Generalized dynamical phase reduction for stochastic oscillators

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Phase reduction is an important tool for studying coupled and driven oscillators. The question of how to generalize phase reduction to stochastic oscillators remains actively debated. In this work, we propose a method to derive a self-contained stochastic phase equation of the form $d\vartheta = a_\vartheta(\vartheta)dt + \sqrt{2D_\vartheta(\vartheta)}\, dW_\vartheta(t)$ that is valid not only for noise-perturbed limit cycles, but also for noise-induced oscillations. We show that our reduction captures the asymptotic statistics of qualitatively different stochastic oscillators, and use it to infer their phase-response properties.

I. INTRODUCTION

Oscillatory behaviour is an ubiquitous phenomenon in physical, biological, chemical and engineering systems [1]. A powerful way of approaching oscillations is by means of a phase variable. In a purely deterministic system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n \tag{1}$$

oscillatory behaviour corresponds to stable T-periodic solutions of system eq. (1) around the attractor of the dynamics: the limit cycle (LC), which we denote as Γ. Typically, the existence of the attractor is used to provide a simpler description of the oscillatory dynamics. Namely, one parameterizes the LC, which is a closed curve in the phase space, by means of an angular *phase* variable θ such that $\Gamma = \{\mathbf{x} \mid \mathbf{x} = \gamma(\theta)\}$. Therefore, assuming the solutions are asymptotically close to the limit cycle, the parameterisation $\gamma(\theta)$ allows to study the system (1) by means of the *phase reduction*

$$d\theta = \frac{2\pi}{T} dt, \tag{2}$$

which is a one-dimensional description of the periodic dynamics. This phase reduction approach is a well-known method to study complex oscillatory phenomena, such as response to perturbations, phase locking or synchronization [2, 3].

Since real-world systems are often intrinsically fluctuating and noisy, it is natural to aim to extend the phase reduction framework to stochastic oscillators. In principle, a meaningful *stochastic phase reduction* should provide a level of understanding of the dynamics similar to the deterministic case, while incorporating the noisy component observed in realistic oscillations.

A first approach to this question is to consider the noise as a weak perturbation of the LC oscillator [4]. In this case, using a perturbative approach, one can describe the stochastic system by means of the deterministic phase [4–6]. Alternatively, extensions of phase reduction to stochastic systems based on variational methods have been proposed [7, 8]. However, perturbative and variational LC approaches require the existence of an underlying LC. Thus, they have trouble generalizing over the important cases when the addition of noise to a non oscillatory deterministic system leads to noise-induced oscillations [9].

Therefore, a fundamental challenge for building a general stochastic phase reduction is to define a phase observable that does not require the existence of an underlying LC, and that is applicable in the wide range of contexts in which LC and noise-induced oscillations can emerge. Overcoming this challenge in a successful way requires going back to the phase definition itself and updating it. The deterministic phase is defined in terms of two equivalent notions: either in terms of Poincaré sections, or of the system's asymptotic behaviour [10]. During the last decade, these two notions of phase have been extended to stochastic oscillators. Ten years ago, Schwabedal and Pikovsky [11] found the natural way of extending Poincaré's approach to noisy oscillators. To this end, they constructed a system of isochrons (curves of "equal timing") with the *mean return time property*, namely, that the *average* time it would take a trajectory to complete one oscillation and return to some point on the original isochron should equal the mean period of the oscillator, a criterion that can be also related to the solution of a partial differential equation [12]. As an alternative to the mean-return-time phase, Thomas and Lindner proposed that a meaningful phase observable (which they denoted as the "stochastic asymptotic phase") can be extracted from the asymptotic behaviour of the conditional density [13].

However, while these two notions of phase solve the problem of finding a phase observable that applies to the many different mechanisms generating stochastic oscillations, a general method for finding a self-contained phase

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equation of the form

$$d\vartheta = a_\vartheta(\vartheta)dt + \sqrt{2D_\vartheta(\vartheta)}\, dW_\vartheta(t) \tag{3}$$

is still missing. While there have been different attempts in the past, they were built ad-hoc for specific classes of stochastic oscillators [8, 14, 15].

In this paper, we aim to fill this gap by developing a generalized reduction procedure: Given a phase observable ϑ, we provide a way to obtain a self-contained phase equation as in eq. (3). We show the generality of our procedure by i) applying it to two different phase observables (the previously mentioned Mean-Return-Time phase and the stochastic asymptotic phase) and ii) finding self-contained phase equations of qualitatively different noisy oscillators.

Our paper is organised as follows. In sec. II, we introduce the mathematical background, which relies on the Kolmogorov backwards operator $\mathcal{L}^\dagger$. Next, in sec. III, we introduce the main result of this work: the stochastic phase reduction procedure. In sec. V, we define the asymptotic statistics we use to evaluate the quality of our reduction procedure. In sec. IV, we introduce two different systems in which oscillations emerge from different mechanisms, and to which we apply our framework. In sec. VI, we show a direct application of our framework: predicting the phase-dependent response of the reduced oscillator to an external perturbation. We end with a discussion of the results in sec. VII.

II. THEORY & MATHEMATICAL PRELIMINARIES

We consider the Itô stochastic differential equation (SDE)

$$d\mathbf{X} = \mathbf{f}(\mathbf{X})dt + \mathbf{g}(\mathbf{X})d\mathbf{W}(t), \tag{4}$$

where $\mathbf{X} \in \mathbb{R}^n$ is the state vector and $\mathbf{W} \in \mathbb{R}^k$ is a collection of IID Wiener processes with increments $d\mathbf{W}(t)$.

Instead of studying system eq. (4) by means of individual realisations (a *pathwise* approach), we adopt an *ensemble* perspective: we consider a collection of trajectories described by the conditional probability density function $P(\mathbf{x}, t|\mathbf{x_0}, t_0)$. This density obeys the Kolmogorov forward and backward equations (the former also known as Fokker-Planck equation) [16]:

$$\begin{aligned}\frac{\partial}{\partial t}P(\mathbf{x}, t|\mathbf{x_0}, t_0) &= \mathcal{L}[P] \\ &= -\nabla_{\mathbf{x}} \cdot (\mathbf{f}(\mathbf{x})P) + \boldsymbol{\nabla}^2_{\mathbf{x}}(\mathcal{G}(\mathbf{x})P), \quad (5)\\ -\frac{\partial}{\partial t_0}P(\mathbf{x}, t|\mathbf{x_0}, t_0) &= \mathcal{L}^\dagger[P] \\ &= \mathbf{f}^\intercal(\mathbf{x_0}) \cdot \nabla_{\mathbf{x_0}} P + \mathcal{G}(\mathbf{x_0})\boldsymbol{\nabla}^2_{\mathbf{x_0}} P, \quad (6)\end{aligned}$$

where $\mathcal{G} = \frac{1}{2}\mathbf{g}\mathbf{g}^\intercal$. Note that $\mathcal{L}^\dagger$ is known as the generator of the Markov process $\mathbf{X}(t)$ and is the infinitesimal generator of the stochastic Koopman operator [17, 18]. If we define the Koopman semigroup of operators $\mathcal{K}^{\Delta t}$ acting on a real valued observable $F(\mathbf{x})$ of system eq. (4) such that

$$\mathcal{K}^{\Delta t}[F(\mathbf{x}(t))] = \langle F(\mathbf{x}(t + \Delta t))\rangle, \tag{7}$$

then [18]

$$\mathcal{L}^\dagger[F] = \lim_{\Delta t \to 0} \frac{\mathcal{K}^{\Delta t}[F(\mathbf{x}(t))] - F(\mathbf{x}(t))}{\Delta t}. \tag{8}$$

We use the standard convention where $\mathbf{X}$ refers to the random variable, while $\mathbf{x}$ refers to the independent argument of the corresponding probability density ($\mathbf{X}$ is stochastic whereas $\mathbf{x}$ is a deterministic object).

We assume that the forward ($\mathcal{L}$) and backward ($\mathcal{L}^\dagger$) Kolmogorov operators possess a discrete spectrum with a one-dimensional null space, and eigenvalues λ and eigenfunctions P_λ, Q^*_λ satisfying

$$\mathcal{L}[P_\lambda] = \lambda P_\lambda, \qquad \mathcal{L}^\dagger[Q^*_\lambda] = \lambda Q^*_\lambda. \tag{9}$$

Biorthogonality of the eigenfunctions under the natural inner product follows:

$$\langle Q_{\lambda'} \mid P_\lambda\rangle = \int d\mathbf{x}\, Q^*_{\lambda'}(\mathbf{x})P_\lambda(\mathbf{x}) = \delta_{\lambda'\lambda}. \tag{10}$$

This relation allows to decompose the conditional probability density as follows [16]: for $t > t_0$,

$$P(\mathbf{x}, t|\mathbf{x}_0, t_0) = P_0(\mathbf{x}) + \sum_{\lambda \neq 0} e^{\lambda(t-t_0)}P_\lambda(\mathbf{x})Q^*_\lambda(\mathbf{x}_0), \tag{11}$$

where P_0 is the eigenfunction associated with eigenvalue 0. Properly normalized, it gives the stationary probability density.

According to Itô's chain rule [16], for any smooth (C^2) observable $F(\mathbf{X})$

$$dF(\mathbf{X}) = \mathcal{L}^\dagger[F(\mathbf{X})]dt + \nabla F(\mathbf{X})^\intercal\mathbf{g}(\mathbf{X})d\mathbf{W}(t). \tag{12}$$

Thus, for any stochastic process, the ensemble properties and pathwise realizations of the system are linked through the Kolmogorov backwards operator.

Following reference [13], we define system (4) to be *robustly oscillatory* if the following conditions are met:

1. there exists a nontrivial eigenvalue of $\mathcal{L}^\dagger$ with least negative real part $\lambda_1 = \mu_1 + i\omega_1$, which is complex valued ($\omega_1 > 0$) and unique;

2. the oscillation is pronounced, i.e. the *quality factor* $|\omega_1/\mu_1|$ is much larger than 1;

3. all other nontrivial eigenvalues λ' are significantly more negative in their real parts, i.e. $|\Re[\lambda']| \geq 2|\Re[\lambda_1]|$.

A. The Stochastic Asymptotic Phase

If these conditions are satisfied, there exists a long-lived oscillatory mode $Q^*_{\lambda_1}$ that dominates the approach to the stationary distribution, even after all other modes in eq. (10) have decayed. As discussed in [13], since $Q^*_{\lambda_1}$ is complex, we can rewrite it in polar form:

$$Q^*_{\lambda_1} = u(\mathbf{x})e^{i\psi(\mathbf{x})}, \tag{13}$$

where $\psi(\mathbf{x})$ is the *stochastic asymptotic phase*: at large times, and provided the robustly oscillatory criterion is met, if one considers the same system at initial time $t = t_0$ with two different initial conditions ($\mathbf{x}(t_0) = \mathbf{x}_1$) and ($\mathbf{x}(t_0) = \mathbf{x}_2$), the respective probability densities $P(\mathbf{y}, t|\mathbf{x}_1, t_0)$ and $P(\mathbf{y}, t|\mathbf{x}_2, t_0)$ will decay to the stationary state with an oscillatory offset given by $\psi(\mathbf{x}_1)-\psi(\mathbf{x}_2)$. Thus, $\psi(\mathbf{x})$ defines level sets,

$$\mathcal{I}_\psi(\mathbf{x}) = \{\mathbf{x} \mid \psi(\mathbf{x}) = \psi\}, \tag{14}$$

corresponding to the sets of initial conditions such that the main oscillatory component of their conditional probability densities will evolve in-phase with each other. In the case of a system consisting of a limit cycle perturbed by noise, we observe that ψ converges to the deterministic asymptotic phase θ in the noise-vanishing limit in each example we have studied. This relationship has already been noted in the context of Koopman theory [19, 20].

Applying the Itô chain rule to this new observable $\psi(\mathbf{x})$, we extract its evolution law [21]

$$\begin{aligned} d\psi(\mathbf{X}) =& \Big(\omega_1 - \overbrace{2\sum_{i,j}\mathcal{G}_{ij}(\mathbf{X})\partial_i \ln(u(\mathbf{X}))\partial_j\psi(\mathbf{X})}^{\Omega(\mathbf{X})}\Big)dt \\ &+ \nabla\psi(\mathbf{X})^\intercal \mathbf{g}(\mathbf{X})d\mathbf{W}(t), \end{aligned} \tag{15}$$

where we introduce the function $\Omega(\mathbf{x})$ to ease notation.

B. The Mean–Return-Time Stochastic Phase

An alternative definition for the phase of stochastic oscillators was proposed by Schwabedal and Pikovsky in [11], who constructed the *Mean-Return-Time phase* in terms of a system of Poincaré sections $\{\ell_{\mathrm{MRT}}(\phi),\ 0 \le \phi \le 2\pi\}$, foliating a domain $\mathcal{R} \subset \mathbb{R}^2$ and possessing a Mean–Return-Time (MRT) property: a section ℓ_{MRT} satisfies the MRT property if for all the points $\mathbf{x} \in \ell_{\mathrm{MRT}}$ the mean return time from $\mathbf{x}$ back to ℓ_{MRT}, having completed one full rotation, is constant.

First defined by [11] by means of an algorithmic numerical procedure, the MRT phase was later related to the solution of a boundary value problem [12]. As the authors in this paper showed, the ℓ_{MRT} sections correspond to the level curves of a function $T(\mathbf{x})$, with appropriate boundary conditions, satisfying the following PDE associated with a first-passage-time problem

$$\mathcal{L}^\dagger T(\mathbf{x}) = -1, \tag{16}$$

where $\mathcal{L}^\dagger$ corresponds to the Kolmogorov backwards operator defined in eq. (6). Imposing a boundary condition amounting to a jump by $\overline{T}$ (the mean period of the oscillator) across an arbitrary section transverse to the oscillation, the *unique* solution of eq. (16), up to an additive constant T_0, is a version of the so-called MRT function,

$$\Theta(\mathbf{x}) = (2\pi/\overline{T})(T_0 - T(\mathbf{x})). \tag{17}$$

Hence, the MRT phase $\Theta(\mathbf{x})$ satisfies

$$\mathcal{L}^\dagger[\Theta(\mathbf{x})] = \frac{2\pi}{\overline{T}}, \tag{18}$$

and the transformation of $\mathbf{X}(t)$ in eq. (4) to the MRT phase Θ obeys the stochastic differential equation

$$d\Theta(\mathbf{X}) = \frac{2\pi}{\overline{T}}dt + \nabla\Theta(\mathbf{X})^\intercal \mathbf{g}(\mathbf{X})d\mathbf{W}(t), \tag{19}$$

so it evolves in the mean in a way which is formally analogous to the dynamics for the deterministic phase (see eq. (2)). As for the stochastic asymptotic phase, it was shown in [12] that the MRT phase converges to the deterministic phase θ as noise vanishes.

III. SELF-CONTAINED PHASE EQUATION

We have introduced two different phase mappings

$$\mathbf{x}(t) \to \psi(\mathbf{x}(t)), \qquad \mathbf{x}(t) \to \Theta(\mathbf{x}(t)), \tag{20}$$

the asymptotic phase and the MRT phase mappings, respectively, which yield two different equations eq. (15) and eq. (19). However, neither of these equations are fully self-contained as they both depend on $\mathbf{X}(t)$ [22].

Ideally, given an arbitrary phase mapping

$$\mathbf{x}(t) \to \vartheta(\mathbf{x}(t)), \tag{21}$$

we would like to have a self-contained equation for the phase dynamics of the form

$$d\vartheta = a_\vartheta(\vartheta)dt + \sqrt{2D_\vartheta(\vartheta)}dW_\vartheta(t), \tag{22}$$

where dW_ϑ is the increment of a single Brownian motion (rather than k of them) and D_ϑ is a (phase-dependent) effective noise intensity. Both D_ϑ and the phase-dependent local frequency a_ϑ should be smooth and periodic in ϑ. While equation eq. (22) is a fully self-contained phase equation, it presents the challenge of estimating these new functions $a_\vartheta(\vartheta)$ and $D_\vartheta(\vartheta)$.

A. Reduction framework

We will start by deriving the general reduction for any phase observable, before applying it to the asymptotic and MRT phases. Consider the general phase observable

$\vartheta(\mathbf{x}(t))$ in eq. (21). Its evolution equation is derived by means of the Itô chain rule:

$$d\vartheta(\mathbf{x}) = \mathcal{L}^\dagger[\vartheta(\mathbf{x})]dt + \nabla\vartheta(\mathbf{x})^\intercal \mathbf{g}(\mathbf{x})d\mathbf{W}(t). \tag{23}$$

Given that the sum of uncorrelated Gaussian white noise processes is Gaussian white noise, we rewrite this process with one dimensional Gaussian white noise $dW_{1\mathrm{D}}$, such that:

$$d\vartheta(\mathbf{x}) = \mathcal{L}^\dagger[\vartheta(\mathbf{x})]dt + \sqrt{2\sigma(\mathbf{x})}dW_{1\mathrm{D}}, \tag{24}$$

where the new noise amplitude term is given by

$$\sigma(\mathbf{x}) = \frac{1}{2}\sum_{ijk} g_{ij}(\mathbf{x})g_{kj}(\mathbf{x})\partial_i\vartheta(\mathbf{x})\partial_k\vartheta(\mathbf{x}).$$

Let us assume there exists a transformation $\mathbf{x} = \mathbf{x}(\eta, \vartheta)$ (where η refers to $n-1$ phase transversal coordinates). In what follows, we show a way in which the system in eq. (24) can be approximated by a reduced, self-contained equation of the form in eq. (22) where the $n-1$ transversal dimensions have been integrated out, leaving only the phase dependency. For simplicity, we assume that we are in the planar case ($n = 2$), so we only have one transverse variable to integrate out.

We rewrite the stationary probability density $P_0(\mathbf{x})$ in terms of this new set of coordinates $\mathbf{x} = \mathbf{x}(\eta, \vartheta)$, and obtain the distribution $\bar{P}_0(\eta, \vartheta)$. We use it to define the following *conditional* probability

$$\bar{P}_0(\eta|\bar{\vartheta}) \equiv \frac{\bar{P}_0(\eta, \bar{\vartheta})}{\bar{P}_0(\bar{\vartheta})}, \tag{25}$$

provided the density $\bar{P}_0(\vartheta) > 0, \ \forall\ 0 \geq \vartheta \geq 2\pi$.

Consider the dynamics for $d\vartheta(\mathbf{X})$ in eq. (24). If we take the expected value of each side, using the stationary probability density, since $dW_{1D}(t)$ is independent of $\mathbf{X}(t)$ (and functions of $\mathbf{X}(t)$), we see that $\langle\sqrt{2\sigma(\mathbf{X}(t))}dW_{1D}\rangle \equiv 0$. This motivates our choice of $a_\vartheta(\vartheta)$: we want $a_\vartheta(\vartheta)$ to represent the average rate of increase of ϑ when we happen to be on a particular isochron. That is,

$$a_\vartheta(\bar{\vartheta}) = \int_{\mathbf{x}\in\mathcal{I}_{\bar{\vartheta}}} \mathcal{L}^\dagger[\vartheta(\mathbf{x}(\eta, \bar{\vartheta}))]\bar{P}_0(\eta|\bar{\vartheta})d\eta, \tag{26}$$

where $\mathcal{L}^\dagger[\vartheta(\mathbf{x}(\eta, \bar{\vartheta}))]$ is the drift of $\vartheta(\mathbf{x})$ in eq. (23), which is averaged over the level curves of ϑ

$$\mathcal{I}_{\bar{\vartheta}} = \{\mathbf{x} \mid \vartheta(\mathbf{x}) = \bar{\vartheta}\}$$

that we parameterize by means of the transverse coordinate η. The choice of $a_\vartheta(\vartheta)$ in eq. (26) is meant to ensure that our reduction captures the first moment of the short term dynamics of ϑ. Indeed, one can easily show that, assuming stationarity, eq. (26) is equivalent to

$$a_\vartheta(\bar{\vartheta}) = \lim_{dt\to 0} \frac{1}{dt}\left\langle \Delta\vartheta(\mathbf{x}(t))\right\rangle_{\vartheta(\mathbf{x}(t))=\bar{\vartheta}}, \tag{27}$$

where $\Delta\vartheta(\mathbf{x}(t)) = \vartheta(\mathbf{x}(t+dt)) - \vartheta(\mathbf{x}(t))$ refers to the increment of the phase variable between t and $t+dt$. Equation eq. (27) makes explicit that with this choice of a_ϑ, we match the *expected* rate of progress of the phase angle for each phase ϑ.

Once our choice of $a_\vartheta(\vartheta)$ is thus made, we choose $D_\vartheta(\vartheta)$ such that we best capture the second moment of the short-term dynamics of the phase:

$$D_\vartheta(\bar{\vartheta}) = \lim_{dt\to 0} \frac{1}{2dt}\left\langle (\Delta\vartheta(\mathbf{x}(t)) - a_\vartheta(\bar{\vartheta})dt)^2\right\rangle_{\vartheta(\mathbf{x}(t))=\bar{\vartheta}}. \tag{28}$$

Expanding this formula yields

$$D_\vartheta(\bar{\vartheta}) = \frac{1}{2}\sum_{ijk}\int_{\mathbf{x}\in\mathcal{I}_{\bar{\vartheta}}} g_{ij}(\mathbf{x})g_{kj}(\mathbf{x})\partial_i\vartheta(\mathbf{x})\partial_j\vartheta(\mathbf{x})\bar{P}_0(\eta|\bar{\vartheta})d\eta + \frac{1}{2}\Big(\int_{\mathbf{x}\in\mathcal{I}_{\bar{\vartheta}}} \mathcal{L}^\dagger[\vartheta(\mathbf{x})]^2\bar{P}_0(\eta|\bar{\vartheta})d\eta - a_\vartheta(\bar{\vartheta})^2\Big),$$

where again $\mathbf{x} = \mathbf{x}(\eta, \bar{\vartheta})$. Note that in this reduction framework, the variability of single realizations around the mean comes from two distinct sources: not only from the average diffusion value, but also from the variance of the drift along a given isochron. We note that equations eq. (27) and eq. (28) show that this reduction procedure makes it possible to extract a_ϑ and D_ϑ from a stationary time series $\vartheta(\mathbf{X}(t))$. Put differently, a_ϑ and D_ϑ are obtained as the first two Kramers-Moyal coefficients of the trajectory $\vartheta(t)$, see [23, 24] (but also [25]) for examples of how to extract them from stochastic trajectories.

B. Stochastic Asymptotic Phase Reduction

Let us now apply the general framework derived above to obtain a reduced evolution equation for the stochastic asymptotic phase ψ, in the form

$$d\psi = a_\psi(\psi)dt + \sqrt{2D_\psi(\psi)}dW_\psi(t). \tag{29}$$

From equation eq. (15), we find that the drift term takes the form

$$a_\psi(\psi) = \omega_1 - \int_{\mathbf{x}\in\mathcal{I}_\psi} \bar{P}_0(\eta|\psi)\Omega(\mathbf{x})d\eta,$$

where $\Omega(\mathbf{x})$ is defined at eq. (15). Similarly, the effective noise term takes the form

$$D_\psi(\bar{\psi}) = \frac{1}{2}\sum_{ijk}\int_{\mathbf{x}\in\mathcal{I}_{\bar{\psi}}} g_{ij}(\mathbf{x})g_{kj}(\mathbf{x})\partial_i\psi(\mathbf{x})\partial_j\psi(\mathbf{x})\bar{P}_0(\eta|\bar{\psi})d\eta + \frac{1}{2}\Big(\int_{\mathbf{x}\in\mathcal{I}_{\bar{\psi}}} (\omega_1 - \Omega(\mathbf{x}))^2\bar{P}_0(\eta|\bar{\psi})d\eta - a_\psi(\bar{\psi})^2\Big),$$

where $\mathcal{I}_\psi$ refers to

$$\mathcal{I}_\psi = \{\mathbf{x} \mid \psi(\mathbf{x}) = \psi\}.$$

As noted above, under the stationarity assumption, those two expressions can be approximated from time series realizations of $\psi(\mathbf{X}(t))$ using expressions eq. (27) and eq. (28).

C. Mean–Return-Time Phase Reduction

Let us now derive a phase equation for the Mean–Return-Time phase Θ in the form

$$d\Theta = a_\Theta(\Theta)dt + \sqrt{2D_\Theta(\Theta)}dW_\vartheta(t). \tag{30}$$

Following the same general reduction procedure, we obtain expressions for the corresponding drift and effective noise functions

$$a_\Theta(\Theta) = \frac{2\pi}{\overline{T}}, \tag{31}$$

and

$$D_\Theta(\Theta) = \sum_{i,j}\int_{\mathbf{x}\in\mathcal{I}_\Theta} \bar{P}_0(\eta|\Theta)\partial_i\Theta(\mathbf{x})\partial_j\Theta(\mathbf{x})\mathcal{G}_{ij}(\mathbf{x})d\eta, \tag{32}$$

where, as for $\mathcal{I}_\psi$, $\mathcal{I}_\Theta$ accounts for the MRT sections

$$\mathcal{I}_\Theta = \{\mathbf{x} \mid \Theta(\mathbf{x}) = \Theta\}.$$

One can also get these coefficients from data series $\Theta(\mathbf{X}(t))$ (see eq. (27) and eq. (28)).

IV. NUMERICAL SIMULATIONS

We apply our framework to two systems exhibiting canonical bifurcations and illustrating various mechanisms of noisy oscillations.

Noisy Hopf bifurcation – We consider the canonical model for a supercritical Hopf bifurcation endowed with Gaussian white noise

$$\begin{aligned} \dot{x} &= \beta x - y - x(x^2+y^2) + \sqrt{2\sigma_x}\xi_x(t), \\ \dot{y} &= x + \beta y - y(x^2+y^2) + \sqrt{2\sigma_y}\xi_y(t), \end{aligned} \tag{33}$$

$\beta \in \mathbb{R}$.

In the deterministic setting, there is a supercritical Hopf bifurcation at $\beta = 0$. For $\beta > 0$, there is a stable limit cycle of radius $\sqrt{\beta}$ and period $T = 2\pi$, which can be parameterized using the polar phase [26]

$$\theta = \arctan\left(\frac{y}{x}\right).$$

The stochastic version has been studied for a long time, especially with respect to its correlation statistics (see e.g. [27–29]). In Fig. 1, we show simulations of the noisy system above the bifurcation ($\beta = 1$). We observe that for low noise ($\sigma_x = \sigma_y = 0.01$, panel 1) and high noise ($\sigma_x = \sigma_y = 0.08$, panel 2), the phase functions $\psi(\mathbf{x})$ and $\Theta(\mathbf{x})$ still show the characteristic spokes of a wheel structure (see panels 1-2 (a) and 1-2 (b)) that would be present without noise.

Panels (1-2(c)) show the stationary probability densities. As these panels show, the trajectories are dispersed around the LC with radial symmetry dispersion which increases with the level of noise. Indeed, because of the symmetry of both the model and the noise, both the drift terms and the effective diffusion terms are constant (see panels (d)). Indeed, we note $a_\psi = a_\Theta = 1$, indicating a $\overline{T} = 2\pi$ mean rotation rate.

For $\beta < 0$, the deterministic system has a stable focus at the origin. Hence, in the absence of noise, the trajectories exhibit damped oscillations decaying towards the origin, and the asymptotic phase would not be well defined [30]. The addition of noise perturbs trajectories away from the stable steady state, creating a quasicycle [14, 31]. This is an example of noise-induced oscillations, leading to a non zero probability of finding the system away from that fixed point. Indeed, as can be seen in Fig. 2, panels c, the probability density has a 2D Gaussian-like profile the maximum of which is located at the origin, where the deterministic fixed point is found. Despite the noise-induced character of the oscillation, the phase functions $\psi(\mathbf{x})$ and $\Theta(\mathbf{x})$ have a 'spokes of a wheel' structure similar to those of the LC case (panels (a) and (b)). Drift and effective noise functions, similarly to the noisy LC case, are constant (Fig. 2 panels 1-2 (d)). However, we note that for the same levels of noise, the effective noise intensities D_ψ and D_Θ are much larger than in the LC case.

Saddle-node on an invariant circle – Next, we consider a system that, in the deterministic case, undergoes a saddle-node bifurcation on an invariant circle (SNIC)

$$\begin{aligned} \dot{x} &= nx - my - x(x^2+y^2) + \frac{y^2}{\sqrt{x^2+y^2}} + \sqrt{2\sigma_x}\xi_x(t), \\ \dot{y} &= mx + ny - y(x^2+y^2) - \frac{xy}{\sqrt{x^2+y^2}} + \sqrt{2\sigma_y}\xi_y(t), \end{aligned} \tag{34}$$

with $m, n \in \mathbb{R}$.

In the noiseless case with $n = 1$, the SNIC bifurcation occurs at $m = 1$ and it separates an oscillatory LC state ($m > 1$) (Fig. 3) from a saddle-node regime ($m < 1$) (Fig. 4). The addition of noise to the saddle-node regime induces a non-zero probability that the system will leave the stable point and jump onto the circle, leading to oscillations. We refer to this state as the *excitable* regime of the system.

Let us now present how our phase reduction applies to system eq. (34) in the oscillatory and excitable cases. We consider parameter values close to the bifurcation ($m = 1.03$ and $m = 0.999$, respectively). In the low noise case (panels 1 in Fig. 3 and Fig. 4), we observe that the phase functions we consider, that is $\psi(\mathbf{x})$ (panel (a)) and $\Theta(\mathbf{x})$ (panel (b)), have a similar structure reflecting the asymmetries in the velocity of the system during a cycle. Indeed, as the trajectories slow down near the ghost of the saddle-node, the phase sections appear more densely packed in this area of the phase space. By contrast, as trajectories speed up far from the ghost, we observe that the phase sections are more broadly separated.

Those velocity variations are also reflected in the drift term of the asymptotic phase reduction a_ψ, which is

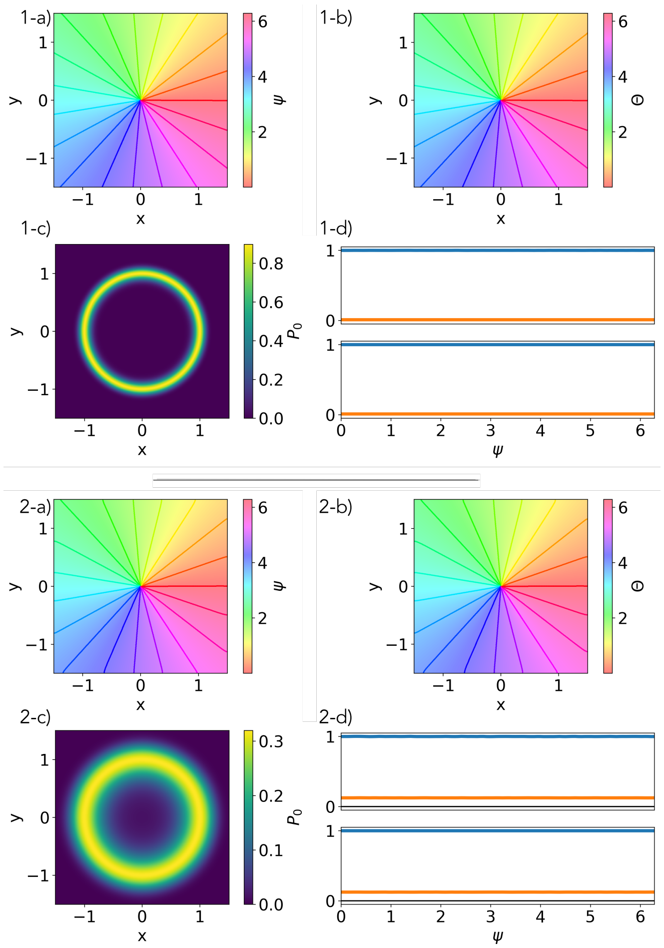

FIG. 1. **Hopf above bifurcation.** For the noisy Hopf bifurcation model in eq. (33) with $\beta = 1$ we show for two levels of noise (panel 1, above $\sigma_x = \sigma_y = 0.01$ and panel 2, below $\sigma_x = \sigma_y = 0.08$): (a) The asymptotic phase function $\psi(\mathbf{x})$. (b) The MRT phase function $\Theta(\mathbf{x})$. (c) The stationary probability distribution. (d) Top panel shows the functions a_ψ and $\sqrt{2D_\psi}$ (blue and orange curves, respectively) and bottom panel shows a_Θ and $\sqrt{2D_\Theta}$ (blue and orange curves, respectively), abscissa shared.

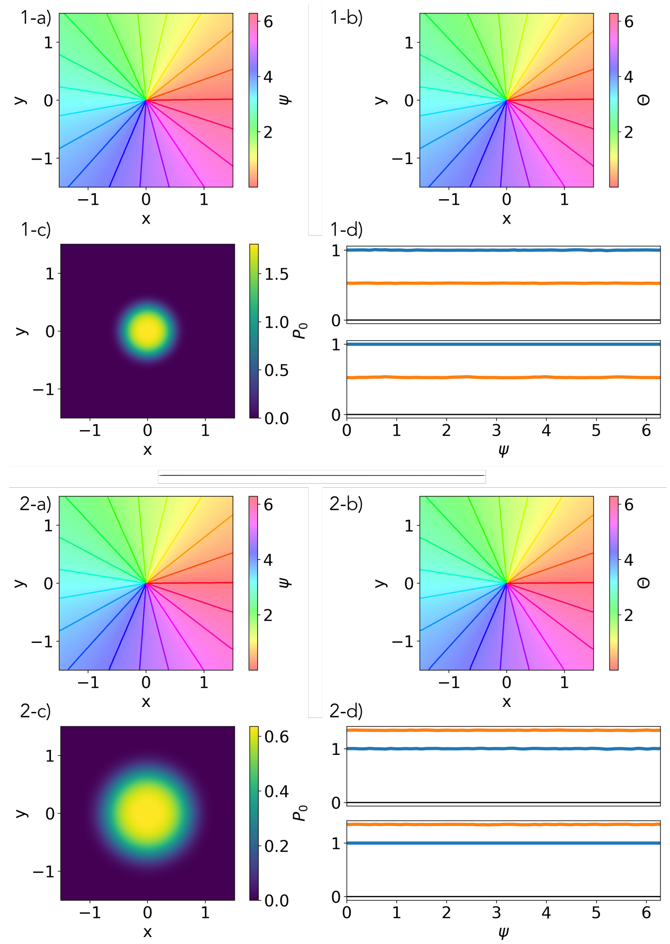

FIG. 2. **Hopf below bifurcation.** For the noisy Hopf bifurcation model in eq. (33) with $\beta = -0.001$ we show for two levels of noise (panel 1, above $\sigma_x = \sigma_y = 0.01$ and panel 2, below $\sigma_x = \sigma_y = 0.08$): (a) The asymptotic phase function $\psi(\mathbf{x})$. (b) The MRT phase function $\Theta(\mathbf{x})$. (c) The stationary probability distribution. (d) Top panel shows the functions a_ψ and $\sqrt{2D_\psi}$ (blue and orange curves, respectively) and bottom panel shows a_Θ and $\sqrt{2D_\Theta}$ (blue and orange curves, respectively), abscissa shared.

smaller near the phase values corresponding to the locations of the saddle node, indicating the slowness of the system. Interestingly, we observe that, below the bifurcation, the asymptotic phase drift a_ψ has two zero crossings (indicating two fixed points) (Fig. 4.1-d), which disappear above bifurcation, where the drift becomes purely positive (Fig. 3.1-d). We interpret this as a sign that, in the case of the stochastic asymptotic phase ψ, our phase reduction technique captures the transition from the excitable regime to the oscillatory regime. This result for the excitable SNIC can be seen as a formulation of noise induced oscillations in excitable systems as a one dimensional escape problem. Indeed, there is a phase interval for which the drift $a_\psi(\psi)$ term becomes negative: in that range, the phase ψ behaves like a particle stuck in a one dimensional potential well and subjected to thermal fluctuations with noise intensity $D_\psi(\psi)$. A full rotation occurs when the fluctuations manage to push the phase out of the well.

When the parameters are varied so that the system goes above the bifurcation, this feature disappears as both a_ψ and $\sqrt{2D_\psi}$ are positive everywhere, consistent with the purely oscillatory regime (Fig. 5-right). We check in Fig. 5 that this transition seems to be continuous, as the drift appears to smoothly move across the 0-line as the bifurcation parameter m is increased.

This distinction between excitable and oscillatory regimes does not hold anymore when noise levels get too large. Indeed, looking at Fig. 3 and 4, panels 2, the drift and effective noise coefficients become qualitatively very similar: the drift terms a_ψ show zero-crossings both below and above bifurcation. We interpret this result as

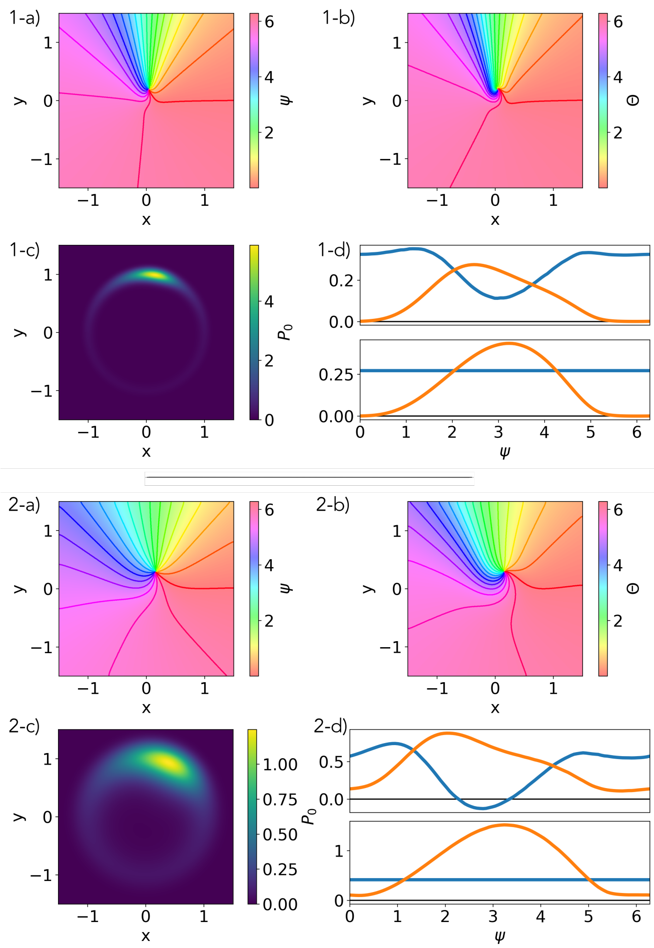

FIG. 3. **SNIC above bifurcation.** For the noisy SNIC bifurcation model in eq. (34) with $n = 1$, $m = 1.03$ we show for two levels of noise (panel 1, above $\sigma_x = \sigma_y = 0.01$ and panel 2, below $\sigma_x = \sigma_y = 0.08$): (a) The asymptotic phase function $\psi(\mathbf{x})$. (b) The MRT phase function $\Theta(\mathbf{x})$. (c) The stationary probability distribution. (d) Top panel shows the functions a_ψ and $\sqrt{2D_\psi}$ (blue and orange curves, respectively) and bottom panel shows a_Θ and $\sqrt{2D_\Theta}$ (blue and orange curves, respectively), abscissa shared.

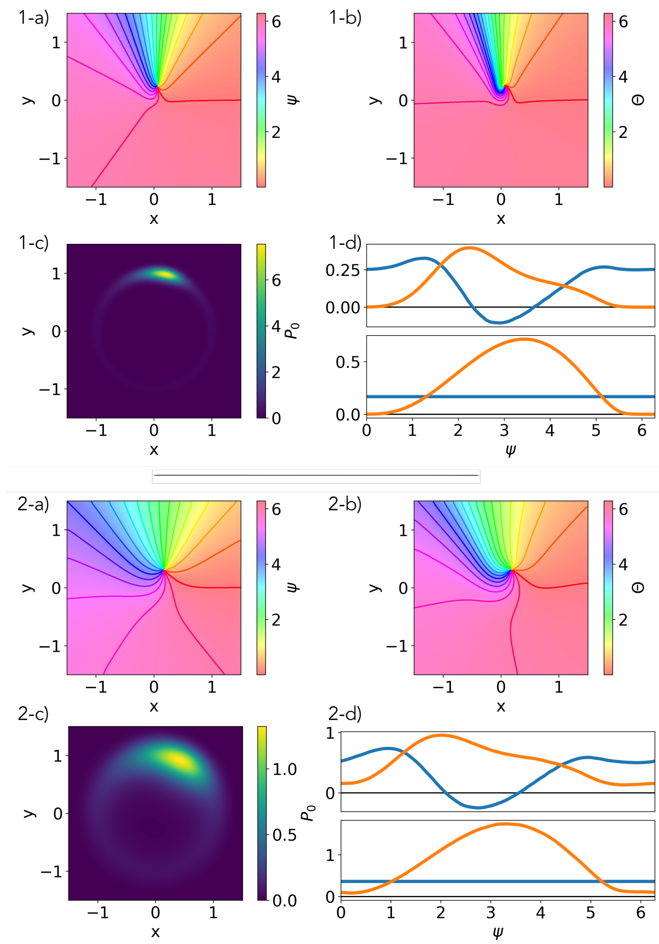

FIG. 4. **SNIC below bifurcation.** For the noisy SNIC bifurcation model in eq. (34) with $n = 1, m = 0.999$ we show for two levels of noise (panel 1, above $\sigma_x = \sigma_y = 0.01$ and panel 2, below $\sigma_x = \sigma_y = 0.08$): (a) The asymptotic phase function $\psi(\mathbf{x})$. (b) The MRT phase function $\Theta(\mathbf{x})$. (c) The stationary probability distribution. (d) Top panel shows the functions a_ψ and $\sqrt{2D_\psi}$ (blue and orange curves, respectively) and bottom panel shows a_Θ and $\sqrt{2D_\Theta}$ (blue and orange curves, respectively), abscissa shared.

follows: we are considering two cases that are very close in the parameter space, therefore, large levels of noise effectively blur the distinction between the two sides of the bifurcation.

Finally, we comment on how the reduction appears through the lens of the MRT phase in place of the asymptotic phase. By construction, the MRT phase defines sections with uniform mean return times. For this reason, one should not be surprised to find a constant drift term. In this case, all the variability in the velocity along the cycle is carried by the effective noise term D_Θ. We also see that the mean-return-time period $\overline{T}$ reflects the difference between regimes: it is larger below bifurcation than above.

V. LONG TERM STATISTICS

Our choice for the drift and effective noise coefficients of the reduced phase equation ensures our reduction accurately captures the short term ($\lim dt \to 0$) statistics of the full phase evolution. However, a meaningful phase reduction should also be able to reliably capture the long term (asymptotic) statistics of the evolution of the full system. For that reason, given a general phase mapping as in eq. (21), we will consider the following statistics: the mean rotation rate

$$\omega_{\text{eff}}^{\vartheta} = \lim_{t\to\infty} \frac{1}{t} \left\langle \vartheta(t) - \vartheta(0) \right\rangle , \tag{35}$$

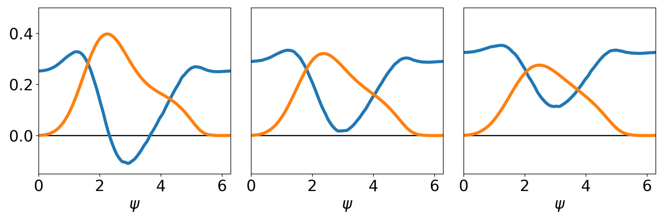

FIG. 5. **Transition across the SNIC bifurcation.** Asymptotic stochastic phase ψ drift function $a_\psi(\bar{\psi})$ (blue) and effective diffusion term $\sqrt{2D_\psi(\bar{\psi})}$ (orange) for the SNIC for $n = 1$ and increasing values of m around the bifurcation. The phase reduction captures the bifurcation from the excitable to the oscillating state. **Left**: m = 0.999. **Middle**: m = 1.0. **Right**: m = 1.03

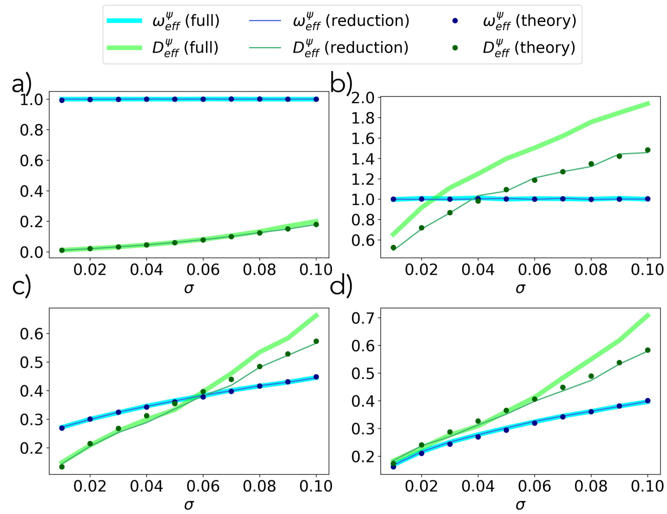

FIG. 6. **Long term statistics of the asymptotic phase ψ.** (a) Hopf bifurcation in the LC case; (b) Hopf bifurcation in the focus case; (c) SNIC in the LC case; (d) SNIC in the excitable case. We compute each statistic for: the full phase equation eq. (15) (thick line), its phase reduction equation eq. (29) (narrow line) and the theoretical formulas eq. (37) and eq. (38) (dots). Results for the MRT are similar and can be found in the Supplemental.

and the phase diffusion coefficient

$$D^{\vartheta}_{\text{eff}} = \lim_{t\to\infty} \frac{1}{2t} \left\langle [\vartheta(t) - \vartheta(0) - \omega^{\vartheta}_{\text{eff}} t]^2 \right\rangle . \tag{36}$$

When considering the long-term statistics, we take ϑ in the preceding expressions to represent the *unwrapped* phase. That is, rather than taking ϑ mod 2π (mapping $\mathbf{x}$ to the circle) we omit any phase "reset" at 2π and instead map $\mathbf{x}$ to the line $\vartheta \in \mathbb{R}$. We will numerically compare those two quantities for the full mapping eq. (23) and the self-contained phase reduction eq. (22). We expect that the more similar those quantities are for the full and reduced system, the better our reduction captures the dynamics of the full system.

Additionally, since the general phase reduction is a 1D SDE with periodic drift and noise coefficients, we can use the results in [32], which give us the following expressions for the mean rotation defined in eq. (35)

$$\omega^{\vartheta}_{\text{eff}} = \frac{2\pi(1 - e^{V(2\pi)})}{\int_0^{2\pi} I_+(\tilde{\vartheta}) d\tilde{\vartheta} / \sqrt{D(\tilde{\vartheta})}}, \tag{37}$$

and the diffusion coefficient in eq. (36)

$$D^{\vartheta}_{\text{eff}} = \frac{4\pi^2 \int_0^{2\pi} I_-(\tilde{\vartheta}) I_+^2(\tilde{\vartheta}) d\tilde{\vartheta} / \sqrt{D(\tilde{\vartheta})}}{\left[\int_0^{2\pi} I_+(\tilde{\vartheta}) d\tilde{\vartheta} / \sqrt{D(\tilde{\vartheta})}\right]^3}, \tag{38}$$

where

$$V(\vartheta) = -\int_0^{\vartheta} \frac{a(\tilde{\vartheta})}{D(\tilde{\vartheta})} d\tilde{\vartheta},$$

and

$$I_{\pm}(\vartheta) = \pm e^{\mp V(\vartheta)} \int_{\vartheta}^{\vartheta \pm 2\pi} \frac{e^{\pm V(\tilde{\vartheta})}}{\sqrt{D(\tilde{\vartheta})}} d\tilde{\vartheta}.$$

The evaluation of the corresponding integrals is feasible as long as the noise intensity is not too small.

Hopf bifurcation – As results in Fig. 6 panel (a) show, for the Hopf system in the LC case, our phase reduction captures both the mean rotation rate ω_{eff} and the diffusion constant D_{eff}: we only observe small deviations in D_{eff} for high values of noise. However, in the quasicyclic regime, our phase reduction underestimates the long term diffusion coefficient of the Hopf below bifurcation, even though it systematically captures its mean rotation rate (see Fig. 6 (b)). We interpret this mismatch as a consequence of the pronounced fluctuations of the system's dynamics around the fixed point. For this particular system, our one-dimensional projection to a line cannot capture all details of the dynamics, as amplitude fluctuations play an important part in the system's dynamics. This leads to an underestimation of the long-term phase fluctuations.

SNIC bifurcation – As our calculations for the long term statistics show (Fig. 6 panels c and d), our phase reduction accurately captures the mean rotation, both above and below the SNIC bifurcation. However, for larger levels of noise, it underestimates the mean diffusion, for reasons similar to those described above: the reduction to the line breaks down when fluctuations are too large, which is the case when the noise strength increases.

VI. INFERRING PHASE RESPONSE PROPERTIES

We will now show how our stochastic phase reduction framework can be applied to infer the phase response to weak external perturbations at linear order.

In the case of a deterministic oscillator parameterized with phase θ, the phase response to a weak perturbation can be linearized around the LC such that it is proportional to the gradient of the phase. This quantity is known as the *infinitesimal phase response curve* (iPRC) [33]

$$\mathrm{iPRC}(\theta) = \nabla\theta(\mathbf{x})|_{\mathbf{x}=\gamma(\theta)}. \tag{39}$$

where we assume the existence of a parameterisation $\mathbf{x} = \gamma(\theta)$ for $\mathbf{x} \in \mathrm{LC}$.

The main obstacle in finding a stochastic analogue of this quantity is the phase variability inherent to stochastic oscillators. As there is no LC, trajectories may visit any point $\mathbf{x}$ of the phase space with a given probability $P_0(\mathbf{x})$. As a consequence, perturbing the system at the same phase will generally yield different phase responses.

Consistent with our averaging approach to obtain a one-dimensional phase description, we postulate that, given a stochastic phase function ϑ, a meaningful curve describing the mean response properties of the system can be obtained by averaging its gradient along a given isochron. We call this quantity the *averaged iPRC (aiPRC)*, and can write it either in integral form

$$\mathrm{aiPRC}(\bar{\vartheta}) = \int_{\mathbf{x}\in\mathcal{I}_{\bar{\vartheta}}} \nabla\vartheta(\mathbf{x}(\eta,\bar{\vartheta}))\bar{P}_0(\eta|\bar{\vartheta})d\eta, \tag{40}$$

or as an average across realizations

$$\mathrm{aiPRC}(\bar{\vartheta}) = \langle\nabla\vartheta(\mathbf{x}(t))\rangle_{\vartheta(\mathbf{x}(t))=\bar{\vartheta}}. \tag{41}$$

We remark that in the vanishing noise limit (as $\sigma \to 0$), for systems with an underlying LC, $P_0(\mathbf{x}) \to 0$ for $\mathbf{x} \notin \mathrm{LC}$. Therefore, in this limit, the aiPRC definition in eq. (40) (or eq. (41)) converges to the deterministic iPRC in eq. (39).

We show that our aiPRC provides the expected unwrapped phase shift $\Delta\vartheta(\bar{\vartheta}) = \vartheta_{\mathrm{new}} - \bar{\vartheta}$ of an oscillator subjected to a weak external pulse $\boldsymbol{\epsilon}\delta(t-t_0)$ as

$$\Delta\vartheta(\bar{\vartheta}) \approx \boldsymbol{\epsilon}\cdot\mathrm{aiPRC}(\bar{\vartheta}). \tag{42}$$

In Fig. 7, we plot the aiPRC and compare it with numerical estimates of the average phase response, obtained by perturbing the system with a weak pulse at random phases, computing the individual phase shift and binning the responses by phase. For each bin, we compute the average response using the circular mean [34]. In panel (a) we compute the aiPRC for the Hopf normal form in the LC case (above the bifurcation). In this case, we observe that the aiPRC shows the characteristic sinusoidal Type II shape. Interestingly, a similar sinusoidal structure is

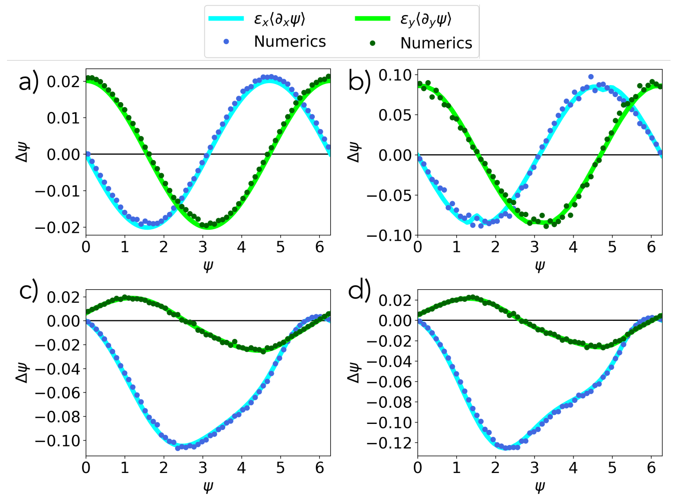

FIG. 7. **Averaged iPRCs for the asymptotic phase** ψ Blue - response to a pulse in the X direction (amplitude $\epsilon_x = 0.02$); Green - response to a pulse in the Y direction ($\epsilon_y = 0.02$) a) Hopf above bifurcation b) Hopf below bifurcation c) SNIC above bifurcation d) SNIC below bifurcation. External level of noise used for all systems: $\sigma = 0.01$

found for the Hopf normal form for $\beta = -0.001$, when no LC exists. We note, however, that the amplitude of the mean response is larger in the excitable case than in the LC case. This result seems to be consistent with our definition based on averages. For the focus (excitable) case, the phase gradients dramatically increase near the origin where the probability density has a pronounced maximum. For the SNIC in the LC case (panel (c)), we observe that the aiPRC exhibits a Type I structure, very similar to the deterministic case (see [35] where this particular example is studied). Interestingly, this structure is not much altered when the same object is studied below the bifurcation.

We observe similar behavior for the aiPRC computed for the MRT phase, see Supplemental Fig. 7.

VII. DISCUSSION

a. Summary. In this paper, we developed a generalized self-contained stochastic phase reduction framework. Specifically, we provided a method for finding an approximate, self-contained phase reduction of stochastic oscillators subjected to Gaussian white noise. To illustrate our framework, we considered two mappings $\vartheta : (\mathbf{x}) \in \mathbb{R}^n \to \mathcal{S}^1 \equiv [0, 2\pi)$, namely the "Mean–return-time" phase $\Theta(\mathbf{x})$ introduced in [11] and the "stochastic asymptotic phase" $\psi(\mathbf{x})$ introduced in [13]. Even though our framework can be applied to n-dimensional systems, for clarity, we focused on examples with two-dimensional stochastic oscillators.

Our reduction is built by considering the short term dynamics of the full phase variable. In order to test the accuracy of our reduction, we considered two well-known long-term statistics: the mean rotation $\omega_{\text{eff}}^{\vartheta}$, and the diffusion constant $D_{\text{eff}}^{\vartheta}$. We consider the reduction to be better, the closer the agreement of these two statistics for i) the full phase system and ii) its reduced version. We observe in Fig. 6 that the accuracy of the reduction decreases as the level of noise increases. This observation is not surprising since, when reducing, we are restricting dynamics to a circle. However, it is important to point out that while our reduction underestimates the effective diffusion constant $D_{\text{eff}}^{\vartheta}$ for larger levels of noise, it consistently captures the mean rotation rate $\omega_{\text{eff}}^{\vartheta}$. We consider this underestimation of the long term diffusion coefficient as a trade-off of our reduction procedure: our framework aims to construct the one dimensional SDE that best captures the local first and second moments of the full oscillatory dynamics. Our method satisfies this condition, at the cost of underestimating the long-term fluctuations. A better estimate of the fluctuations might be obtained by incorporating an amplitude term, which would come at the cost of having a higher dimensional reduction (see Future Directions). Indeed, while it is generally possible to match the short-term (local) and long-term (global) mean phase dynamics, it is generally not possible to match both the short-term variability and the long-term phase diffusion statistics, without taking into account additional degrees of freedom transverse to the phase variable.

b. Alternative phase descriptions. As previously mentioned, we have studied our reduction method for two different phase functions $\Theta(\mathbf{x})$ and $\psi(\mathbf{x})$. Even if the long term statistics are almost identical for both phases, we have found important differences in their drift functions (a_Θ and a_ψ). The MRT phase shows a constant drift term, so it carries all the variability in the effective noise term $\sqrt{2D_\Theta(\Theta)}$. By contrast, the stochastic asymptotic phase ψ, shows a non-constant drift term. As we have shown in Fig. 5, this variable drift term is able to encode important *dynamical* information about the system, namely the transition from an excitable to an oscillatory regime.

A way to understand this difference between drift functions is by relating our approach to the noise-induced frequency shift (NIFS) phenomenon [5]: in a deterministic LC oscillator with phase θ, adding white noise typically causes a shift in the average frequency [36]. The new average frequency is given by the ensemble average, $\overline{\omega} = \langle \dot{\theta} \rangle$. Generalizing to stochastic oscillators, we see that, by construction, the MRT phase Θ takes the effect of noise on the frequency into account by setting its instantaneous frequency to the average frequency: $a_\Theta(\Theta) = \overline{\omega} = \frac{2\pi}{\overline{T}}, \forall \Theta$. By contrast, the asymptotic phase ψ has an additional degree of variability, as it has an instantaneous average frequency value $a_\psi(\psi)$ for all ψ which need not equal $\overline{\omega}$. Thus, we can understand the MRT phase approach as a "coarser grained" description of the oscillation cycle. In contrast, the asymptotic phase ψ keeps track of finer details arising from the interaction between noise and deterministic dynamics, at the cost of added complexity in the equation.

c. Future Directions. In this paper, we have applied our method to systems whose SDE was known. However, both the MRT phase and the stochastic asymptotic phase can be extracted from data. For example, the original procedure to extract the MRT was built upon an iterative method that can accommodate both simulated and real-world data [11]. The stochastic asymptotic phase was first extracted from data by fitting the oscillatory-exponential asymptotic decay of the probability density to its stationary state [13]. Dynamic Mode Decomposition (DMD) based methods, based on an eigenfunction decomposition of the Koopman operator (which is closely related to the Kolmogorov backwards operator for stochastic systems) offer an alternative approach to obtaining these functions, see [18]). Thus, they should allow one to recover an estimation of the spectral properties of $\mathcal{L}^\dagger$ from data, most particularly of the $Q_{\lambda_1}^*$ eigenfunction that carries the stochastic asymptotic phase. This connection would allow application of our framework to real world oscillatory data, to be explored in future work.

In the deterministic case, adding an amplitude variable can extend the domain of accuracy of the phase description [19, 26, 37, 38]. We believe our construction may benefit from a similar approach. Recently, the stochastic amplitude has been found [20, 39]. In related work, it has been shown that a different observable, the slowest decaying complex eigenfunction $Q_{\lambda_1}^*$ of the Kolmogorov backwards operator, yields a universal description of stochastic oscillators [40]. This complex phase function, $Q_{\lambda_1}^*$, allows comparison of stochastic oscillators regardless of their underlying oscillatory mechanism [40]. Written in polar form, the complex phase function $Q_{\lambda_1}^* = ue^{i\psi}$ defines both a notion of phase $\psi \in [0, 2\pi)$ and an amplitude u that captures the concentration or coherence of an oscillator's probability density. Both the stochastic analogues of the phase-amplitude description and the complex phase ideas appear as interesting targets for future research.

An additional interesting question arising from this work is the exploration of the *averaged infinitesimal phase response curve (aiPRC)* function defined in sec. VI. We have shown that it provides a meaningful estimation of the average phase response of a stochastic oscillator to a small pulselike perturbation. Being able to compute the average response of stochastic oscillators to external perturbations by means of the aiPRC is a first step towards the analysis of complex noisy oscillatory phenomena, such as synchronization among oscillators connected on networks [2, 41, 42]. In the past, defining those phenomena, such as noisy phase and frequency synchronization [43], or noise-enhanced phase-locking [44], required using a deterministic notion of phase, such as the Hilbert phase, and extending it to the noisy case. The work we put forward in this manuscript builds upon recent notions of stochastic phase [11, 13]. Thus, obtaining a reduction for those stochastic phases will allow to revisit those earlier results in a purely stochastic setting. Moreover, Adams and MacLaurin have recently proposed a formal approach to deriving a self-contained stochastic differential equation for what they term the "isochronal phase", for systems that have a particular invariant manifold structure (such as system with an underlying LC) see [45]. Application of their methods, drawn from rigorous analysis of stochastic partial differential equations, to the examples we present here, is an interesting opportunity for future investigation.

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