BOREL AND PARABOLIC-TYPE SUBALGEBRAS OF THE LATTICE VERTEX OPERATOR ALGEBRA

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Abstract. In this paper, we introduce and study some new classes of subalgebras of the lattice vertex operator algebras, which we call the Borel-type and parabolic-type subVOAs. For the lowest-rank examples of Borel-type subVOAs V_B of $V_{\mathbb{Z}\alpha}$, and one nontrivial lowest-rank example of the parabolic-type subVOA V_P of the lattice VOA V_{A_2} associated to the root lattice A_2, we explicitly determine their Zhu's algebras $A(V_B)$ and $A(V_P)$ in terms of generators and relations. Using the descriptions of $A(V_B)$ and $A(V_P)$, we classify the irreducible modules over V_B and V_P.

Contents

1. Introduction

The lattice vertex operator algebra (VOA for short) V_L associated to a positive-definite even lattice L was introduced by Borcherds in [Bor86] and Frenkel, Lepowsky, and Meurman in [FLM88], whose construction was based on the vertex operator realization of irreducible representations over the affine Kac-Moody algebras on the Fock space $S(\hat{\mathfrak{h}}^{<0}) \otimes \mathbb{C}^{\epsilon}[Q]$ given by Frenkel and Kac in [FK80]. As the first example of vertex operator algebras, the lattice VOAs play a fundamental role in the theory of VOAs. The famous moonshine module vertex operator algebra $V^{\natural}$, which connects the j-invariant and the monster simple group $\mathbb{M}$, was constructed using the orbifold and the simple current extension methods for the lattice VOA V_{Λ} associated to the Leech lattice Λ by Frenkel, Lepowsky, and Meurman in [FLM88]. Motivated by the understanding of the moonshine module $V^{\natural}$ and conformal field theory [MS89, S93], the lattice VOA V_L has been studied extensively: Dong and Lepowsky computed the fusion rules for V_L in [DL93]. Dong classified the irreducible modules and irreducible θ-twisted modules over V_L in [D93, D94]. The $\mathbb{Z}_2$-orbifold V_L^+ of the lattice VOA V_L, which was used to construct $V^\natural$, was studied in detail in a series of papers by Abe, Dong, Griess, Li, and Nagatomo [AD04, ADL05, DG98, DN99(1), DN99(2)]. Using the $\mathbb{Z}_2$ and $\mathbb{Z}_3$-orbifold constructions for the lattice VOAs, the Schellekens' conjecture on 71 holomorphic VOAs with central charge 24 [S93] was solved recently by van Ekeren, Lam, Möller, Scheithauer, and Shimakura [vEMS20, Lam11, LS15, LS16].

In the classical theory of finite or infinite-dimensional Lie algebras, a Lie algebra $\mathfrak{g}$ that has interesting representation theory is normally equipped with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The Borel (or parabolic) subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is used in a standard technique to construct Verma modules and irreducible highest-weight modules over $\mathfrak{g}$. Although vertex operator algebras have more resemblance with semisimple associative algebras from the representation theoretical point of view, their structural theory actually has more resemblance with the semisimple Lie algebras. For instance, the Jacobi identity for VOAs is a formal variable generalization of the usual Jacobi identity for Lie algebras [FLM88]; the vertex operator Y satisfies the skew-symmetry axiom similar to a Lie bracket [FHL93]; and a CFT-type simple VOA such that $L(1)V_1 = 0$ always has a non-degenerate symmetric invariant bilinear form [FHL93, L94] similar to a Cartan-Killing form, etc.

In this paper, we provide more evidence in the lattice VOAs showing that the structural theory of VOAs resembles semisimple Lie algebras. To outline our results, we set some notation. Let $M_{\hat{\mathfrak{h}}}(1,0)$ be the Heisenberg VOA associated to $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$, where $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. By the construction of lattice VOA V_L in [FLM88], one has the following decomposition of V_L as a module over the Heisenberg subVOA $M_{\hat{\mathfrak{h}}}(1,0)$:

$$V_L = \bigoplus_{\alpha \in L} M_{\hat{\mathfrak{h}}}(1,\alpha). \tag{1.1}$$

The vertex operator Y of V_L is given by intertwining operators among irreducible Heisenberg modules, which satisfies $Y(M_{\hat{\mathfrak{h}}}(1,\alpha), z)M_{\hat{\mathfrak{h}}}(1,\beta) \subset M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)((z))$, for any $\alpha, \beta \in L$. In particular, for any abelian submonoid $M \leq L$, the subspace $V_M = \bigoplus_{\alpha \in M} M_{\hat{\mathfrak{h}}}(1,\alpha)$ is a subVOA of V_L. Let $B \leq L$ be the submonoid $\mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$, where $\{\alpha_1, \ldots, \alpha_r\}$ is a basis of L, and let $P \leq L$ be a submonoid containing B. We call $V_B = \bigoplus_{\alpha \in B} M_{\hat{\mathfrak{h}}}(1,\alpha)$ (resp. $V_P = \bigoplus_{\alpha \in P} M_{\hat{\mathfrak{h}}}(1,\alpha)$) a Borel (resp. parabolic)-type subVOA of V_L. These subVOAs of V_L first appeared in a recent study of Rota-Baxter operators and classical Yang-Baxter equations for vertex operator algebras by Bai, Guo, the author, and Wang in [BGL, BGLW]. V_B and V_P give rise to natural examples of Rota-Baxter operators for lattice VOAs.

In the classical Lie theory, a Borel subgroup B of a connected linear algebraic group G is defined to be a closed connected solvable subgroup of G that is maximal subject to these conditions. A parabolic subgroup can be equivalently characterized as a closed subgroup P that contains a Borel subgroup (see [B56, BT65, Hum1]). The Lie algebra $\mathfrak{b} = \mathrm{Lie}(B)$ is a Borel subalgebra of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ that contains a Borel subalgebra $\mathfrak{b}$. If a Lie algebra $\mathfrak{g}$ is semisimple, then it has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and Φ is the root system associated to $\mathfrak{h}$. In this case, a Borel subalgebra is given by $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$, where Φ_+ is the set of positive roots (see [BT65, Hum2]). In our Borel-type subVOA $V_B = M_{\hat{\mathfrak{h}}}(1,0) \oplus \bigoplus_{\alpha \in B \backslash \{0\}} M_{\hat{\mathfrak{h}}}(1,\alpha)$, we can view $M_{\hat{\mathfrak{h}}}(1,0)$ as an analog of the Cartan-part $\mathfrak{h}$ in $\mathfrak{b}$, and view $M_{\hat{\mathfrak{h}}}(1,\alpha)$ with $\alpha \in B \backslash \{0\}$ as an analog of the positive root space $\mathfrak{g}_\alpha$ in $\mathfrak{b}$.

Although the "Cartan part" $M_{\hat{\mathfrak{h}}}(1,0)$ of V_B is non-commutative, and the "positive-root parts" $M_{\hat{\mathfrak{h}}}(1,\alpha)$ of V_B are not one-dimensional, we can still show that V_B and V_P satisfy many similar properties as Borel and parabolic-subalgebras of a semisimple Lie algebra. We discuss these properties in detail in Section 2. First, we show that V_B and V_P are all irrational as VOAs

(Proposition 2.9), which corresponds to the fact that $\mathfrak{b}$ and $\mathfrak{p}$ of a semisimple Lie algebra $\mathfrak{g}$ are no longer semisimple as Lie algebras. Next, we show that certain Borel-type subVOAs V_{B_Δ} of an A, D, or E-type root lattice VOA V_Q are conjugate under the automorphism group $\mathrm{Aut}(V_Q)$ (Proposition 2.10), which corresponds to that fact that all Borel subgroups B of G and Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ are conjugate under G. Furthermore, after introducing a notion of normalizer $N_V(W)$ of a sub-vertex algebra W in a VOA V (Definition 2.11), we show that the normalizer $N_{V_L}(V_P)$ of a parabolic-type subVOA V_P in V_L is equal to V_P itself (Proposition 2.13), which corresponds to the normalizer property of parabolic subgroups or subalgebras. Finally, we show that all Borel-type subVOA V_B of rank 1 or 2 are C_1-cofinite (Proposition 2.16).

In Section 3, we focus on the lowest-rank examples of Borel-type subVOAs $V_B \leq V_{\mathbb{Z}\alpha}$, where $(\alpha|\alpha) = 2N$ and $B = \mathbb{Z}_{\geq 0}\alpha$. To study the representation theory of V_B, we first study the Zhu's algebra $A(V_B)$ for V_B. For any CFT-type VOA V, Zhu found an associative algebra $A(V)$ attached to V by the calculation of recursive formulas of genus-zero correlation functions restricted onto the bottom levels of admissible V-modules [Z96]. One of the most significant properties of Zhu's algebra is the one-two-one correspondence between irreducible V-modules and irreducible $A(V)$-modules, see Theorem 2.2.2 in [Z96]. By construction, $A(V) = V/O(V)$, where $O(V) \subset V$ is spanned by elements of the form $a \circ b = \sum_{j\geq 0} \binom{\mathrm{wt} a}{j} a_{j-2}b$, where $a, b \in V$ are homogeneous. Our main result in this Section is a concrete description of $O(V_B)$ of the VOA V_B (Theorem 3.8):

Theorem 1.1. *Let $V_B = V_{\mathbb{Z}_{\geq 0}\alpha}$ be the Borel-type subVOA of $V_{\mathbb{Z}\alpha}$, with $(\alpha|\alpha) = 2N$. Then*

$$\begin{aligned} O(V_B) = \mathrm{span}\Big\{&\alpha(-n-2)u + \alpha(-n-1)u,\ \alpha(-1)v + v,\ M_{\hat{\mathfrak{h}}}(1, k\alpha) : \\ &n \geq 0,\ u \in V_B,\ v \in \bigoplus_{m\geq 1} M_{\hat{\mathfrak{h}}}(1, m\alpha),\ k \geq 2\Big\}. \end{aligned} \tag{1.2}$$

The proof of Theorem 1.1 uses a sequence of inductions on the length of spanning elements $u = \alpha(-n_1)\ldots\alpha(-n_r)e^{m\alpha}$ of V_B, which are carried out in details by Lemma 3.4, Proposition 3.5, Lemma 3.6, and Proposition 3.7. As an immediate Corollary, we prove that $A(V_B)$ is isomorphic to the associative algebra $\mathbb{C}[x] \oplus \mathbb{C}y$, where $y^2 = 0$, $yx = -Ny$, and $xy = Ny$. Using this description of $A(V_B)$ and Theorem 2.2.2 in [Z96], we show that the irreducible modules of V_B are in one-to-one correspondence with the irreducible modules over the Heisenberg VOA $M_{\widehat{\mathbb{C}\alpha}}(1, 0)$, and the fusion rules among irreducible modules over V_B are also the same as fusion rules among irreducible modules over $M_{\widehat{\mathbb{C}\alpha}}(1, 0)$ (Theorem 3.15). This result is also parallel to the semisimple Lie algebra case, namely, the irreducible modules over a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$ are the same as irreducible modules over the Cartan part $\mathfrak{h}$ on which $\mathfrak{n}_+$ acts as 0.

To better understand the difference between Borel-type and parabolic-type subVOAs, in Section 4 and 5, we study a lowest-rank example of parabolic-type subVOA V_P that is not of Borel-type. We take $L = A_2 = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ to be the type A_2-root lattice, and take $P = \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$. Note that there is no difference between Borel-type and parabolic-type subVOAs in a rank-one lattice VOA. Our main result in Section 4 is a concrete description of the spanning elements of $O(V_P)$ (see Definition 4.4 and Theorem 4.9):

Theorem 1.2. *Let $V_P = V_{\mathbb{Z}\alpha\oplus\mathbb{Z}\beta}$ be the parabolic-type subVOA of V_{A_2}. Then $O(V_P)$ is spanned by the following elements:*

$$\begin{cases} h(-n-2)u + h(-n-1)u, & u \in V_P,\ h \in \mathfrak{h},\ n \geq 0; \\ \gamma(-1)v + v, & v \in M_{\hat{\mathfrak{h}}}(1, \gamma),\ \gamma \in \{\alpha, -\alpha, \beta, \alpha+\beta\}; \\ \gamma(-1)^2 v + \gamma(-1)v, & v \in M_{\hat{\mathfrak{h}}}(1, \gamma+\gamma'),\ \gamma, \gamma' \in \{\alpha, -\alpha, \beta, \alpha+\beta\},\ \gamma+\gamma' \in \{\alpha+\beta, \beta\}; \\ M_{\hat{\mathfrak{h}}}(1, m\alpha+n\beta), & m\alpha+n\beta \in (\mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta)\backslash\{0, \alpha, -\alpha, \beta, \alpha+\beta\}; \\ \alpha(-1)^3 w - \alpha(-1)w, & w \in M_{\hat{\mathfrak{h}}}(1, 0). \end{cases}$$

The proof of Theorem 1.2 uses a similar induction process on the length of spanning elements $h^1(-n_1)\dots h^r(-n_r)e^\gamma$ of V_P as in the proof of Theorem 1.1, with details carried out in Lemmas 4.2 and 4.3, and Propositions 4.6, 4.7, and 4.8. With the spanning elements of $O(V_P)$, we find a presentation of $A(V_P)$ by generators and relations (Theorem 4.11):

$$A(V_P)\cong \mathbb{C}\langle x,y,x_\alpha,x_{-\alpha},x_\beta,x_{\alpha+\beta}\rangle/R, \tag{1.3}$$

where R is the two-sided ideal of the tensor algebra $\mathbb{C}\langle x,y,x_\alpha,x_{-\alpha},x_\beta,x_{\alpha+\beta}\rangle$ generated by the relations (4.21)–(4.26). Using these relations, we give a concrete description for the structure of $A(V_P)$. In particular, we show that as associative algebras, there is an isomorphism:

$$A(V_P)\cong A(V_{\mathbb{Z}\alpha})[y;\mathrm{Id};\delta]\oplus J, \tag{1.4}$$

where $A(V_{\mathbb{Z}\alpha})[y;\mathrm{Id};\delta]$ is the skew-polynomial algebra (see [O33, GW04]) on the Zhu's algebra $A(V_{\mathbb{Z}\alpha})$ of the rank-one lattice VOA $V_{\mathbb{Z}\alpha}$, which is isomorphic to $U(sl_2(\mathbb{C}))/\langle e^2\rangle$, and $J\subset A(V_P)$ is a two-sided ideal such that $J^2=0$, see Corollaries 4.12 and 4.15.

In Section 5, we use the identifications (1.3) and (1.4) of $A(V_P)$, together with Theorem 2.2.1 in [Z96] again, to classify irreducible modules over V_P. Our approach is to first construct two irreducible modules over V_P associated to a $\lambda\in(\mathbb{C}\alpha)^\perp\subset\mathfrak{h}$:

$$L^{(0,\lambda)}=\bigoplus_{n\in\mathbb{Z}}M_{\hat{\mathfrak{h}}}(1,n\alpha)\otimes\mathbb{C}e^\lambda,\qquad L^{(\frac{1}{2}\alpha,\lambda)}=\bigoplus_{n\in\mathbb{Z}}M_{\hat{\mathfrak{h}}}(1,n\alpha+\frac{1}{2}\alpha)\otimes\mathbb{C}e^\lambda,$$

using a slight variation of the lattice vertex operators in [FLM88] (Definition 5.3). Then show that the bottom levels $U^{(0,\lambda)}$ and $U^{(\frac{1}{2}\alpha,\lambda)}$ of these irreducible modules exhaust all the possible irreducible modules over $A(V_P)$ when λ varies in $(\mathbb{C}\alpha)^\perp$. The following is our main result in Section 5 (see Theorem 5.6 and Corollary 5.7):

Theorem 1.3. *The set* $\Sigma(P)=\left\{(L^{(0,\lambda)},Y_M),(L^{(\frac{1}{2}\alpha,\lambda)},Y_M):\lambda\in(\mathbb{C}\alpha)^\perp\subset\mathfrak{h}\right\}$ *is a complete list of irreducible modules over the rank-two parabolic-type subVOA* V_P *of* V_{A_2}.

Our constructions for the Borel and parabolic-type subVOAs have some further applications, which are briefly discussed in Section 6. The first application of these subalgebras is the existence of an analog of triangular decomposition for vertex operator algebras, which we call a quasi-triangular decomposition. A VOA V, equipped with a non-degenerated symmetric invariant bilinear form $(\cdot|\cdot)$, is said to admit a quasi-triangular decomposition if V has a subspace decomposition: $V=V_-\oplus V_H\oplus V_+$, where $V_\pm$ and V_H are invariant under the action of $sl_2(\mathbb{C})=\mathbb{C}L(-1)+\mathbb{C}L(0)+\mathbb{C}L(1)$, $V_\pm$ are sub-vertex algebras without vacuum, V_H is a sub-vertex algebra of V, and $(V_\pm|V_\pm)=(V_H|V_\pm)=0$, see Definition 6.1. These axioms and properties of $V=V_-\oplus V_H\oplus V_+$ in Lemma 6.2 are also parallel to the properties of $\mathfrak{n}_\pm$ and $\mathfrak{h}$ with respect to the Cartan-Killing form in a triangular decomposition of a semisimple Lie algebra $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ The subVOAs V_B and V_P in Sections 3–5 naturally give rise to distinct examples of the quasi-triangular decomposition for lattice VOAs. Our definitions for Borel and parabolic-type subVOAs can also be generalized to the case of affine vertex operator algebras $V_{\hat{\mathfrak{g}}}(k,0)$ and $L_{\hat{\mathfrak{g}}}(k,0)$ of arbitrary positive integral level k [FZ92]. We briefly discuss it by the end of Section 6.

This paper is organized as follows: we introduce the concepts of Borel and parabolic-type subVOAs of lattice VOAs and prove some basic properties in Section 2. In Section 3, we focus on the rank-one Borel-type subVOA $V_{\mathbb{Z}_{\geq0}\alpha}$ of $V_{\mathbb{Z}\alpha}$ and determine its Zhu's algebra by presenting a concrete description of $O(V_{\mathbb{Z}_{\geq0}\alpha})$. In Section 4, we determine the structure of Zhu's algebra of a typical rank-two parabolic-type subVOA V_P of V_{A_2} that is not of Borel-type. In Section 5, we classify the irreducible modules over V_P using the results from previous sections. Finally, in Section 6, we introduce the notion of quasi-triangular decomposition for VOAs and the Borel and parabolic-type subVOAs for affine VOAs.

Conventions. Throughout this paper, all vector spaces are defined over $\mathbb{C}$, the complex number field. $\mathbb{N}$ represents the set of all natural numbers, including 0.

2. The Borel-type and parabolic-type subVOAs of V_L

In this section, we first review the construction of lattice VOAs in [FLM88] and some related results, then define the Borel-type and parabolic type sub-algebras of a lattice VOA V_L, based on the decomposition of V_L as an irreducible module over the Heisenberg subVOA. We will prove that these subVOAs are irrational, and some of them are C_1-cofinite; they also share some similar properties as the usual Borel and parabolic subalgebras of a semisimple Lie algebra.

2.1. **Preliminaries.** For the general definitions of vertex operator algebras (VOAs), modules over VOAs, and examples of VOAs, we refer to [DL93, FLM88, FHL93, LL04].

2.1.1. *The lattice vertex operator algebras.* Let L be a positive definite even lattice of rank $d \geq 1$, equipped with $\mathbb{Z}$-bilinear form $(\cdot|\cdot) : L \times L \to \mathbb{Z}$. Let $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} L$, extend $(\cdot|\cdot)$ to a nondegenerate $\mathbb{C}$-bilinear form $(\cdot|\cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$, and let $M_{\hat{\mathfrak{h}}}(1,0)$ be the level-one Heisenberg VOA associated to $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$. Recall that the Lie bracket on the affine algebra $\hat{\mathfrak{h}}$ is given by:

$$[h_1(m), h_2(n)] = m(h_1|h_2)\delta_{m+n,0}K, \quad h_1, h_2 \in \mathfrak{h}, \text{ and } m, n \in \mathbb{Z}, \tag{2.1}$$

where we denote $h \otimes t^m \in \hat{\mathfrak{h}}$ by $h(m)$. Then $\hat{\mathfrak{h}} = (\hat{\mathfrak{h}})_+ \oplus (\hat{\mathfrak{h}})_0 \oplus (\hat{\mathfrak{h}})_-$, where $(\hat{\mathfrak{h}})_\pm = \bigoplus_{n \in \mathbb{Z}_\pm} \mathfrak{h} \otimes \mathbb{C}t^n$, and $(\hat{\mathfrak{h}})_0 = \mathfrak{h} \otimes \mathbb{C}1 \oplus \mathbb{C}K$. Let $(\hat{\mathfrak{h}})_{\geq 0} = (\hat{\mathfrak{h}})_+ \oplus (\hat{\mathfrak{h}})_0$, which is a Lie subalgebra of $\hat{\mathfrak{h}}$.

For each $\lambda \in \mathfrak{h}$, let e^λ be a formal symbol associated to λ. Then $\mathbb{C}e^\lambda$ is a module over $(\hat{\mathfrak{h}})_{\geq 0}$, with the module actions given by $h(0)e^\lambda = (h|\lambda)e^\lambda$, $K.e^\lambda = e^\lambda$, and $h(n)e^\lambda = 0$, for all $h \in \mathfrak{h}$ and $n > 0$. $M_{\hat{\mathfrak{h}}}(1,\lambda)$ is defined to be the induced module:

$$M_{\hat{\mathfrak{h}}}(1,\lambda) := \mathrm{Ind}_{\hat{\mathfrak{h}}_{\geq 0}}^{\hat{\mathfrak{h}}} \mathbb{C}e^\lambda = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_{\geq 0})} \mathbb{C}e^\lambda. \tag{2.2}$$

Then $M_{\hat{\mathfrak{h}}}(1,\lambda) \cong U(\hat{\mathfrak{h}}_{<0}) \otimes_{\mathbb{C}} \mathbb{C}e^\lambda = \mathrm{span}\{h_1(-n_1)\dots h_k(-n_k)e^\lambda : k \geq 0,\ h_1,\dots,h_k \in \mathfrak{h},\ n_1 \geq \cdots \geq n_k \geq 0\}$ as vector spaces. It was proved in [FLM88, DL93] that $M_{\hat{\mathfrak{h}}}(1,0)$ has a VOA structure, called the level-one Heisenberg VOA, and $M_{\hat{\mathfrak{h}}}(1,\lambda)$ with different $\lambda \in \mathfrak{h}$, are all the irreducible modules over $M_{\hat{\mathfrak{h}}}(1,0)$ up to isomorphism.

Write $\mathbb{C}^\epsilon[L] = \bigoplus_{\alpha \in L} \mathbb{C}e^\alpha$, where e^α is a formal symbol associated to α for each $\alpha \in L$ (e^α was denoted by $\iota(\alpha)$ in [FLM88]), and $\epsilon : L \times L \to \langle \pm 1 \rangle$ is a 2-cocycle of the abelian group L such that $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)}$ for any $\alpha, \beta \in L$. Let $V_L = M_{\hat{\mathfrak{h}}}(1,0) \otimes_{\mathbb{C}} \mathbb{C}^\epsilon[L]$, then

$$V_L = \mathrm{span}\{h_1(-n_1)\dots h_k(-n_k)e^\alpha : k \geq 0, \alpha \in L,\ h_1,\dots,h_k \in \mathfrak{h},\ n_1 \geq \cdots \geq n_k \geq 0\},$$

where we omit the tensor sign $\otimes$ in the term $h_1(-n_1)\dots h_k(-n_k)e^\alpha$. The vertex operator $Y : V_L \to \mathrm{End}(V_L)[[z,z^{-1}]]$ on the spanning elements of V_L is given as follows:

$$Y(h(-1)\mathbf{1}, z) := h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1} \quad \left(h(n)e^\alpha := 0,\ h(0)e^\lambda := (h|\alpha)e^\alpha\right), \tag{2.3}$$

$$Y(e^\alpha, z) := E^-(-\alpha,z)E^+(-\alpha,z)e_\alpha z^\alpha \quad \left(z^\alpha(e^\beta) := z^{(\alpha|\beta)}e^\beta,\ e_\alpha(e^\beta) := \epsilon(\alpha,\beta)e^{\alpha+\beta}\right), \tag{2.4}$$

$$Y(h_1(-n_1-1)\dots h_k(-n_k-1)e^\alpha, z) := {\circ \atop \circ}(\partial_z^{(n_1)}h_1(z))\dots(\partial_z^{(n_k)}h_k(z))Y(e^\alpha,z){\circ \atop \circ}, \tag{2.5}$$

for any $k \geq 1$, $n_1 \geq \cdots \geq n_k \geq 0$, $h, h_1, \dots h_k \in \mathfrak{h}$, and $\alpha, \beta \in L$, where $E^\pm$, $\partial_z^{(n)}$, e^α, and z^α in (2.3)–(2.5) are given as follows:

$$E^\pm(-\alpha,z) = \exp\left(\sum_{n \in \mathbb{Z}_\pm} \frac{-\alpha(n)}{n} z^{-n}\right), \quad \partial_z^{(n)} = \frac{1}{n!}, \quad e_\alpha(e^\beta) = \epsilon(\alpha,\beta)e^{\alpha+\beta}, \quad z^\alpha(e^\beta) = z^{(\alpha|\beta)}z^\beta.$$

The normal ordering in (2.5) rearranges the terms in such a way that the right hand side of (2.5) has the following expression:

$$\sum_{m_1>0,\dots,m_k>0} \sum_{n_1 \geq 0,\dots,n_k \geq 0} c_{m_1,\dots,n_k} h_1(-m_1)\dots h_k(-m_k)E^-(-\alpha,z)e_\alpha z^\alpha E^+(-\alpha,z)h_1(n_1)\dots h_k(n_k). \tag{2.6}$$

Let $\{\alpha_1,\dots,\alpha_d\}$ be an orthonormal basis of $\mathfrak{h}$, and let $\omega=\frac{1}{2}\sum_{i=1}^d \alpha_i(-1)^2\mathbf{1}\in M_{\hat{\mathfrak{h}}}(1,0)\subset V_L$.

It was proved in the appendix A.2 in [FLM88] that $(V_L,Y,\omega,\mathbf{1})$ is a VOA such that $M_{\hat{\mathfrak{h}}}(1,0)\subset V_L$ is a subVOA. In particular, V_L has the same Virasoro element ω and the vacuum element $\mathbf{1}$ with the Heisenberg subVOA $M_{\hat{\mathfrak{h}}}(1,0)$. Recall that V_L has the following decomposition as a module over the Heisenberg VOA $M_{\hat{\mathfrak{h}}}(1,0)$ (see [FLM88, D93]):

$$V_L=\bigoplus_{\alpha\in L} M_{\hat{\mathfrak{h}}}(1,\alpha), \tag{2.7}$$

where $M_{\hat{\mathfrak{h}}}(1,\alpha)=M_{\hat{\mathfrak{h}}}(1,0)\otimes\mathbb{C}e^\alpha$ for each $\alpha\in L$.

Let $L^\circ:=\{h\in\mathfrak{h}:(h|\alpha)\in\mathbb{Z},\ \forall\alpha\in L\}$ be the dual lattice of L. For each element $\lambda\in L^\circ$, it was proved in [FLM88] that $V_{L+\lambda}=M_{\hat{\mathfrak{h}}}(1,0)\otimes_{\mathbb{C}}\mathbb{C}^\epsilon[L+\lambda]$ is a module over V_L, with the module vertex operator $Y_M: V_L\to \mathrm{End}(V_{L+\lambda})[[z,z^{-1}]]$ given by formulas similar with (2.3)–(2.5), the only differences are $h(0)e^{\beta+\lambda}:=(h|\beta+\lambda)e^{\beta+\lambda}$, $z^\alpha(e^{\beta+\lambda}):=z^{(\alpha|\beta+\lambda)}e^{\beta+\lambda}$, and $e_\alpha e^{\beta+\lambda}:=\epsilon(\alpha,\beta)e^{\alpha+\beta+\lambda}$, for any $h\in\mathfrak{h}$, $\alpha,\beta\in L$ and $\lambda\in L^\circ$.

Furthermore, Dong classified the irreducible modules over V_L in [D93]. The main result is the following: Let $L^\circ/L=\bigsqcup_{i=0}^p(L+\lambda_i)$ be the coset decomposition of the subgroup L in L°. Then $\{V_{L+\lambda_0},\dots,V_{L+\lambda_p}\}$ are all the irreducible module over V_L up to isomorphism (see Theorem 3.1 in [D93]). Furthermore, V_L is a rational VOA.

2.1.2. *subVOAs of V_L associated to submonoids of L.* Observe that a lattice L is an abelian monoid, with the dentity element 0. An **(abelian) submonoid** of L is a subset $M\subset L$ such that $0\in M$, and M is closed under the addition of L. An **(abelian) sub-semigroup** of L is a subset $S\subset L$ such that S is closed under addition of L.

The following notion was introduced by Huang and Lepowsky in [HL96]:

Definition 2.1. A **vertex algebra without vacuum** is a triple (V,Y,D), where V is a vector spaces, $Y:V\to\mathrm{End}(V)[[z,z^{-1}]]$, $a\mapsto Y(a,z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}$, is a linear map, and $D:V\to V$ is a linear map, satisfying the following axioms:

(1) (Truncation property) For any $a,b\in V$, $a_nb=0$ for $n\gg 0$.
(2) (D-derivative property) For any $a\in V$, $[D,Y(a,z)]=\frac{d}{dz}Y(a,z)$.
(3) (Skew-symmetry) For any $a,b\in V$, $Y(a,z)b=e^{zD}Y(b,-z)a$.
(4) (Jacobi identity) For any $a,b,c\in V$,

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(a,z_1)Y(b,z_2)-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)Y(b,z_2)Y(a,z_1)=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(a,z_0)b,z_2).$$

In particular, a VOA $(V,Y,\mathbf{1},\omega)$ is a vertex algebra without vacuum with $D=L(-1)$, and a sub-vertex algebra without vacuum is a subspace $W\subset V$ that is closed under Y and $L(-1)$.

Proposition 2.2. *Let $M\le L$ be an abelian submonoid, with the identity element $0\in L$, and let $V_M:=\bigoplus_{\alpha\in M}M_{\hat{\mathfrak{h}}}(1,\alpha)$. Then $(V_M,Y,\omega,\mathbf{1})$ is a CFT-type subVOA of $(V_L,Y,\omega,\mathbf{1})$.*

Let $S\subset L$ be a sub-semigroup, and $V_S:=\bigoplus_{\alpha\in S}M_{\hat{\mathfrak{h}}}(1,\alpha)$. Then $(V_S,Y,L(-1))$ is a sub-vertex algebra without vacuum of $(V_L,Y,L(-1))$. If, furthermore, $S\subset M$ and $M+S\subseteq S$, then V_S is an ideal of V_M.

Proof. By (2.3) and (2.4), for any $\alpha,\beta\in M$, we have

$$\begin{aligned}Y(e^\alpha,z)e^\beta&=E^-(-\alpha,z)E^+(-\alpha,z)e_\alpha z^\alpha(e^\beta)=E^-(-\alpha,z)E^+(-\alpha,z)\epsilon(\alpha,\beta)z^{(\alpha|\beta)}e^{\alpha+\beta},\\&=\exp\left(\sum_{n<0}-\frac{\alpha(n)}{n}z^{-n}\right)\epsilon(\alpha,\beta)z^{(\alpha|\beta)}e^{\alpha+\beta}\end{aligned}$$

which is contained in $M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)((z))\subset V_M((z))$, in view of the decomposition (2.7). More generally, for any $h_1(-n_1-1)\dots h_k(-n_k-1)e^\alpha\in M_{\hat{\mathfrak{h}}}(1,\alpha)$ and $h_1'(-m_1-1)\dots h_r'(-m_r-1)e^\beta\in$

$M_{\hat{\mathfrak{h}}}(1,\beta)$, with $\alpha,\beta \in M$, it is easy to see from (2.5) and (2.6) that

$$Y(h_1(-n_1-1)\dots h_k(-n_k-1)e^{\alpha}, z)h_1'(-m_1-1)\dots h_r'(-m_r-1)e^{\beta} \in M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)((z)).$$

Since M is closed under addition and $M_{\hat{\mathfrak{h}}}(1,0) \subset V_M$, it follows that V_M is a sub-VOA of V_L. Since V_M has the same Virasoro element as V_L, we have $(V_M)_n \subseteq (V_L)_n$ for each $n \geq 0$, and $(V_M)_0 = (V_L)_0 = \mathbb{C}\mathbf{1}$. Thus V_M is of the CFT-type. The second statement is also clear since S is closed under addition, and $L(-1)M_{\hat{\mathfrak{h}}}(1,\alpha) \subseteq M_{\hat{\mathfrak{h}}}(1,\alpha)$ for any $\alpha \in S$. □

The proof of Proposition 2.2 essentially relies on the fact that

$$Y(M_{\hat{\mathfrak{h}}}(1,\alpha), z)M_{\hat{\mathfrak{h}}}(1,\beta) \subset M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)((z)), \quad \alpha,\beta \in L, \tag{2.8}$$

where Y is the vertex operator of the lattice VOA V_L.

Definition 2.3. Given an abelian submonoid $M \leq L$ (resp. sub-semigroup $S \leq L$). We call $(V_M, Y, \omega, \mathbf{1})$ (resp. $(V_S, Y, L(-1))$) in Proposition 2.2 the **subVOA (resp. sub-vertex algebra without vacuum) of V_L associated to M (resp. S)**.

Remark 2.4. When L is a rank one lattice $\mathbb{Z}\alpha$, it was observed by Dong (see Proposition 4.1 in [D93]) that $V_{\mathbb{N}\alpha}$ is a subVOA of $V_{\mathbb{Z}\alpha}$. Proposition 2.2 is a generalization of this result, noting that $\mathbb{N}\alpha$ is an abelian sub-monoid of $\mathbb{Z}\alpha$.

2.2. **Definitions of Borel-type and parabolic-type subVOAs of V_L.** We use Proposition 2.2 and define the Borel and parabolic-type subVOAs of V_L by taking the subVOAs $V_M \leq V_L$ associated to special submonoids $M \leq L$.

Definition 2.5. Let L be a positive-definite even lattice of rank r, and let V_L be the lattice VOA associated to L.

(1) An abelian submonoid $B \leq L$ is called a **Borel-type submonoid** if there exists a basis $\{\alpha_1, \dots, \alpha_r\}$ of L such that $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \dots \oplus \mathbb{Z}_{\geq 0}\alpha_r$. An abelian submonoid $P \leq L$ is called a **parabolic-type submonoid** if there exists a Borel-type submonoid $B \leq L$ such that $B \subseteq P$ (So any parabolic-type submonoid is automatically of Borel-type).
(2) A **Borel-type subalgebra (or subVOA)** V_B of the lattice VOA V_L is the subVOA associated to a Borel-type submonoid $B \leq L$. i.e., $V_B = \bigoplus_{\alpha\in B} M_{\hat{\mathfrak{h}}}(1,\alpha)$. A **parabolic-type subalgebra (or subVOA)** V_P of V_L is the subVOA associated to a parabolic-type submonoid $P \leq L$. i.e., $V_P = \bigoplus_{\alpha\in P} M_{\hat{\mathfrak{h}}}(1,\alpha)$.

Observe that both the Borel-type and parabolic-type subVOAs of V_L are of the CFT-type, and have the same vacuum element $\mathbf{1}$ and Virasoro element ω with the lattice VOA V_L. Moreover, they are not simple VOAs. In fact, $S = \mathbb{Z}_{>0}\alpha_1 \oplus \dots \oplus \mathbb{Z}_{>0}\alpha_r$ is an obvious sub-semigroup of $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \dots \oplus \mathbb{Z}_{\geq 0}\alpha_r$. By Proposition 2.2, V_B has an ideal V_S.

Remark 2.6. For a Borel-type sub-VOA $V_B = \bigoplus_{\alpha\in B} M_{\hat{\mathfrak{h}}}(1,\alpha)$, where $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \dots \oplus \mathbb{Z}_{\geq 0}\alpha_r$, we may view $M_{\hat{\mathfrak{h}}}(1,0) \leq V_B$ as an analog of the "Cartan subalgebra", and view $M_{\hat{\mathfrak{h}}}(1,\alpha)$ with $\alpha \in B - \{0\}$ as an analog of a "root space" associated to a "positive-root" $\alpha \in B$. However, unlike the Lie algebra case, the Cartan-part $M_{\hat{\mathfrak{h}}}(1,0)$ is not commutative.

Example 2.7. Certain parabolic-type subVOAs can give rise to the decomposition of the lattice VOA V_L into a direct sum of two sub-vertex algebras without vacuum.

(1) Let L be the rank one positive definite even lattice $L = \mathbb{Z}\alpha$, with $(\alpha|\alpha) = 2N$ for some fixed positive integer N. Clearly, $B = \mathbb{Z}_{\geq 0}\alpha$ is a Borel-type submonoid, $\mathbb{Z}\alpha_{<0}$ is a sub-semigroup of L, and $L = B \bigsqcup \mathbb{Z}\alpha_{<0}$. Then $V_B = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} M_{\hat{\mathfrak{h}}}(1,m\alpha)$ is a Borel-type subalgebra. Moreover, $V_{\mathbb{Z}_{<0}\alpha} = \bigoplus_{m\in\mathbb{Z}_{<0}} M_{\hat{\mathfrak{h}}}(1,m\alpha)$ is a sub-vertex algebra without vacuum, and $V_{\mathbb{Z}\alpha} = V_B \oplus V_{\mathbb{Z}_{<0}\alpha}$.

(2) $L = A_2$ be the root lattice of type A_2. Note that $A_2 = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, with $(\alpha|\alpha) = (\beta|\beta) = 2$ and $(\alpha|\beta) = -1$. Then $P = \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$ is a parabolic-type submonoid of A_2 as $B = \mathbb{Z}_{\geq 0}\alpha \oplus \mathbb{Z}_{\geq 0}\beta \subset P$, and $A_2 = P \sqcup P^-$, where $P^- = \mathbb{Z}\alpha \oplus \mathbb{Z}_{<0}\beta$. See Figure 4 for an illustration. By Proposition 2.2, $V_P = \bigoplus_{\gamma\in P} M_{\hat{\mathfrak{h}}}(1,\gamma)$ is a parabolic subalgebra of V_{A_2}, V_{P^-} is a sub-vertex algebra without vacuum of V_{A_2}, and $V_{A_2} = V_P \oplus V_{P^-}$.

(3) There is another example parabolic-type subalgebra of V_{A_2}. Let $P_1 := \mathbb{Z}\alpha \oplus \mathbb{Z}_{>0}\beta + \mathbb{Z}_{\geq 0}\alpha = \{m\alpha + n\beta, k\alpha : m \in \mathbb{Z}, n > 0, k \geq 0\}$. See Figure 3 for an illustration. It is easy to check that $P_1 \leq A_2$ is a submonoid, and $B = \mathbb{Z}_{\geq 0}\alpha \oplus \mathbb{Z}_{\geq 0}\beta \subset P_1$. By Proposition 2.2 again, $V_{P_1} = \bigoplus_{\gamma\in P_1} M_{\hat{\mathfrak{h}}}(1,\gamma) \leq V_{A_2}$ is a parabolic-type subalgebra.

(4) More generally, let L be a positive-definite even lattice of rank r with a basis $\{\alpha_1, \dots, \alpha_r\}$, and let

$$P := \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{\geq 0}\alpha_r, \quad \text{and} \quad P^- := \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{<0}\alpha_r. \tag{2.9}$$

Then P is a parabolic-type submonoid of L since it contains $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$, P^- is a sub-semigroup of L, and $L = P \sqcup P^-$. Then $V_P = \bigoplus_{\alpha\in P} M_{\hat{\mathfrak{h}}}(1,\alpha) \leq V_L$ is a parabolic-type subVOA, and $V_{P^-} = \bigoplus_{\beta\in P^-} M_{\hat{\mathfrak{h}}}(1,\beta) \leq V_L$ is a sub-vertex algebra without vacuum. Moreover, $V_L = V_P \oplus V_{P^-}$.

Remark 2.8. The decomposition $V_{\mathbb{Z}\alpha} = V_B \oplus V_{\mathbb{Z}_{<0}\alpha}$ and $V_L = V_P \oplus V_{P^-}$ first appeared in the study of Rota-Baxter operators and classical Yang-Baxter equations for VOAs by Bai, Guo, the author, and Wang in [BGL, BGLW]. The projections $p : V_{\mathbb{Z}\alpha} \to V_B \subset V_{\mathbb{Z}\alpha}$ and $p : V_L \to V_P \subset V_L$ are natural examples of weight -1 Rota-Baxter operators for VOAs.

Berman, Dong, and Tan studied the representations of another class of sub-vertex algebras of V_L, which are generated by half of the lattice basis elements of L in [BDT02]. The sub-vertex algebras they studied are different with Borel-type or parabolic-type subVOAs since the structures of V_B and V_P are not symmetrical as the structure of the subalgebras in [BDT02].

Unlike the lattice VOA itself, the Borel-type and parabolic-type subVOAs that are not equal to V_L itself are irrational. This is parallel to the fact that the parabolic subalgebras of a semisimple Lie algebra are not semisimple.

Proposition 2.9. *The proper parabolic-type sub-algebras of a lattice VOA V_L are all irrational.*

Proof. By Definition 2.5, it suffices to show that a proper parabolic subVOA $V_P \lneq V_L$ is irrational. Assume P contains a Borel-type submonoid $\mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$. First, we note that there must exist some index $1 \leq j \leq r$ such that for any $n_j < 0$, the element $n_1\alpha_1 + \cdots + n_j\alpha_j + \cdots + n_r\alpha_r$, with $n_k \in \mathbb{Z}$ for any $k \neq j$, is not in P, as P would be the whole lattice L if otherwise. Without loss of generality, we assume $j = 1$, then elements in P are of the form $m\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r$, for some $m \geq 0$, and $n_2, \dots, n_r \in \mathbb{Z}$. In particular, $\mathbb{Z}_{\geq 0}\alpha_1 \subset P$. Let

$$P^1 := \{m\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : m \geq 1, n_i \in \mathbb{Z}\} \cup \{0 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : n_i \in \mathbb{Z}\}.$$

It is clear that P^1 is a submonoid of P. i.e., $P + P^1 \subseteq P^1$. Then by (2.8), $V_{P^1} := \bigoplus_{\alpha\in P^1} M_{\hat{\mathfrak{h}}}(1,\alpha)$ is a submodule of the adjoint module V_P, and $V_P/V_{P^1} \cong M_{\hat{\mathfrak{h}}}(1,0)$, which is an irreducible V_P-module. Similarly, if we let

$$P^2 := \{m\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : m \geq 2, n_i \in \mathbb{Z}\} \cup \{0 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : n_i \in \mathbb{Z}\},$$

then P^2 is a submonoid of both P^1 and P; $V_{P^2} \subset V_{P^1}$ is V_P-submodule such that $V_{P^1}/V_{P^2} \cong M_{\hat{\mathfrak{h}}}(1,\alpha_1)$, which is an irreducible V_P-module. Proceed like this, and we can construct a composition series of V_P-modules:

$$V_P \supset V_{P^1} \supset V_{P^2} \supset \ldots V_{P^m} \supset V_{P^{m+1}} \supset \ldots,$$

such that the consecutive quotient is $V_{P^m}/V_{P^{m+1}} \cong M_{\hat{\mathfrak{h}}}(1,m\alpha_1)$, which is an irreducible V_P-module, for any $m \geq 0$. Note that $M_{\hat{\mathfrak{h}}}(1,m\alpha_1)$ is not isomorphic to $M_{\hat{\mathfrak{h}}}(1,m'\alpha_1)$ if $m \neq m'$ as V_P-modules, since they are not isomorphic as $M_{\hat{\mathfrak{h}}}(1,0)$-modules and $M_{\hat{\mathfrak{h}}}(1,0) \leq V_P$. Thus, V_P

has infinitely many non-isomorphic irreducible modules, and so V_P is irrational (see [DLM98] Theorem 8.1). $\square$

2.3. **Basic properties.** In this subsection, we show that the Borel and parabolic-type subVOAs share some similar properties as a Borel subgroup of a linear algebraic group and Borel subalgebra of a semisimple Lie algebra. Although not rational, we show that all Borel-type subVOAs of a rank 1 or 2 lattice are C_1-cofinite.

2.3.1. *Conjugacy property.* In Lie theory, we know that Borel subgroups of a linear algebra group are conjugate (see [Hum1] Chapter 21); Borel subalgebras of a semisimple Lie algebra $\mathfrak{g}$ are conjugate under inner automorphisms of $\mathfrak{g}$ (see [Hum2] Chapter 16).

Let L be a root lattice Q associated to an A, D or E-type root system Φ of rank r. Then for each set of simple roots $\Delta = \{\alpha_1, \dots \alpha_r\}$, we have a Borel-type submonoid $B_\Delta = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \dots \oplus \mathbb{Z}_{\geq 0}\alpha_r$ and an associated Borel-type subVOA $V_{B_\Delta} \leq V_Q$.

Given a lattice isometry $\sigma \in O(L) = \{\sigma \in \mathrm{Aut}(L) : (\sigma\alpha|\sigma\beta) = (\alpha|\beta),\ \alpha, \beta \in L\}$. We can first lift it to $\mathbb{C}$-linear isomorphism $\sigma : \mathfrak{h} \to \mathfrak{h}$, $\sigma(\alpha\alpha) = \lambda\sigma(\alpha)$, where $\lambda \in \mathbb{C}$ and $\alpha \in L$, and then lift it to an automorphism $\hat{\sigma} : V_L \to V_L$ of the lattice VOA V_L:

$$\hat{\sigma}(h_1(-n_1)\dots h_r(-n_r)e^{\alpha}) = (\sigma h_1)(-n_1)\dots(\sigma h_r)(-n_r)e^{\sigma\alpha}, \tag{2.10}$$

where $h_i \in \mathfrak{h}$, $n_1 \geq \dots \geq n_r \geq 1$, and $\alpha \in L$ (see Section 2.4 in [DN99(2)]). On the other hand, if Δ and Δ' are two sets of simple roots of Φ, then there exists $w \in W(\Phi)$, the Weyl group of Φ, such that $w(\Delta) = \Delta'$. Since w is generated by simple reflections, we have $w \in O(Q)$ and $w(B_\Delta) = B_{\Delta'}$. Then by (2.10) and the definition of V_M in Proposition 2.2, we have $\hat{w} \in \mathrm{Aut}(V_L)$ and $\hat{w}(V_{B_\Delta}) = V_{B_{\Delta'}}$. Thus we proved the following:

Proposition 2.10. *Let Q be a root lattice of type $A, D,$ or E. Then the Borel-type subVOAs of the form V_{B_Δ}, where $\Delta \subset \Phi$ is a basis of Φ, are conjugate under* $\mathrm{Aut}(V_Q)$.

2.3.2. *Normalizer property.* In Lie theory, we know that if $P \leq G$ is a Borel or parabolic subgroup of a linear algebraic group G, then $N_G(P) = P$; and if $\mathfrak{b} \leq \mathfrak{g}$ is a Borel subalgebra of a semisimple Lie algebra $\mathfrak{g}$, then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b}) = \mathfrak{b}$ (see [Hum1] Chapter 23).

Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and $W \leq V$ be a sub-vertex algebra. There is an analog of the "centralizer" of W in V, defined in [FZ92], called the commutant. By definition, $\mathrm{Com}_V(W) = \{a \in V : w_j a = 0,\ \text{for all } j \geq 0,\ w \in W\}$. We define the normalizer of W in V as follows:

Definition 2.11. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and $W \leq V$ be a sub-vertex algebra without vacuum.

$$N_V(W) := \{a \in V : a_j W \subseteq W,\ \text{for any } j \geq 0\} \tag{2.11}$$

is called the **normalizer of** W **in** V.

Lemma 2.12. *$N_V(W) \leq V$ is a sub-vertex algebra of V with $W \subseteq N_V(W)$. Moreover, $N_V(W)$ can also be characterized as follows:*

$$N_V(W) := \{a \in V : b_j a \in W,\ \text{for any } b \in W, j \geq 0\}. \tag{2.12}$$

In particular, we have $\mathrm{Com}_V(W) \subseteq N_V(W)$.

Proof. Since $\mathbf{1}_j W = 0$ for any $j \geq 0$, we have $\mathbf{1} \in N_V(W)$. Since W is closed under Y, clearly $W \subseteq N_V(W)$. Let $a, b \in N_V(W)$, $n \in \mathbb{Z}$, $m \geq 0$, and $w \in W$. By the associativity we have

$$(a_n b)_m w = \sum_{j\geq 0} \binom{n}{j} (-1)^j a_{n-j} b_{m+j} w - (-1)^n \sum_{j \geq 0} \binom{n}{j} (-1)^j b_{m+n-j} a_j w \in W$$

since $b_{m+j}w \in W$ and $a_j w \in W$ for any $j \geq 0$. Thus, $N_V(W)$ is a sub-vertex algebra of V. Finally, by the skew-symmetry of Y we have $a_n b = \sum_{i \geq 0} 1/i!(-1)^{n+i+1} L(-1)^i b_{n+i} a$. Since $L(-1)W \subseteq W$ by Definition 2.1, it follows that for given $a \in V$, $a_n w \in W$ for any $n \geq 0$ and $w \in W$ if and only if $w_n a \in W$ for any $n \geq 0$ and $w \in W$. This shows (2.12). $\square$

Proposition 2.13. *Let $M \leq L$ be an abelian submonoid. Then $N_{V_L}(V_M) = V_M$. In particular, for any Borel-type or parabolic-type submonoid $P \leq L$, we have $N_{V_L}(V_P) = V_P$.*

Proof. Note that $V_L = \bigoplus_{\gamma\in L\backslash M} M_{\hat{\mathfrak{h}}}(1,\gamma) \oplus V_M$, in view of (2.7). Let $a = u + v \in N_{V_L}(V_M)$ with $u = \sum_{\gamma\in L\backslash M} u_\gamma$ and $v \in V_M$, where $u_\gamma = 0$ for all but finitely many $\gamma \in L\backslash M$. Since $V_M \subseteq N_{V_L}(V_M)$ by Lemma 2.12, we have $u \in N_{V_L}(V_M)$.

Now we show that $u = 0$. Since $0 \in M$, we have $M_{\hat{\mathfrak{h}}}(1,0) \subseteq V_M$. Then by (2.12), we have $(h(-1)\mathbf{1})_0 u = \sum_{\gamma\in L\backslash M} h(0)u_\gamma = \sum_{\gamma\in L\backslash M}(h|\gamma)u_\gamma \in V_M \cap \bigoplus_{\gamma\in L\backslash M} M_{\hat{\mathfrak{h}}}(1,\gamma) = 0$. Hence $(h|\gamma)u_\gamma = 0$ for all $\gamma \in L\backslash M$. For a fixed $\gamma \in L\backslash M$, choose $h \in \mathfrak{h}$ s.t. $(h|\gamma) \neq 0$, then we have $u_\gamma = 0$ for all γ, and so $u = \sum_{\gamma\in L\backslash M} u_\gamma = 0$. □

Remark 2.14. In general the normalizer $N_V(W)$ might not be equal to W. For example, let $S \subset L$ be a sub-semigroup and $0 \notin S$. Then $V_S \leq V_L$ is a sub-vertex algebra without vacuum by Proposition 2.2. It is easy to see that $M_{\hat{\mathfrak{h}}}(1,0) \subset N_{V_L}(V_S)$ but $M_{\hat{\mathfrak{h}}}(1,0) \not\subset V_S$.

2.3.3. *Strongly generation property.* Recall that a CFT-type VOA V is called strongly generated by a subset $U \subseteq V$ if V is spanned by elements of the following form:

$$u^1_{-n_1}u^2_{-n_2}\dots u^k_{-n_k}u, \quad \text{where} \quad u^1,\dots,u^k,u \in U, \quad n_1 \geq n_2 \geq \cdots \geq n_k \geq 1. \tag{2.13}$$

(see [K97]). It was proved by Karel and Li in [Li05, KL99] that a VOA V is strongly generated by a finite-dimensional subspace $U \subset V$ if and only if V is C_1-cofinite.

Lemma 2.15. *Let $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$ be a Borel-type submonoid of L such that $(\alpha_i|\alpha_j) \geq 0$, for all $1 \leq i \neq j \leq r$. Then V_B is strongly generated by $U := \{\mathbf{1}, \alpha_i(-1)\mathbf{1}, e^{\alpha_i} : 1 \leq i \leq r\}$. In particular, V_B is C_1-cofinite.*

Proof. Let W be the subspace of $V_{\mathbb{Z}_{\geq 0}\alpha}$ spanned by elements of the form (2.13), with $u^j, u \in U$. We need to show that each $M_{\hat{\mathfrak{h}}}(1, n_1\alpha_1 + \cdots + n_r\alpha_r)$ is contained in W, for all non-negative integers $n_j \geq 0$. Clearly, $M_{\hat{\mathfrak{h}}}(1,0) \subset W$. Since $M_{\hat{\mathfrak{h}}}(1,\alpha) = M_{\hat{\mathfrak{h}}}(1,0) \otimes_{\mathbb{C}} \mathbb{C}e^\alpha$, and $M_{\hat{\mathfrak{h}}}(1,0)$ is strongly generated by $\{\alpha_1(-1)\mathbf{1},\dots,\alpha_r(-1)\mathbf{1}\}$, we only need to show $e^{n_1\alpha_1+\cdots+n_r\alpha_r} \in W$.

Observe that if $e^\alpha \in W$ and $e^\beta \in W$, with $(\alpha|\beta) \geq 0$, then

$$\begin{aligned} e^\alpha_{-(\alpha|\beta)-1}e^\beta &= \mathrm{Res}_z\, z^{-(\alpha|\beta)-1}E^-(-\alpha,z)E^+(-\alpha,z)e_\alpha z^\alpha e^\beta \\ &= \mathrm{Res}_z\, z^{-(\alpha|\beta)-1}E^-(-\alpha,z)z^{(\alpha|\beta)}\epsilon(\alpha,\beta)e^{\alpha+\beta} \\ &= \epsilon(\alpha,\beta)e^{\alpha+\beta} \equiv 0 \pmod W. \end{aligned} \tag{2.14}$$

Furthermore, since $(\alpha_i|\alpha_j) \geq 0$ for all $1 \leq i \neq j \leq r$ and $(\alpha_i|\alpha_i) = 2N_i > 0$ for all i, we have $(m_1\alpha_1 + \cdots + m_r\alpha_r|n_1\alpha_1 + \cdots + n_r\alpha_r) = \sum_{i,j=1}^r m_in_j(\alpha_i|\alpha_j) \geq 0$, for any $m_i, n_j \geq 0$. In particular, if $e^{m_1\alpha_1+\cdots+m_r\alpha_r} \in W$ and $e^{n_1\alpha_1+\cdots+n_r\alpha_r} \in W$, we have $e^{(n_1+m_1)\alpha_1+\cdots+(n_r+m_r)\alpha_r} \in W$ by (2.14). Then it follows from an easy induction that $e^{n_1\alpha_1+\cdots+n_r\alpha_r} \in W$ for any $n_i \geq 0$. □

Proposition 2.16. *Assume that L is a positive-definite even lattice of rank at most 2. Let $B \leq L$ be a Borel-type submonoid. Then the VOA V_B is C_1-cofinite.*

Proof. If $L = \mathbb{Z}\alpha$ or $L = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, with $(\alpha|\beta) \geq 0$, the conclusion follows from Lemma 2.15.

Now assume that $(\alpha|\beta) = -n$ with $n > 0$, and $(\beta|\beta) \geq (\alpha|\alpha)$. Without loss of generality, we may also assume $(\beta|\beta) = 2k$ with $k \geq 1$ and $(\alpha|\alpha) = 2$. Since $L = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ is positive-definite, for any $p, q \in \mathbb{Z}$, we have $(p\beta + q\alpha|p\beta + q\alpha) = 2(kq^2 - npq + p^2) \geq 0$. Hence the discriminant $n^2 - 4k \leq 0$, and so $n \leq 2\sqrt{k}$. Now we fix $p := \lfloor \frac{n+1}{2} \rfloor > 0$. Then $2p - n = 0$ or 1.

Let $B = \mathbb{Z}_{\geq 0}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$, and let $U = \{\alpha(-1)\mathbf{1}, \beta(-1)\mathbf{1}, e^\beta, e^{\beta+\alpha}, \dots, e^{\beta+p\alpha}\}$. Let W be the subspace spanned by elements of the form (2.13), with $u^j, u \in U$. By the proof of Lemma 2.15, it suffices to show that $e^{r\beta+s\alpha} \in W$, for any $r \geq 0$ and $s \geq 0$.

Let $i \geq 0$. Since $(\beta + p\alpha|i\alpha) = i(-n + 2p) \geq 0$, we have $e^{\beta+p\alpha+i\alpha} \in W$ for all $i \geq 0$ by (2.14). This shows $e^{\beta+s\alpha} \in W$ for all $s \geq 0$. Now we show that $e^{2\beta+s\alpha} \in W$. Indeed, for any $q \geq 0$ with $q \leq p - 1$, we have $q \leq \frac{n}{2} \leq \frac{2k}{n}$ as $n \leq 2\sqrt{k}$, then $(\beta|\beta + q\alpha) = 2k - qn \geq 0$, and $e^{2\beta+q\alpha} \in W$

for $q \leq p-1$ by (2.14). On the other hand, it is easy to check that $(\beta+\alpha|\beta+(p-1)\alpha) = 2k-np-2(p-1) \geq 0$, as $p \leq \frac{n+1}{2}$ and $n \leq 2\sqrt{k}$. Hence $e^{2\beta+p\alpha} \in W$ by (2.14). Moreover, since the angle between $\beta+j\alpha$ and $\beta+(p-1)\alpha$ is less than the angle between $\beta+\alpha$ and $\beta+(p-1)\alpha$ for any $2 \leq j \leq p-1$. It follows that $(\beta+j\alpha|\beta+(p-1)\alpha) \geq 0$ for any $2 \leq j \leq p-1$, and so $e^{2\beta+s\alpha} \in W$ for $0 \leq s \leq 2p-2$.

By discussing the parity of n, we can show that $(\beta+p\alpha|\beta+(p-1)\alpha) \geq \frac{n^2}{2}-(2p-1)n+2p(p-1) = 0$ or $\frac{3}{2}$. Thus, $e^{2\beta+(2p-1)\alpha} \in W$ by (2.14). Since $(\beta+p\alpha|\beta+p\alpha) \geq 0$, we have $e^{2\beta+2p\alpha} \in W$. Finally, since $(2\beta+2p\alpha|i\alpha) = 2i(-n+2p) \geq 0$ for any $i \geq 0$, then $e^{2\beta+2p\alpha+i\alpha} \in W$ for any $i \geq 0$. This shows $e^{2\beta+s\alpha} \in W$ for all $s \geq 0$. By adopting a similar argument, we can show that $e^{t\beta+s\alpha} \in W$ for all $t \geq 0$ and $s \geq 0$. Hence $V_B = W$ is strongly generated by U, and V_B is C_1-cofinite. □

Remark 2.17. However, the Borel or parabolic-type subVOAs are **not** C_2-cofinite. We will see this in the following sections. The non-C_2-cofiniteness of these subVOAs is also indicated by the irrationality in Proposition 2.9. We conjecture that all parabolic-type subVOAs of a lattice VOA V_L are C_1-cofinite.

3. The rank-one Borel-type subVOA V_B of $V_{\mathbb{Z}\alpha}$

In this Section, we fix the rank-one lattice $L = \mathbb{Z}\alpha$, with $(\alpha|\alpha) = 2N$ for some $N \geq 1$, and $\epsilon(m\alpha, n\alpha) = 1$, for all $m, n \in \mathbb{Z}$, and study its Borel (and parabolic)-type subVOA V_B, where $B = \mathbb{Z}_{\geq 0}\alpha$. Note that the only Borel-type subVOAs of $V_{\mathbb{Z}\alpha}$ are of this form by Definition 2.5

We will show that the Zhu's algebra (see [Z96]) $A(V_B)$ of the VOA V_B isomorphic to the following associative algebra:

$$\mathbb{C}\langle x, y\rangle/\langle y^2, yx+Ny, xy-Ny\rangle,$$

where $\mathbb{C}\langle x, y\rangle$ is the tensor algebra on generators x and y.

First, we recall the construction of Zhu's algebra $A(V)$ in [Z96, FZ92]. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA. For homogeneous elements $a, b \in V$, define

$$a \circ b := \operatorname{Res}_z Y(a,z)b\frac{(1+z)^{\mathrm{wt}a}}{z^2} = \sum_{j\geq 0}\binom{\mathrm{wt}a}{j}a_{j-2}b, \tag{3.1}$$

$$a * b := \operatorname{Res}_z Y(a,z)b\frac{(1+z)^{\mathrm{wt}a}}{z} = \sum_{j\geq 0}\binom{\mathrm{wt}a}{j}a_{j-1}b. \tag{3.2}$$

Let $O(V) = \mathrm{span}\{a \circ b : a, b \in V\}$, and let $A(V) := V/O(V)$. By Theorem 2.1.1 in [Z96], $O(V)$ is a two-sided ideal with respect to $*$, and $A(V)$ is an associative algebra with respect to $*$, with the unit element $[\mathbf{1}]$. By Lemma 2.1.3 in [Z96], we also have the following formulas:

$$a * b \equiv \operatorname{Res}_z Y(b,z)a\frac{(1+z)^{\mathrm{wt}b-1}}{z} \pmod{O(V)}, \tag{3.3}$$

$$a * b - b * a \equiv \operatorname{Res}_z Y(a,z)b(1+z)^{\mathrm{wt}a-1} \pmod{O(V)}, \tag{3.4}$$

for any homogeneous $a, b \in V$. Furthermore, if $m \geq n \geq 0$, one has

$$\operatorname{Res}_z Y(a,z)b\frac{(1+z)^{\mathrm{wt}a+n}}{z^{2+m}} \equiv 0 \pmod{O(V)}. \tag{3.5}$$

3.1. A spanning set of $O(V_B)$.

It is easy to establish a morphism between the associative algebra $\mathbb{C}\langle x, y\rangle/\langle y^2, yx+Ny, xy-Ny\rangle$ and $A(V_B)$:

Proposition 3.1. *There exists an epimorphism of associative algebras:*

$$F : \mathbb{C}\langle x, y\rangle/\langle y^2, yx+Ny, xy-Ny\rangle \to A(V_B), \tag{3.6}$$

such that $F(x) = [\alpha(-1)\mathbf{1}]$ *and* $F(y) = [e^{\alpha}]$.

Proof. By the definition of $Y(e^\alpha, z)$ in (2.4), for any $n \geq 0$, we have

$$e^\alpha_n e^\alpha = 0, \quad e^\alpha_{-1}e^\alpha = \cdots = e^\alpha_{-2N}e^\alpha = 0, \quad n \geq 0, \quad \text{and} \quad e^\alpha_{-2N-1}e^\alpha = e^{2\alpha}. \tag{3.7}$$

Since $\mathrm{wt}e^\alpha = N$, by (3.2) and (3.7), we have $e^\alpha * e^\alpha = \sum_{j\geq 0}\binom{N}{j}e^\alpha_{j-1}e^\alpha = 0$. Hence $[e^\alpha]*[e^\alpha] = 0$ in $A(V_B)$. By (3.5), we have

$$\alpha(-n-2)u + \alpha(-n-1)u \equiv 0 \pmod{O(V_B)},$$

and $[\alpha(-1)u] = [u] * [\alpha(-1)\mathbf{1}]$, for all $n \geq 0$ and $u \in V$. Thus

$$[\alpha(-n_1-1)\alpha(-n_2-1)\dots\alpha(-n_k-1)u] = (-1)^{n_1+\cdots+n_k}[u]*[\alpha(-1)\mathbf{1}]*\cdots*[\alpha(-1)\mathbf{1}],$$

for any $n_1, \dots, n_k \geq 0$ and $u \in V$. This shows $A(V_B)$ is generated by $[\alpha(-1)\mathbf{1}]$ and $[e^{m\alpha}]$, for all $m \geq 1$. We claim that $[e^{k\alpha}] = 0$ for any $k \geq 2$.

Indeed, for $m \geq 1$, since we have $e^\alpha_{-2Nm-1}e^{m\alpha} = e^{(m+1)\alpha}$, $e^\alpha_{-n}e^{m\alpha} = 0$ for any $n \leq 2Nm$, and $2Nm + 1 \geq 2$, then it follows from (3.5) that for any $m \geq 1$,

$$\begin{aligned} e^{(m+1)\alpha} &= e^\alpha_{-2Nm-1}e^{m\alpha} + \binom{N}{1}e^\alpha_{-2Nm}e^{m\alpha} + \cdots + \binom{N}{N}e^\alpha_{-2Nm-1+N}e^{m\alpha} \\ &= \mathrm{Res}_z Y(e^\alpha, z)e^{m\alpha}\frac{(1+z)^N}{z^{2Nm+1}} \equiv 0 \pmod{O(V_B)}. \end{aligned}$$

Hence $[e^{k\alpha}] = 0$ in $A(V_B)$ for all $k \geq 2$, and $A(V_B)$ is generated by $[\alpha(-1)\mathbf{1}]$ and $[e^\alpha]$. Then we have an epimorphism $F : \mathbb{C}\langle x, y\rangle \to A(V_B)$, such that $F(x) = [\alpha(-1)\mathbf{1}]$ and $F(y) = [e^\alpha]$. Moreover, by (3.1) and the definition of $Y(e^\alpha, z)$ in (2.4), we have

$$\begin{aligned} e^\alpha \circ \mathbf{1} &= e^\alpha_{-2}\mathbf{1} + \binom{N}{1}e^\alpha_{-1}\mathbf{1} + \sum_{j\geq 2}\binom{N}{j}e^\alpha_{j-2}\mathbf{1} = \mathrm{Res}_z z^{-2}\exp\left(-\sum_{n<0}\frac{\alpha(n)}{n}z^{-n}\right)e^\alpha + Ne^\alpha + 0 \\ &= \alpha(-1)e^\alpha + Ne^\alpha \equiv 0 \pmod{O(V_B)}, \end{aligned}$$

and so $[e^\alpha]*[\alpha(-1)\mathbf{1}] + N[e^\alpha] = 0$ in $A(V_B)$. By (3.4), we also have

$$[\alpha(-1)\mathbf{1}]*[e^\alpha] - [e^\alpha]*[\alpha(-1)\mathbf{1}] = [\mathrm{Res}_z Y(\alpha(-1)\mathbf{1}, z)e^\alpha] = [\alpha(0)e^\alpha] = 2N[e^\alpha],$$

and so $[\alpha(-1)\mathbf{1}]*[e^\alpha] - N[e^\alpha] = 0$ in $A(V_B)$. Therefore, the epimorphism $F : \mathbb{C}\langle x, y\rangle \to A(V_B)$ factors through $\mathbb{C}\langle x, y\rangle/\langle y^2, yx + Ny, xy - Ny\rangle$, and induces an epimorphism F in (3.6). □

Our goal is to show that epimorphism (3.6) is an isomorphism. We can achieve this goal by finding a concrete description of $O(V_B)$.

Let O' be the subspace of V_B spanned by the following elements:

$$\begin{cases} \alpha(-n-2)u + \alpha(-n-1)u, & u \in V_B, \text{ and } n \geq 0, \\ \alpha(-1)v + Nv, & v \in \bigoplus_{m\geq 1} M_{\hat{\mathfrak{h}}}(1, m\alpha), \\ M_{\hat{\mathfrak{h}}}(1, k\alpha), & k \geq 2. \end{cases} \tag{3.8}$$

Our goal in this Section is to show that $O(V_B) = O'$.

First, we prove the easier part: $O' \subseteq O(V_B)$. By (3.5), clearly we have $\alpha(-n-2)u+\alpha(-n-1)u \in O(V_B)$, for all $u \in V_B$ and $n \geq 0$. By Theorem 2.1.1 in [Z96], we also have

$$a * O(V_B) \subset O(V_B), \quad \text{and} \quad O(V_B) * a \subset O(V_B), \quad a \in V. \tag{3.9}$$

Lemma 3.2. *For any $k \geq 2$, we have $M_{\hat{\mathfrak{h}}}(1, k\alpha) \subset O(V_B)$.*

Proof. By the proof of Proposition 3.1, we have $e^{k\alpha} \in O(V_B)$, for any $k \geq 2$. By (3.3), we have $u * \alpha(-1)\mathbf{1} \equiv \alpha(-1)u \pmod{O(V_B)}$. Now by (3.5) and (3.9), we have:

$$\begin{aligned} \alpha(-n_1-1)\dots\alpha(-n_r-1)e^{k\alpha} &\equiv (-1)^{n_1+\cdots+n_r}\alpha(-1)^r e^{k\alpha} \pmod{O(V_B)} \\ &\equiv (-1)^{n_1+\cdots+n_r}e^{k\alpha}*(\alpha(-1)\mathbf{1})*\cdots*(\alpha(-1)\mathbf{1}) \pmod{O(V_B)} \\ &\equiv 0 \pmod{O(V_B)}, \end{aligned}$$

for any $k \geq 2$, $r \geq 1$, and $n_1, \dots n_r \geq 0$, where the last congruence follows from (3.9). Thus we have $M_{\hat{\mathfrak{h}}}(1, k\alpha) \subset O(V_B)$, for all $k \geq 2$. □

Lemma 3.3. *For any $v \in \bigoplus_{m \geq 1} M_{\hat{\mathfrak{h}}}(1, m\alpha)$, we have $\alpha(-1)v + Nv \in O(V_B)$.*

Proof. If $m \geq 2$ and $v \in M_{\hat{\mathfrak{h}}}(1, m\alpha)$, then by Lemma 3.2, we have $v \in O(V_B)$, and

$$\alpha(-1)v + Nv \equiv v * (\alpha(-1)\mathbf{1}) + Nv \equiv 0 \pmod{O(V_B)},$$

by (3.9). Now let $m = 1$, by the proof of Proposition 3.1, we have $\alpha(-1)e^\alpha + Ne^\alpha = e^\alpha \circ \mathbf{1} \equiv 0 \pmod{O(V_B)}$. Let $v = \alpha(-n_1 - 1) \dots \alpha(-n_r - 1)e^\alpha$ be a general spanning element of $M_{\hat{\mathfrak{h}}}(1, \alpha)$, where $r \geq 1$, and $n_1, \dots n_r \geq 0$. Since $[\alpha(-1), \alpha(-p)] = 0$ for all $p \geq 1$, we have:

$$\begin{aligned}
\alpha(-1)v + Nv &= \alpha(-n_1 - 1) \dots \alpha(-n_r - 1)(\alpha(-1)e^\alpha + Ne^\alpha) \\
&\equiv (-1)^{n_1 + \dots + n_r} \alpha(-1)^r (\alpha(-1)e^\alpha + Ne^\alpha) \pmod{O(V_B)} \\
&\equiv (-1)^{n_1 + \dots + n_r} (\alpha(-1)e^\alpha + Ne^\alpha) * (\alpha(-1)\mathbf{1}) * \dots * (\alpha(-1)\mathbf{1}) \pmod{O(V_B)} \\
&\equiv 0 \pmod{O(V_B)},
\end{aligned}$$

where the last congruence follows from (3.9) and the fact that $\alpha(-1)e^\alpha + Ne^\alpha \in O(V_B)$. □

By Lemma 3.2 and Lemma 3.3, we have $O' \subseteq O(V_B)$.

3.2. **Proof of the main Theorem.** Conversely, we need to show that $a \circ u = \mathrm{Res}_z Y(a, z)u((1 + z)^{\mathrm{wt} a}/z^2) \in O'$, for any homogeneous $a, u \in V$. First, note that if $a \in M_{\hat{\mathfrak{h}}}(1, m\alpha)$ and $u \in M_{\hat{\mathfrak{h}}}(1, n\alpha)$ for some $m, n \geq 1$, then by (2.3) and (2.8), we have:

$$\mathrm{Res}_z Y(a, z)u \frac{(1 + z)^{\mathrm{wt} a}}{z^2} \in M_{\hat{\mathfrak{h}}}(1, (m + n)\alpha)((z)) \subset O'((z)),$$

since $m + n \geq 2$, and $M_{\hat{\mathfrak{h}}}(1, k\alpha) \subset O'$ for any $k \geq 2$ by (3.8). Thus, we only need to show:

$$a \circ u \in O', \quad \text{for} \quad \begin{cases} a \in M_{\hat{\mathfrak{h}}}(1, \alpha) \quad \text{and} \quad u \in M_{\hat{\mathfrak{h}}}(1, 0), \\ \text{or} \\ a \in M_{\hat{\mathfrak{h}}}(1, 0) \quad \text{and} \quad u \in M_{\hat{\mathfrak{h}}}(1, \alpha). \end{cases} \tag{3.10}$$

First, we consider the case when $a \in M_{\hat{\mathfrak{h}}}(1, \alpha)$ and $u \in M_{\hat{\mathfrak{h}}}(1, 0)$.

Our strategy is to show $\mathrm{Res}_z Y(e^\alpha, z)u((1 + z)^N/z^{2+n}) \in O'$ first, where $u \in M_{\hat{\mathfrak{h}}}(1, 0)$ and $n \geq 0$, then prove $\mathrm{Res}_z Y(a, z)u((1+z)^{\mathrm{wt} a}/z^2) \in O'$, for $a = \alpha(-n_1) \dots \alpha(-n_r)e^\alpha \in M_{\hat{\mathfrak{h}}}(1, \alpha)$ by induction.

Lemma 3.4. *For any $m \geq 1$, we have $\alpha(-m)O' \subset O'$. For any $u \in M_{\hat{\mathfrak{h}}}(1, \alpha)$, we have $L(-1)u + L(0)u \in O'$.*

Proof. Since $[\alpha(-m), \alpha(-n)] = 0$ for any $m, n \geq 1$, and $\alpha(-m)M_{\hat{\mathfrak{h}}}(1, k\alpha) \subset M_{\hat{\mathfrak{h}}}(1, k\alpha)$, for any $k \geq 0$, we have $\alpha(-m)O' \subset O'$, in view of (3.8).

Let $u = \alpha(-n_1) \dots \alpha(-n_r)e^\alpha \in M_{\hat{\mathfrak{h}}}(1, \alpha)$, where $r \geq 0$ and $n_1, \dots, n_r \geq 1$. Since $L(-1)e^\alpha = (e^\alpha)_{-2}\mathbf{1} = \alpha(-1)e^\alpha$, and $[L(-1), \alpha(-m)] = ma(-m - 1)$, we have

$$\begin{aligned}
&L(-1)\alpha(-n_1) \dots \alpha(-n_r)e^\alpha + L(0)\alpha(-n_1) \dots \alpha(-n_r)e^\alpha \\
&= \alpha(-n_1) \dots \alpha(-n_r)\alpha(-1)e^\alpha + \sum_{j=1}^r n_j \cdot \alpha(-n_1) \dots \alpha(-n_j - 1) \dots \alpha(-n_r)e^\alpha \\
&\quad + (n_1 + \dots + n_k + N)\alpha(-n_1) \dots \alpha(-n_r)e^\alpha \\
&= \alpha(-n_1) \dots \alpha(-n_r)(\alpha(-1)e^\alpha + Ne^\alpha) \\
&\quad + \sum_{j=1}^r (\alpha(-n_j - 1) + \alpha(-n_j))\alpha(-n_1) \dots \widehat{\alpha(-n_j)} \dots \alpha(-n_r)e^\alpha \\
&\equiv 0 \pmod{O'},
\end{aligned}$$

where the last congruence follows from $\alpha(-1)e^{\alpha}+Ne^{\alpha}\in O'$, $\alpha(-m)O'\subset O'$ for any $m\geq 1$, and $\alpha(-n-1)v+\alpha(-n)v\in O'$ for all $v\in M_{\hat{\mathfrak{h}}}(1,\alpha)$ and $n\geq 1$, in view of (3.8). □

Proposition 3.5. *Let $u\in M_{\hat{\mathfrak{h}}}(1,0)$, and $n\geq 0$. We have*

$$\operatorname{Res}_z Y(e^{\alpha},z)u\frac{(1+z)^N}{z^{2+n}}\in O'. \tag{3.11}$$

Proof. We use induction on the length r of a spanning element $u=\alpha(-n_1)\dots\alpha(-n_r)\mathbf{1}$ of $M_{\hat{\mathfrak{h}}}(1,0)$, where $n_1,\dots,n_r\geq 1$. The base case is $u=\mathbf{1}$. First, we note that

$$e^{\alpha}_{-j-1}\mathbf{1}=\frac{1}{j!}(L(-1)^je^{\alpha})_{-1}\mathbf{1}=\frac{1}{j!}L(-1)^je^{\alpha},\quad j\geq 0.$$

Since $e^{\alpha}_{-j-1}\mathbf{1}\in M_{\hat{\mathfrak{h}}}(1,\alpha)$ for any $j\geq 0$, by Lemma 3.4 we have

$$\begin{aligned}L(-1)^je^{\alpha}&\equiv -L(0)L(-1)^{j-1}e^{\alpha}\pmod{O'}\\&=(-1)(N+j-1)L(-1)^{j-1}e^{\alpha}\\&\vdots\\&\equiv(-1)^j(N+j-1)(N+j-2)\dots(N+1)Ne^{\alpha}\pmod{O'}.\end{aligned}$$

Then it follows from the definition of binomial coefficients that

$$\begin{aligned}Y(e^{\alpha},z)\mathbf{1}&=\sum_{j\geq 0}e^{\alpha}_{-j-1}\mathbf{1}z^j=\sum_{j\geq 0}\frac{1}{j!}L(-1)^jz^je^{\alpha}\\&\equiv\sum_{j\geq 0}(-1)^j\frac{(N+j-1)(N+j-2)\dots(N+1)N}{j!}z^je^{\alpha}\pmod{O'}\\&=\sum_{j\geq 0}\frac{(-N-j+1)(-N-j+2)\dots(-N-1)(-N)}{j!}z^je^{\alpha}\\&=\sum_{j\geq 0}\binom{-N}{j}z^je^{\alpha}=(1+z)^{-N}e^{\alpha}.\end{aligned}\tag{3.12}$$

Now by (3.11) and (3.12), and the assumption that $n\geq 0$, we have

$$\operatorname{Res}_z Y(e^{\alpha},z)\mathbf{1}\frac{(1+z)^N}{z^{2+n}}\equiv\operatorname{Res}_z(1+z)^{-N}\frac{(1+z)^N}{z^{2+n}}e^{\alpha}=\operatorname{Res}_z\frac{1}{z^{2+n}}e^{\alpha}=0\pmod{O'}.$$

This finishes the proof of base case. Assume the conclusion holds for smaller r. Note that for any $m\geq 1$, we have

$$[\alpha(-m),Y(e^{\alpha},z)]=\sum_{i\geq 0}\binom{-m}{i}Y(\alpha(i)e^{\alpha},z)z^{-m-i}=2NY(e^{\alpha},z)z^{-m}.$$

Then by $\alpha(-m)O'\subset O'$ in Lemma 3.4, the base case and the induction hypothesis, we have

$$\begin{aligned}&\operatorname{Res}_z Y(e^{\alpha},z)\alpha(-n_1)\dots\alpha(-n_r)\mathbf{1}\frac{(1+z)^N}{z^{2+n}}\\&=\operatorname{Res}_z\alpha(-n_1)\dots\alpha(-n_r)Y(e^{\alpha},z)\mathbf{1}\frac{(1+z)^N}{z^{2+n}}\\&\quad-\sum_{j=1}^r\operatorname{Res}_z\alpha(-n_1)\dots\alpha(-n_{j-1})[\alpha(-n_j),Y(e^{\alpha},z)]\alpha(n_{j+1})\dots\alpha(-n_r)\mathbf{1}\frac{(1+z)^N}{z^{2+n}}\\&\equiv-\sum_{j=1}^r 2N\operatorname{Res}_z\alpha(-n_1)\dots\alpha(-n_{j-1})Y(e^{\alpha},z)\alpha(-n_{j+1})\dots\alpha(-n_r)\mathbf{1}\frac{(1+z)^N}{z^{2+n+n_j}}\pmod{O'}\\&\equiv 0\pmod{O'}\end{aligned}$$

where the last congruence follows from the induction hypothesis, which indicates that

$$\mathrm{Res}_z\, Y(e^{\alpha}, z)\alpha(-n_{j+1})\dots\alpha(-n_r)\mathbf{1}((1+z)^N)/z^{2+n+n_j} \in O'.$$

Hence (3.11) holds for any $u \in M_{\hat{\mathfrak{h}}}(1,0)$ and $n \geq 0$. □

Consider an arbitrary spanning element a in $M_{\hat{\mathfrak{h}}}(1,\alpha)$, we can write

$$a = \alpha(-n_1)\dots\alpha(-n_r)e^{\alpha}, \tag{3.13}$$

for some $r \geq 0$ and $n_1,\dots,n_r \geq 1$. We want to show that $a \circ u \in O'$, for any $u \in M_{\hat{\mathfrak{h}}}(1,0)$. If $r = 0$, we have $a = e^{\alpha}$, and $a \circ u \in O'$ by Proposition 3.5.

Assume $r \geq 1$, we will use induction on the length r of a to show that $a \circ u \in O'$. The base case $a = \alpha(-k)e^{\alpha}$ with $\mathrm{wt} a = N + k$ is given by the following Lemma:

Lemma 3.6. *For any $k \geq 1$, $n \geq 0$, and $u \in M_{\hat{\mathfrak{h}}}(1,0)$, we have*

$$\mathrm{Res}_z\, Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}} \in O'. \tag{3.14}$$

Proof. By the Jacobi identity of VOA, it is easy to derive the following formula:

$$Y(\alpha(-1)v, z) = \sum_{j\geq 0}\alpha(-j-1)Y(v,z)z^j + \sum_{j\geq 0} Y(v,z)\alpha(j)z^{-j-1}, \quad v \in V_{\mathbb{Z}\alpha}. \tag{3.15}$$

Now we prove (3.14) by induction on k. When $k = 1$, by (3.15) we have

$$\begin{aligned}
&\mathrm{Res}_z\, Y(\alpha(-1)e^{\alpha}, z)u\frac{(1+z)^{N+1}}{z^{2+n}}\\
&= \mathrm{Res}_z \sum_{j\geq 0}\alpha(-j-1)Y(e^{\alpha},z)uz^j\frac{(1+z)^{N+1}}{z^{2+n}} + \mathrm{Res}_z\sum_{j\geq 0} Y(e^{\alpha},z)\alpha(j)u\frac{(1+z)^{N+1}}{z^{2+n+j+1}}\\
&= \mathrm{Res}_z\left(\sum_{j\geq 0}\alpha(-j-1)Y(e^{\alpha},z)uz^j\frac{(1+z)^{N}}{z^{2+n}} + \sum_{j\geq 0}\alpha(-j-1)Y(e^{\alpha},z)uz^{j+1}\frac{(1+z)^{N}}{z^{2+n}}\right)\\
&\quad + \mathrm{Res}_z\sum_{j\geq 0} Y(e^{\alpha},z)\alpha(j)u\frac{(1+z)^{N}}{z^{2+n+j+1}} + \mathrm{Res}_z\sum_{j\geq 0} Y(e^{\alpha},z)\alpha(j)u\frac{(1+z)^{N}}{z^{2+n+j}}\\
&\equiv \mathrm{Res}_z\,\alpha(-1)Y(e^{\alpha},z)u\frac{(1+z)^{N}}{z^{2+n}}\\
&\quad + \sum_{j\geq 0}(\alpha(-j-2)+\alpha(-j-1))\,\mathrm{Res}_z\, Y(e^{\alpha},z)uz^{j+1}\frac{(1+z)^{N}}{z^{2+n}} + 0 \pmod{O'}\\
&\equiv \mathrm{Res}_z\,\alpha(-1)Y(e^{\alpha},z)u\frac{(1+z)^{N}}{z^{2+n}} \pmod{O'},
\end{aligned}$$

where the first congruence follows from Proposition 3.5, as $n + j \geq 0$, and the second congruence follows from (3.8). Furthermore, by Proposition 3.5 again, we have

$$\begin{aligned}
&\mathrm{Res}_z\,\alpha(-1)Y(e^{\alpha},z)u\frac{(1+z)^{N}}{z^{2+n}}\\
&= \mathrm{Res}_z\left(Y(e^{\alpha},z)\alpha(-1)u\frac{(1+z)^{N}}{z^{2+n}} + \sum_{j\geq 0}\binom{-1}{j}z^{-1-j}Y(\alpha(j)e^{\alpha},z)u\frac{(1+z)^{N}}{z^{2+n}}\right)\\
&\equiv 0 + \mathrm{Res}_z\, 2NY(e^{\alpha},z)u\frac{(1+z)^{N}}{z^{2+n+1}} \pmod{O'}\\
&\equiv 0 \pmod{O'}.
\end{aligned}$$

This proves (3.14) for $k = 1$. Assume (3.14) holds for smaller k. Note that $[L(-1), \alpha(-k)] = \frac{1}{k}\alpha(-k-1)$ and $\alpha(-k)L(-1)e^{\alpha} = \alpha(-k)\sum_{i\geq 0} \frac{\alpha}{\sqrt{2N}}(-1-i)\frac{\alpha}{\sqrt{2N}}(i)e^{\alpha} = \alpha(-1)\alpha(-k)e^{\alpha}$. By (3.15),

$$\begin{aligned}
&\mathrm{Res}_z\, Y(\alpha(-k-1)e^{\alpha}, z)u\frac{(1+z)^{N+k+1}}{z^{2+n}}\\
&= \frac{1}{k}\,\mathrm{Res}_z\left(Y(L(-1)\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k+1}}{z^{2+n}} - Y(\alpha(-k)L(-1)e^{\alpha}, z)u\frac{(1+z)^{N+k+1}}{z^{2+n}}\right)\\
&= -\frac{1}{k}\,\mathrm{Res}_z\left(Y(\alpha(-k)e^{\alpha}, z)u\frac{d}{dz}\left(\frac{(1+z)^{N+k+1}}{z^{2+n}}\right) + Y(\alpha(-1)\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k+1}}{z^{2+n}}\right)\\
&= -\frac{N+k+1}{k}\,\mathrm{Res}_z\, Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&\quad + \frac{2+n}{k}\,\mathrm{Res}_z\, Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{3+n}} + \frac{2+n}{k}\,\mathrm{Res}_z\, Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&\quad - \frac{1}{k}\,\mathrm{Res}_z \sum_{j\geq 0} \alpha(-j-1)Y(\alpha(-k)e^{\alpha}, z)uz^{j}\frac{(1+z)^{N+k+1}}{z^{2+n}}\\
&\quad - \frac{1}{k}\,\mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-k)e^{\alpha}, z)\alpha(j)u\frac{(1+z)^{N+k+1}}{z^{2+n+j+1}}\\
&\equiv 0 - \frac{1}{k}\,\mathrm{Res}_z \sum_{j\geq 0} \alpha(-j-1)\left(Y(\alpha(-k)e^{\alpha}, z)uz^{j} + Y(\alpha(-k)e^{\alpha}, z)uz^{j+1}\right)\frac{(1+z)^{N+k}}{z^{2+n}}\\
&\quad - \frac{1}{k}\,\mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-k)e^{\alpha}, z)\alpha(j)u\frac{(1+z)^{N+k}(1+z)}{z^{2+n+1+j}} \pmod{O'}\\
&= -\frac{1}{k}\,\mathrm{Res}_z\, \alpha(-1)Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&\quad - \frac{1}{k}\,\mathrm{Res}_z \sum_{j\geq 0} (\alpha(-j-2)+\alpha(-j-1))\, Y(\alpha(-k)e^{\alpha}, z)uz^{j+1}\frac{(1+z)^{N+k}}{z^{2+n}}\\
&\quad - \frac{1}{k}\,\mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-k)e^{\alpha}, z)\alpha(j)u\frac{(1+z)^{N+k}(1+z)}{z^{2+n+1+j}}\\
&\equiv -\frac{1}{k}\,\mathrm{Res}_z\, \alpha(-1)Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}} \pmod{O'},
\end{aligned}$$

where the first congruence follows from the induction hypothesis (3.14), and the second congruence follows from (3.8) and the induction hypothesis. By the Jacobi identity and the Heisenberg relation $[\alpha(j), \alpha(-k)] = \delta_{j,k}kK$ for any $j \geq 0$, we have

$$\begin{aligned}
&-\frac{1}{k}\,\mathrm{Res}_z\, \alpha(-1)Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&= -\frac{1}{k}\,\mathrm{Res}_z\, Y(\alpha(-k)e^{\alpha}, z)\alpha(-1)u\frac{(1+z)^{N+k}}{z^{2+n}} - \frac{1}{k}\,\mathrm{Res}_z \sum_{j\geq 0}\binom{-1}{j} z^{-1-j}Y(\alpha(j)\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&\equiv 0 - \frac{1}{k}\,\mathrm{Res}_z(-1)^k kY(e^{\alpha}, z)u(1+z)^k\frac{(1+z)^{N}}{z^{2+n+1+k}} \pmod{O'}\\
&= -\mathrm{Res}_z(-1)^k\sum_{i\geq 0}\binom{k}{i}Y(e^{\alpha}, z)u\frac{(1+z)^{N}}{z^{2+n+1+i}}\\
&\equiv 0 \pmod{O'},
\end{aligned}$$

where the first congruence follows from the induction hypothesis, and the second congruence follows from Proposition 3.5. Therefore, we have

$$\mathrm{Res}_z\, Y(\alpha(-k-1)e^\alpha, z)u\frac{(1+z)^{N+k+1}}{z^{2+n}} \in O'.$$

So (3.14) holds for $k+1$, and the inductive step is complete. □

Proposition 3.7. *For any $u \in M_{\hat{\mathfrak{h}}}(1,0)$ and $a \in M_{\hat{\mathfrak{h}}}(1,\alpha)$, we have:*

$$\mathrm{Res}_z\, Y(a,z)u\frac{(1+z)^{\mathrm{wt}a}}{z^{2+n}} \in O', \tag{3.16}$$

for any $n \geq 0$. In particular, we have $a \circ u \in O'$.

Proof. Write $a = \alpha(-n_1)\dots\alpha(-n_r)e^\alpha$ as (3.13), where $r \geq 0$ and $n_1,\dots,n_r \geq 1$. We prove (3.16) by induction on r. By Proposition 3.5 and Lemma 3.6, (3.16) holds when $a = e^\alpha$ or $a = \alpha(-k)e^\alpha$. Now let $r \geq 2$. The induction hypothesis is the assumption that

$$\mathrm{Res}_z\, Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)u\frac{(1+z)^{N+n_2+\dots+n_r}}{z^{2+n}} \in O', \tag{3.17}$$

for $n_2,\dots,n_r \geq 1$, $n \geq 0$, and $u \in M_{\hat{\mathfrak{h}}}(1,0)$. First, we claim that

$$\mathrm{Res}_z\, Y(\alpha(-1)\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)u\frac{(1+z)^{N+n_2+\dots+n_r+1}}{z^{2+n}} \in O'. \tag{3.18}$$

Denote $N+n_2+\dots+n_r$ by m, note that $\mathrm{wt}(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha) = m$. Then by (3.15), (3.8), and the induction hypothesis, we have

$$\begin{aligned}
&\mathrm{Res}_z\, Y(a(-1)\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)u\frac{(1+z)^{1+m}}{z^{2+n}}\\
&= \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)uz^j\frac{(1+z)^{1+m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)(a(j)u)\frac{(1+z)^{1+m}}{z^{2+n+j+1}}\\
&\equiv \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)uz^j\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(a_2(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}} \pmod{O'}\\
&= \mathrm{Res}_z\, a(-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)u\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} a(-j-2)Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}}\\
&= \mathrm{Res}_z[a(-1), Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)]u\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z\, Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)a(-1)u\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0}(a(-j-2)+a(-j-1))Y(\alpha(-n_2)\dots\alpha(-n_r)e^\alpha, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}}
\end{aligned}$$

$$
\begin{aligned}
&\equiv \sum_{i\geq 0}\binom{-1}{i} z^{-1-i}\operatorname{Res}_z Y(a(i)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n}} \pmod{O'}\\
&= \operatorname{Res}_z 2Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^m}{z^{3+n}}\\
&\quad + \sum_{i\geq 0}\sum_{s=2}^{r}\binom{-1}{i}\operatorname{Res}_z Y(\alpha(-n_2)\dots[\alpha(i),\alpha(-n_s)]\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n+i+1}}\\
&\equiv \sum_{s=2}^{r}(-1)^{n_s}n_s\operatorname{Res}_z Y(\alpha(-n_2)\dots\widehat{\alpha(-n_s)}\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n+n_s+1}} \pmod{O'}.
\end{aligned}
$$

Denote $\alpha(-n_2)\dots\widehat{\alpha(-n_s)}\dots\alpha(-n_r)e^{\alpha}$ by a_s. Then $m = \mathrm{wt} a_s + n_s$, and by the induction hypothesis (3.17), with r replaced by $r-1$, we have

$$
\begin{aligned}
&\sum_{s=2}^{r}(-1)^{n_s}n_s\operatorname{Res}_z Y(\alpha(-n_2)\dots\widehat{\alpha(-n_s)}\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n+n_s+1}}\\
&= \sum_{s=2}^{r}(-1)^{n_s}n_s\operatorname{Res}_z Y(a_s,z)u(1+z)^{n_s}\frac{(1+z)^{\mathrm{wt} a_s}}{z^{2+n+n_s+1}}\\
&= \sum_{s=2}^{r}\sum_{j\geq 0}\binom{n_s}{j}(-1)^{n_s}n_s\operatorname{Res}_z Y(a_s,z)u\frac{(1+z)^{\mathrm{wt} a_s}}{z^{2+n_s+j+1}}\\
&\equiv 0 \pmod{O'}
\end{aligned}
$$

since $n_s + j + 1 \geq 1$. This proves (3.18). Now assume that

$$
\operatorname{Res}_z Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+n_2+\dots+n_r}}{z^{2+n}} \in O', \tag{3.19}
$$

for some fixed $k\geq 1$, $n_2,\dots,n_r\geq 1$, and $n\geq 0$. We want to show that

$$
\operatorname{Res}_z Y(\alpha(-k-1)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}} \in O'. \tag{3.20}
$$

Indeed, by adopting a similar calculation as the proof of Lemma 3.6, we have

$$
\begin{aligned}
&\operatorname{Res}_z Y(\alpha(-k-1)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\\
&= \operatorname{Res}_z \frac{1}{k}Y(L(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\\
&\quad + \operatorname{Res}_z \frac{1}{k}Y(\alpha(-k)[L(-1),\alpha(-n_2)\dots\alpha(-n_r)]e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\\
&\quad + \operatorname{Res}_z \frac{1}{k}Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\\
&= -\operatorname{Res}_z \frac{1}{k}Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{d}{dz}\left(\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\right)\\
&\quad + \sum_{s=2}^{r}\operatorname{Res}_z \frac{n_s}{k}Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_s-1)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+\dots(1+n_s)\dots+n_r}}{z^{2+n}}\\
&\quad + \operatorname{Res}_z \frac{1}{k}Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\\
&= -\operatorname{Res}_z \frac{1}{k}(N+k+1+\dots+n_r)Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+n_2+\dots+n_r}}{z^{2+n}}
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
&+\operatorname{Res}_z \frac{2+n}{k} Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)u\frac{(1+z)^{N+k+n_2+\dots+n_r}(1+z)}{z^{2+n+1}}\\
&+\sum_{s=2}^{r} \operatorname{Res}_z \frac{n_s}{k} Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_s-1)\dots\alpha(-n_r)e^{\alpha}, z)u\frac{(1+z)^{N+k+\dots(1+n_s)\dots+n_r}}{z^{2+n}}\\
&+\operatorname{Res}_z \frac{1}{k} Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\\
\equiv\ & 0+\operatorname{Res}_z \frac{1}{k} Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}} \pmod{O'},
\end{aligned}
$$

where the congruences follow from the induction (on $k \geq 1$) hypothesis (3.19). Moreover, by adopting a similar argument as our previous proof of (3.18), with the given assumption (3.17), the following inclusion holds:

$$
\operatorname{Res}_z \frac{1}{k} Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}} \in O',
$$

with the given assumption (3.19). Thus, (3.20) is true. Now the induction step for $k \geq 1$ and the induction step for the length $r \geq 1$ of $a \in M_{\hat{\mathfrak{h}}}(1,\alpha)$ are both complete. □

Now we have finished the proof of $a \circ u \in O'$ for the first case in (3.10). The second case when $a \in M_{\hat{\mathfrak{h}}}(1,0)$ and $u \in M_{\hat{\mathfrak{h}}}(1,\alpha)$ follows from a similar induction process as Lemma 3.6 and Proposition 3.7 (see also (3.1.5) and (3.1.6) in [FZ92]), we omit the details.

In conclusion, we proved the following theorem:

Theorem 3.8. *Let $V_B = V_{\mathbb{Z}_{\geq 0}\alpha}$ be the Borel-type subVOA of $V_{\mathbb{Z}\alpha}$, with $(\alpha|\alpha) = 2N$. Then*

$$
\begin{aligned}
O(V_B) = O' = \operatorname{span}\Big\{&\alpha(-n-2)u + \alpha(-n-1)u,\ \alpha(-1)v + v,\ M_{\hat{\mathfrak{h}}}(1,k\alpha) :\\
&n \geq 0,\ u \in V_B,\ v \in \bigoplus_{m\geq 1} M_{\hat{\mathfrak{h}}}(1,m\alpha),\ k \geq 2\Big\}.
\end{aligned} \tag{3.22}
$$

Corollary 3.9. *With the settings in Theorem 3.8, the epimorphism F given by* (3.6) *is an isomorphism of associative algebras. In particular, we have $A(V_B) \cong \mathbb{C}[x] \oplus \mathbb{C}y$, with*

$$
y^2 = 0, \quad yx = -Ny, \quad xy = Ny. \tag{3.23}
$$

Proof. We construct an inverse map of F in (3.6) as follows:

$$
\begin{aligned}
G : V &\to \mathbb{C}\langle x, y\rangle / \langle y^2, yx + Ny, xy - Ny\rangle,\\
\alpha(-n_1-1)\dots\alpha(-n_r-1)\mathbf{1} &\mapsto (-1)^{n_1+\dots n_r}x^r,\\
\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha} &\mapsto (-1)^{n_1+\dots n_r}yx^r = (-1)^{r+n_1+\dots n_r}y,\\
M_{\hat{\mathfrak{h}}}(1,k\alpha) &\mapsto 0, \quad k \geq 2,
\end{aligned} \tag{3.24}
$$

where $r \geq 0$ and $n_1, \dots, n_r \geq 0$, and we use the same symbols x and y to denote their image in the quotient space. Note that G is well-defined since $V = \bigoplus_{k\geq 0} M_{\hat{\mathfrak{h}}}(1,k\alpha)$, and $\alpha(-n_1 - 1)\dots\alpha(-n_r-1)\mathbf{1}$ and $\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}$ are basis elements of $M_{\hat{\mathfrak{h}}}(1,0)$ and $M_{\hat{\mathfrak{h}}}(1,\alpha)$, respectively. We claim that $G(O(V_B)) = 0$.

Indeed, it suffices to show that G vanishes on the spanning elements of $O(V_B)$ in (3.22). By the definition (3.24) of G, we already have $G(M_{\hat{\mathfrak{h}}}(1,k\alpha)) = 0$ for any $k \geq 2$. In particular, we have $G(\alpha(-n-2)u + \alpha(-n-1)u) = G(\alpha(-1)v + Nv) = 0$ if $u, v \in M_{\hat{\mathfrak{h}}}(1,k\alpha)$ for some $k \geq 2$. If $u = \alpha(-n_1-1)\dots\alpha(-n_r-1)\mathbf{1} \in M_{\hat{\mathfrak{h}}}(1,0)$, by (3.24) we have

$$
\begin{aligned}
&G(\alpha(-n-2)u + \alpha(-n-1)u)\\
&= G(\alpha(-n-2)\alpha(-n_1-1)\dots\alpha(-n_r-1)\mathbf{1}) + G(\alpha(-n-1)\alpha(-n_1-1)\dots\mathbf{1})\\
&= (-1)^{n+1+n_1+\dots+n_r}x^{r+1} + (-1)^{n+n_1+\dots+n_r}x^{r+1} = 0.
\end{aligned}
$$

If $u = \alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha} \in M_{\hat{\mathfrak{h}}}(1,\alpha)$, by (3.24) we have:

$$\begin{aligned}
&G(\alpha(-n-1)u + \alpha(-n-1)u)\\
&= G(\alpha(-n-2)\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}) + G(\alpha(-n-1)\alpha(-n_1-1)\dots e^{\alpha})\\
&= (-1)^{n+1+n_1+\dots+n_r}yx^{r+1} + (-1)^{n+n_1+\dots+n_r}yx^{r+1} = 0.
\end{aligned}$$

Thus, $G(\alpha(-n-2)u+\alpha(-n-1)u) = 0$ for any $u \in V$. Finally, if $v = \alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha} \in M_{\hat{\mathfrak{h}}}(1,\alpha)$, by (3.24) we have

$$\begin{aligned}
&G(\alpha(-1)v + Nv)\\
&= G(\alpha(-1)\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}) + NG(\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha})\\
&= (-1)^{n_1+\dots n_r}yx^{r+1} + (-1)^{n_1+\dots n_r}Nyx^r\\
&= (-1)^{n_1+\dots n_r}(yx+Ny)x^r = 0,
\end{aligned}$$

as $yx + Ny = 0$. Thus, G in (3.24) induces a linear map

$$\begin{aligned}
G : A(V) = V/O(V_B) \to \mathbb{C}\langle x,y\rangle/\langle y^2, yx+Ny, xy-Ny\rangle, \quad \text{such that}\\
G([\alpha(-n_1-1)\dots\alpha(-n_r-1)\mathbf{1}]) = G((-1)^{n_1+\dots+n_r}[\alpha(-1)\mathbf{1}]^r) = (-1)^{n_1+\dots+n_r}x^r,\\
G([\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}]) = G((-1)^{n_1+\dots+n_r}[e^{\alpha}]*[\alpha(-1)\mathbf{1}]^r) = (-1)^{n_1+\dots+n_r}yx^r,
\end{aligned} \tag{3.25}$$

for any $r \geq 0$, $n_1,\dots,n_r \geq 0$, and $k \geq 2$. Since $A(V_B)$ is spanned by elements of the form $[\alpha(-n_1-1)\dots\alpha(-n_r-1)\mathbf{1}]$ and $[\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}]$ because of (3.22), it is clear that $G\circ F = \mathrm{Id}$ and $F\circ G = \mathrm{Id}$, in view of (3.6) and (3.25). □

3.3. **Applications of the main theorem.** The following example gives a comparison between the Zhu's algebras $A(V_B)$ and $A(V_{\mathbb{Z}\alpha})$:

Example 3.10. Let $L = \mathbb{Z}\alpha$, with $(\alpha|\alpha) = 2$. Then $L = A_1$ is the root lattice of type A_1. Then V_{A_1} is isomorphic to the affine VOA $L_{\hat{sl_2}}(1,0)$, where $sl_2 = \mathbb{C}e+\mathbb{C}h+\mathbb{C}f$, and $e^{\alpha}\mapsto e$, $\alpha(-1)\mathbf{1}\mapsto h$, $e^{-\alpha}\mapsto f$ (see [FK80, FZ92]).

Recall that $A(L_{\hat{sl_2}}(1,0)) \cong U(sl_2)/\langle e^2\rangle$, where $\langle e^2\rangle$ is the two-sided ideal of $A(L_{\hat{sl_2}}(1,0))$ generated by e^2, and $[a(-1)\mathbf{1}] \mapsto a + \langle e^2\rangle$ for all $a \in sl_2$ (see [FZ92]). By applying the Lie bracket $[a,\cdot]$ to e^2 repeatedly, it is easy to show that the following relations hold in $A(L_{\hat{sl_2}}(1,0))$:

$$eh+e=0;\quad h^2-h-2fe=0;\quad fh-f=0;\quad e^2=f^2=0, \tag{3.26}$$

where we used the same symbol to denote the equivalent classes. It follows that $A(L_{\hat{sl_2}}(1,0))$ has a basis $\{1,e,f,h,fe\}$.

Now let A be the subalgebra of $A(L_{\hat{sl_2}}(1,0))$ generated by the Borel subalgebra $\mathfrak{b} = \mathbb{C}e+\mathbb{C}h \leq \mathfrak{g}$. By (3.26), we have $A = \langle 1,e,h,fe\rangle$. Moreover, by Corollary 3.9, we have an epimorphism of associative algebras:

$$A(V_B) \twoheadrightarrow A = \langle 1,e,h,fe\rangle \leq A(V_{A_1}),\quad x\mapsto h,\ y\mapsto e,\ x^2-x\mapsto fe. \tag{3.27}$$

For the lattice VOA V_{A_1}, the following Corollary presents a spanning set of $O(V_{A_1})$, which will be used in the next Section.

Corollary 3.11. *Let $L = A_1 = \mathbb{Z}\alpha$. Then $O(V_{A_1})$ is spanned by the following elements:*

$$\begin{cases}
\alpha(-n-2)u+\alpha(-n-1)u, & u\in V_{A_1},\ \text{and } n\geq 0,\\
\pm\alpha(-1)v+Nv, & v\in M_{\hat{\mathfrak{h}}}(1,\pm\alpha),\\
M_{\hat{\mathfrak{h}}}(1,\pm k\alpha), & k\geq 2.\\
\alpha(-1)^3w-\alpha(-1)w, & w\in M_{\hat{\mathfrak{h}}}(1,0).
\end{cases} \tag{3.28}$$

Proof. Let O'' be the subspace of V_{A_1} spanned by the elements (3.28). Since both $O(V_{\mathbb{Z}_{\geq 0}\alpha})$ and $O(V_{\mathbb{Z}_{\geq 0}(-\alpha)})$ are contained in $O(V_{A_1})$, by Lemma 3.2 and 3.3, we have $\alpha(-n-2)u+\alpha(-n-1)u \in O(V_{A_1})$ for any $u \in V_{A_1}$, $\pm\alpha(-1)v+Nv \in O(V_{A_1})$ for any $v \in M_{\hat{\mathfrak{h}}}(1, \pm\alpha)$, and $M_{\hat{\mathfrak{h}}}(1, \pm k\alpha) \subset O(V_{A_1})$ for all $k \geq 2$. Moreover, since $\mathrm{wt} e^{\alpha} = 2$ and $(\alpha| -\alpha) = -2$, we have

$$\begin{aligned}
e^{\alpha} \circ e^{-\alpha} &= e^{\alpha}_{-2}e^{-\alpha} + e^{\alpha}_{-1}e^{-\alpha} \\
&= \frac{1}{3}\alpha(-3)\mathbf{1} + \frac{1}{4}(\alpha(-2)\alpha(-1) + \alpha(-1)\alpha(-2))\mathbf{1} + \frac{1}{3!}\alpha(-1)^3\mathbf{1} + \frac{1}{2}\alpha(-2)\mathbf{1} + \frac{1}{2!}\alpha(-1)^2\mathbf{1} \\
&\equiv \frac{1}{3}\alpha(-1)\mathbf{1} - \frac{1}{2}\alpha(-1)^2\mathbf{1} + \frac{1}{6}\alpha(-1)^3\mathbf{1} - \frac{1}{2}\alpha(-1)\mathbf{1} + \frac{1}{2}\alpha(-1)^2\mathbf{1} \\
&\equiv \frac{1}{6}(\alpha(-1)^3\mathbf{1} - \alpha(-1)\mathbf{1}) \pmod{O(V_{A_1})}.
\end{aligned}$$

Hence $\alpha(-1)^3w - \alpha(-1)w \in O(V_{A_1})$ for any $w \in M_{\hat{\mathfrak{h}}}(1, 0)$, in view of the proof of Lemma 3.3. This shows $O'' \subseteq O(V_{A_1})$. Conversely, with a similar argument as Proposition 3.1 and Corollary 3.9, we can easily show that

$$V_{A_1}/O'' \cong \mathbb{C}z \oplus \mathbb{C}[x]/\langle x^3 - x\rangle \oplus \mathbb{C}y, \quad [e^{-\alpha}] \mapsto z,\ [\alpha(-1)\mathbf{1}] \mapsto x,\ [e^{\alpha}] \mapsto y.$$

In particular, we have $\dim V_{A_1}/O'' = 5$. On the other hand, we have $\dim V/O(V_{A_1}) = 5$, since V_{A_1} has two irreducible modules $V_{\mathbb{Z}\alpha}$ and $V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}$ with bottom levels of dimensions 1 and 2. respectively (see [D93] Theorem 3.1), and there is a one-to-one correspondence between irreducible $A(V_{A_1})$-modules and irreducible V_{A_1}-modules (see [Z96] Theorem 2.2.2). Then we have $O'' = O(V_{A_1})$ since there is an epimorphism $V_{A_1}/O'' \to A(V_{A_1})$. □

Remark 3.12. There is a description of Zhu's algebra $A(V_L)$ of the lattice VOAs V_L in [DLM97] by a quotient algebra of $U(\widehat{sl_2(\mathbb{C})})$. Such a description was obtained by the classification theorem on irreducible modules over V_L in [D93]. Since we do not have a classification of irreducible modules over V_B (or V_P) to begin with, we have to first determine $O(V_B)$ (or $O(V_P)$) in order to determine the structures of $A(V_B)$ (or $A(V_P)$).

In the rest of this subsection, we assume that $L = \mathbb{Z}\alpha$ with $(\alpha|\alpha) = 2N$ and $B = \mathbb{Z}_{\geq 0}\alpha$. Now we classify the irreducible modules over the Borel-type subVOA V_B.

Lemma 3.13. *If $U \neq 0$ is an irreducible module over $A(V_B) \cong \mathbb{C}[x] \oplus \mathbb{C}y$, then we must have $y.U = 0$, and $U \cong \mathbb{C}e^{\lambda}$ for some $\lambda \in \mathfrak{h} = \mathbb{C}\alpha$, with $x.e^{\lambda} = (\alpha|\lambda)e^{\lambda}$.*

Proof. By (3.23), $\mathbb{C}y$ is an ideal of $A(V_{\mathbb{Z}_{\geq 0}\alpha})$. Then $y.U \leq U$ is an $A(V_B)$-submodule, and so $y.U$ is either U or 0. If $y.U = U$, then we have $0 = y^2.U = y.U = U$, a contradiction. Thus, $y.U = 0$ and U is an irreducible module over $\mathbb{C}[x]$. We have $U \cong \mathbb{C}[x]/\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of $\mathbb{C}[x]$. By Hilbert's Nullstellensatz, we have $\mathfrak{m} = \langle x - \mu\rangle$ for some $\mu \in \mathbb{C}$. Choose $\lambda \in \mathfrak{h}$ so that $(\alpha|\lambda) = \mu$. Then $U \cong \mathbb{C}[x]/\langle x - (\alpha|\lambda)\rangle \cong \mathbb{C}e^{\lambda}$, with $x.e^{\lambda} = (\alpha|\lambda)e^{\lambda}$. □

Lemma 3.14. *For any irreducible module $W = M_{\hat{\mathfrak{h}}}(1, \lambda)$ over the Heisenberg VOA $M_{\hat{\mathfrak{h}}}(1, 0)$, W is also an irreducible module over the Borel-type subVOA V_B, where $Y_W : V_B \to \mathrm{End}(W)[[z, z^{-1}]]$ satisfies $Y_W(a, z) = 0$, for any $a \in M_{\hat{\mathfrak{h}}}(1, n\alpha)$ and $n \geq 1$, and $Y_W|_{M_{\hat{\mathfrak{h}}}(1,0)}$ is given by the action of the Heisenberg VOA $M_{\hat{\mathfrak{h}}}(1, 0)$.*

Proof. By (2.8), for any $a \in M_{\hat{\mathfrak{h}}}(1, n\alpha)$ and $b \in M_{\hat{\mathfrak{h}}}(1, m\alpha)$, where $m, n \geq 0$, $Y_W(a, z)b$ is either 0 or given by the Heisenberg module vertex operator. Hence (W, Y_W) is a well-defined module over the Borel-type subVOA V_B. It is clear that W is irreducible. □

Theorem 3.15. $\Sigma = \left\{(W = M_{\hat{\mathfrak{h}}}(1, \lambda), Y_W) : \lambda \in \mathfrak{h} = \mathbb{C}\alpha\right\}$, *with Y_W defined by Lemma 3.14, is a complete list of irreducible modules over the rank-one Borel-type subVOA V_B.*

Moreover, the fusion rule between the irreducible V_B-modules $M_{\hat{\mathfrak{h}}}(1, \lambda)$, $M_{\hat{\mathfrak{h}}}(1, \mu)$, and $M_{\hat{\mathfrak{h}}}(1, \gamma)$ is the same as the fusion rule of these modules as modules over the Heisenberg VOA. In other words, $N\binom{M_{\hat{\mathfrak{h}}}(1,\gamma)}{M_{\hat{\mathfrak{h}}}(1,\lambda)\ M_{\hat{\mathfrak{h}}}(1,\mu)} \cong \delta_{\lambda+\mu,\gamma}$.

Proof. Given a module $(W = M_{\hat{\mathfrak{h}}}(1,\lambda), Y_W)$ in Σ, the bottom level $W(0) = \mathbb{C}e^{\lambda}$ is an $A(V_B)$-module, with the actions of $x = [\alpha(-1)\mathbf{1}]$ and $y = [e^{\alpha}]$ given by

$$x.e^{\alpha} = o(\alpha(-1)\mathbf{1})e^{\lambda} = (\alpha|\lambda)e^{\lambda}, \quad y.e^{\lambda} = o(e^{\alpha})e^{\lambda} = \mathrm{Res}_z z^{N-1}Y_W(e^{\alpha},z)e^{\lambda} = 0.$$

(see [FZ92, Z96]). By Lemma 3.14, such $A(V_B)$-modules $W(0)$, with W varies in Σ, are all the irreducible modules over $A(V_B)$ up to isomorphism. Then by Theorem 2.2.2 in [Z96], Σ is the complete list of irreducible modules over V_B. Finally, note that any intertwining operator between modules over the Heisenberg VOA $I \in I\binom{M_{\hat{\mathfrak{h}}}(1,\gamma)}{M_{\hat{\mathfrak{h}}}(1,\lambda)\ M_{\hat{\mathfrak{h}}}(1,\mu)}$ can be naturally lifted up to an intertwining operator $\tilde{I}$ of V_B, since the Jacobi identity of I is

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{W^3}(a,z_1)I(v,z_2)u - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)I(v,z_2)Y_{W^2}(a,z_1)u \\ &= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)I(Y_{W^1}(a,z_0)v,z_2)u, \end{aligned} \tag{3.29}$$

and $Y_{W^i}(a,z) = 0$ for $i = 1,2,3$, if $a \in M_{\hat{\mathfrak{h}}}(1,n\alpha)$ with $n \geq 1$. Therefore, we can replace I in (3.29) by the intertwining operator $\tilde{I}$ of V_B. Conversely, we can also restrict any intertwining operator $\mathcal{Y} \in I\binom{M_{\hat{\mathfrak{h}}}(1,\gamma)}{M_{\hat{\mathfrak{h}}}(1,\lambda)\ M_{\hat{\mathfrak{h}}}(1,\mu)}$ of V_B to an intertwining operator $\mathcal{Y}$ of the same type between modules over the Heisenberg VOA. Therefore, the fusion rules of the Borel-type subVOA V_B is the same as the fusion rules between modules over the Heisenberg VOA $M_{\hat{\mathfrak{h}}}(1,0)$. □

Remark 3.16. Theorem 3.15 is also parallel to the semisimple Lie algebra case. Note that a Borel subalgebra $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ has the same irreducible modules as its Cartan part $\mathfrak{h}$, and the irreducible modules over (the abelian Lie algebra) $\mathfrak{h}$ are all one-dimensional.

4. The rank-two parabolic-type subVOA V_P of V_{A_2}

In this and the next Sections, we study a nontrivial lowest rank example of parabolic-type subVOAs in V_L, where $\mathrm{rank}(L) = 2$. Note that if $\mathrm{rank}(L) = 1$, by Definition 2.5, there is no difference between Borel-type and parabolic-type submonoids, and the rank-one case has been dealt with in Section 3.

Let $L = A_2 = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ be the root lattice of type A_2: $(\alpha|\alpha) = (\beta|\beta) = 2$, and $(\alpha|\beta) = -1$, equipped with the 2-cocycle $\epsilon : L \times L \to \langle \pm 1 \rangle$, such that

$$\epsilon(\alpha,\alpha) = 1, \quad \epsilon(\beta,\beta) = 1, \quad \epsilon(\alpha,\beta) = 1, \quad \epsilon(\beta,\alpha) = -1, \tag{4.1}$$

Since ϵ is bi-multiplicative, $\epsilon(\alpha,\alpha)\epsilon(\alpha,-\alpha) = \epsilon(\alpha,0) = 1$ and $\epsilon(\beta,\beta)\epsilon(\beta,-\beta) = \epsilon(\beta,0) = 1$, it follows that

$$\epsilon(\alpha,-\alpha) = 1, \quad \epsilon(\beta,-\beta) = 1. \tag{4.2}$$

Let $P := \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$, which is a parabolic-type submonoid of A_2. See Example 2.7 and Figure 1 for an illustration. Then $V_P = \bigoplus_{\gamma\in P} M_{\hat{\mathfrak{h}}}(1,\gamma)$, where $\mathfrak{h} = \mathbb{C}\alpha \oplus \mathbb{C}\beta$, is a parabolic-type subVOA of V_{A_2}. Similar to the argument in Proposition 2.16, by using (2.14) repeatedly, we can easily prove the following:

Proposition 4.1. *V_P is strongly generated by $U = \{e^{\alpha}, e^{-\alpha}, e^{\beta}, e^{\alpha+\beta}, \alpha(-1)\mathbf{1}, \beta(-1)\mathbf{1}\}$. In particular, V_P is C_1-cofinite.*

We will determine the structure of Zhu's algebra $A(V_P)$ in this Section, and classify the irreducible modules over V_P in the next Section. In this Section and the next, P will always represent the parabolic-type submonoid $P = \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$ of the root lattice A_2.

4.1. **A spanning set of $O(V_P)$.** Similar to our approach in Section 3, we first give a concrete description of $O(V_P)$, and then use it to determine $A(V_P)$.

Lemma 4.2. *Suppose $\gamma, \theta \in P$ such that $\frac{(\gamma|\gamma)}{2} = N \geq 1$ and $(\gamma|\theta) = n \geq 1$. Then $e^{\gamma+\theta} \in O(V_P)$.*

Proof. By the definition of $Y(e^\gamma, z)e^\theta$ in (2.4), we have

$$e^\gamma_{-n-1}e^\theta = \mathrm{Res}_z\, z^{-n-1}E^-(-\gamma, z)z^{(\gamma|\theta)}\epsilon(\gamma, \theta)e^{\gamma+\theta} = \epsilon(\gamma, \theta)e^{\gamma+\theta},$$

and $e^\gamma_{-m}e^\theta = 0$ for any $m \leq n$. Since $n \geq 1$ and $\mathrm{wt}\, e^\gamma = \frac{(\gamma|\gamma)}{2} = N \geq 1$, we have

$$\mathrm{Res}_z\, Y(e^\gamma, z)e^\theta \frac{(1+z)^N}{z^{1+n}} = e^\gamma_{-n-1}e^\theta + \binom{N}{1}e^\gamma_{-n}e^\theta + \cdots + \binom{N}{N}e^\gamma_{-n-1+N}e^\theta = \epsilon(\gamma, \theta)e^{\gamma+\theta} \in O(V_P).$$

This shows $e^{\gamma+\theta} \in O(V_P)$ since $\epsilon(\gamma, \theta) \in \{\pm 1\}$. □

Lemma 4.3. *Let $S := \{e^{m\alpha+n\beta} : m \in \mathbb{Z}, n \in \mathbb{N}\}\backslash\{e^{\pm\alpha}, e^\beta, e^{\alpha+\beta}\}$. We have $S \subset O(V_P)$.*

Proof. For $m \geq 1$, since $(\alpha|m\alpha + \beta) = 2m - 1 \geq 1$, by Lemma 4.2 and an easy induction on m, we have $e^{(m+1)\alpha+\beta} \in O(V_P)$ for any $m \geq 1$. Similarly, since $(\beta|\alpha + n\beta) = 2n - 1 \geq 1$ when $n \geq 1$, we have

$$e^{\alpha+(n+1)\beta} \in O(V_P), \quad n \geq 1. \tag{4.3}$$

Now let $n \geq 2$, and assume that $e^{m\alpha+n\beta} \in O(V_P)$, for all $m \geq 1$. We want to show that $e^{m\alpha+(n+1)\beta} \in O(V_P)$, for all $m \geq 1$.

Indeed, since $(m\alpha + n\beta|\alpha + \beta) = m((\alpha|\alpha) + (\alpha|\beta)) + n((\beta|\alpha) + (\beta|\beta)) = m + n \geq 1$, by Lemma 4.2 we have $e^{(m+1)\alpha+(n+1)\beta} \in O(V_P)$ for all $m \geq 1$. Thus, $e^{m\alpha+(n+1)\beta} \in O(V_P)$ for all $m \geq 2$, and by (4.3), we see that it is also true for $m = 1$. This finishes the induction step and shows that $e^{m\alpha+n\beta} \in O(V_P)$, for all $m \geq 1$ and $n \geq 2$. Hence we have

$$S_1 := \{e^{m\alpha+n\beta} : m \geq 1, n \geq 2\} \cup \{e^{m\alpha+\beta} : m \geq 2\} \subset O(V_P). \tag{4.4}$$

On the other hand, for any $m \geq 1$, since $(-m\alpha|\beta) = m \geq 1$, we have $e^{-m\alpha+\beta} \in O(V_P)$ for all $m \geq 1$. Since $(-\alpha + n\beta|\beta) = 1 + 2n \geq 1$ for any $n \geq 0$, we have

$$e^{-\alpha+(n+1)\beta} \in O(V_P), \quad n \geq 0. \tag{4.5}$$

Using the fact that $(-m\alpha + n\beta| - \alpha + \beta) = 3m + 3n \geq 1$ for $m, n \geq 1$, together with (4.5), we can similarly show that

$$S_2 := \{e^{-m\alpha+n\beta} : m \geq 1, n \geq 1\} \subset O(V_P). \tag{4.6}$$

Finally, for any $m \geq 1$ and $n \geq 1$, since $(\alpha|m\alpha) = (-\alpha| - m\alpha) = 2m > 1$ and $(\beta|n\beta) = 2n > 1$, by Lemma 4.2 again, we can easily show that $e^{\pm m\alpha} \in O(V_P)$ and $e^{n\beta} \in O(V_P)$, for all $m \geq 2$ and $n \geq 2$. Then by (4.4) and (4.6), we have $S = S_1 \cup S_2 \cup \{e^{\pm m\alpha} : m \geq 2\} \cup \{e^{n\beta} : n \geq 2\} \subset O(V_P)$. □

Inspired by (3.8), (3.28), and Lemma 4.3, we give the following definition:

Definition 4.4. Let O be the subspace of V_P spanned by the following elements:

$$\begin{cases} h(-n-2)u + h(-n-1)u, & u \in V_P,\ h \in \mathfrak{h},\ n \geq 0; \\ \gamma(-1)v + v, & v \in M_{\hat{\mathfrak{h}}}(1, \gamma),\ \gamma \in \{\alpha, -\alpha, \beta, \alpha+\beta\}; \\ \gamma(-1)^2v + \gamma(-1)v, & v \in M_{\hat{\mathfrak{h}}}(1, \gamma+\gamma'),\ \gamma, \gamma' \in \{\alpha, -\alpha, \beta, \alpha+\beta\},\ \gamma+\gamma' \in \{\alpha+\beta, \beta\}; \\ M_{\hat{\mathfrak{h}}}(1, m\alpha+n\beta), & m\alpha+n\beta \in (\mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta)\backslash\{0, \alpha, -\alpha, \beta, \alpha+\beta\}; \\ \alpha(-1)^3w - \alpha(-1)w, & w \in M_{\hat{\mathfrak{h}}}(1, 0). \end{cases} \tag{4.7}$$

Note that the only possible ordered pairs (γ, γ') such that $\gamma, \gamma' \in \{\alpha, -\alpha, \beta, \alpha+\beta\}$ and $\gamma+\gamma' \in \{\alpha+\beta, \beta\}$ as in (4.7) are contained in the following set:

$$\{(\alpha, \beta), (\beta, \alpha), (-\alpha, \alpha+\beta), (\alpha+\beta, -\alpha)\}. \tag{4.8}$$

So the set of elements $\gamma(-1)^2v + \gamma(-1)v$ in (4.7) can be written more explicitly as follows:

$$\alpha(-1)^2v + \alpha(-1)v, \qquad \beta(-1)^2v + \beta(-1)v, \quad v \in M_{\hat{\mathfrak{h}}}(1, \alpha+\beta), \tag{4.9}$$

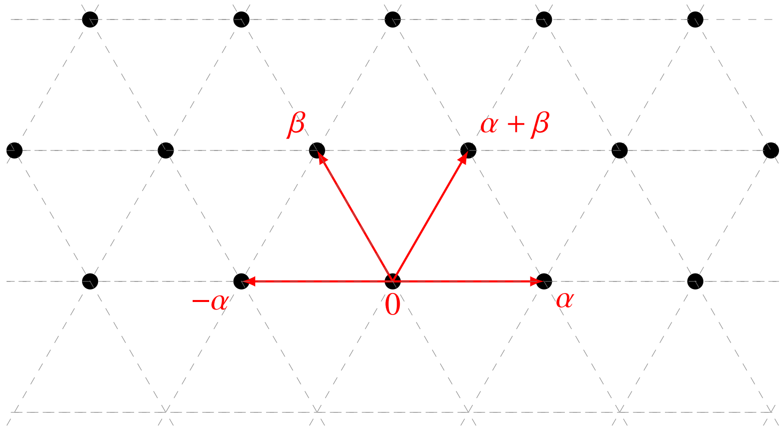

FIGURE 1.

$$\alpha(-1)^2 v - \alpha(-1)v, \qquad (\alpha+\beta)(-1)^2 v + (\alpha+\beta)(-1)v, \quad v \in M_{\hat{\mathfrak{h}}}(1,\beta). \tag{4.10}$$

Moreover, we observe that $O(V_{\mathbb{Z}\alpha}) \subset O$, and $O(V_{\mathbb{Z}_{\geq 0}\gamma}) \subset O$ for any $\gamma \in \{\alpha, -\alpha, \beta, \alpha+\beta\}$, since $M_{\widehat{\mathbb{C}\alpha}}(1, \pm\alpha) \subset M_{\hat{\mathfrak{h}}}(1, \pm\alpha)$ and $M_{\widehat{\mathbb{C}\gamma}}(1,\gamma) \subset M_{\hat{\mathfrak{h}}}(1,\gamma)$ (see (3.22) and Corollary 3.11).

Remark 4.5. We use Figure 1 to illustrate our definition for O. The black dots in this diagram represent the elements in parabolic-type submonoid $P = \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$ of A_2. Except for the roots represented by the red vectors, the Heisenberg modules $M_{\hat{\mathfrak{h}}}(1,\gamma)$ associated to all other dots γ are contained in the subspace O of $V_P = \bigoplus_{\gamma \in P} M_{\hat{\mathfrak{h}}}(1,\gamma)$.

In the rest of this subsection, we will show that O is equal to $O(V_P)$.

Proposition 4.6. *Let O be the subspace given by Definition 4.4. Then $O \subseteq O(V_P)$.*

Proof. It is clear that $h(-n-2)u + h(-n-1)u \in O(V_P)$ for any $h \in \mathfrak{h}$, $u \in V_P$, and $n \geq 0$. Then by the congruence $h(-m)v \equiv (-1)^{m-1} v * (h(-1)\mathbf{1}) \pmod{O(V_P)}$ and (3.9), we have $h(-m)O(V_P) \subseteq O(V_P)$ for any $h \in \mathfrak{h}$ and $m \geq 1$. Now it follows from Lemma 4.3 that $M_{\hat{\mathfrak{h}}}(1, m\alpha + n\beta) \subseteq O(V_P)$ for $m\alpha + n\beta \in P \backslash \{\alpha, -\alpha, \beta, \alpha+\beta\}$.

Moreover, given $\gamma \in \{\alpha, -\alpha, \beta, \alpha+\beta\}$, since $(\gamma|\gamma) = 2$, we have

$$\gamma(-1)e^{\gamma} + e^{\gamma} = e^{\gamma}_{-2}\mathbf{1} + e^{\gamma}_{-1}\mathbf{1} = e^{\gamma} \circ \mathbf{1} \in O(V_P),$$

and so $\gamma(-1)v + v \in O(V_P)$ for any $v \in M_{\hat{\mathfrak{h}}}(1,\gamma)$ and $\gamma \in \{\alpha, -\alpha, \beta, \alpha+\beta\}$. On the other hand, suppose $\gamma, \gamma' \in \{\alpha, -\alpha, \beta, \alpha+\beta\}$ such that $\gamma + \gamma' \in \{\beta, \alpha+\beta\}$. As $h(-n-2)u + h(-n-1)u \equiv 0 \pmod{O(V_P)}$ for any $h \in \mathfrak{h}$, $n \geq 0$, and $u \in V_P$, we have

$$\begin{aligned} 0 \equiv e^{\gamma} \circ e^{\gamma'} &= \frac{1}{2}\epsilon(\gamma,\gamma')\gamma(-2)e^{\gamma+\gamma'} + \frac{1}{2!}\epsilon(\gamma,\gamma')\gamma(-1)^2 e^{\gamma+\gamma'} + \epsilon(\gamma,\gamma')\gamma(-1)e^{\gamma+\gamma'} \\ &\equiv \frac{1}{2}\epsilon(\gamma,\gamma')\gamma(-1)^2 e^{\gamma+\gamma'} + \frac{1}{2}\epsilon(\gamma,\gamma')\gamma(-1)e^{\gamma+\gamma'} \pmod{O(V_P)}. \end{aligned}$$

Thus, $\gamma(-1)^2 v + \gamma(-1)v \in O(V_P)$ for any $v \in M_{\hat{\mathfrak{h}}}(1, \gamma+\gamma')$.

Finally, since we have $O(V_{\mathbb{Z}\alpha}) \subset O(V_P)$, it follows from Corollary 3.11 that $\alpha(-1)^3\mathbf{1} - \alpha(-1)\mathbf{1} \in O(V_P)$. Thus $\alpha(-1)^3 w - \alpha(-1)w \in O(V_P)$ for any $w \in M_{\hat{\mathfrak{h}}}(1,0)$, as $h(-m)O(V_P) \subseteq O(V_P)$ for any $h \in \mathfrak{h}$ and $m \geq 1$. $\square$

Conversely, in order to prove $O(V_P) \subseteq O$, we need to show that

$$M_{\hat{\mathfrak{h}}}(1,\eta) \circ M_{\hat{\mathfrak{h}}}(1,\theta) \subset O, \quad \text{for any } \eta, \theta \in P = \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta. \tag{4.11}$$

By the construction of subspace O in (4.7) and $Y(e^{\eta}, z)e^{\theta} \in M_{\hat{\mathfrak{h}}}(1, \eta+\theta)((z))$, we have

$$M_{\hat{\mathfrak{h}}}(1,\eta) \circ M_{\hat{\mathfrak{h}}}(1,\theta) \subset M_{\hat{\mathfrak{h}}}(1,\eta+\theta) \subset O, \quad \text{if} \quad \eta+\theta \in P \backslash \{0, \alpha, -\alpha, \beta, \alpha+\beta\}.$$

Hence we only need to show that

$$M_{\hat{\mathfrak{h}}}(1,0)\circ M_{\hat{\mathfrak{h}}}(1,\gamma)\subset O \quad\text{and}\quad M_{\hat{\mathfrak{h}}}(1,\gamma)\circ M_{\hat{\mathfrak{h}}}(1,0)\subset O; \tag{4.12}$$

$$M_{\hat{\mathfrak{h}}}(1,\alpha)\circ M_{\hat{\mathfrak{h}}}(1,\beta)\subset O \quad\text{and}\quad M_{\hat{\mathfrak{h}}}(1,\beta)\circ M_{\hat{\mathfrak{h}}}(1,\alpha)\subset O; \tag{4.13}$$

$$M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)\circ M_{\hat{\mathfrak{h}}}(1,-\alpha)\subset O \quad\text{and}\quad M_{\hat{\mathfrak{h}}}(1,-\alpha)\circ M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)\subset O; \tag{4.14}$$

$$M_{\hat{\mathfrak{h}}}(1,\alpha)\circ M_{\hat{\mathfrak{h}}}(1,-\alpha)\subset O \quad\text{and}\quad M_{\hat{\mathfrak{h}}}(1,-\alpha)\circ M_{\hat{\mathfrak{h}}}(1,\alpha)\subset O, \tag{4.15}$$

where $\gamma\in\{\alpha,-\alpha,\beta,\alpha+\beta\}$.

(4.12) can be proved by a similar argument as Proposition 3.5, Lemma 3.6, and Proposition 3.7, we omit the details.

4.1.1. *Proof of* (4.13) *and* (4.14). Let (γ,γ') be an ordered pair in the set (4.8). Given a spanning element $u=h^1(-n_1)\dots h^r(-n_r)e^{\gamma}$ of $M_{\hat{\mathfrak{h}}}(1,\gamma)$ and $v=h^1(-m_1)\dots h^s(-m_s)e^{\gamma'}$ of $M_{\hat{\mathfrak{h}}}(1,\gamma')$, we need to show that $\mathrm{Res}_z\, Y(u,z)v(1+z)^{\mathrm{wt}u}/z^2\in O$. For $u=e^{\gamma}$, this is true by the following (stronger) statement:

Proposition 4.7. *Let (γ,γ') be an ordered pair in the set* (4.8)*, and $n\geq 0$. We have*

$$\mathrm{Res}_z\, Y(e^{\gamma},z)\left(h^1(-n_1)\dots h^r(-n_r)e^{\gamma'}\right)\frac{(1+z)}{z^{2+n}}\in O, \tag{4.16}$$

where $r\geq 0$, $h^i\in\mathfrak{h}$ for all i, and $n_1\geq\cdots\geq n_r\geq 1$.

Proof. It is clear from (4.7) that $h(-m)O\subset O$, for any $m\geq 1$ and $h\in\mathfrak{h}$. Similar to Lemma 3.4, we also have $L(-1)u+L(0)u\in O$ for any $u\in V_P$. Indeed, if $u\in M_{\hat{\mathfrak{h}}}(1,m\alpha+n\beta)$, with $m\alpha+n\beta\in P\backslash\{\alpha,-\alpha,\beta,\alpha+\beta\}$, then $L(-1)u+L(0)u\in M_{\hat{\mathfrak{h}}}(1,m\alpha+n\beta)\subset O$ by (4.7). Now let $u=h^1(-n_1)\dots h^r(-n_r)e^{\gamma}$, with $\gamma\in\{\alpha,-\alpha,\beta,\alpha+\beta\}$, $h^i\in\mathfrak{h}$ for all i, and $n_1\geq\cdots\geq n_r\geq 1$. Since $L(-1)e^{\gamma}=\gamma(-1)e^{\gamma}$, a similar calculation as Lemma 3.4 shows

$$\begin{aligned}L(-1)u+L(0)u&=h^1(-n_1)\dots h^r(-n_r)(\gamma(-1)e^{\gamma}+e^{\gamma})\\&\quad+\sum_{j=1}^{r}(h^j(-n_j-1)+h^j(-n_j))h^1(-n_1)\dots\widehat{h^j(-n_j)}\dots h^r(-n_r)e^{\gamma}\\&\equiv 0 \pmod{O},\end{aligned}$$

in view of (4.7). Hence $L(-1)u+L(0)u\in O$ for any $u\in V_P$.

Now let $(\gamma,\gamma')\in\{(\alpha,\beta),(\beta,\alpha),(-\alpha,\alpha+\beta),(\alpha+\beta,-\alpha)\}$. Note that $\epsilon(\gamma,\gamma')=-1$ by (4.1). First, we use induction on $n\geq 0$ to show

$$\mathrm{Res}_z\, Y(e^{\gamma},z)e^{\gamma'}\frac{(1+z)}{z^{2+n}}\in O, \tag{4.17}$$

which is the base case for an induction on r of (4.16). If $n=0$, we have

$$\mathrm{Res}_z\, Y(e^{\gamma},z)e^{\gamma'}\frac{(1+z)}{z^2}=e^{\gamma}_{-2}e^{\gamma'}+e^{\gamma}_{-1}e^{\gamma'}=-\frac{1}{2}\gamma(-1)^2e^{\gamma+\gamma'}-\frac{1}{2}\gamma(-1)e^{\gamma+\gamma'}\equiv 0 \pmod{O}.$$

Suppose (4.17) holds for smaller n. Then

$$\begin{aligned}&(n+1)(n+2)\left(e^{\gamma}_{-n-3}e^{\gamma'}+e^{\gamma}_{-n-2}e^{\gamma'}\right)=(n+1)(L(-1)e^{\gamma})_{-n-2}e^{\gamma'}+(n+2)(L(-1)e^{\gamma})_{-n-1}e^{\gamma'}\\&=(n+1)(\gamma(-1)e^{\gamma})_{-n-2}e^{\gamma'}+(n+2)(\gamma(-1)e^{\gamma})_{-n-1}e^{\gamma'}\\&=(n+1)\left(\sum_{j\geq 0}\gamma(-1-j)e^{\gamma}_{-n-2+j}e^{\gamma'}+\sum_{j\geq 0}e^{\gamma}_{-n-3-j}\gamma(j)e^{\gamma'}\right)\\&\quad+(n+2)\left(\sum_{j\geq 0}\gamma(-1-j)e^{\gamma}_{-n-1+j}e^{\gamma'}+\sum_{j\geq 0}e^{\gamma}_{-n-2-j}\gamma(j)e^{\gamma'}\right)\end{aligned}$$

$$= (n+1)\sum_{j\geq 0}\gamma(-1-j)e^{\gamma}_{-n-2+j}e^{\gamma'} + (n+1)(\gamma|\gamma')e^{\gamma}_{-n-3}e^{\gamma'}$$
$$+ (n+2)\sum_{j\geq 0}\gamma(-1-j)e^{\gamma}_{-n-1+j}e^{\gamma'} + (n+2)(\gamma|\gamma')e^{\gamma}_{-n-2}e^{\gamma'}.$$

Since $(\gamma|\gamma') = -1$ and $\gamma(-2-j)v + \gamma(-1-j)v \in O$ for any $j \geq 0$, we have

$$\begin{aligned}
&(n+1)(n+3)\left(e^{\gamma}_{-n-3}e^{\gamma'} + e^{\gamma}_{-n-2}e^{\gamma'}\right) = (n+1)\gamma(-1)e^{\gamma}_{-n-2}e^{\gamma'}\\
&+ (n+1)\sum_{t\geq 0}\gamma(-2-t)e^{\gamma}_{-n-1+t}e^{\gamma'} + (n+2)\sum_{j\geq 0}\gamma(-1-j)e^{\gamma}_{-n-1+j}e^{\gamma'} - e^{\gamma}_{-n-2}e^{\gamma'}\\
&\equiv (n+1)\gamma(-1)e^{\gamma}_{-n-2}e^{\gamma'} + \sum_{j\geq 0}\gamma(-1-j)e^{\gamma}_{-n-1+j}e^{\gamma'} - e^{\gamma}_{-n-2}e^{\gamma'} \pmod{O}\\
&\equiv (n+1)\gamma(-1)e^{\gamma}_{-n-2}e^{\gamma'} + \sum_{j\geq 0}(-1)^j\gamma(-1)e^{\gamma}_{-n-1+j}e^{\gamma'} - e^{\gamma}_{-n-2}e^{\gamma'} \pmod{O}.
\end{aligned}$$

Since $\gamma(-1)O \subset O$, then by the induction hypothesis we have

$$\gamma(-1)e^{\gamma}_{-n-1+j}e^{\gamma'} \equiv (-1)e^{\gamma}_{-n-1+j-1}e^{\gamma'} \equiv \cdots \equiv (-1)^j\gamma(-1)e^{\gamma}_{-n-1}e^{\gamma'} \pmod{O},$$

for any $0 \leq j \leq n$, and $e^{\gamma}_{-n-2}e^{\gamma'} \equiv \cdots \equiv (-1)^{n+1}e^{\gamma}_{-1}e^{\gamma'} = (-1)^{n+1}\epsilon(\gamma,\gamma')\gamma(-1)e^{\gamma+\gamma'} \pmod{O}$. Moreover, we have $e^{\gamma}_0e^{\gamma'} = \epsilon(\gamma,\gamma')e^{\gamma+\gamma'}$, and $e^{\gamma}_me^{\gamma'} = 0$ for $m \geq 1$. It follows that

$$\begin{aligned}
&(n+1)\gamma(-1)e^{\gamma}_{-n-2}e^{\gamma'} + \sum_{j=0}^{n+1}(-1)^j\gamma(-1)e^{\gamma}_{-n-1+j}e^{\gamma'} - e^{\gamma}_{-n-2}e^{\gamma'}\\
&\equiv (n+1)\gamma(-1)e^{\gamma}_{-n-2}e^{\gamma'} + (n+1)\gamma(-1)e^{\gamma}_{-n-1}e^{\gamma'} + (-1)^{n+1}\gamma(-1)e^{\gamma}_0e^{\gamma'} - e^{\gamma}_{-n-2}e^{\gamma'}\\
&\equiv 0 + (-1)^{n+1}\gamma(-1)\epsilon(\gamma,\gamma')e^{\gamma+\gamma'} - (-1)^{n+1}\epsilon(\gamma,\gamma')\gamma(-1)e^{\gamma+\gamma'}\\
&\equiv 0 \pmod{O}.
\end{aligned}$$

Hence we have $(n+1)(n+3)\left(e^{\gamma}_{-n-3}e^{\gamma'} + e^{\gamma}_{-n-2}e^{\gamma'}\right) \in O$, and the proof of (4.17) is complete.

Finally, we use induction on the length r to prove (4.16). The base case $r = 0$ follows from (4.17). Suppose (4.16) holds for smaller $r \geq 1$. Then

$$\begin{aligned}
&\mathrm{Res}_z\, Y(e^{\gamma},z)h^1(-n_1)\dots h^r(-n_r)e^{\gamma'}\frac{(1+z)}{z^{2+n}}\\
&= h^1(-n_1)\dots h^r(-n_r)\,\mathrm{Res}_z\, Y(e^{\gamma},z)e^{\gamma'}\frac{(1+z)}{z^{2+n}}\\
&-\sum_{j=1}^{r}(h^j|\gamma)\,\mathrm{Res}_z\, h^1(-n_1)\dots h^{j-1}(-n_{j-1})Y(e^{\gamma},z)h^{j+1}(-n_{j+1})\dots h^r(-n_r)e^{\gamma'}\frac{(1+z)}{z^{2+n+n_j}}\\
&\equiv 0 \pmod{O},
\end{aligned}$$

where the last congruence follows from the induction hypothesis and the fact that $h(-m)O \subset O$ for any $h \in \mathfrak{h}$ and $m \geq 1$. □

By a slight modification of our induction arguments in Lemma 3.6 and Proposition 3.7, we can show that

$$\mathrm{Res}_z\, Y(h^1(-n_1)\dots h^r(-n_r)e^{\gamma})h^1(-m_1)\dots h^s(-m_s)e^{\gamma'}\frac{(1+z)^{n_1+\dots n_r+1}}{z^{2+n}} \in O, \tag{4.18}$$

for any $n \geq 0$. Note that the only properties of O' we used in the proof of Lemma 3.6 and Proposition 3.7 are $\alpha(-n-2)v + \alpha(-n-1)v \in O'$ and Proposition 3.5, which, in our rank-two parabolic case, are satisfied by (4.7) and Proposition 4.7.

Now (4.13) and (4.14) follow from (4.18).

4.1.2. *Proof of* (4.15). Given a spanning element $u = h^1(-n_1)\dots h^r(-n_r)e^\alpha$ of $M_{\hat{\mathfrak{h}}}(1,\alpha)$ and $v = h^1(-m_1)\dots h^s(-m_s)e^{-\alpha}$ of $M_{\hat{\mathfrak{h}}}(1,-\alpha)$, we need to show that $u\circ v \in O$. Again, we only prove the base case when $u = e^\alpha$ since the induction steps are similar to the proof of Lemma 3.6 and Proposition 3.7.

Proposition 4.8. *For any $n \geq 0$, we have*

$$\operatorname{Res}_z Y(e^\alpha, z)\left(h^1(-n_1)\dots h^r(-n_r)e^{-\alpha}\right)\frac{(1+z)}{z^{2+n}} \in O, \tag{4.19}$$

where $r \geq 0$, $h^i \in \mathfrak{h}$ for all i, and $n_1 \geq \cdots \geq n_r \geq 1$.

Proof. Again, we first prove (4.19) for $r = 0$ by induction on $n \geq 0$. The base case $n = 0$ follows from our calculation in Corollary 3.11:

$$e^\alpha \circ e^{-\alpha} = e^\alpha_{-2}e^{-\alpha} + e^\alpha_{-1}e^{-\alpha} \equiv \frac{1}{6}(\alpha(-1)^3\mathbf{1} - \alpha(-1)\mathbf{1}) \equiv 0 \pmod{O}.$$

Suppose the conclusion holds for smaller $n \geq 1$. Then by a similar calculation as Proposition 4.7, with $\gamma = \alpha$ and $\gamma' = -\alpha$, noting that $e^\alpha_m e^{-\alpha} = 0$ for $m \geq 2$, we have

$$\begin{aligned}
&(n+1)(n+4)\left(e^\alpha_{-n-3}e^{-\alpha} + e^\alpha_{-n-2}e^{-\alpha}\right)\\
&\equiv (n+1)\alpha(-1)e^\alpha_{-n-2}e^{-\alpha} + \sum_{j=0}^{n+2}(-1)^j\alpha(-1)e^\alpha_{-n-1+j}e^{-\alpha} - 2e^\alpha_{-n-2}e^{-\alpha} \pmod{O}\\
&\equiv ((n+1)\alpha(-1)e^\alpha_{-n-2}e^{-\alpha} + (n+1)\alpha(-1)e^\alpha_{-n-1}e^{-\alpha})\\
&\quad + (-1)^{n+1}\alpha(-1)e^\alpha_0 e^{-\alpha} + (-1)^{n+2}\alpha(-1)e^\alpha_1 e^{-\alpha} - 2(-1)^{n+1}e^\alpha_{-1}e^{-\alpha} \pmod{O}\\
&\equiv 0 + (-1)^{n+1}\epsilon(\alpha,-\alpha)\left(\alpha(-1)^2\mathbf{1} - \alpha(-1)\mathbf{1} - \alpha(-2)\mathbf{1} - \alpha(-1)^2\mathbf{1}\right)\\
&\equiv 0 \pmod{O}.
\end{aligned}$$

This proves (4.19) with $r = 0$. The induction step for (4.19) with $r \geq 1$ is also similar to Proposition 4.7, we omit the details. □

With (4.12)–(4.15) and Proposition 4.6, we have our conclusion in this subsection:

Theorem 4.9. *Let P be the parabolic-type submonoid $\mathbb{Z}\alpha\oplus\mathbb{Z}_{\geq 0}\beta$ of the root lattice $A_2 = \mathbb{Z}\alpha\oplus\mathbb{Z}\beta$. The subspace $O(V_P)$ of V_P is equal to O in Definition 4.4.*

4.2. **The Zhu's algebra of V_P.** With the explicit expression of $O(V_P)$ by (4.7) and Proposition 4.9, we give a concrete description of Zhu's algebra $A(V_P)$.

4.2.1. *Generators and relations.* Since $M_{\hat{\mathfrak{h}}}(1, m\alpha+n\beta) \subset O$ for $m\alpha+n\beta \in P\backslash\{0,\alpha,-\alpha,\beta,\alpha+\beta\}$ by (4.7), we have

$$A(V_P) = V_P/O = [M_{\hat{\mathfrak{h}}}(1,0)] + [M_{\hat{\mathfrak{h}}}(1,\alpha)] + [M_{\hat{\mathfrak{h}}}(1,-\alpha)] + [M_{\hat{\mathfrak{h}}}(1,\beta)] + [M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)].$$

Moreover, it is easy to see that the relations in $A(V_P)$ given by (4.7) indicate that

$$\begin{aligned}
&[M_{\hat{\mathfrak{h}}}(1,0)] = \mathbb{C}[[\alpha(-1)\mathbf{1}],[\beta(-1)\mathbf{1}]]/\langle[\alpha(-1)\mathbf{1}]^3 - [\alpha(-1)\mathbf{1}]\rangle,\\
&[M_{\hat{\mathfrak{h}}}(1,\pm\alpha)] = \operatorname{span}\{[\beta(-1)^n e^{\pm\alpha}] : n \in \mathbb{N}\},\\
&[M_{\hat{\mathfrak{h}}}(1,\beta)] = \mathbb{C}[e^\beta] + \mathbb{C}[\alpha(-1)e^\beta] + \mathbb{C}[\alpha(-1)^2e^\beta],\\
&[M_{\hat{\mathfrak{h}}}(1,\alpha+\beta)] = \mathbb{C}[e^{\alpha+\beta}] + \mathbb{C}[\alpha(-1)e^{\alpha+\beta}] + \mathbb{C}[\alpha(-1)^2e^{\alpha+\beta}].
\end{aligned} \tag{4.20}$$

Definition 4.10. Let A_P be the associative (unital) algebra defined by

$$A_P := \mathbb{C}\langle x, y, x_\alpha, x_{-\alpha}, x_\beta, x_{\alpha+\beta}\rangle/R,$$

where $\mathbb{C}\langle x, y, x_\alpha, x_{-\alpha}, x_\beta, x_{\alpha+\beta}\rangle$ is the tensor algebra on (six) generators $x, y, x_\alpha, x_{-\alpha}, x_\beta, x_{\alpha+\beta}$, and R is the two-sided ideal generated by the following sets of relations:

$$xx_{\pm\alpha} = \pm x_{\pm\alpha}, \quad x_{\pm\alpha}x = \mp x_{\pm\alpha}, \quad x_\alpha x_{-\alpha} = \frac{1}{2}x^2 + \frac{1}{2}x, \quad x_{-\alpha}x_\alpha = \frac{1}{2}x^2 - \frac{1}{2}x; \tag{4.21}$$

$$xy = yx, \quad x^3 - x = 0, \quad yx_{\pm\alpha} = x_{\pm\alpha}y \mp x_{\pm\alpha}, \quad x_\beta y + x_\beta = 0, \quad yx_\beta - x_\beta = 0; \tag{4.22}$$

$$x_{\alpha+\beta}(x + y) + x_{\alpha+\beta} = 0, \quad (x + y)x_{\alpha+\beta} - x_{\alpha+\beta} = 0; \tag{4.23}$$

$$xx_\beta - x_\beta x + x_\beta = 0, \quad xx_{\alpha+\beta} - x_{\alpha+\beta}x - x_{\alpha+\beta} = 0; \tag{4.24}$$

$$x_\alpha x_\beta = -x_{\alpha+\beta}y, \quad x_\beta x_\alpha = -x_{\alpha+\beta}y - x_{\alpha+\beta}, \quad x_{-\alpha}x_{\alpha+\beta} = -x_\beta x + x_\beta, \quad x_{\alpha+\beta}x_{-\alpha} = -x_\beta x; \tag{4.25}$$

$$x_{\pm\alpha}^2 = x_\beta^2 = x_{\alpha+\beta}^2 = x_\alpha x_{\alpha+\beta} = x_{\alpha+\beta}x_\alpha = x_\beta x_{\alpha+\beta} = x_{\alpha+\beta}x_\beta = x_\beta x_{-\alpha} = x_{-\alpha}x_\beta = 0. \tag{4.26}$$

Note that relations in (4.21) are similar to the relations of Zhu's algebra $A(V_{A_1})$ of the rank-one lattice VOA V_{A_1} (see (3.26) and Corollary 3.11). Relations in (4.22), (4.23), and (4.24) are the product relations between $\{x, y\}$ and $\{x_\alpha, x_{-\alpha}, x_\beta, x_{\alpha+\beta}\}$. Relations in (4.25) and (4.26) are the product relations among $\{x_\alpha, x_{-\alpha}, x_\beta, x_{\alpha+\beta}\}$.

4.2.2. *The isomorphism.* Now we have our main theorem in this Section:

Theorem 4.11. *There is an isomorphism of the (unital) associative algebras:*

$$\begin{aligned} &F : A_P = \mathbb{C}\langle x, y, x_\alpha, x_{-\alpha}, x_\beta, x_{\alpha+\beta}\rangle/R \to A(V_P),\\ &x \mapsto [\alpha(-1)\mathbf{1}], \quad y \mapsto [\beta(-1)\mathbf{1}], \quad x_{\pm\alpha} \mapsto [e^{\pm\alpha}], \quad x_\beta \mapsto [e^\beta], \quad x_{\alpha+\beta} \mapsto [e^{\alpha+\beta}], \end{aligned} \tag{4.27}$$

where we use the same notations for the equivalent classes of $x, y, x_\alpha, x_{-\alpha}, x_\beta$, and $x_{\alpha+\beta}$ in A_P.

Proof. First we show that F is well-defined. i.e., F preserves the relations given by (4.21)–(4.26). Indeed, by (3.26) and the fact that there is an algebra homomorphism $A(V_{\mathbb{Z}\alpha}) \to A(V_P)$, F preserves (4.21). Note that the following relations hold in $A(V_P)$:

$$\begin{aligned} &[\alpha(-1)\mathbf{1}] * [\beta(-1)\mathbf{1}] = [\beta(-1)\alpha(-1)\mathbf{1}] = [\alpha(-1)\beta(-1)\mathbf{1}] = [\beta(-1)\mathbf{1}] * [\alpha(-1)\mathbf{1}],\\ &[\beta(-1)\mathbf{1}] * [e^{\pm\alpha}] - [\beta(-1)\mathbf{1}] * [e^{\pm\alpha}] = [\beta(0)e^{\pm\alpha}] = \mp[e^{\pm\alpha}],\\ &[e^\beta] * [\beta(-1)\mathbf{1}] = [\beta(-1)e^\beta] = -[e^\beta], \quad [\beta(-1)\mathbf{1}] * [e^\beta] = [(\beta(0) + \beta(-1))e^\beta] = [e^\beta], \end{aligned}$$

where the last equality follows from $\beta(-1)e^\beta + e^\beta \in O = O(V_P)$ by (4.7) and Proposition 4.9. Hence F preserves (4.22). Similarly, we can prove F preserves (4.23). The preservation of (4.24) under F follows from

$$[[\alpha(-1)\mathbf{1}], [e^\beta]] = [\alpha(0)e^\beta] = -[e^\beta], \quad [[\alpha(-1)\mathbf{1}], [e^{\alpha+\beta}]] = [\alpha(0)e^{\alpha+\beta}] = [e^{\alpha+\beta}];$$

and the preservation of (4.25) under F follows from

$$\begin{aligned} &[e^\alpha] * [e^\beta] = [e^\beta_{-1}e^\alpha] = [\epsilon(\beta, \alpha)\beta(-1)e^{\alpha+\beta}] = -[e^{\alpha+\beta}] * [\beta(-1)\mathbf{1}],\\ &[e^\beta] * [e^\alpha] = [\epsilon(\alpha, \beta)\alpha(-1)e^{\alpha+\beta}] = -[e^{\alpha+\beta}] * [\beta(-1)\mathbf{1}] - [e^{\alpha+\beta}],\\ &[e^{-\alpha}] * [e^{\alpha+\beta}] = [\epsilon(\alpha + \beta, -\alpha)(\alpha + \beta)(-1)e^\beta] = -[e^\beta] * [\alpha(-1)\mathbf{1}] + [e^\beta],\\ &[e^{\alpha+\beta}] * [e^{-\alpha}] = [\epsilon(-\alpha, \alpha + \beta)(-\alpha(-1))e^\beta] = -[e^\beta] * [\alpha(-1)\mathbf{1}], \end{aligned}$$

where we used (4.1) and the fact that $[\beta(-1)e^\beta] = -[e^\beta]$ in $A(V_P)$.

Finally, for $\gamma, \gamma' \in \{\alpha, -\alpha, \beta, \alpha + \beta\}$ such that $\gamma + \gamma' \notin \{0, \alpha, -\alpha, \beta, \alpha + \beta\}$, by (4.7) and Proposition 4.9, we have $e^\gamma * e^{\gamma'} \in M_{\hat{\mathfrak{h}}}(1, \gamma + \gamma') \subset O$, and so $[e^\gamma] * [e^{\gamma'}] = 0$ in $A(V_P)$. This shows F preserves (4.26). Hence F is well-defined.

By (4.20), it is easy to see that F is surjective. To show F is an isomorphism, we adopt a similar idea as the proof of Corollary 3.9 and construct an inverse map of F. In order to properly define the inverse of F, we introduce the following linear map:

$$\bar{(\cdot)} : \mathfrak{h} = \mathbb{C}\alpha \oplus \mathbb{C}\beta \to A_P, \quad h = \lambda\alpha + \mu\beta \mapsto \bar{h} = \lambda x + \mu y, \quad \lambda, \mu \in \mathbb{C}. \tag{4.28}$$

Again, we use the same symbol x and y to denote their image in A_P. Now we define

$$\begin{aligned} G : V_P &\to A_P = \mathbb{C}\langle x, y, x_\alpha, x_{-\alpha}, x_\beta, x_{\alpha+\beta}\rangle / R, \\ h^1(-n_1-1)\dots h^r(-n_r-1)e^\gamma &\mapsto (-1)^{n_1+\cdots+n_r} x_\gamma \cdot \overline{h^1}\cdot\overline{h^2}\cdots\overline{h^r}, \quad \gamma\in\{\alpha,-\alpha,\beta,\alpha+\beta\} \\ h^1(-n_1-1)\dots h^r(-n_r-1)\mathbf{1} &\mapsto (-1)^{n_1+\cdots+n_r}\overline{h^1}\cdot\overline{h^2}\cdots\overline{h^r}, \\ M_{\hat{\mathfrak{h}}}(1, m\alpha+n\beta) &\mapsto 0, \quad m\alpha+n\beta\in P\backslash\{0,\alpha,-\alpha,\beta,\alpha+\beta\}, \end{aligned} \tag{4.29}$$

where $r\geq 0$, $n_1\geq\cdots\geq n_r\geq 0$, and $\overline{h^i}$ is the image of $h^i\in\mathfrak{h}$ under $\bar{(\cdot)}$ in (4.28) for all i.

Next, we show that G vanishes on $O(V_P)=O$ given by (4.7). Indeed, clearly $G(h(-n-2)u + h(-n-1)u) = 0$ for any $h\in\mathfrak{h}$, $u\in V_P$, and $n\geq 0$ (see also Theorem 3.8).

To show $G(\gamma(-1)v+v)=0$, where $v = h^1(-n_1-1)\dots h^r(-n_r-1)e^\gamma\in M_{\hat{\mathfrak{h}}}(1,\gamma)$, note that

$$\begin{aligned} &G(\gamma(-1)h^1(-n_1-1)\dots h^r(-n_r-1)e^\gamma + h^1(-n_1-1)\dots h^r(-n_r-1)e^\gamma) \\ &= (-1)^{n_1+\cdots+n_r}x_\gamma\cdot\overline{\gamma}\cdot\overline{h^1}\cdots\overline{h^r} + (-1)^{n_1+\cdots+n_r}x_\gamma\cdot\overline{h^1}\cdots\overline{h^r} \\ &= (x_\gamma\cdot\overline{\gamma}+x_\gamma)(-1)^{n_1+\cdots+n_r}\overline{h^1}\cdots\overline{h^r} \\ &= 0, \end{aligned}$$

since $x_\alpha x + x_\alpha = 0$, $x_{-\alpha}x - x_{-\alpha} = 0$, $x_\beta y + x_\beta = 0$, and $x_{\alpha+\beta}(x+y)+x_{\alpha+\beta}=0$, in view of (4.21), (4.22), (4.23), and (4.28).

To show $G(\gamma(-1)^2v+\gamma(-1)v)=0$, where $(\gamma,\gamma')\in\{(\alpha,\beta),(\beta,\alpha),(-\alpha,\alpha+\beta),(\alpha+\beta,-\alpha)\}$ as in (4.8), and $v\in M_{\hat{\mathfrak{h}}}(\gamma+\gamma')$, we claim the following relations hold in A_P:

$$x_{\alpha+\beta}x^2 + x_{\alpha+\beta}x = 0 \quad\text{and}\quad x_{\alpha+\beta}y^2 + x_{\alpha+\beta} = 0, \tag{4.30}$$

$$x_\beta(x+y)^2 + x_\beta(x+y) = 0 \quad\text{and}\quad x_\beta x^2 - x_\beta x = 0. \tag{4.31}$$

Indeed, by (4.23), (4.25), (4.24), and (4.21), we have

$$(x_{\alpha+\beta}x + x_{\alpha+\beta})x = -x_{\alpha+\beta}yx = x_\alpha x_\beta x = x_\alpha x x_\beta + x_\alpha x_\beta = -x_\alpha x_\beta + x_\alpha x_\beta = 0.$$

The second equality of (4.30) can be proved by a similar method, we omit the details. Furthermore, since $x_\beta y = -x_\beta$ in (4.22) and $x_\beta x = xx_\beta + x_\beta$ in (4.24), we have

$$\begin{aligned} x_\beta(x+y)^2 + x_\beta(x+y) &= x_\beta x^2 + x_\beta xy + x_\beta yx + x_\beta y^2 + x_\beta x + x_\beta y \\ &= x_\beta x^2 + (-x_\beta)x - x_\beta x + (-1)^2x_\beta + x_\beta x - x_\beta \\ &= (x_\beta x - x_\beta)x = xx_\beta x. \end{aligned}$$

On the other hand, since $x_{\alpha+\beta}x_{-\alpha} = -x_\beta x$ in (4.25) and $xx_{\alpha+\beta} = x_{\alpha+\beta}x + x_{\alpha+\beta}$ in (4.24), we have

$$xx_\beta x = -xx_{\alpha+\beta}x_{-\alpha} = -x_{\alpha+\beta}(xx_{-\alpha}) - x_{\alpha+\beta}x_{-\alpha} = -x_{\alpha+\beta}(-x_{-\alpha}) - x_{\alpha+\beta}x_{-\alpha} = 0.$$

This proves both the equalities in (4.31).

Now let (γ,γ') be an ordered pair in (4.8). By (4.30), (4.31), and (4.28), we have $x_{\gamma+\gamma'}\overline{\gamma}^2 + x_{\gamma+\gamma'}\overline{\gamma} = 0$ in A_P. Thus, for $v = h^1(-n_1-1)\dots h^r(-n_r-1)e^{\gamma+\gamma'}\in M_{\hat{\mathfrak{h}}}(1,\gamma+\gamma')$, we have

$$\begin{aligned} &G(\gamma(-1)^2h^1(-n_1-1)\dots h^r(-n_r-1)e^{\gamma+\gamma'} + \gamma(-1)h^1(-n_1-1)\dots h^r(-n_r-1)e^{\gamma+\gamma'}) \\ &= (x_{\gamma+\gamma'}\overline{\gamma}^2 + x_{\gamma+\gamma'}\overline{\gamma})(-1)^{n_1+\cdots+n_r}\overline{h^1}\cdots\overline{h^r} \\ &= 0. \end{aligned}$$

Furthermore, we have $G(M_{\hat{\mathfrak{h}}}(1,m\alpha+n\beta))=0$ for $m\alpha+n\beta\in P\backslash\{0,\alpha,-\alpha,\beta,\alpha+\beta\}$ by definition (4.29) of G. Finally, for $w = h^1(-n_1-1)\dots h^r(-n_r-1)\mathbf{1}\in M_{\hat{\mathfrak{h}}}(1,0)$, by (4.22) we have

$$\begin{aligned} &G(\alpha(-1)^3h^1(-n_1-1)\dots h^r(-n_r-1)\mathbf{1} - \alpha(-1)h^1(-n_1-1)\dots h^r(-n_r-1)\mathbf{1}) \\ &= (-1)^{n_1+\cdots+n_r}\overline{h^1}\cdots\overline{h^r}\cdot x^3 - (-1)^{n_1+\cdots+n_r}\overline{h^1}\cdots\overline{h^r}\cdot x \\ &= 0. \end{aligned}$$

This shows $G(O(V_P)) = 0$, and so G induces $G : A(V_P) = V_P/O(V_P) \to A_P$. It is easy to see from (4.29) and (4.27) that $G : A(V_P) \to A_P$ and $F : A_P \to A(V_P)$ are mutually inverse on the generators of A_P and $A(V_P)$. Hence G is an inverse of F, and F is an isomorphism of associative algebras. □

4.2.3. *Structure of A_P and the skew-polynomial algebra.* With (4.20), the identification (4.27), and the relations (4.21)–(4.26), we have the following direct sum decomposition of the Zhu's algebra A_P:

$$\begin{aligned} A_P = &\left(\bigoplus_{n=0}^{\infty} \mathbb{C}(x_{-\alpha}y^n)\right) \oplus \mathbb{C}[x,y]/\langle x^3 - x\rangle \oplus \left(\bigoplus_{n=0}^{\infty} \mathbb{C}(x_{\alpha}y^n)\right) \\ &\oplus \left(\mathbb{C}x_\beta \oplus \mathbb{C}x_\beta x \oplus \mathbb{C}x_\beta x^2\right) \oplus \left(\mathbb{C}x_{\alpha+\beta} \oplus \mathbb{C}x_{\alpha+\beta}x \oplus \mathbb{C}x_{\alpha+\beta}x^2\right), \end{aligned} \tag{4.32}$$

where the products of the spanning elements are given by (4.21)–(4.26). With the decomposition (4.32), we have the following Corollary which will be used in next Section:

Corollary 4.12. $J = \left(\mathbb{C}x_\beta \oplus \mathbb{C}x_\beta x \oplus \mathbb{C}x_\beta x^2\right) \oplus \left(\mathbb{C}x_{\alpha+\beta} \oplus \mathbb{C}x_{\alpha+\beta}x \oplus \mathbb{C}x_{\alpha+\beta}x^2\right)$ *is a two-sided ideal of A_P such that $J^2 = 0$, and the quotient algebra $A^P = A_P/J$ is isomorphic to $A(V_{\mathbb{Z}\alpha}) \otimes_{\mathbb{C}} \mathbb{C}[y]$ as vector spaces, with $A_P = A^P \oplus J$. Furthermore, both $A(V_{\mathbb{Z}\alpha}) = A(V_{\mathbb{Z}\alpha}) \otimes \mathbb{C}1$ and $\mathbb{C}[y] = \mathbb{C}[\mathbf{1}] \otimes \mathbb{C}[y]$ are subalgebras of A_P and A^P.*

Proof. Let $J_1 = \mathbb{C}x_\beta \oplus \mathbb{C}x_\beta x \oplus \mathbb{C}x_\beta x^2$ and $J_2 = \mathbb{C}x_{\alpha+\beta} \oplus \mathbb{C}x_{\alpha+\beta}x \oplus \mathbb{C}x_{\alpha+\beta}x^2$. Then $J = J_1 \oplus J_2$. By (4.21)–(4.26), it is easy to check that J_1, J_2 satisfy the following properties:

$$\begin{aligned} &xJ_1,\ yJ_1,\ J_1x,\ J_1y \subset J_1, \\ &xJ_2,\ yJ_2,\ J_2x,\ J_2y \subset J_2, \\ &x_\alpha J_1,\ J_1x_\alpha \subset J_2, \quad x_\gamma J_1 = J_1 x_\gamma = 0, \quad \gamma \in \{-\alpha, \beta, \alpha+\beta\}, \\ &x_{-\alpha}J_2,\ J_2x_{-\alpha} \subset J_1, \quad x_{\gamma'}J_2 = J_2x_{\gamma'} = 0, \quad \gamma' \in \{\alpha, \beta, \alpha+\beta\}. \end{aligned}$$

This shows $J = J_1 \oplus J_2$ is a two-sided ideal of A_P. Moreover, using $x_{\pm\alpha}^2 = x_\beta^2 = x_{\alpha+\beta}^2 = 0$, together with by (4.21) and (4.24), we can show $J^2 = 0$. By the decomposition (4.32), we have

$$\begin{aligned} A_P/J &= \left(\bigoplus_{n=0}^{\infty} \mathbb{C}(x_{-\alpha}y^n)\right) \oplus \mathbb{C}[x,y]/\langle x^3 - x\rangle \oplus \left(\bigoplus_{n=0}^{\infty} \mathbb{C}(x_{\alpha}y^n)\right) \\ &\cong \left(\mathbb{C}x_{-\alpha} \oplus \mathbb{C}[x]/\langle x^3 - x\rangle \oplus \mathbb{C}x_\alpha\right) \otimes_{\mathbb{C}} \mathbb{C}[y] \end{aligned}$$

as vector spaces. Hence $A_P = A^P \oplus J$. Moreover, by (4.21) and (4.22), the subspace $\mathbb{C}x_{-\alpha} \oplus \mathbb{C}[x]/\langle x^3 - x\rangle \oplus \mathbb{C}x_\alpha$ is closed under the product of A_P. Moreover, the product relations among x_α, $x_{-\alpha}$, and x are exactly the same as products relations among $[e^\alpha]$, $[e^{-\alpha}]$, and $[\alpha(-1)\mathbf{1}]$ of $A(V_{\mathbb{Z}\alpha})$, in view of Examples 3.10 and Corollary 3.11. Thus, the subalgebra $\mathbb{C}x_{-\alpha} \oplus \mathbb{C}[x]/\langle x^3 - x\rangle \oplus \mathbb{C}x_\alpha$ is isomorphic to $A(V_{\mathbb{Z}\alpha})$. □

In fact, the subalgebra $A^P = A_P/J$ of Zhu's algebra A_P is a skew-polynomial ring over $A(V_{\mathbb{Z}\alpha})$. Recall the following definition in [GW04]:

Definition 4.13. Let R be a ring (not necessarily commutative), $\sigma : R \to R$ be a homomorphism, and let $\delta : R \to R$ be a σ-derivation, that is, σ is an abelian group homomorphism that satisfies $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in R$.

Then the **skew-polynomial ring**, or **Ore-extension**, $R[x; \sigma; \delta]$ is the free left R-module on a basis $\{1, x, x^2, x^3, \dots\}$, with the multiplication given by $xa = \sigma(a)x + \delta(a)$, for all $a \in R$.

Lemma 4.14. *Let R be the subalgebra $A(V_{\mathbb{Z}\alpha}) = \mathrm{span}\{1, x_\alpha, x_{-\alpha}, x, x^2\} \leq A_P$, and $\sigma = \mathrm{Id} : R \to R$. Then the derivation $\delta := [y, \cdot] : A_P \to A_P$ preserves R, and it satisfies:*

$$\delta(1) = \delta(x) = \delta(x^2) = 0, \quad \delta(x_\alpha) = -x_\alpha, \quad \text{and} \quad \delta(x_{-\alpha}) = x_{-\alpha}. \tag{4.33}$$

In particular, δ restricts to a σ-derivation of R.

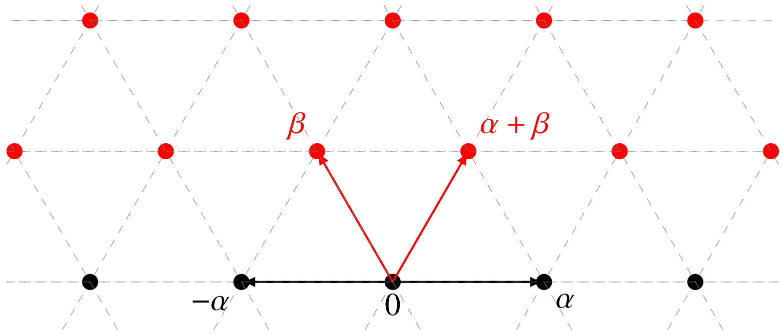

FIGURE 2.

Proof. Since $xy = yx$, clearly $[y, 1] = [y, x] = [y, x^2] = 0$. We also have $yx_{\pm\alpha} = x_{\pm\alpha}y \mp x_{\pm\alpha}$ by (4.22), and so $[y, x_\alpha] = -x_\alpha$ and $[y, x_{-\alpha}] = x_{-\alpha}$. This shows $\delta = [y, \cdot]$ preserves R, and δ satisfies (4.33). Since $\delta(ab) = \delta(a)b + \mathrm{Id}(a)\delta(b)$ for any $a, b \in R$, δ is a $\sigma = \mathrm{Id}$-derivation. □

Corollary 4.15. *$A^P = A_P/J$ is isomorphic to the skew-polynomial algebra $A(V_{\mathbb{Z}\alpha})[y; \mathrm{Id}; \delta]$, where $\delta = [y, \cdot]|_{A(V_{\mathbb{Z}\alpha})}$.*

Proof. By Corollary 4.12, we have $A^P \cong A(V_{\mathbb{Z}\alpha}) \otimes_{\mathbb{C}} \mathbb{C}[y] = A(V_{\mathbb{Z}\alpha})[y]$ as vector spaces. By Lemma 4.14, $\delta[y, \cdot]|_{A(V_{\mathbb{Z}\alpha})}$ is a Id-derivation on $A(V_{\mathbb{Z}\alpha})$ such that $ya = \mathrm{Id}(a)y + \delta(a)$ for all $a \in A(V_{\mathbb{Z}\alpha})$. Hence $A^P \cong A(V_{\mathbb{Z}\alpha})[y; \mathrm{Id}; \delta]$, in view of Definition 4.13. □

5. REPRESENTATION OF THE RANK-TWO PARABOLIC-TYPE SUBVOA V_P OF V_{A_2}

In this Section, we use our main results in the Section 4 to classify the irreducible modules over the parabolic-type subVOA of the lattice VOA V_{A_2}. Again, we fix the parabolic-type submonoid $P = \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$ of the root lattice A_2 in this Section.

5.1. **Construction of irreducible modules of V_P.** Note that $P = \mathbb{Z}\alpha \oplus \mathbb{Z}_{\geq 0}\beta$ is also an abelian semigroup. Let $I \leq P$ be sub-semigroup $\mathbb{Z}\alpha \oplus \mathbb{Z}_{>0}\beta$. In Figure 2, the dots represent elements in P, and the red dots represent the elements in I.

Lemma 5.1. *$V_I = \bigoplus_{\gamma \in I} M_{\hat{\mathfrak{h}}}(1, \gamma)$ is an ideal of the parabolic-type VOA V_P. The quotient VOA $V_P/V_I \cong \bigoplus_{n\in\mathbb{Z}} M_{\hat{\mathfrak{h}}}(1, n\alpha)$ is a subVOA of V_P, and $V_P = (V_P/V_I) \oplus V_I$ as vertex Leibniz algebras.*

Proof. The diagram indicates that $P + I \subseteq I$ and $P = I \oplus \mathbb{Z}\alpha$ as an abelian semigroup. Then by Proposition 2.2 and (2.8), the subspace V_I is an ideal of V_P. Furthermore, since $Y(M_{\hat{\mathfrak{h}}}(1, n\alpha), z)M_{\hat{\mathfrak{h}}}(1, m\alpha) \subset M_{\hat{\mathfrak{h}}}(1, (m + n)\alpha)((z))$ for any $m, n \in \mathbb{Z}$, and $M_{\hat{\mathfrak{h}}}(1, 0) \subset V_P/V_I$. It follows that V_P/V_I is a subVOA of V_P with the same Virasoro element. □

Remark 5.2. The quotient VOA V_P/V_I also has the following identification as a vector space:

$$\begin{aligned} &V_P/V_I \cong \bigoplus_{n\in\mathbb{Z}} M_{\hat{\mathfrak{h}}}(1, n\alpha) = \bigoplus_{n\in\mathbb{Z}} M_{\widehat{\mathbb{C}\alpha}}(1, n\alpha) \otimes M_{\widehat{\mathbb{C}\beta}}(1, 0) = V_{\mathbb{Z}\alpha} \otimes M_{\widehat{\mathbb{C}\beta}}(1, 0), \\ &\alpha(-n_1)\dots\alpha(-n_k)\beta(-m_1)\dots\beta(-m_l)e^{n\alpha} \mapsto \alpha(-n_1)\dots\alpha(-n_k)e^{n\alpha} \otimes \beta(-m_1)\dots\beta(-m_l)\mathbf{1}. \end{aligned} \tag{5.1}$$

However, the identification (5.1) is **not** an isomorphism between the (quotient) VOA V_P/V_I and the tensor product of the VOAs $V_{\mathbb{Z}\alpha}$ and $M_{\widehat{\mathbb{C}\beta}}(1, 0)$ defined in [FHL93]. This is essentially because $(\alpha|\beta) \neq 0$, and so the operators $E^+(-\alpha, z)$ and $\beta(-n)$, where $n \geq 1$, are not commutative.

But it is easy to see from the spanning elements (4.7) and Theorem 4.9 that the Zhu's algebra $A(V_P/V_I) = A(\bigoplus_{n\in\mathbb{Z}} M_{\hat{\mathfrak{h}}}(1, n\alpha))$ is isomorphic to the skew-polynomial algebra $A^P = A_P/J = A(V_{\mathbb{Z}\alpha})[y; \mathrm{Id}; \delta]$ in Corollary 4.12 and 4.15.

Note that the rank-one lattice VOA $V_{A_1} = V_{\mathbb{Z}\alpha}$ is clearly a subVOA of both V_P and V_P/V_I (see (5.1)). By Theorem 3.1 in [D93], V_{A_1} has two irreducible modules $V_{\mathbb{Z}\alpha}$ and $V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}$. We will use these irreducible V_{A_1}-modules to construct irreducible V_P-modules.

5.1.1. *Construction of $L^{(0,\lambda)}$ and $L^{(\frac{1}{2}\alpha,\lambda)}$.* For the root lattice $A_2 = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, recall that $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} A_2$ has a nondegenerate symmetric bilinear form $(\cdot|\cdot) : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$.

Definition 5.3. Let $\lambda \in (\mathbb{C}\alpha)^{\perp} \subset \mathfrak{h}$. Define $L^{(0,\lambda)}$ and $L^{(\frac{1}{2}\alpha,\lambda)}$ to be the following vector spaces:

$$L^{(0,\lambda)} := \bigoplus_{n\in\mathbb{Z}} M_{\hat{\mathfrak{h}}}(1, n\alpha)\otimes \mathbb{C}e^{\lambda} \cong V_{\mathbb{Z}\alpha}\otimes M_{\widehat{\mathbb{C}\beta}}(1,\lambda), \tag{5.2}$$

$$L^{(\frac{1}{2}\alpha,\lambda)} := \bigoplus_{n\in\mathbb{Z}} M_{\hat{\mathfrak{h}}}(1, n\alpha+\frac{1}{2}\alpha)\otimes \mathbb{C}e^{\lambda} \cong V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}\otimes M_{\widehat{\mathbb{C}\beta}}(1,\lambda), \tag{5.3}$$

where $M_{\hat{\mathfrak{h}}}(1, n\alpha)$ and $M_{\hat{\mathfrak{h}}}(1, n\alpha+\frac{1}{2}\alpha)$ are modules over the Heisenberg Lie algebra $\hat{\mathfrak{h}}$ of level 1.

Define the actions of operators $h(m)$, $e_{n\alpha}$, and $z^{n\alpha}$ with $m,n\in\mathbb{Z}$ on the tensor product spaces $\bigoplus_{n\in\mathbb{Z}} M_{\hat{\mathfrak{h}}}(1,n\alpha)\otimes\mathbb{C}e^{\lambda}$ and $\bigoplus_{n\in\mathbb{Z}} M_{\hat{\mathfrak{h}}}(1,n\alpha+\frac{1}{2}\alpha)\otimes\mathbb{C}e^{\lambda}$ as follows:

$$h(0) := h(0)\otimes \mathrm{Id} + \mathrm{Id}\otimes h(0),\quad h(m) := h(m)\otimes\mathrm{Id},\quad m\neq 0, \tag{5.4}$$

$$e_{n\alpha} := e_{n\alpha}\otimes\mathrm{Id},\quad z^{n\alpha} := z^{n\alpha}\otimes\mathrm{Id},\quad n\in\mathbb{Z}. \tag{5.5}$$

where we let $h(0)e^{\lambda} := (\lambda|h)e^{\lambda}$. In particular, for any $m,n\in\mathbb{Z}$, and $n_1\geq\cdots\geq n_r\geq 1$, we have

$$h(m)(h^1(-n_1)\dots h^r(-n_r)e^{n\alpha}\otimes e^{\lambda}) := h(m)h^1(-n_1)\dots h^r(-n_r)e^{n\alpha}\otimes e^{\lambda},\quad m\neq 0, \tag{5.6}$$

$$h(0)(e^{m\alpha}\otimes e^{\lambda}) = (m\alpha+\lambda|h)e^{m\alpha}\otimes e^{\lambda},\ \ h(0)(e^{m\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}) = (m\alpha+\frac{1}{2}\alpha+\lambda|h)e^{m\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}, \tag{5.7}$$

$$e_{n\alpha}(e^{m\alpha}\otimes e^{\lambda}) = \epsilon(n\alpha,m\alpha)e^{(m+n)\alpha}\otimes e^{\lambda},\quad e_{n\alpha}(e^{m\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}) = \epsilon(n\alpha,m\alpha)e^{(m+n)\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}, \tag{5.8}$$

$$z^{n\alpha}(e^{m\alpha}\otimes e^{\lambda}) = z^{(n\alpha|m\alpha)}e^{m\alpha}\otimes e^{\lambda},\quad z^{n\alpha}(e^{m\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}) = z^{(n\alpha|m\alpha+\frac{1}{2}\alpha)}e^{m\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}. \tag{5.9}$$

Define the module vertex operators $Y_M : V_P \to \mathrm{End}(L^{(\epsilon,\lambda)})[[z,z^{-1}]]$, where $\epsilon = 0$ or $\frac{1}{2}\alpha$, by the following common formula:

$$Y_M(h^1(-n_1-1)\dots h^r(-n_r-1)e^{n\alpha},z) := {}^{\circ}_{\circ}(\partial_z^{(n_1)}h^1(z))\dots(\partial_z^{(n_r)}h^r(z))Y(e^{n\alpha},z){}^{\circ}_{\circ},\quad n\in\mathbb{Z}, \tag{5.10}$$

$$Y_M(h^1(-n_1-1)\dots h^r(-n_r-1)e^{\gamma},z) := 0,\quad \gamma\in I = \mathbb{Z}\alpha\oplus\mathbb{Z}_{>0}\beta. \tag{5.11}$$

where $h^i\in\mathfrak{h}$ for all i, $n_1\geq\cdots\geq n_r\geq 0$, and $Y(e^{n\alpha},z) = E^-(-n\alpha,z)E^+(-n\alpha,z)e_{n\alpha}z^{n\alpha}$, and the normal ordering is given by (2.6).

By (5.6) and (5.7), it is clear that the subspace $M_{\hat{\mathfrak{h}}}(1,n\alpha+\epsilon)\otimes\mathbb{C}e^{\lambda}\subset L^{(\epsilon,\lambda)}$ is a $\hat{\mathfrak{h}}$-module, and it is isomorphic to the $\hat{\mathfrak{h}}$-module $M_{\hat{\mathfrak{h}}}(1,n\alpha+\epsilon+\lambda)$, where $\epsilon = 0$ or $\frac{1}{2}\alpha$, and $n\in\mathbb{Z}$.

5.1.2. *Irreducibility of $L^{(0,\lambda)}$ and $L^{(\frac{1}{2}\alpha,\lambda)}$.*

Lemma 5.4. *For any $\lambda\in(\mathbb{C}\alpha)^{\perp}\subset\mathfrak{h}$, the vector spaces $L^{(\epsilon,\lambda)}$, where $\epsilon = 0$ or $\frac{1}{2}\alpha$, equipped with Y_M in Definition 5.3, are weak V_P-modules.*

Proof. We need to show that the operator Y_M satisfies the truncation property and Jacobi identity. For the truncation property, we fix a spanning element $v = h^1(-n_1)\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda}$ of $L^{(\epsilon,\lambda)}$, where $h^i\in\mathfrak{h}$ for all i, and $n_1\geq\cdots\geq n_r\geq 1$, and show that $Y_M(a,z)v\in L^{(\epsilon,\lambda)}((z))$ for any $a\in V_P$.

Indeed, if $a\in V_I$, then by (5.11) we have $a_n v = 0$ for any $n\in\mathbb{Z}$, there is noting to prove. Now assume that $a\in M_{\hat{\mathfrak{h}}}(1,m\alpha)$. If $a = e^{m\alpha}$, then by (5.4), (5.5), and (5.10), we have

$$Y_M(a,z)(e^{n\alpha+\epsilon}\otimes e^{\lambda}) = (E^-(-m\alpha,z)E^+(-m\alpha,z)e_{m\alpha}z^{m\alpha}e^{n\alpha+\epsilon})\otimes e^{\lambda}\in L^{(\epsilon,\lambda)}((z)), \tag{5.12}$$

since the V_{A_1}-modules $V_{\mathbb{Z}\alpha}$ and $V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}$ satisfy the truncation property. Furthermore, since $[h(-p),E^-(-m\alpha,z)] = 0$ and $[h(-p),E^+(-m\alpha,z)] = -(h|m\alpha)z^{-p}E^+(-m\alpha,z)$ for any $h\in\mathfrak{h}$ and $p>0$ (see [FLM88, LL04]), it follows that

$$Y_M(a,z)v = h^1(-n_1)Y_M(a,z)(h^2(-n_2)\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda})$$

$$- (h^1|m\alpha)z^{-n_1}Y_M(a,z)(h^2(-n_2)\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda}).$$

Then by an induction on the length r of v, with base case given by (5.12), we have

$$Y(e^{m\alpha},z)\left(h^1(-n_1)\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda}\right)\in L^{(\epsilon,\lambda)}((z)). \tag{5.13}$$

Now let a be a general spanning element $a = h_1(-m_1)\dots h_s(-m_s)e^{m\alpha}$ of $M_{\hat{\mathfrak{h}}}(1,m\alpha)$, where $h_j\in\mathfrak{h}$ for all j, and $m_1\geq\cdots\geq m_s\geq 1$. We show that $a_kv = 0$ if $k\gg 0$. Again, by induction on the length s of a, it suffices to consider the case when $s = 1$. The proof of the general case is similar, we omit it. Note that

$$\begin{aligned}
&(h_1(-m_1)e^{m\alpha})_kv\\
&=\sum_{j\geq 0}\binom{-m_1}{j}(-1)^jh_1(-m_1-j)(e^{m\alpha})_{k+j}v-\sum_{j\geq 0}\binom{-m_1}{j}(-1)^{m_1+j}(e^{m\alpha})_{-m_1+k-j}(h_1(j)v)\\
&=\sum_{j\geq 0}\binom{-m_1}{j}(-1)^jh_1(-m_1-j)(e^{m\alpha})_{k+j}v\\
&-(n\alpha+\epsilon+\lambda|h)(e^{m\alpha})_{-m_1+k}\left(h^1(-n_1)\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda}\right)\\
&-\binom{-m_1}{n_r}n_r(h_1|h^r)(-1)^{m_1+n_r}(e^{m\alpha})_{-m_1+k-n_r}\left(h^1(-n_1)\dots\widehat{h^r(-n_r)}e^{n\alpha+\epsilon}\otimes e^{\lambda}\right)\\
&-\binom{-m_1}{n_{r-1}}n_{r-1}(h_1|h^{r-1})(-1)^{m_1+n_{r-1}}(e^{m\alpha})_{-m_1+k-n_{r-1}}\left(h^1(-n_1)\dots\widehat{h^{r-1}(-n_{r-1})}h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda}\right)\\
&\vdots\\
&-\binom{-m_1}{n_1}n_1(h_1|h^1)(-1)^{m_1+n_1}(e^{m\alpha})_{-m_1+k-n_1}\left(\widehat{h^1(-n_1)}\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda}\right).
\end{aligned}$$

By (5.13), it is clear that we can choose $k\gg 0$ large enough so that each term on the right hand side of the equation above is equal to 0. This shows the truncation property of Y_M.

It remains to show the Jacobi identity of Y_M. Let $a\in M_{\hat{\mathfrak{h}}}(1,\gamma)$ and $b\in M_{\hat{\mathfrak{h}}}(1,\eta)$, where $\gamma,\eta\in P$. We need to show that

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(a,z_1)Y_M(b,z_2)-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)Y_M(b,z_2)Y_M(a,z_1)\\
&=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(a,z_0)b,z_2).
\end{aligned}\tag{5.14}$$

Note that $Y(a,z_0)b\in M_{\hat{\mathfrak{h}}}(1,\gamma+\eta)((z_0))$. If either γ or η are contained in $I\subset P$, then by Lemma 5.1, (5.11), and the fact that $I+P=P+I\subseteq I$, both sides of the Jacobi identity (5.14) are 0. Now assume $a = h^1(-n_1-1)\dots h^r(-n_r-1)e^{n\alpha}$ and $b = h_1(-m_1-1)\dots h_s(-m_s-1)e^{m\alpha}$ for some $m,n\in\mathbb{Z}$, $h^i,h_j\in\mathfrak{h}$ for all i,j, $n_1\geq\cdots\geq n_r\geq 0$, and $m_1\geq\cdots\geq m_s\geq 0$. By adopting a similar argument as the proof of Theorem 8.6.1 in [FLM88], we can show that

$$[Y_M(a,z_1),Y_M(b,z_2)]=\mathrm{Res}_{z_0}z_2^{-1}Y_M(Y(a,z_0)b,z_2)e^{-z_0(\partial/\partial z_1)}\left((z_1/z_2)^{m\alpha}\delta(z_1/z_2)\right).$$

This commutator relation also (essentially) follows from the fact that the V_{A_1}-module vertex operators for $V_{\mathbb{Z}\alpha}$ and $V_{\mathbb{Z}\alpha+\frac{1}{2}\alpha}$ satisfy the Jacobi identity. Then by Theorem 8.8.9 in [FLM88], the Jacobi identity (5.14) holds for Y_M. □

Lemma 5.5. *Given $\lambda\in(\mathbb{C}\alpha)^{\perp}\subset\mathfrak{h}$, the weak V_P-modules $(L^{(0,\lambda)},Y_M)$ and $(L^{(\frac{1}{2}\alpha,\lambda)},Y_M)$ are irreducible ordinary V_P-modules, whose bottom levels are $\mathbb{C}(\mathbf{1}\otimes e^{\lambda})$ and $\mathbb{C}(e^{\frac{1}{2}\alpha}\otimes e^{\lambda})\oplus\mathbb{C}(e^{-\frac{1}{2}\alpha}\otimes e^{\lambda})$, respectively.*

Proof. Note that $\mathrm{Res}_z zY_M(\omega,z) = L_M(0) = \frac{1}{2}\sum_{i=1}^{2}\sum_{s\geq 0} u^i(-s)u^i(s)$, where $\{u^1,u^2\}$ is an orthonormal basis of $\mathfrak{h}$. By (5.4) and the fact that $(\lambda|\alpha)=0$, we have

$$\begin{aligned}L_M(0)(e^{n\alpha+\epsilon}\otimes e^{\lambda}) &= \frac{1}{2}(n\alpha+\epsilon+\lambda|n\alpha+\epsilon+\lambda)e^{n\alpha+\epsilon}\otimes e^{\lambda}\\ &= \left(\frac{1}{2}(n\alpha+\epsilon|n\alpha+\epsilon)+\frac{(\lambda|\lambda)}{2}\right)e^{n\alpha+\epsilon}\otimes e^{\lambda}.\end{aligned}$$

Moreover, by (5.4) again, it is easy to show that $[L_M(0),h(-n)] = nh(-n)$, for any $h\in\mathfrak{h}$ and $n>0$. Hence we have

$$\begin{aligned}&L_M(0)\left(h^1(-n_1)\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda}\right)\\ &=\left(n_1\cdots+n_r+\frac{1}{2}(n\alpha+\epsilon|n\alpha+\epsilon)+\frac{(\lambda|\lambda)}{2}\right)h^1(-n_1)\dots h^r(-n_r)e^{n\alpha+\epsilon}\otimes e^{\lambda},\end{aligned}\tag{5.15}$$

where $\epsilon=0$ or $\frac{1}{2}\alpha$, $h^i\in\mathfrak{h}$ for all i, $n\in\mathbb{Z}$, and $n_1\geq\cdots\geq n_r\geq 1$. Since $(\pm\frac{1}{2}\alpha|\pm\frac{1}{2}\alpha)=\frac{1}{2}$, then it follows from (5.2) , (5.3), and (5.15) that $L^{(0,\lambda)}$ and $L^{(\frac{1}{2}\alpha,\lambda)}$ are graded vector spaces, with the grading subspaces given by $L_M(0)$-eigenspaces:

$$L^{(0,\lambda)}=\bigoplus_{m=0}^{\infty}\left(L^{(0,\lambda)}\right)_{\frac{(\lambda|\lambda)}{2}+m},\quad L^{(\frac{1}{2}\alpha,\lambda)}=\bigoplus_{m=0}^{\infty}\left(L^{(\frac{1}{2}\alpha,\lambda)}\right)_{\frac{(\lambda|\lambda)}{2}+\frac{1}{2}+m}.\tag{5.16}$$

By (5.15) and (5.16), it is easy to see that the bottom levels ($m=0$) of $L^{(0,\lambda)}$ and $L^{(\frac{1}{2}\alpha,\lambda)}$ are given by $\mathbb{C}(\mathbf{1}\otimes e^{\lambda})$ and $\mathbb{C}(e^{\frac{1}{2}\alpha}\otimes e^{\lambda})\oplus\mathbb{C}(e^{-\frac{1}{2}\alpha}\otimes e^{\lambda})$, respectively.

Now we show that $L^{(0,\lambda)}$ and $L^{(\frac{1}{2}\alpha,\lambda)}$ are irreducible. We only prove the irreducibility of $L^{(\frac{1}{2}\alpha,\lambda)}$, the other one is similar. Let $W\neq 0$ be a submodule of $L^{(\frac{1}{2}\alpha,\lambda)}$. Consider a nonzero element $0\neq u\in W$. By the decomposition 5.3, u can be written as follows:

$$u=u_{-m}+u_{-m+1}+\cdots+u_0+\cdots+u_n\in\bigoplus_{n\in\mathbb{Z}}M_{\hat{\mathfrak{h}}}(1,n\alpha+\frac{1}{2}\alpha)\otimes\mathbb{C}e^{\lambda},$$

where $u_j\in M_{\hat{\mathfrak{h}}}(1,j\alpha+\frac{1}{2}\alpha)\otimes\mathbb{C}e^{\lambda}$ for all $-m\leq j\leq n$. By (5.4) and (5.7), we have

$$\beta(0)u_j=\left(j\alpha+\frac{1}{2}\alpha+\lambda|\beta\right)u_j=\left((\lambda|\beta)-j-\frac{1}{2}\right)u_j,\quad -m\leq j\leq n.$$

i.e., u_j with $-m\leq j\leq n$ are eigenvectors of $\beta(0)$ of distinct eigenvalues. Since $\beta(0)^k u\in W$ for any $k\geq 0$, it follows that $u_j\in W$ for all j (using the Vandermonde determinant).

Since $u\neq 0$, we may assume that $0\neq u_j\in W$ for some fixed j. Since $M_{\hat{\mathfrak{h}}}(1,j\alpha+\frac{1}{2}\alpha)\otimes\mathbb{C}e^{\lambda}$ is isomorphic to $\hat{\mathfrak{h}}$-module $M_{\hat{\mathfrak{h}}}(1,j\alpha+\frac{1}{2}\alpha+\lambda)$ by the remark after Definition 5.3, then by applying $h(m)$, with $h\in\mathfrak{h}$ and $m\geq 0$, repeatedly onto u_j, we can show that $e^{j\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}\in W$. Hence

$$e^{(j+n)\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}=\epsilon(j\alpha,n\alpha)^{-1}e_{n\alpha}\left(e^{j\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}\right)\in W,\quad n\in\mathbb{Z},$$

in view of (5.8). This shows $e^{m\alpha+\frac{1}{2}\alpha}\otimes e^{\lambda}\in W$ for all $m\in\mathbb{Z}$. Now it follows from (5.6) that $M_{\hat{\mathfrak{h}}}(1,m\alpha+\frac{1}{2}\alpha)\otimes\mathbb{C}e^{\lambda}\subseteq W$ for all $m\in\mathbb{Z}$. Hence we have $L^{(\frac{1}{2}\alpha,\lambda)}=W$. □

5.2. **Classification of irreducible modules of V_P.** By Lemma 5.4 and 5.5,

$$\Sigma(P)=\left\{(L^{(0,\lambda)},Y_M),(L^{(\frac{1}{2}\alpha,\lambda)},Y_M):\lambda\in(\mathbb{C}\alpha)^{\perp}\subset\mathfrak{h}\right\}\tag{5.17}$$

is a set of irreducible modules over the parabolic-type subVOA V_P of V_{A_2}, where Y_M is given by Definition 5.3. In this subsection, using the description of the Zhu's algebra $A_P=A(V_P)$ of V_P in Theorem 4.11 and Corollary 4.12, we show that $\Sigma(P)$ gives a complete list of irreducible modules over V_P.

By Lemma 5.5 and Theorem 2.1.2 in [Z96], $U^{(0,\lambda)} = \mathbb{C}(\mathbf{1}\otimes e^{\lambda})$ and $U^{(\frac{1}{2}\alpha,\lambda)} = \mathbb{C}(e^{\frac{1}{2}\alpha}\otimes e^{\lambda}) \oplus \mathbb{C}(e^{-\frac{1}{2}\alpha}\otimes e^{\lambda})$ are irreducible modules over $A(V_P)$. We use the following notations for simplicity:

$$U^{(0,\lambda)} := \mathbb{C}e, \quad \text{where } e = \mathbf{1}\otimes e^{\lambda},$$

$$U^{(\frac{1}{2}\alpha,\lambda)} := \mathbb{C}e^{+}\oplus\mathbb{C}e^{-}, \quad \text{where } e^{+} = e^{\frac{1}{2}\alpha}\otimes e^{\lambda}, \quad e^{-} = e^{-\frac{1}{2}\alpha}\otimes e^{\lambda}.$$

By Corollary 4.13 and (4.32), we have $A_P = (A(V_{\mathbb{Z}\alpha})\otimes\mathbb{C}[y])\oplus J$ as vector spaces, wherein $A(V_{\mathbb{Z}\alpha})$ is a subalgebra of A_P, and J is a two-sided nilpotent ideal of A_P.

Since the action of an element $[a]\in A(V_P)$ on $U^{(\epsilon,\lambda)}$ is given by $o(a) = \mathrm{Res}_z z^{\mathrm{wt}a-1}Y_M(a,z)$, then by (5.4)–(5.9), the following relations hold for the spanning elements of $U^{(\epsilon,\lambda)}$:

$$J.e = J.e^{+} = J.e^{-} = 0, \tag{5.18}$$

$$x_{\alpha}.e = x_{-\alpha}.e = x.e = 0, \quad y.e = (\lambda|\beta)e, \tag{5.19}$$

$$x_{\alpha}.e^{+} = 0, x_{\alpha}.e^{-} = 0;\ x_{-\alpha}.e^{+} = e^{-}, x_{-\alpha}.e^{-} = 0;\ x.e^{\pm} = \pm e^{\pm};\ y.e^{\pm} = ((\lambda|\beta)\mp\frac{1}{2})e^{\pm}. \tag{5.20}$$

By Theorem 2.2.2 in [Z96], to show $\Sigma(P)$ is a complete list of irreducible V_P-modules, it suffices to show that the following set

$$\Sigma_0(P) := \left\{U^{(0,\lambda)}, U^{(\frac{1}{2}\alpha,\lambda)} : \lambda\in(\mathbb{C}\alpha)^{\perp}\subset\mathfrak{h}\right\} \tag{5.21}$$

is a complete list of irreducible A_P-modules.

Theorem 5.6. *Let $U\neq 0$ be an irreducible A_P-module. Then U is isomorphic to either $U^{(0,\lambda)}$ or $U^{(\frac{1}{2}\alpha,\lambda)}$, for some $\lambda\in(\mathbb{C}\alpha)^{\perp}$.*

Proof. Since $J\leq A_P$ is an nilpotent ideal, we must have $J.U = 0$. So U is an irreducible module over the quotient algebra $A^P = A_P/J = \left(\bigoplus_{n=0}^{\infty}\mathbb{C}(x_{-\alpha}y^n)\right)\oplus\mathbb{C}[x,y]/\langle x^3-x\rangle\oplus\left(\bigoplus_{n=0}^{\infty}\mathbb{C}(x_{\alpha}y^n)\right)$. By Corollary 4.12, $A^P\cong A(V_{\mathbb{Z}\alpha})\otimes\mathbb{C}[y]$ as a vector space, and $A(V_{\mathbb{Z}\alpha})$ is a subalgebra of A^P. Hence U is also a $A(V_{\mathbb{Z}\alpha})$-module. Recall that $A(V_{\mathbb{Z}\alpha})$ is a semisimple associative algebra with two irreducible modules $W^0 = \mathbb{C}\mathbf{1}$ and $W^{\frac{1}{2}\alpha} = \mathbb{C}e^{\frac{1}{2}\alpha}+\mathbb{C}e^{-\frac{1}{2}\alpha}$ up to isomorphism (see Example 3.10). Then U has the following decomposition as an $A(V_{\mathbb{Z}\alpha})$-module:

$$U = \bigoplus_{i\in I}W^0\oplus\bigoplus_{j\in J}W^{\frac{1}{2}\alpha},$$

where the irreducible modules W^0 and $W^{\frac{1}{2}\alpha}$ occur $|I|$ and $|J|$-times, respectively.

Case I: $I\neq\emptyset$. In this case, there exists a nonzero copy of $W^0\subset U$.

Consider $W = \mathbb{C}[y].W^0 = \mathbb{C}[y].\mathbf{1}$. Since $x_{\pm\alpha}.\mathbf{1} = x.\mathbf{1} = 0$, and

$$xy = yx, \quad yx_{\alpha} = x_{\alpha}y - x_{\alpha}, \quad yx_{-\alpha} - x_{-\alpha}y + x_{-\alpha}, \tag{5.22}$$

in view of (4.22), W is an $A^P = A(V_{\mathbb{Z}\alpha})\cdot\mathbb{C}[y]$-submodule. Hence we have $U = W = \mathbb{C}[y].\mathbf{1}$ as U is irreducible. Moreover, in this case, U is also an irreducible $\mathbb{C}[y]$-module. Then by Hilbert's Nullstellensatz, we have $U\cong\mathbb{C}[y]/\langle y-\lambda_0\rangle$ for some $\lambda_0\in\mathbb{C}$. Choose $\lambda\in\mathfrak{h}$ so that $(\lambda|\alpha) = 0$ and $(\lambda|\beta) = \lambda_0$, then $U\cong U^{(0,\lambda)}$, in view of (5.19).

Case II: $J\neq\emptyset$. In this case, there exists a nonzero copy of $W^{\frac{1}{2}\alpha}\subset U$.

Again, by (5.22), the subspace $W = \mathbb{C}[y].W^{\frac{1}{2}\alpha}\subset U$ is a A^P-submodule of U. Hence $U = W = \mathbb{C}[y].e^{\frac{1}{2}\alpha}+\mathbb{C}[y].e^{-\frac{1}{2}\alpha}$. We want to show that $U\cong U^{(\frac{1}{2}\alpha,\lambda)}$ for some $\lambda\in(\mathbb{C}\alpha)^{\perp}$.

For the simplicity of our notations, we denote $e^{\frac{1}{2}\alpha}$ and $e^{-\frac{1}{2}\alpha}$ in $W^{\frac{1}{2}\alpha}$ by e^{+} and e^{-}, respectively. Similar to (5.20), we have the following relations:

$$x_{\alpha}.e^{+} = 0, \quad x_{\alpha}.e^{-} = e^{+}, \quad x_{-\alpha}.e^{+} = e^{-}, \quad x_{-\alpha}.e^{-} = 0, \quad x.e^{\pm} = \pm e^{\pm}.$$

Moreover, it follows from (5.22) that $x_{\alpha}y^n = (y+1)^n x_{\alpha}$ and $x_{-\alpha}y^n = (y-1)^n x_{-\alpha}$, for any $n\geq 0$. Then for any $f(y), g(y)\in\mathbb{C}[y]$, we have

$$x_{\alpha}.\left((y-1)f(y).e^{-}\right) = yf(y+1)e^{+}, \quad x_{\alpha}.\left(yg(y).e^{+}\right) = (y+1)g(y+1)e.e^{+} = 0, \tag{5.23}$$

$$x_{-\alpha}.\left(yg(y).e^+\right) = (y-1)g(y-1).e^-,\quad x_{-\alpha}.\left((y-1)f(y).e^-\right) = (y-2)f(y-1)x_{-\alpha}.e^- = 0. \tag{5.24}$$

Now consider the subspace

$$N := y\mathbb{C}[y].e^+ + (y-1)\mathbb{C}[y].e^- \subseteq U.$$

By (5.23) and (5.24), we have $x_{\pm\alpha}.((y-1)\mathbb{C}[y].e^-) \subseteq y\mathbb{C}[y].e^+ \subset N$ and $x_{\pm\alpha}.(y\mathbb{C}[y].e^+) \subseteq (y-1)\mathbb{C}[y].e^- \subset N$. Moreover, since $xy = yx$ and $x.e^\pm = \pm e^\pm$, it follows that N is an $A^P = A(V_{\mathbb{Z}\alpha})\cdot\mathbb{C}[y]$-submodule of U. By the irreducibility of U, we have $N = 0$ or $N = U$.

If $N = 0$, then $y.e^+ = y.e^- = 0$, and it is clear that $U \cong U^{(\frac{1}{2}\alpha,0)}$. If $N = U$, then there exists $f(y), g(y) \in \mathbb{C}[y]$ such that

$$e^+ = yf(y).e^+ + (y-1)g(y).e^-. \tag{5.25}$$

Apply x_α to (5.25), we have $0 = yg(y+1).e^+$ by (5.23). Apply $x_{-\alpha}$ to this equation, we have $0 = (y-1)g(y).e^-$ by (5.24). Hence $e^+ = yf(y).e^+$, and

$$0 = (yf(y)-1).e^+ = (y-\lambda_k)\dots(y-\lambda_1).e^+,\quad \lambda_1,\dots,\lambda_k \in \mathbb{C},$$

where k is the degree of the polynomial $yf(y)$. Note that $\lambda_1,\dots,\lambda_k$ are nonzero since their product is $(-1)^{k+1}$. Let $1\le j\le k$ be the smallest index such that

$$(y-\lambda_{j-1})\dots(y-\lambda_1).e^+ \neq 0,\quad \text{and}\quad (y-\lambda_j).\left((y-\lambda_{j-1})\dots(y-\lambda_1).e^+\right) = 0.$$

Let $\tilde{e^+} := (y-\lambda_{j-1})\dots(y-\lambda_1).e^+$. Then $\tilde{e^+}\neq 0$ and $y.\tilde{e^+} = \lambda_j\tilde{e^+}$. Moreover, we have $x_\alpha.\tilde{e^+} = (y+1-\lambda_{j-1})\dots(y+1-\lambda_1)x_\alpha.e^+ = 0$ and $x.\tilde{e^+} = \tilde{e^+}$, in view of (5.23) and (5.22).

On the other hand, let $\tilde{e^-} := x_{-\alpha}.\tilde{e^+} = (y-1-\lambda_{j-1})\dots(y-1-\lambda_1).e^-$ (see (5.24)). We have $U = A^P.\tilde{e^+} = \mathbb{C}\tilde{e^+} + \mathbb{C}\tilde{e^-}$ since $A^P = A(V_{\mathbb{Z}\alpha})\cdot\mathbb{C}[y]$. Moreover, we have $x.e^- = -e^-$ since $xx_{-\alpha} = -x_{-\alpha}$, and $y.\tilde{e^-} = (\lambda_j+1)\tilde{e^-}$ by applying $x_{-\alpha}$ to the equation $(y-\lambda_j)(y-\lambda_{j-1})\dots(y-\lambda_1).e^+ = 0$.

If $\tilde{e^-} = 0$, then $U = \mathbb{C}\tilde{e^+}$ is isomorphic to some $U^{(0,\lambda)}$. This contradicts our assumption that $W^{\frac{1}{2}\alpha} \subset U$. Hence $\tilde{e^-}\neq 0$, and we have

$$\begin{aligned}
&x_\alpha.\tilde{e^+} = 0,\quad x_{-\alpha}.\tilde{e^+} = \tilde{e^-},\quad x.\tilde{e^+} = \tilde{e^+},\quad y.\tilde{e^+} = \lambda_j\tilde{e^+},\\
&x_\alpha.\tilde{e^-} = (y+1-1-\lambda_{j-1})\dots(y+1-1-\lambda_1).x_\alpha.e^- = (y-\lambda_{j-1})\dots(y-\lambda_1).e^+ = \tilde{e^+},\\
&x_{-\alpha}.\tilde{e^-} = (y-1-1-\lambda_{j-1})\dots(y-1-1-\lambda_1)x_{-\alpha}.e^- = 0,\\
&x.\tilde{e^-} = -\tilde{e^-},\quad y.\tilde{e^-} = (\lambda_j+1)\tilde{e^-}.
\end{aligned}$$

Note that $\tilde{e^+}$ and $\tilde{e^-}$ are linearly independent since they are eigenvectors of x (or y) of distinct eigenvalues. Now choose $\lambda\in\mathfrak{h}$ such that $(\lambda|\alpha) = 0$ and $(\lambda|\beta) - \frac{1}{2} = \lambda_j$. Then by (5.20), we have $U = \mathbb{C}\tilde{e^+}\oplus\mathbb{C}\tilde{e^-} \cong U^{(\frac{1}{2}\alpha,\lambda)}$ as A^P-modules. By (5.18) and Corollary 4.12, they are also isomorphic as $A_P = A^P\oplus J$-modules. □

Corollary 5.7. *The set* $\Sigma(P) = \left\{(L^{(0,\lambda)}, Y_M), (L^{(\frac{1}{2}\alpha,\lambda)}, Y_M) : \lambda\in(\mathbb{C}\alpha)^\perp\subset\mathfrak{h}\right\}$ *is a complete list of irreducible modules over the rank-two parabolic-type subVOA* V_P *of* V_{A_2}.

6. Further constructions and questions

In this Section, we introduce some new constructions motivated by the Borel and parabolic-type subVOAs of lattice VOAs, and ask a few questions arising from our constructions.

6.1. Quasi-triangular decomposition of vertex operator algebras.

According to [FHL93] Section 5.3, a non-degenerate symmetric bilinear form $(\cdot|\cdot) : V\times V\to\mathbb{C}$ on a vertex operator algebra V is called **invariant** if

$$(Y(a,z)b|c) = (b|Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})c),\quad a,b,c\in V. \tag{6.1}$$

Li proved in [L94] that the vector space of symmetric invariant bilinear forms of a VOA V is isomorphic $(V_0/L(1)V_1)^*$, and any invariant bilinear form on V is automatically symmetric.

6.1.1. *Definition and first properties of Quasi-triangular decomposition.*

Definition 6.1. Let V be a vertex operator algebra, equipped with a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot) : V \times V \to \mathbb{C}$ such that $(\mathbf{1}|\mathbf{1}) = 1$. A subspace decomposition $V = V_+ \oplus V_H \oplus V_-$ is called a **quasi-triangular decomposition** of V if

(1) V_H and $V_\pm$ are invariant under the action of $sl_2(\mathbb{C}) = \mathbb{C}L(-1) + \mathbb{C}L(0) + \mathbb{C}L(1)$;
(2) V_H is a sub-vertex algebra of V such that $\mathbf{1}_V \in V_H$;
(3) V_+ and V_- are sub-vertex algebras without vacuum of V;
(4) $(V_\pm|V_\pm) = (V_H|V_+) = (V_H|V_-) = 0$.

As an immediate consequence of Definition 6.1, the following properties, which resembles the properties of usual triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of a semisimple Lie algebra $\mathfrak{g}$, hold for a quasi-triangular decomposition of VOAs:

Lemma 6.2. *Let $V = V_+ \oplus V_H \oplus V_-$ be a quasi-triangular decomposition of a VOA V. Then*

(a) $(\cdot|\cdot)|_{V_H\times V_H}$ is non-degenerate on V_H;
(b) $V_\pm$ and V_H are graded subalgebras, with $V_n = (V_+)_n \oplus (V_H)_n \oplus (V_-)_n$ for all $n \in \mathbb{Z}$;
(c) V_+ and V_- are isotropic with respect to $(\cdot|\cdot)$;
(d) $Y(V_H, z)V_\pm \subseteq V_\pm$ and $Y(V_\pm, z)V_H \subseteq V_\pm$.

Proof. (a) follows from the non-degeneracy of $(\cdot|\cdot)$ and (4) in Definition 6.1. Since $L(0)V_H \subseteq V_H$ and $L(0)V_\pm \subseteq V_\pm$, it follows that $V_H = \bigoplus_{n\in\mathbb{Z}}(V_H)_n$ and $V_\pm = \bigoplus_{n\in\mathbb{Z}}(V_\pm)_n$, with $(V_H)_n = V_H \cap V_n$ and $(V_\pm)_n = V_\pm \cap V_n$ for all n. This proves (b). Furthermore, since $(V_m|V_n) = 0$ unless $m = n$ (see [FHL93] Section 5.3), the restriction $(\cdot|\cdot)|_{V_n\times V_n}$ is non-degenerate for any n. Then it follows from (a) and (b) that $(V_+)_n$ and $(V_-)_n$ are isotropic for all $n \in \mathbb{Z}$. Hence V_+ and V_- are isotropic.

Finally, given $h \in V_H$ and $a \in V_+$, assume $Y(h, z)a = b_- + h' + b_+$, with $b_\pm \in V_\pm$ and $h' \in V_H$. We claim that $b_- = 0$ and $h' = 0$.

Indeed, by the invariance (6.1), the axioms (2) and (4), and the fact that $L(1)h \subseteq V_H$, we have

$$(b_- + h' + b_+|V_H) = (Y(h, z)a|V_H) = (a|Y(e^{zL(1)}(-z^{-2})^{L(0)}h, z^{-1})V_H) = (a|V_H) = 0.$$

In particular, we have $(h'|V_H) = 0$ since $(b_\pm|V_H) = 0$. Then it follows from (a) that $h' = 0$. Furthermore, by axioms (3) and (4), together with $L(1)V_+ \subseteq V_+$, we have

$$\begin{aligned}(b_- + b_+|V_+) &= (Y(h, z)a|V_+) = (e^{zL(-1)}Y(a, -z)h|V_+) = (h|Y(e^{-zL(1)}(-z^{-2})^{L(0)}a, -z^{-1})e^{zL(1)}V_+)\\ &= (h|V_+) = 0.\end{aligned}$$

In particular, we have $(b_-|V_+) = 0$ since $(b_+|V_+) = 0$. Then we have $b_- = 0$ since V_+ and V_- are isotropic. This shows $Y(V_H, z)V_+ \subseteq V_+$. Similarly, we can show that $Y(V_H, z)V_- \subseteq V_-$. Finally, $Y(V_\pm, z)V_H \subseteq V_\pm$ follows from the skew-symmetry of Y and the fact that $L(-1)V_\pm \subseteq V_\pm$. □

6.1.2. *Examples of quasi-triangular decomposition.* There are natural examples of quasi triangular decomposition for lattice VOAs arising from the Borel-type subVOA V_B in Section 3 and the parabolic-type subVOA V_P in Sections 4 and 5. Note that a lattice VOA V_L has a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot) : V_L \times V_L \to \mathbb{C}$ extended from the bilinear form on L. It satisfies $(\mathbf{1}|\mathbf{1}) = 1$ (see, for example, [FLM88, D93, L94]).

Lemma 6.3. *Let $L = \mathbb{Z}\alpha$ with $(\alpha|\alpha) = 2N$, and let $V_+ := V_{\mathbb{Z}_{>0}\alpha}$, $V_H := M_{\hat{\mathfrak{h}}}(1, 0)$, and $V_- := V_{\mathbb{Z}_{<0}\alpha}$. Then $V_{\mathbb{Z}\alpha} = V_+ \oplus V_H \oplus V_-$ is a quasi-triangular decomposition.*

Proof. Since $\omega = \frac{1}{4}\alpha(-1)^2$, it is clear that $L(n)M_{\hat{\mathfrak{h}}}(1, m\alpha) \subseteq M_{\hat{\mathfrak{h}}}(1, m\alpha)$ for all $m \in \mathbb{Z}$ and $n = -1, 0, 1$. It suffices to prove (4) in Definition 6.1. We claim that for any $m, n \in \mathbb{Z}$,

$$\Big(M_{\hat{\mathfrak{h}}}(1, m\alpha)|M_{\hat{\mathfrak{h}}}(1, n\alpha)\Big) = 0 \quad \text{unless} \quad m + n = 0. \tag{6.2}$$

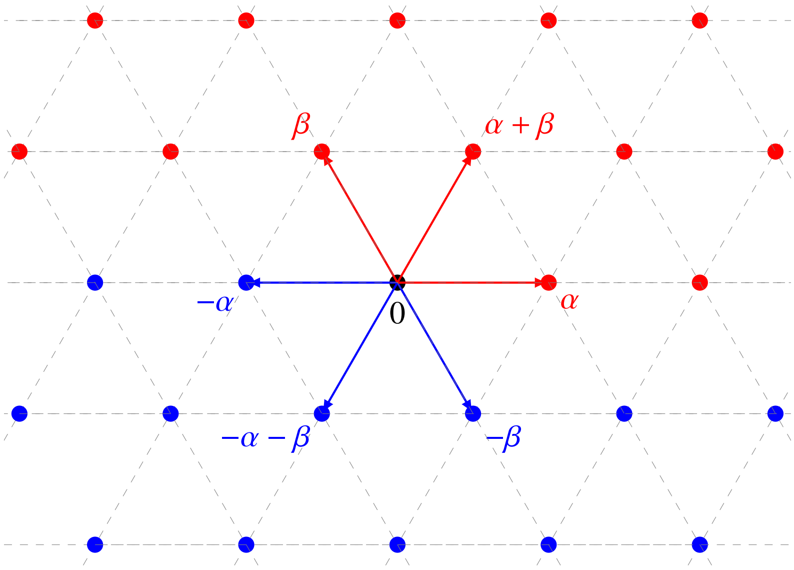

FIGURE 3.

Indeed, note that $\alpha(0)\alpha(-n_1)\dots\alpha(-n_k)e^{m\alpha} = 2Nm(\alpha(-n_1)\dots\alpha(-n_k)e^{m\alpha})$ for any $m \in \mathbb{Z}$. Then by the invariance (6.1) of the bilinear form and the fact that $L(1)\alpha(-1)\mathbf{1} = 0$, we have

$$2Nm(u|v) = (\alpha(0)u|v) = (u|\sum_{j\geq 0}\frac{(-1)}{j!}(L(1)^j\alpha(-1)\mathbf{1})_{2-0-j-2}v) = (u|-\alpha(0)v) = -2Nn(u|v),$$

where $u \in M_{\hat{\mathfrak{h}}}(1, m\alpha)$ and $v \in M_{\hat{\mathfrak{h}}}(1, n\alpha)$. This proves (6.2). Since $\mathbb{Z}_{>0}\alpha$ and $\mathbb{Z}_{<0}\alpha$ are sub-semigroups of $\mathbb{Z}\alpha$, it is clear that $(V_\pm|V_\pm) = (V_H|V_+) = (V_H|V_-) = 0$. □

Now let $L = A_2 = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ be the root lattice of type A_2. There are many examples of quasi-triangular decompositions on the lattice VOA V_{A_2}. Here we present two different examples.

Consider the following subsets of the lattice A_2:

$$N_+ := \{m\alpha + n\beta : m \in \mathbb{Z}, n > 0\} + \mathbb{Z}_{>0}\alpha, \tag{6.3}$$
$$N_- := \{m\alpha + n\beta : m \in \mathbb{Z}, n < 0\} + \mathbb{Z}_{<0}\alpha. \tag{6.4}$$

Clearly, $N_\pm$ are sub-semigroups of A_2, and so $V_{N_\pm} = \bigoplus_{\gamma\in N_\pm} M_{\hat{\mathfrak{h}}}(1,\gamma)$ are sub-vertex algebras without vacuum of V_L, in view of Proposition 2.2. Moreover, we have $L = N_+ \sqcup \{0\} \sqcup N_-$.

In figure 3, the red dots represent elements in N_+, and the blue dots represent elements in N_-. The black dot is 0 of A_2.

Lemma 6.4. *With the notations as above, let $V_+ := V_{N_+}$, $V_H := M_{\hat{\mathfrak{h}}}(1,0)$, and $V_- := V_{N_-}$. Then $V_{A_2} = V_+ \oplus V_H \oplus V_-$ is a quasi-triangular decomposition of the lattice VOA V_{A_2}.*

Proof. Again, it suffices to prove (4) in Definition 6.1 since $\omega = \frac{1}{2}\sum_{i=1}^2 u^i(-1)^2\mathbf{1}$, where $\{u^1, u^2\}$ is an orthonormal basis of $\mathfrak{h}$. For $\gamma, \theta \in A_2$, similar to (6.2), we have

$$\left(M_{\hat{\mathfrak{h}}}(1,\gamma)|M_{\hat{\mathfrak{h}}}(1,\theta)\right) = 0, \quad \text{unless} \quad \gamma + \theta = 0. \tag{6.5}$$

The proof of (6.5) is also similar to (6.2), we omit it. Now it is clear from figure 3 that $(V_\pm|V_\pm) = (V_H|V_+) = (V_H|V_-) = 0$. □

There is another quasi-triangular decomposition of V_{A_2} that corresponds to the parabolic-type subVOA V_P in Section 4 and 5. Consider the following sub-semigroups of A_2:

$$N^+ := \mathbb{Z}\alpha \oplus \mathbb{Z}_{>0}\beta, \quad T := \mathbb{Z}\alpha, \quad N^- := \mathbb{Z}\alpha \oplus \mathbb{Z}_{<0}\beta. \tag{6.6}$$

Again, it follows from Proposition 2.2 that $V_{N^\pm} = \bigoplus_{\gamma\in N^\pm} M_{\hat{\mathfrak{h}}}(1,\gamma)$ are sub-vertex algebras without vacuum of V_{A_2}, and $V_T = \bigoplus_{\theta\in T} M_{\hat{\mathfrak{h}}}(1,\theta)$ is a subVOA of V_{A_2}.

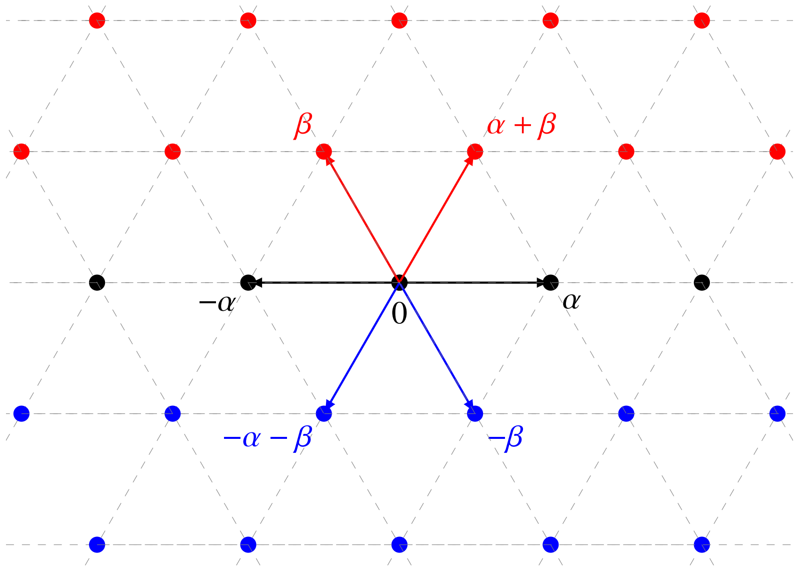

FIGURE 4.

In figure 4, the red dots represent elements in N^+, blue dots represent elements in N^-, and black dots represent elements in T. The following lemma follows immediately from (6.5) and figure 4:

Lemma 6.5. *With notations as above, let $V^\pm := V_{N^\pm}$ and $V^H := V_T$. Then $V_{A_2} = V^+ \oplus V^H \oplus V^-$ is a quasi-triangular decomposition of V_{A_2}, with $V_P = V^+ \oplus V^H$.*

Remark 6.6. The quasi-triangular decompositions in Lemma 6.3 and 6.4 are compatible with the triangular decomposition of the first-level Lie algebras of the corresponding lattice VOAs.

Indeed, assume $(\alpha|\alpha) = 2$ for the rank-one lattice $\mathbb{Z}\alpha$, then by figure 3 we have

$$\begin{aligned}(V_{\mathbb{Z}\alpha})_1 &= sl_2(\mathbb{C}) = \mathbb{C}e^\alpha \oplus \mathbb{C}\alpha(-1)\mathbf{1} \oplus \mathbb{C}e^{-\alpha} = (V_+)_1 \oplus (V_H)_1 \oplus (V_-)_1,\\ (V_{A_2})_1 &= sl_3(\mathbb{C}) = \left(\mathbb{C}e^\alpha + \mathbb{C}e^\beta + \mathbb{C}e^{\alpha+\beta}\right) \oplus \mathfrak{h} \oplus \left(\mathbb{C}e^{-\alpha} + \mathbb{C}e^{-\beta} + \mathbb{C}e^{-\alpha-\beta}\right) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-\\ &\qquad\qquad\quad = (V_+)_1 \oplus (V_H)_1 \oplus (V_-)_1.\end{aligned}$$

However, the quasi-triangular decomposition of $V_{A_2} = V^+ \oplus V^H \oplus V^-$ given by figure 4 and Lemma 6.3 is not compatible with the triangular decomposition of first-level Lie algebra, since the "Cantan-part" V^H is not the Heisenberg subVOA $M_{\hat{\mathfrak{h}}}(1,0)$.

In the representation theory of Lie algebras (finite or infinite-dimensional), the triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is used to construct Verma modules and irreducible highest-weight modules over $\mathfrak{g}$. With our definition and examples, it is natural to ask the following:

Question 6.7. Is there a construction of "Verma-type" modules for VOAs arising from the quasi-triangular decomposition in Definition 6.1?

There is a notion of generalized Verma module for a VOA V given by Dong, Li, and Mason in [DLM98]. A generalized Verma module $\bar{M}(U)$ was constructed from a module U over Zhu's algebra $A(V)$ (see [DLM98] Section 5). We suspect that one can use modules over Zhu's algebra $A(V_H)$ of V_H in the quasi-triangular decomposition of V to construct Verma-type modules.

6.2. **Borel-type and parabolic-type subalgebras of affine vertex operator algebras.** The vertex operator realization of the highest weight representations of affine Kac-Moody algebras in [FK80] indicates that there is an isomorphism $L_{\hat{\mathfrak{g}}}(1,0) \cong V_Q$ of VOAs, where V_Q is the lattice VOA associated to the root lattice Q of $\mathfrak{g}$, and Q is of A, D, E-type.

Inspired by this isomorphism of VOAs and our previous examples, it is reasonable to define Borel and parabolic-type subalgebras for general affine VOAs of arbitrary level.

6.2.1. *Definition of affine vertex operator algebras.* We first recall the definition of affine vertex operator algebras of positive integer level k defined by Frenkel and Zhu in [FZ92].

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$, and let Δ be the root system associated to $\mathfrak{h}$ with root lattice $Q \subset \mathfrak{h}^*$. Normalize the invariant bilinear form on $\mathfrak{g}$ so that $(\theta|\theta) = 2$, where θ is the longest root of Δ. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be its affinization, with Lie bracket given by

$$[K, \hat{\mathfrak{g}}] = 0, \quad [a(m), b(n)] = [a, b](m+n) + m\delta_{m+n,0}(a|b)K, \quad a, b \in \mathfrak{g},\ m, n \in \mathbb{Z}.$$

Let $\mathbb{C}\mathbf{1}$ be a $\hat{\mathfrak{g}}_{\geq 0}$-module with $K.\mathbf{1} = k\mathbf{1}$ and $\mathfrak{g} \otimes \mathbb{C}[t].\mathbf{1} = 0$. The Weyl vacuum module

$$V_{\hat{\mathfrak{g}}}(k, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} \mathbb{C}\mathbf{1}$$

is a VOA, with $\mathbf{1} = 1 \otimes \mathbf{1}$ and $\omega_{\mathrm{aff}} = \frac{1}{2(h^\vee + k)} \sum_{i=1}^{\dim \mathfrak{g}} u^i(-1)u_i(-1)\mathbf{1}$, called the **vacuum module vertex operator algebra of level** k, where $h^\vee$ is the dual Coxeter number of Δ, and $\{u^i\}$ and $\{u_i\}$ are dual orthonormal basis of $\mathfrak{g}$.

$V_{\hat{\mathfrak{g}}}(k, 0)$ has a unique maximal $\hat{\mathfrak{g}}$-submodule $W(k, 0) = U(\hat{\mathfrak{g}})x_\theta^{k+1}(-1)\mathbf{1}$, where $x_\theta \in \mathfrak{g}_\theta$. The quotient module $L_{\hat{\mathfrak{g}}}(k, 0) = V_{\hat{\mathfrak{g}}}(k, 0)/W(k, 0)$ is also a VOA called the **affine vertex operator algebra of level** k.

In the study of parafermion vertex operator algebras (see, for example, [DW10, DR17]), the following decomposition of $V_{\hat{\mathfrak{g}}}(k, 0)$ and $L_{\hat{\mathfrak{g}}}(k, 0)$ were frequently used:

$$V_{\hat{\mathfrak{g}}}(k, 0) = \bigoplus_{\lambda \in Q} V_{\hat{\mathfrak{g}}}(k, 0)(\lambda) = \bigoplus_{\lambda \in Q} M_{\hat{\mathfrak{h}}}(k, 0) \otimes N_\lambda, \tag{6.7}$$

$$L_{\hat{\mathfrak{g}}}(k, 0) = \bigoplus_{\lambda \in Q} L_{\hat{\mathfrak{g}}}(k, 0)(\lambda) = \bigoplus_{\lambda \in Q} M_{\hat{\mathfrak{h}}}(k, 0) \otimes K_\lambda, \tag{6.8}$$

where $N_\lambda = \{v \in V_{\hat{\mathfrak{g}}}(k, 0) : h(m)v = \delta_{m,0}v,\ \text{for } h \in \mathfrak{h}, m \in \mathbb{Z}\}$ and $K_\lambda = \{v \in L_{\hat{\mathfrak{g}}}(k, 0) : h(m)v = \delta_{m,0}v,\ \text{for } h \in \mathfrak{h}, m \in \mathbb{Z}\}$ are subspaces of $\hat{\mathfrak{h}}$-highest-weight vectors of weight $\lambda \in Q$ in $V_{\hat{\mathfrak{g}}}(k, 0)$ and $L_{\hat{\mathfrak{g}}}(k, 0)$, respectively, and

$$V_{\hat{\mathfrak{g}}}(k, 0)(\lambda) = \{v \in V_{\hat{\mathfrak{g}}}(k, 0) : h(0)v = \lambda(h)v,\ \text{for } h \in \mathfrak{h}\} = M_{\hat{\mathfrak{h}}}(k, 0) \otimes N_\lambda, \tag{6.9}$$

$$L_{\hat{\mathfrak{g}}}(k, 0)(\lambda) = \{v \in L_{\hat{\mathfrak{g}}}(k, 0) : h(0)v = \lambda(h)v,\ \text{for } h \in \mathfrak{h}\} = M_{\hat{\mathfrak{h}}}(k, 0) \otimes K_\lambda, \tag{6.10}$$

for any $\lambda \in Q$. Note that $K(\mathfrak{g}, k) := K_0$ is the parafermion vertex operator algebra.

Similar to Proposition 2.2, we have the following fact about the decomposition (6.7) and (6.8):

Proposition 6.8. *Let k be a positive integer, and let V be $V_{\hat{\mathfrak{g}}}(k, 0)$ or $L_{\hat{\mathfrak{g}}}(k, 0)$, with $V(\lambda) = V_{\hat{\mathfrak{g}}}(k, 0)(\lambda)$ or $L_{\hat{\mathfrak{g}}}(k, 0)(\lambda)$ for $\lambda \in Q$, respectively.*

Assume $M \leq Q$ is an abelian sub-monoid of the root lattice Q, with identity $0 \in Q$. Then $V^M := \bigoplus_{\lambda \in M} V(\lambda)$ is a subVOA of $(V, Y, \omega_{\mathrm{aff}}, \mathbf{1})$. We call V^M the **subVOA associated to** M**.**

Proof. Let $\lambda, \mu \in M$, and $a \in V(\lambda)$, $b \in V(\mu)$. For any $h \in \mathfrak{h}$, since

$$h(0)(a_n b) = a_n h(0)b + (h(0)a)_n b = (\lambda + \mu|h)a_n b, \quad n \in \mathbb{Z},$$

and M is a sub-monoid, we have $a_n b \in V(\lambda + \mu) \subset V^M$. Moreover, since $\omega_{\mathrm{aff}}, \mathbf{1} \in V(0) \subset V^M$, it follows that V_M is a subVOA of V. □

Remark 6.9. Note that $V(0)$ is a tensor product of $(M_{\hat{\mathfrak{h}}}(k, 0), Y, \omega_{\mathfrak{h}}, \mathbf{1})$ and $(N_0(\text{or } K_0), Y, \omega_{\mathrm{aff}} - \omega_h, \mathbf{1})$. Unlike the lattice case in Proposition 2.2, the sub-vertex algebra $V^M := \bigoplus_{\lambda \in M} V(\lambda)$ could have multiple Virasoro elements. We say that V^M a subVOA of V only if it has ω_{aff} as the Virasoro element.

6.2.2. *Borel and parabolic-type subalgebras of* $V_{\hat{\mathfrak{g}}}(k,0)$ *and* $L_{\hat{\mathfrak{g}}}(k,0)$. Recall the definitions of a Borel-type sub-monoid B and a parabolic-type sub-monoid P of the root lattice Q in Definition 2.5.

Definition 6.10. Let k be a positive integer, and let V be the vacuum module or affine VOA $V_{\hat{\mathfrak{g}}}(k,0)$ or $L_{\hat{\mathfrak{g}}}(k,0)$. A subVOA V^B (resp. V^P) associated to a Borel-type (resp. parabolic-type) sub-monoid B (resp. P) $\leq Q$ is called a **Borel-type (resp. parabolic-type) subVOA** of V.

Example 6.11. Similar to the Borel-type subVOA $V_{\mathbb{Z}_{\geq 0}\alpha}$ of lattice VOA V_{A_1} in Sections 3 and the parabolic-type subVOA $V_{\mathbb{Z}\alpha\oplus\mathbb{Z}_{\geq 0}\alpha}$ of lattice VOA V_{A_2} Sections 4 and 5, we have the following examples of Borel and parabolic-type subVOAs of affine VOAs:

$$V^B = L_{\widehat{sl_2(\mathbb{C})}}(k,0)^{\mathbb{Z}_{\geq 0}\alpha} = \bigoplus_{n\geq 0} L_{\widehat{sl_2(\mathbb{C})}}(k,0)(n\alpha), \tag{6.11}$$

$$V^P = L_{\widehat{sl_3(\mathbb{C})}}(k,0)^{\mathbb{Z}\alpha\oplus\mathbb{Z}_{\geq 0}\beta} = \bigoplus_{m\in\mathbb{Z},n\geq 0} L_{\widehat{sl_3(\mathbb{C})}}(k,0)(m\alpha+n\beta). \tag{6.12}$$

Clearly, the first-level Lie algebra $V_1^B = \mathbb{C}h(-1)\mathbf{1} + \mathbb{C}e(-1)\mathbf{1}$ is a Borel subalgebra of $sl_2(\mathbb{C})$, and first-level Lie algebra $V_1^P = \mathbb{C}h_\alpha(-1)\mathbf{1} + \mathbb{C}h_\beta(-1)\mathbf{1} + \mathbb{C}x_\alpha(-1)\mathbf{1} + \mathbb{C}x_{-\alpha}(-1)\mathbf{1} + \mathbb{C}x_\beta(-1)\mathbf{1} + \mathbb{C}x_{\alpha+\beta}(-1)\mathbf{1}$ is a parabolic subalgebra of $sl_3(\mathbb{C})$.

Given our main results in Sections 3–5, it is natural to ask the following:

Question 6.12. Find a concrete description (by generators and relations) of the Zhu's algebra of V^B in (6.11) and V^P in (6.12). Moreover, find and classify all the irreducible modules over these Borel and parabolic-type subVOAs of the affine VOAs.

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References

[AD04] T. Abe and C. Dong, Classification of irreducible modules for the vertex operator algebra V_L^+: general case, *J. Alg.* **273** (2004), no.2, 657–685.

[ADL05] T. Abe, C Dong and H. Li, Fusion rules for the vertex operator algebra $M(1)^+$ and V_L^+, *Comm. Math. Phys.* **253** (2005), no.1, 171–219.

[BDT02] S. Berman, C. Dong and S. Tan, Representations of a class of lattice type vertex algebras, *J. Pure Appl. Alg.* **176** (2002), no.1, 27–47.

[Bor86] R.E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[BGL] C. Bai, L. Guo and J. Liu, Classical Yang-Baxter equation for vertex operator algebras and its operator forms, math.QA/2307.01977.

[BGLW] C. Bai, L. Guo, J. Liu and X. Wang, On Rota-Baxter vertex operator algebras, math.QA/2307.09826.

[B56] A. Borel, Groupes linéaires algébriques, *Ann. of Math. (2)* **64** (1956), 20–82.

[BT65] A. Borel and J. Tits, Groupes réductifs, *Inst. Hautes Études Sci. Publ. Math.* **27** (1965), 55–150.

[D93] C. Dong, Vertex algebras associated with even lattices, *J. Alg.* **160** (1993), 245–265.

[D94] C. Dong, Twisted modules for vertex algebras associated with even lattices, *J. Alg.* **165** (1994), no. 1, 91–112. MR1272580

[DG98] C. Dong and R. Jr. Griess, Rank one lattice type vertex operator algebras and their automorphism groups, *J. Alg.* **208** (1998), no.1, 262–275.

[DL93] C. Dong and J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, *Progress in Math.*, **Vol. 112**, Birkhäuser, Boston, MA, 1993.

[DLM97] C. Dong, H. Li and G. Mason, Certain associative algebras similar to $U(sl_2)$ and Zhu's algebra $A(V_L)$, *J. Alg.* **196** (1997), no. 2, 532–551.

[DLM98] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras. *Math. Ann.* **310**, (1998) 571–600.

[DN99(1)] C. Dong and K. Nagatomo, Representations of vertex operator algebra V_L^+ for rank one lattice L, *Comm. Math. Phys.* **202** (1999), no. 1, 169–195.

[DN99(2)] C. Dong and K. Nagatomo, Automorphism groups and twisted modules for lattice vertex operator algebras, in Recent Developments in Quantum Affine Algebras and Related Topics (Raleigh, NC, 1998), Contemporary Mathematics, American, Mathematical Society, Providence, RI, 1999, **Vol. 248**, 117–133.

[DR17] C. Dong and L. Ren, Representations of the parafermion vertex operator algebras, *Adv. Math.* **315** (2017), 88–101.

[DW10] C. Dong and Q. Wang, The structure of parafermion vertex operator algebras: general case, *Comm. Math. Phys.* **299** (2010), 783–792.

[vEMS20] J. van Ekeren, S. Möller and N. R. Scheithauer, Construction and classification of holomorphic vertex operator algebras, *J. Reine Angew. Math.* **759** (2020), 61–99.

[FHL93] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules. *Memoirs American Math. Soc.* **104**, (1993) 1–64.

[FLM88] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, *Pure and Applied Math.*, **Vol. 134**, Academic Press, Boston MA, 1988.

[FK80] I. B. Frenkel and V. G Kac, Basic representations of affine Lie algebras and dual resonance models, *Invent. Math.* **62** (1980), 23–66.

[FZ92] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.

[GW04] K. R. Goodearl and R. B. Jr. Warfield, An Introduction to Noncommutative Noetherian Rings, 2nd ed., *London Mathematical Society Student Texts*, **Vol. 61**, Cambridge University Press, Cambridge, 2004.

[Hum1] J. E. Humphreys, Linear Algebraic Groups, *Graduate Texts in Mathematics*, **Vol. 21**, Springer-Verlag New York, NY, 1995.

[Hum2] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, *Graduate Texts in Mathematics*, **Vol. 9**, Springer-Verlag New York, NY, 1972.

[HL96] Y.-Z. Huang and J. Lepowsky, On the $\mathcal{D}$-module and formal-variable approaches to vertex algebras, in Topics in Geometry: In Memory of Joseph D'Atri, ed. S. Gindikin, Progress in Nonlinear Differential Equations, Birkhäuser, Boston, 1996, **Vol. 20**, 175–202.

[K97] V. G. Kac, Vertex Algebras for Beginners, *University Lecture Series*, **10**, Amer. Math. Soc., 1997.

[L94] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, *J. Pure Appl. Alg.* **96** (1994), 279–297.

[KL99] M. Karel and H. Li, Certain generating subspaces for vertex operator algebras, *J. Alg.* **217** (1999), 393–421.

[Lam11] C. H. Lam, On the constructions of holomorphic vertex operator algebras of central charge 24, *Comm. Math. Phys.* **305** (2011), 153–198.

[LS15] C. H. Lam and H. Shimakura, Classification of holomorphic framed vertex operator algebras of central charge 24, *Amer. J. Math.* **137** (2015), 111–137.

[LS16] C. H. Lam and H. Shimakura, Orbifold construction of holomorphic vertex operator algebras associated to inner automorphisms, *Comm. Math. Phys.* **342** (2016), 803–841.

[Li05] H. Li, Abelianizing vertex algebras, *Comm. Math. Phys.* **259** (2005), 391–411.

[LL04] J. Lepowsky and H. Li, Introduction to Vertex Operator Algebras and Their Representations, *Progress in Math.*, **Vol. 227**, Birkhäuser, Boston, 2004.

[MS89] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

[O33] O. Ore, Theory of non-commutative polynomials, *Ann. of Math. (2)* **34** (1933), 480–508.

[S93] A. N. Schellekens, Meromorphic $c = 24$ conformal field theories, *Comm. Math. Phys.* **153** (1993), 159–185.

[Z96] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer, Math. Soc.* **9** (1996), 237–302.

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