in costs that occurs with respect to the minimum unit price.

The contribution of this paper to the literature of lot-sizing games is twofold: firstly, a new family of cost allocations in cooperative lot-sizing models with backlogging and heterogeneous costs is presented: The unitary Owen points. It is an extension of the Owen set which enjoys very-good properties in production-inventory problems. Necessary and sufficient conditions are provided for unitary Owen points to be fair allocations. Furthermore, we empirically show that these conditions are satisfied for almost any SI-situation, resulting in an explicit quasi-solution for this class of games. Secondly, a relationship between lot-sizing games and a certain family of production-inventory games, through Owen's points of the latter, is described. This interesting relationship enables us to analyze cooperative lot-sizing models using properties of the much simpler and well-known class of PI-games.

The rest of this paper is organized as follows: the next section formulates SI-problems and shows that SI-games are totally balanced resorting to a result of Pochet and Wolsey (1988). Section three describes the unitary Owen points, provides a necessary and sufficient condition for those points to be core allocations and gives empirical evidence to consider the unitary Owen point as a quasi-solution for SI-games. Section four presents a relationship between SI-games and a certain family of PI-games through the Owen's points of the latter. This interesting relationship simplifies the analysis and construction of core allocations for SI-games. Finally, the fifth section presents a research summary and some conclusions.

# 2 SI-games: reformulation and balancedness

We begin by formulating the setup-inventory problems with backlogging (SI-problems).

Consider T periods, numbered from 1 to T, where the demand for a single product occurs

in each of them. This demand is satisfied by own production, and can be done during the production periods, in previous periods (inventory) or later periods (backlogging). In each production period a fixed cost must be paid. Therefore, the model includes production, inventory holding, backlogging and setup costs. The aim is to find an optimal ordering plan, that is a feasible ordering plan that minimizes the sum of setup, production, inventory holding and backlogging cost. Although the model assumes that companies produce their demand, we can interchangeably consider the case where demand is satisfied either by producing or purchasing. One has simply to interpret that the purchasing costs can be ordering costs (set up costs) and unit purchasing costs (variable costs). The goal is to establish an operational plan in order to satisfy demand at minimum total cost. Formally, for each period  $t = 1, \ldots, T$  we define  $d_t$  as the integer demand to be satisfied, and  $k_t$ ,  $h_t$ ,  $b_t$ ,  $p_t$ , respectively, as the setup, inventory carrying, backlogging and unit production costs.

The decision variables of the model for each period t are the order size  $q_t$ , the inventory at the end of the period  $I_t$  and the backlogged demand  $E_t$ . In addition, the order size must be integer. In the following, we present a first mathematical programming formulation for the setupinventory problem (SI-problem). Let  $M = \sum_{t=1}^{T} d_t$  and  $z_t$  be a decision variable that assumes the value one if an order is placed at the beginning of period t and denote by C(d, k, h, b, p) the minimum overall operation cost during the planning horizon, then

$$C(d, k, h, b, p) := \min \sum_{t=1}^{T} (p_t q_t + h_t I_t + b_t E_t + k_t z_t)$$
  
s.t.  $I_0 = I_T = E_0 = E_T = 0,$   
 $I_t - E_t = I_{t-1} - E_{t-1} + q_t - d_t, \quad t = 1, \dots, T$   
 $q_t \le M \cdot z_t, \quad t = 1, \dots, T,$   
 $q_t, I_t, E_t, \text{ non-negative, integer}, \ t = 1, \dots, T,$   
 $z_t \in \{0, 1\}$ 

The above objective function minimizes the sum of all the considered costs over the planning horizon. The first constraint imposes that the model must start and finish with empty inventory. The second group of constraints are flow conservation constraints ensuring the right transition of inventory and backlogged demand among periods. The third group of constraints model that setup cost is only charged whenever an order is placed.

In order to simplify the notation we define Z as a matrix in which all costs are included, that is, Z := (K, H, B, P) being K, H, B and P the matrices containing the setup, inventory carrying, backlogging and unit production costs for all periods  $t = 1, \ldots, T$ . A cost TU-game is a pair (N, c), where N is the finite player set,  $\mathcal{P}(N)$  is the power set of N (i.e. the set of coalitions in N), and  $c : \mathcal{P}(N) \to \mathbb{R}$  the characteristic function satisfying  $c(\emptyset) = 0$ . A cost allocation will be  $x \in \mathbb{R}^n$  and, for every coalition  $S \subseteq N$  we denote by x(S) the aggregate cost-sharing for coalition S, i.e.,  $x(S) := \sum_{i \in S} x_i$  with  $x(\emptyset) = 0$ . For each SI-situation represented by its cost matrices (N, D, Z), we associate a cost TU-game (N, c) where, for any nonempty coalition  $S \subseteq N, c(S) := C(d^S, k^S, h^S, b^S, p^S)$  with  $d^S = \sum_{i \in S} d^i$ , where  $d^i = (d_1^i, ..., d_T^i)$ , and the rest of the costs will be the minimum value among all the costs of the players in the coalition S at each one of the periods, serve as an example  $p^S = [p_1^S, \ldots, p_T^S]'$  where  $p_t^S = \min_{i \in S} \{p_t^i\}$  for  $t = 1, \ldots, T$ . Every cost TU-game defined in this way is what we call a setup-inventory game (SI-game). The reader may notice that every PI-game (as introduced by Guardiola et al., 2009) is a SI-game with  $k_t = 0$ , for all  $t \in T$ . Moreover, as mentioned above, although the model assumes that companies produce their demand, we can interchangeably consider the case where demand is satisfied either by producing or purchasing. One has simply to interpret that the purchasing costs can be ordering costs (set up costs) and unit purchasing costs (variable costs).

Recall that the core of a game (N, c) consists of those cost allocations which divide the cost of the grand coalition, c(N), in such a way that any other coalition pays at most its cost by the characteristic function. Formally,

$$Core(N,c) = \{x \in \mathbb{R}^n / x(N) = c(N) \text{ and } x(S) \le c(S) \text{ for all } S \subset N\}.$$

In the following, fair allocations of the total cost will be called core allocations. Bondareva (1963) and Shapley (1967) independently provide a general characterization of games with a non-empty core by means of balancedness. They prove that (N, c) has a nonempty core if and only if it is balanced. In addition, it is a totally balanced game<sup>1</sup> if the core of every subgame is nonempty.

Our goal is to show that SI-games are totally balanced resorting. To do so we use an easy proof which is based on duality resorting to a result by Pochet and Wolsey (1988). Observe that the characteristic function c(S) of these games can be written as the optimal value of the following LSI(S) problem:

$$c(S) = \min \sum_{1 \le t \le T} \sum_{1 \le \tau \le T} d_{\tau}^{S} p_{t\tau}^{S} \lambda_{t\tau} + \sum_{1 \le t \le T} k_{t}^{S} z_{t} \qquad (LSI(S))$$
  
s.t.  $d_{\tau}^{S} \sum_{1 \le t \le T} \lambda_{t\tau} = d_{\tau}^{S}, \quad \forall \tau = 1, \dots, T,$   
 $\lambda_{t\tau} \le z_{t}, \quad \forall t, \tau = 1, \dots, T,$   
 $\lambda_{t\tau}, z_{t} \in \{0, 1\}.$ 

The variables  $\lambda_{t\tau}$  are equal to 1 if and only if demand in period  $\tau$  is produced in period t and zero otherwise. Likewise, the variables  $z_t$  are equal to 1 if and only if there is some production

<sup>&</sup>lt;sup>1</sup>Totally balanced games were introduced by Shapley and Shubik in the study of market games (see Shapley and Shubik, 1969).

at period t. The cost of covering the demand in period  $\tau$  if the production is done in period t is given by

$$p_{t\tau}^{S} = \begin{cases} p_{t}^{S} & \text{if } t = \tau, \\ p_{t}^{S} + \sum_{i=t}^{\tau-1} h_{i}^{S} & \text{if } t < \tau, \\ p_{t}^{S} + \sum_{i=\tau}^{t-1} b_{i}^{S} & \text{if } t > \tau. \end{cases}$$
(1)

This is the facility location reformulation by Pochet and Wolsey (1988) of the SI problem. This formulation has a strong dual if the underlying graph of the location problem is a tree (Tamir 1992). In this case, the graph is a line and thus the mentioned result applies. Let  $y_{\tau}$  be the dual variable associated with the first constraints and  $\beta_{t\tau}$  those associated with the second family of constraints, then the dual is:

$$c(S) = \max \sum_{1 \le \tau \le T} d_{\tau}^{S} y_{\tau}$$
(DSI(S))  
s.t. 
$$\sum_{1 \le \tau \le T} d_{\tau}^{S} \beta_{t\tau} \le k_{t}^{S}, \forall t,$$
$$y_{\tau} - \beta_{t\tau} \le p_{t\tau}^{S}, \quad \forall t, \tau,$$
$$\beta_{t,\tau} \ge 0, \ y_{t} \text{ free.}$$

Pochet and Wolsey (1988) proved that the linear relaxation of a SI-problem, LSI(S), has an integral optimal solution. Hence, the optimal value of its dual problem matches that of the primal one, that is,  $v(DSI(S)) = C(d^S, k^S, h^S, b^S, p^S) = c(S)$  for all  $S \subseteq N$ .

#### **Theorem 2.1** Every SI-game is totally balanced.

**Proof.** Take a SI-situation (N, D, Z) and the associated SI-game (N, c). Consider  $(y^*, \beta^*)$  an optimal solution to dual DSI(N) where  $y^* = (y_1^*, ..., y_T^*)$  and  $\beta^* = (\beta_{11}^*, ..., \beta_{TT}^*)$ . It is known from optimality that

$$\sum_{t=1}^{T} y_t^* d_t^N = v(DSI(N)) = c(N)$$

Note that the solution  $(y^*, \beta^*)$  is also feasible for any dual problem with  $S \subseteq N$  since  $p^N \leq p^S$ ,  $h^N \leq h^S$ ,  $b_t^N \leq b_t^S$  and  $k_t^N \leq k_t^S$ . Therefore,

$$\sum_{t=1}^{T} y_t^* d_t^S \le v(DSI(S)) = c(S)$$

Thus, the allocation  $(\sum_{t=1}^{T} y_t^* d_t^i)_{i \in N} \in Core(N, c).$ 

Note that every subgame of a SI-game is also a SI-game. Hence, we can also conclude that every SI-game is totally balanced. ■

### 3 Unitary Owen points

In this section we introduce a new family of cost allocations on the class of SI-games. This family is inspired by the flavour of the Owen point and its relationship with the shadow prices of the dual problems associated with SI-problems. To define those cost allocations, it is necessary to describe the set of optimal plans and the unitary prices.

Consider a SI-situation (N, D, Z). A feasible ordering plan for such a situation is defined by  $\sigma \in \mathbb{R}^T$  where  $\sigma_t \in T \cup \{0\}$  denotes the period where demand of period t is ordered. We assume the convention that  $\sigma_t = 0$  if and only if  $d_t = 0$ . It means that there is no order placed to satisfy demand at period t since demand at this period is null. Moreover,  $P^S(\sigma) \in \mathbb{R}^T$  is defined as the operation cost vector associated to the ordering plan  $\sigma$  (henceforth: cost-plan vector) for any coalition  $S \subseteq N$ , where

$$P_t^S(\sigma) = \begin{cases} 0 & \text{if } \sigma_t = 0, \\ p_{\sigma_t t}^S & \text{if } \sigma_t \in \{1, ..., T\}. \end{cases}$$

Given an optimal ordering plan,  $\sigma^S$ , for the SI-problem  $C(d^S, k^S, h^S, b^S, p^S)$ , the characteristic function is rewritten as follows: for any non-empty coalition  $S \subseteq N$ ,

$$c(S) = P^S(\sigma^S)'d^S + \delta(\sigma^S)'k^S = \sum_{t=1}^T \left( P_t^S(\sigma^S)d_t^S + \delta_t(\sigma^S)k_t^S \right),$$

where,  $\delta(\sigma^S) = (\delta_t(\sigma^S))_{t \in T}$  and

$$\delta_t(\sigma^S) = \begin{cases} 1 & \text{if } \exists r \in T/\sigma_r^S = t \\ 0 & \text{otherwise.} \end{cases}$$

The set of optimal plans is denoted by  $\Lambda(N, D, Z) := \left\{ \left( \sigma^S \right)_{S \in \mathcal{P}(N)} \right\}$  where  $\sigma^S$  is an optimal ordering plan associated to LSI(S). Note that the set of optimal plans may be large since often there are multiple optimal solutions for the program LSI(S). Core allocations built from optimal dual variables are known to exhibit some questionable properties as pointed out for instance by Perea et al. 2012 or Fernández et. al. 2002. For this reason, whenever the core is larger than the set of allocations coming from dual variables, it is interesting to provide some alternative core allocations. In the following we derive alternatives which under mild conditions are stable, i.e. core allocations for these situations.

We define the *unitary prices* as the sum of the production, inventory and backlogging costs plus a proportion of the fixed order cost which depends on the total demand satisfied in each period.

**Definition 3.1** Let (N, D, Z) be a SI-situation and  $(\sigma^S)_{S \in \mathcal{P}(N)} \in \Lambda(N, D, Z)$ . For each period  $t \in T$  and each coalition  $S \subseteq N$ , the unitary price is defined as follows:

$$y_t\left(\sigma^S, d^S, z^S\right) := \begin{cases} 0 & \text{if } \sigma_t^S = 0, \\ P_t^S(\sigma^S) + \frac{k_{\sigma_t^S}^S}{\sum_{m \in Q^S(\sigma_t^S)} d_m^S} & \text{if } \sigma_t^S \neq 0, \end{cases}$$

where  $Q^{S}(t) := \{k \in T : \sigma_{k}^{S} = t\}$  and  $z^{S}$  represents the cost matrix  $(k^{S}, h^{S}, b^{S}, p^{S})$ .

The reader should observe that  $Q^{S}(t)$  is the set of periods that satisfy the demand in period t, for the optimal plan  $\sigma^{S}$ . Note that for any coalition  $S \subseteq N$ ,  $\sum_{t=1}^{T} y_t \left(\sigma^{S}, d^{S}, z^{S}\right) \cdot d_t^{S} = c(S)$ .

The next proposition shows that we may construct core allocations from the unitary prices of the grand coalition as long as they are the cheapest in each period with positive demand. We shall call them *unitary Owen points*.

**Definition 3.2** Let (N, D, Z) be a SI-situation and  $(\sigma^S)_{S \in \mathcal{P}(N)} \in \Lambda(N, D, Z)$ . The unitary Owen point is given by

$$\theta\left(\sigma^{N}, d^{N}, z^{N}\right) := \left(\sum_{t=1}^{T} y_{t}\left(\sigma^{N}, d^{N}, z^{N}\right) \cdot d_{t}^{i}\right)_{i \in N}$$

Note that every optimal plan generates a unit price for each period of time and hence, a unitary Owen point.

Observe that from  $y(\sigma^N, d^N, z^N)$  we can build a solution  $(y(\sigma^N, d^N, z^N), \beta(\sigma^N))$  with  $\beta_{\sigma^N(\tau),\tau} = 0$  if  $\sigma^N(\tau) = 0$  and  $\beta_{\sigma^N(\tau),\tau} = \frac{k_{\sigma^N(\tau)}^N}{\sum_{m \in Q^N(\sigma^N(\tau))} d_m^N}$  if  $\sigma^N(\tau) \neq 0$  satisfying  $c(N) = \sum_{\tau=1}^T d_\tau^N y_t(\sigma^N, d^N, z^N)$ . However, it may not be a feasible solution of the dual for the grand coalition whenever  $P_\tau^N(\sigma^N(\tau)) > p_{t\tau}^N$  for some t. Still, the unitary Owen point associated with this dual solution can be a core allocation.

The following example elaborates on a SI-situation with 3 players and 2 periods. The unitary Owen point for the corresponding SI-game is a core allocation but this allocation does not come from optimal dual prices.

**Example 3.3** Consider the following SI-situation with two periods and three players and the

associated SI-game:

	$d_1^S$	$d_2^S$	$p_1^S$	$p_2^S$	$h_1^S$	$b_1^S$	$k_1^S$	$k_2^S$	С
{1}	2	1	9	9	6	4	6	8	39
$\{2\}$	8	2	9	6	9	7	7	9	100
{3}	6	1	5	6	3	5	6	10	44
$\{1, 2\}$	10	3	9	6	6	4	6	8	122
$\{1, 3\}$	8	2	5	6	3	4	6	8	62
$\{2,3\}$	14	3	5	6	3	5	6	9	100
$\{1, 2, 3\}$	16	4	5	6	3	4	6	8	118

The optimal plan for coalition N is  $\sigma^N = (1,1)$  with  $p^N(\sigma^N) = (5,8)$  and  $y(\sigma^N, d^N, z^N) = (5+\frac{3}{10}, 8+\frac{3}{10})$ . The unitary Owen point  $\theta(\sigma^N, d^N, z^N) = (18+\frac{9}{10}, 59, 40+\frac{1}{10}) \in Core(N,c)$ . Note that  $(y(\sigma^N, d^N, z^N), \beta(\sigma^N))$  with  $\beta_{21}(\sigma^N) = \frac{3}{10}, \beta_{22}(\sigma^N) = \frac{3}{10}$  and  $\beta_{t\tau}(\sigma^N) = 0$  for all the remaining t and  $\tau$ , is not feasible for the dual problem DSI(N). Indeed, it violates the constraint  $y_2(\sigma^N, d^N, z^N) - \beta_{22}(\sigma^N) \leq p_{22}^N$ , since this is equivalent to  $p_2^N(\sigma^N) \leq p_{22}^N$  but  $p_2^N(\sigma^N) = 8$  and  $p_{22}^N = 6$ .

Therefore, it is clear that the unitary Owen point can provide core allocation which do not come from optimal dual prices, although it is not clear under which conditions this unitary price fulfills this property. The following result provides an easy sufficient condition for this to happen.

**Proposition 3.4** Let (N, D, Z) be a SI-situation,  $(\sigma^S)_{S \in \mathcal{P}(N)} \in \Lambda(N, D, Z)$ , and (N, c) the associated SI-game. If  $y_t(\sigma^N, d^N, z^N) \leq y_t(\sigma^S, d^S, z^S)$  for all  $t \in T$  and for all  $S \subset N$  with  $d_t^S \neq 0$ , then

$$\theta\left(\sigma^{N}, d^{N}, z^{N}\right) \in Core(N, c).$$

**Proof.** It is straightforward from the definition of the unitary Owen point.

It would be reasonable that the larger a coalition the lower its unit prices, since its members operate with the best technology available in the group. Unfortunately, this condition is not always satisfied as Example 3.7 shows. Therefore, we are interested in finding stronger conditions than the one given in Proposition 3.4. In the following we address this question.

In order to simplify the notation, for each  $t \in T$ , we define:

• Cost difference per demand unit between coalition S and R in a period t:

$$a_t^{SR} := P_t^S(\sigma^S) - P_t^R(\sigma^R)$$

Note that  $a_t^{SR} + a_t^{RS} = 0$ .

• Aggregate demand of coalition  $S \subseteq N$  in all those periods that satisfy its demand in period t:

$$\alpha_t(S) := \sum_{m \in Q^N(t)} d_m^S.$$

• Aggregate order cost of coalition  $S \subseteq N$ :

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$$k(S) := \sum_{t \in T^S} k_t^S,$$

where  $T^S := \{t \in T \ \delta_t(\sigma^S) = 1\}$  is the set of ordering periods.

The next theorem provides necessary and sufficient conditions for the unitary Owen point to be a core allocation. These conditions state an upper bound for the average cost savings per unit demand in the grand coalition, for those periods where an order is placed. Such an upper bound is related to the savings in fixed order costs.

**Theorem 3.5** Let (N, D, Z) be a SI-situation,  $(\sigma^S)_{S \in \mathcal{P}(N)} \in \Lambda(N, D, Z)$ , and (N, c) the associated SI-game.  $\theta(\sigma^N, d^N, z^N) \in Core(N, c)$  if and only if there are real weights  $\beta_t^S$ , for any  $S \subsetneq N$  and any  $t \in T^N$  with  $\alpha_t(S) > 0$ , satisfying that

$$\sum_{\substack{\in Q^N(t)}} \frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} \le \beta_t^S \cdot \frac{k(S)}{\alpha_t(S)} - \frac{k_t^N}{\alpha_t(N)}$$

with  $\sum_{t \in T^N: \alpha_t(S) > 0} \beta_t^S \leq 1$ .

**Proof.** (if) Take  $(\sigma^S)_{S \in \mathcal{P}(N)} \in \Lambda(N, D, Z)$  and consider a coalition  $S \subsetneq N$ . We must prove that  $\theta(\sigma^N, d^N, z^N) \in Core(N, c)$ , e.g.  $\sum_{i \in S} \theta_i (\sigma^N, d^N, z^N) - c(S) \leq 0$ . Indeed,

$$\sum_{i \in S} \theta_i \left( \sigma^N, d^N, z^N \right) - c(S)$$

$$= \sum_{t=1}^T y_t \left( \sigma^N, d^N, z^N \right) \cdot d_t^S - \sum_{t=1}^T y_t \left( \sigma^S, d^S, z^S \right) \cdot d_t^S$$

$$= \sum_{t=1}^T \left( a_t^{NS} \cdot d_t^S + \frac{k_{\sigma_t^N}^N \cdot d_j^S}{\sum_{m \in Q^N(\sigma_t^N)} d_m^N} - \frac{k_{\sigma_t^S}^S \cdot d_j^S}{\sum_{m \in Q^S(\sigma_t^S)} d_m^S} \right)$$

$$= \sum_{t \in T^N} \left( \sum_{j \in Q^N(t)} \left( a_j^{NS} \cdot d_j^S + \frac{k_t^N \cdot d_j^S}{\alpha_t(N)} \right) \right) - k(S)$$

$$= \sum_{t \in T^N: \alpha_t(S) > 0} \left( \alpha_t(S) \cdot \sum_{j \in Q^N(t)} \left( \frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} \right) + \frac{k_t^N \cdot \alpha_t(S)}{\alpha_t(N)} \right) - k(S)$$
  
$$\leq \sum_{t \in T^N: \alpha_t(S) > 0} \left( \beta_t^S \cdot \frac{\alpha_t(S) \cdot k(S)}{\alpha_t(S)} - \frac{\alpha_t(S) \cdot k_t^N}{\alpha_t(N)} + \frac{k_t^N \cdot \alpha_t(S)}{\alpha_t(N)} \right) - k(S)$$
  
$$= k(S) \cdot \sum_{t \in T^N: \alpha_t(S) > 0} \beta_t^S - k(S) \le 0$$

(only if) Consider now that  $\theta(\sigma^N, d^N, z^N) \in Core(N, c)$ . Then, for all  $S \subset N$ ,  $\sum_{i \in S} \theta_i(\sigma^N, d^N, z^N) - c(S) \leq 0$  which is equivalent to

$$\sum_{t \in T^N} \left( \sum_{j \in Q^N(t)} \left( a_j^{NS} \cdot d_j^S + \frac{k_t^N \cdot d_j^S}{\alpha_t(N)} \right) \right) \le k(S).$$

For all  $t \in T^N$  and every coalition  $S \subsetneq N$  with  $\alpha_t(S) > 0$  there are always real weights  $\beta_t^S$  with  $\sum_{t \in T^N} \beta_t^S \leq 1$ , satisfying

$$\begin{split} \sum_{j \in Q^N(t)} \left( a_j^{NS} \cdot d_j^S + \frac{k_t^N \cdot d_j^S}{\alpha_t(N)} \right) &\leq \beta_t^S \cdot k(S), \\ \sum_{j \in Q^N(t)} \left( \frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} \right) + \frac{k_t^N \cdot \alpha_t(S)}{\alpha_t(N) \cdot \alpha_t(S)} &\leq \frac{\beta_t^S \cdot k(S)}{\alpha_t(S)}, \\ \sum_{j \in Q^N(t)} \frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} &\leq \beta_t^S \cdot \frac{k(S)}{\alpha_t(S)} - \frac{k_t^N}{\alpha_t(N)}. \end{split}$$

At first glance, the reader might think that the conditions of the previous theorem are too restrictive, i.e. they are only satisfied by a small family of SI-situations. However, an empirical analysis simulating SI-situations shows that most of the instances satisfy those conditions. Indeed, we start by randomly generating (using the uniform probability distribution) a first set of 100,000 instances of SI-situations so that for every player and for each period the data range in  $d_t^i \in [0, 30], p_t^i, h_t^i, b_t^i \in [0, 10]$  and  $k_t^i \in [0, 50]$ . The percentage of SI-situations for which the Unitary Owen point belongs to the core of the corresponding SI-game is shown in Table 1.

It can be seen that the larger the number of players and periods the higher the percentage that some unitary Owen point belongs to the core. In case that we impose that the demand and the costs are greater than zero:  $d_t^i \in [1, 30]$ ,  $p_t^i, h_t^i, b_t^i \in [1, 10]$  and  $k_t^i \in [1, 50]$ , the results even improve significantly as table 2 shows.

Players	T=2	T = 3	T = 4	T = 5
2	99.934%	99.979%	99.993%	100%
3	99.942%	99.983%	99.989%	99.995%
4	99.950%	99.991%	99.996%	99.999%
5	99.974%	99.982%	99.992%	99.998%
6	99.974%	99.993%	99.998%	99.999%
7	99.985%	99.996%	99.999%	100%

Table 1: Percentage of instances fulfilling the condition of Theorem 3.5 for the first set of instances

Players	T=2	T = 3	T = 4	T = 5
2	99.984%	99.996%	99.999%	100%
3	99.997%	99.995%	99.999%	99.999%
4	99.998%	99.996%	99.999%	99.998%
5	100%	99,999%	100%	100%
6	100%	100%	100%	100%

Table 2: Percentage of instances fulfilling the condition of Theorem 3.5 for instances with positive costs

In the previous simulation, the range of variation for the costs have been chosen so that those costs are actually relevant to determine the optimal plans for each coalition. In addition, if the the set up costs are large compare to the other costs, as for instance for  $d_t^i \in [0, 10]$ ,  $p_t^i, h_t^i, b_t^i \in [0, 10]$  and  $k_t^i \in [50, 500]$  the percentage of instances where the unitary Owen point is a core allocation is close to 99, 995%, even for the case of two players and two periods. Moreover, if the demand is larger as it happens in the following situation  $d_t^i \in [10, 50], p_t^i, h_t^i, b_t^i \in [0, 10]$ and  $k_t^i \in [0, 50]$ , percentages of "success" also increase close to 1 (99, 999%).

The next example illustrates Proposition 3.4 and Theorem 3.5. It shows how unitary Owen points are calculated by using unitary prices.

Example 3.6 Consider the following SI-situation with three periods and three players an the

associated SI-game:

	$d_1^S$	$d_2^S$	$d_3^S$	$p_1^S$	$p_2^S$	$p_3^S$	$h_1^S$	$h_2^S$	$b_1^S$	$b_2^S$	$k_1^S$	$k_2^S$	$k_3^s$	c
{1}	1	3	1	1	1	1	1	1	1	1	3	1	5	8
{2}	2	1	1	2	3	4	1	1	1	1	1	4	8	12
{3}	2	1	3	2	3	5	1	1	1	1	1	1	7	20
$\{1, 2\}$	3	4	2	1	1	1	1	1	1	1	1	1	5	13
$\{1,3\}$	3	4	4	1	1	1	1	1	1	1	1	1	5	17
$\{2,3\}$	4	2	4	2	3	4	1	1	1	1	1	1	7	31
$\{1, 2, 3\}$	5	5	5	1	1	1	1	1	1	1	1	1	5	22

An optimal plan is the following:

	$\sigma_1^S$	$\sigma_2^S$	$\sigma_3^S$	$P_1^S(\sigma^S)$	$P_2^S(\sigma^S)$	$P_3^S(\sigma^S)$	k(S)
{1}	2	2	2	2	1	2	1
{2}	1	1	1	2	3	4	1
{3}	1	1	1	2	3	4	1
$\{1, 2\}$	1	2	2	1	1	2	2
$\{1,3\}$	1	2	2	1	1	2	2
$\{2,3\}$	1	1	1	2	3	4	1
$\{1, 2, 3\}$	1	2	2	1	1	2	2

Thus, the unitary prices for the optimal plan above are:

	$y_1\left(\sigma^S, d^S, z^S\right)$	$y_2\left(\sigma^S, d^S, z^S\right)$	$y_3\left(\sigma^S, d^S, z^S\right)$
{1}	$2 + \frac{1}{5}$	$1 + \frac{1}{5}$	$2 + \frac{1}{5}$
$\{2\}$	$2 + \frac{1}{4}$	$3 + \frac{1}{4}$	$4 + \frac{1}{4}$
{3}	$2 + \frac{1}{6}$	$3 + \frac{1}{6}$	$4 + \frac{1}{6}$
$\{1, 2\}$	$1 + \frac{1}{3}$	$1 + \frac{1}{6}$	$2 + \frac{1}{6}$
$\{1, 3\}$	$1 + \frac{1}{3}$	$1 + \frac{1}{8}$	$2 + \frac{1}{8}$
$\{2, 3\}$	$2 + \frac{1}{10}$	$3 + \frac{1}{10}$	$4 + \frac{1}{10}$
$\{1, 2, 3\}$	$1 + \frac{1}{5}$	$1 + \frac{1}{10}$	$2 + \frac{1}{10}$

One can observe that  $y_t(\sigma^N, d^N, z^N) \leq y_t(\sigma^S, d^S, z^S)$  for all  $t \in T$  and so, by Proposition 3.4,  $\theta(\sigma^N, d^N, z^N) = (6.6, 5.6, 9.8) \in Core(N, c)$ .

On the other hand, the ordering plan for the grand coalition  $\tilde{\sigma}^N = (1, 2, 3)$  belongs to an optimal plan and the associated unit prices are  $y_1(\tilde{\sigma}^N, d^N, z^N) = 1 + \frac{1}{5}, y_2(\tilde{\sigma}^N, d^N, z^N) = 1 + \frac{1}{5}$ 

and  $y_3(\tilde{\sigma}^N, d^N, z^N) = 1 + \frac{5}{5}$ . Note that for this plan  $T^N = \{1, 2, 3\}$ . Theorem 3.5 is here applied for the weights given in the next table:

	$\beta_1^{\scriptscriptstyle S} \geq$	$\beta_2^{\scriptscriptstyle S} \geq$	$\beta_3^{\scriptscriptstyle S} \geq$	$\sum_{t \in T^N} \beta_t^S$
{1}	$\frac{-4}{5}$	$\frac{3}{5}$	0	$\frac{-1}{5}$
$\{2\}$	$\frac{-8}{5}$	$\frac{-9}{5}$	-2	$\frac{-27}{5}$
{3}	$\frac{-8}{5}$	$\frac{-9}{5}$	-6	$\frac{-47}{5}$
$\{1, 2\}$	$\frac{3}{10}$	$\frac{4}{10}$	0	$\frac{7}{10}$
$\{1, 3\}$	$\frac{3}{10}$	$\frac{4}{10}$	0	$\frac{7}{10}$
$\{2,3\}$	$\frac{-16}{5}$	$\frac{-18}{5}$	-8	$\frac{-27}{5}$

Hence, it follows that  $\theta\left(\tilde{\sigma}^{N}, d^{N}, z^{N}\right) = (6'8, 5'6, 9'6) \in Core(N, c).$ 

This section is completed with a third example which shows that if any of the conditions either of the Proposition 3.4 or Theorem 3.5 fail, the unitary Owen points are not core allocations any more.

**Example 3.7** Consider now the following SI-situation with three periods, two players, and the associated 2-player SI-game:

	$d_1^S$	$d_2^S$	$d_3^S$	$p_1^s$	$p_2^S$	$p_3^S$	$h_1^s$	$h_2^s$	$b_1^S$	$b_2^S$	$k_1^s$	$k_2^s$	$k_3^s$	С
{1}	0	10	10	1	1	1	1	1	1	1	1	50	15	46
$\{2\}$	0	35	0	1	1	1	1	1	1	1	1	50	15	71
$\{1, 2\}$	0	45	10	1	1	1	1	1	1	1	1	50	15	115

There is a single optimal ordering plan which is

	$\sigma_1^S$	$\sigma_2^S$	$\sigma_3^S$	$P_1^S(\sigma^S)$	$P_2^S(\sigma^S)$	$P_3^S(\sigma^S)$	k(S)
{1}	0	1	3	0	2	1	16
$\{2\}$	0	1	0	0	2	0	1
$\{1, 2\}$	0	2	2	0	1	2	50

The unitary prices for the optimal plan above are:

	$y_1\left(\sigma^S, d^S\right)$	$y_2\left(\sigma^S, d^S\right)$	$y_3\left(\sigma^S, d^S\right)$
{1}	0	$2 + \frac{1}{10}$	$1 + \frac{15}{10}$
$\{2\}$	0	$2 + \frac{1}{35}$	0
$\{1, 2\}$	0	$1 + \frac{50}{55}$	$2 + \frac{50}{55}$

Note that  $\theta(\sigma^N, d^N, z^N) = (30 + \frac{200}{11}, 35 + \frac{350}{11}) = (48'\hat{18}, 66'\hat{81})$  is not a core allocation. Theorem 3.5 fails here because  $T^N = \{2\}$  and  $\beta_2^{\{1\}} \ge \frac{5}{4}$ .

# 4 SI-games and PI-games

To complete the paper we provide a relationship between a generic SI-game and a specific family of PI-games through Owen's points of the latter. We use Owen points from an *ad hoc* family of PI-situations constructed from core allocations of the so called *surplus game* which measures the excess in costs that occurs with respect to the minimum unit price. This interesting relationship simplifies the analysis and construction of core allocations for SI-games.

First, we introduce the minimum unitary prices for every optimal plan. Denote by  $\Delta := (\sigma^S)_{S \in \mathcal{P}(N)}$  an optimal plan in  $\Lambda(N, D, Z)$ .

**Definition 4.1** Let (N, D, Z) be a SI-situation. The minimum unitary price for  $\Delta$ , in each period  $t \in T$ , is

$$y_t^*(\Delta) = \min_{\substack{S \subseteq N \\ d_t^S \neq 0}} \{ y_t \left( \sigma^S, d^S, z^S \right) \}.$$

Second, for each coalition we measure the excess in costs that occurs with respect to the minimum unit prices. The resulting cost game is what we have called *surplus game*.

**Definition 4.2** Let (N, D, Z) be a SI-situation and (N, c) the associated SI-game. For any  $\Delta \in \Lambda(N, D, Z)$ , the surplus game  $(N, c^{\Delta})$  is defined for all  $S \subseteq N$ , as

$$c^{\Delta}(S) := c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S.$$

Note that the surplus game is a non-negative cost game which measures the increase in costs by the influence of set-up costs. The first result of this section shows that surplus games are always balanced.

**Proposition 4.3** Every surplus game  $(N, c^{\Delta})$  is balanced.

**Proof.** It follows from Theorem 2.1 that every SI-game (N, c) is balanced. Take a core

allocation  $x \in \mathbb{R}^N$  for it. For each  $S \subset N$  it holds that

$$\begin{aligned} x(S) &\leq c(S) \Longleftrightarrow x(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \leq c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \\ &\iff x(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \leq c^{\Delta}(S) \\ &\iff \sum_{i \in S} \left( x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right) \leq c^{\Delta}(S). \end{aligned}$$

Moreover x(N) = c(N) what easily implies that  $\sum_{i \in N} \left( x_i - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i \right) = c^{\Delta}(N)$ . Hence, we conclude that  $\left( x_i - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i \right)_{i \in N} \in Core(N, c^{\Delta})$ .

In the following we use this game to construct core allocations for SI-games by means of the Owen points of the surplus game which is an easy PI-game. Next result provides a necessary and sufficient condition for this purpose: the set-up costs cannot contribute to any increase in costs for the grand coalition. In other words, there are no cost exceeding the unit prices of the grand coalition.

**Proposition 4.4** Let (N, c) be a SI-game. For any  $\Delta \in \Lambda(N, D, Z)$ ,

$$c^{\Delta}(N) = 0$$
 if and only if  $\left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i\right)_{i \in N} \in Core(N, c)$ .

**Proof.** (If) If  $c^{\Delta}(N) = 0$  then  $\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^N = c(N)$ . For each  $S \subset N$ ,  $\sum_{i \in S} \left( \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right) = \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \leq \sum_{t=1}^{T} y_t(\sigma^S, d^S) \cdot d_t^S = c(S)$ . Thus,  $\left( \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right)_{i \in N} \in Core(N, c)$ . (Only if) If  $\left( \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right)_{i \in N} \in Core(N, c)$ , it is satisfy that  $\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^N = c(N)$ , and so  $c^{\Delta}(N) = 0$ .

The main theorem in this section shows that the core of any SI-game consists of the Owen points of certain PI-games obtained from core allocations of surplus games. To state this theorem, it is necessary to describe a procedure to construct a PI-situation from core allocations of surplus games.

Consider a SI-situation (N, D, Z), the associated SI-game (N, c), and the surplus game  $(N, c^{\Delta})$ , for  $\Delta \in \Lambda(N, D, Z)$ . For any  $\alpha \in Core(N, c^{\Delta})$ ,  $(N, \overline{D}(\alpha), \overline{Z})$  is a PI-situation with  $\overline{Z} = (\overline{K}, \overline{H}, \overline{B}, \overline{P})$  and  $\overline{D}(\alpha) = [\overline{d}^1, \ldots, \overline{d}^n]', \overline{K} = 0, \overline{H} = [M, \ldots, M]', \overline{B} = [M, \ldots, M]', \overline{P} = [\overline{p}, \ldots, \overline{p}]'$ , with  $\overline{p} = (y_1^*(\Delta), \ldots, y_T^*(\Delta), 1), \ \overline{d}^i = (d_1^i, \ldots, d_T^i, \alpha_i)$  for all  $i \in N$  and  $M \in \mathbb{R}^N$  large enough. This procedure shows that any SI-situation can be transformed into multiple PI-situations just by using the core of the surplus games.

**Theorem 4.5** Let (N, c) be a SI-game and  $(N, c^{\Delta})$  the associated surplus game for  $\Delta \in \Lambda(N, D, Z)$ . Thus,

$$Core(N,c) = \left\{ Owen\left(N,\overline{D}(\alpha),\overline{Z}\right) : \alpha \in Core(N,c^{\Delta}) \right\}.$$

**Proof.** As  $(N, c^{\Delta})$  is balanced, there is at least one  $\alpha \in \mathbb{R}^N$ , such that  $\alpha(S) \leq c^{\Delta}(S) = c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S$  for all  $S \subset N$  and  $\alpha(N) = c(N) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^N$ . Consider a PI-situation  $(N, \overline{D}(\alpha), \overline{Z})$  with T + 1 periods, where

$$\overline{D}(\alpha) = [\overline{d}^1, \dots, \overline{d}^n]', \overline{K} = 0, \overline{H} = [M, \dots, M]', \overline{B} = [M, \dots, M]', \overline{P} = [\overline{p}, \dots, \overline{p}]'$$

with  $\overline{p} = (y_1^*(\Delta), ..., y_T^*(\Delta), 1), \ \overline{d}^i = (d_1^i, ..., d_T^i, \alpha_i)$  for all  $i \in N$  and  $M \in \mathbb{R}^N$  large enough. For each  $i \in N$ ,  $Owen_i\left(N, \overline{D}(\alpha), \overline{Z}\right) = \sum_{t=1}^{T+1} y_t^*(N) d_t^i = \left(\sum_{t=1}^T y_t^*(\Delta) d_t^i\right) + \alpha_i$ . Then, for all  $S \subset N$ :

$$\sum_{i \in S} Owen_i \left( N, \overline{D}(\alpha), \overline{Z} \right) = \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^S + \alpha(S)$$
  
$$\leq \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^S + c(S) - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^S = c(S).$$

Moreover,  $\sum_{i\in N} Owen_i\left(N,\overline{D}(\alpha),\overline{Z}\right) = \sum_{t=1}^T y_t^*(\Delta)d_t^N + \alpha(N) = \sum_{t=1}^T y_t^*(\Delta)d_t^N + c(N) - \sum_{t=1}^T y_t^*(\Delta)d_t^N = c(N).$  Thus  $Owen\left(N,\overline{D}(\alpha),\overline{Z}\right) \in Core(N,c).$ 

On the other hand, if  $x \in Core(N, c)$ , for each  $S \subset N$  it holds

$$\begin{aligned} x(S) &\leq c(S); \\ x(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S &\leq c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \\ \sum_{i \in S} x_i - \sum_{i \in S} \left( \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^{\{i\}} \right) &\leq c^{\Delta}(S); \\ \sum_{i \in S} \left( x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^{\{i\}} \right) &\leq c^{\Delta}(S); \end{aligned}$$

Moreover  $x(N) = c(N) \Leftrightarrow \sum_{i \in N} \left( x_i - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i \right) = c^{\Delta}(N)$ . Thus for each  $x \in Core(N, c)$  we can take  $\alpha_i := x_i - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i$  for all  $i \in N$  such that  $\alpha \in Core(N, c^{\Delta})$ . From here it easily follows that  $Owen\left(N, \overline{D}(\alpha), \overline{Z}\right) = \left( \left( \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i \right) + \alpha_i \right)_{i \in N} = x$ .

We illustrate the procedure above with the Example 3.7 shown above.

**Example 4.6** Consider the 2-player SI-game given in Example 3.7. We have shown that the unitary Owen point is not a core allocation for this example. It can be easily checked that the minimal unit prices are:

$y_1^*(\Delta)$	$y_2^*(\Delta)$	$y_3^*(\Delta)$
0	$1 + \frac{50}{55}$	$1 + \frac{15}{10}$

Thus, the surplus game is given by

	С	$c^{\Delta}$
{1}	46	$1 + \frac{10}{11}$
$\{2\}$	71	$4 + \frac{2}{11}$
$\{1, 2\}$	115	$4 + \frac{1}{11}$

Consider a core allocation from the surplus game, for instance the nucleolus  $\eta(N, c^{\Delta}) = (\frac{10}{11}, 3\frac{2}{11})$ . We obtain a core allocation for the SI-game just by calculating the Owen point of the associated PI-situation  $(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z})$ . Thus, Owen  $(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}) = (45, 70)$ . One can conclude that

$$\eta(N,c) = Owen\left(N,\overline{D}(\eta(N,c^{\Delta})),\overline{Z}\right).$$

In the above example, the nucleolus of the surplus game leads to the nucleolus of the SI-game through the Owen point. The last result in the paper shows that this close relationship between both nucleoli always holds, e.g. the nucleolus of any SI-game matches the Owen point for the PI-situation obtained from the nucleolus of the surplus game.

**Proposition 4.7** Let (N, D, Z) be a SI-situation, (N, c) the associated SI-game, and  $(N, c^{\Delta})$  the surplus game for  $\Delta \in \Lambda(N, D, Z)$ . Thus,

$$Owen\left(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\right) = \eta(N, c).$$

**Proof.** It is known that  $x \in Core(N, c)$  if and only if  $x^{\Delta} := \left(x_i - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i\right)_{i \in N} \in Core(N, c^{\Delta})$ . Thus the excess vectors, e(S, x), and  $e_{\Delta}(S, x^{\Delta})$  coincide.

For each coalition  $S \subseteq N$ , it holds that:

$$e(S, \eta(N, c)) = c(S) - \sum_{i \in S} \eta_i(N, c)$$
  
=  $c^{\Delta}(S) + \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^S - \sum_{i \in S} \eta_i(N, c)$   
=  $c^{\Delta}(S) - \sum_{i \in S} \left( \eta_i(N, c) - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i \right)$ 

Therefore,  $\eta_i(N, c^{\Delta}) = \eta_i(N, c) - \sum_{t=1}^T y_t^*(\Delta) \cdot d_t^i$  for all  $i \in N$  because otherwise  $\eta(N, c)$  would not be the nucleolus. Moreover,

$$c^{\Delta}(S) - \eta_i(N, c^{\Delta}) = c(S) - \left(\sum_{t=1}^T y_t^*(\Delta) \cdot d_t^S + \sum_{i \in S} \eta_i(N, c^{\Delta})\right)$$
$$= c(S) - \sum_{i \in S} Owen_i \left(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\right)$$
$$= e(S, Owen \left(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\right)).$$

This implies that  $Owen\left(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\right) = \eta(N, c).$ 

#### 5 Concluding Remarks

The study of cooperation in lot-sizing problems with backlogging and heterogeneous costs begins in Guardiola et al. (2020). The authors prove that there are always fair allocations of the overall operation cost among the firms so that no group of agents profit from leaving the consortium. They propose a parametric family of cost allocations and provide sufficient conditions for this to be a stable family against coalitional defections of firms and focus on those periods of the time horizon that are consolidated analyzying their effect on the stability of cost allocations.

To complete the study of those cooperative lot-sizing models, this paper presents the unitary Owen point. As mentioned, the Owen point works great for constructing core-allocations in the class of PI-games. Unfortunately, this does not work for SI-problems any more. In spite of that, we manage here to construct a particular kind of prices, that we call unitary prices, based on the production, inventory and backlogging costs and a proportion of the fixed order cost which depends on the total demand satisfied in each period. These unit prices enable one to replicate the construction of the Owen point so that these allocations "a la Owen" are called unitary Owen points. We provide necessary and sufficient conditions for unitary Owen points to be core-allocations. Moreover, we show a relationship between SI-games and a certain family of PI-games through Owen's points of the latter. Specifically, we use Owen points from a family of PI-situations, constructed from core allocations of an *ad hoc* surplus game which measures the excess in costs that occurs with respect to the minimum unit prices. This amazing relationship enables one to easily construct another family of fair allocations for SI-games.

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