POISSON-LIE ANALOGUES OF SPIN SUTHERLAND MODELS REVISITED

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Abstract. Some generalizations of spin Sutherland models descend from 'master integrable systems' living on Heisenberg doubles of compact semisimple Lie groups. The master systems represent Poisson–Lie counterparts of the systems of free motion modeled on the respective cotangent bundles and their reduction relies on taking quotient with respect to a suitable conjugation action of the compact Lie group. We present an enhanced exposition of the reductions and prove rigorously for the first time that the reduced systems possess the property of degenerate integrability on the dense open subset of the Poisson quotient space corresponding to the principal orbit type for the pertinent group action. After restriction to a smaller dense open subset, degenerate integrability on the generic symplectic leaves is demonstrated as well. The paper also contains a novel description of the reduced Poisson structure and a careful elaboration of the scaling limit whereby our reduced systems turn into the spin Sutherland models.

Contents

1. Introduction

It is well known that many important integrable Hamiltonian systems can be viewed as low dimensional 'shadows' of higher dimensional manifestly integrable master systems. The integrability of the master systems is due to their rich symmetries, and their shadows result by projection onto the quotient space of the pertinent master phase space with respect to the symmetry group. This is the essence of the method of Hamiltonian reduction and its variants [50]. For example, in the pioneering paper by Kazhdan, Kostant and Sternberg [34] the higher dimensional phase space was the cotangent bundle of the unitary group $\mathrm{U}(n)$, and Marsden–Weinstein reduction at a specific moment map value for the Hamiltonian action of $\mathrm{U}(n)$ by conjugations was applied to reduce the master system of free geodesic motion to the Sutherland model of n interacting particles on the circle. Several generalizations of this construction were later investigated in which the group $\mathrm{U}(n)$ was replaced by other Lie groups or their symmetric spaces [49, 51]. It turned out that reductions at generic moment map values often lead to many-body systems possessing internal 'spin' degrees of freedom [27, 53]. Infinite dimensional master phase spaces built on loop groups [7, 31], and on spaces of flat connections on Riemann surfaces [5, 29] were also utilized for constructing integrable systems. Moreover, there appeared interesting applications of the reduction method [23, 24, 25, 29, 41, 48] in which the nature of the underlying symmetry had been promoted from Hamiltonian group actions to their generalizations based on Poisson–Lie groups [10, 11, 62, 63] and on quasi–Poisson/quasi–Hamiltonian geometry [3, 4]. The reviews [6, 13, 47, 52] and the recent papers [8, 9, 16, 17, 35, 55, 56] show that this subject possesses close connections to important areas of physics and mathematics, and enjoys considerable current activity.

The principal goal of the present paper is to complement and enhance our previous results [19, 21] on the structure of Poisson–Lie analogues of those spin Sutherland models that result by reductions of cotangent bundles of semisimple Lie groups via the conjugation action. Here, we consider these models in association with every (connected and simply connected) compact Lie group G having a simple Lie algebra. The relevant master system is a generalization of the Hamiltonian system on the cotangent bundle T^*G governed by the kinetic energy of a 'free particle' moving on G in the bi-invariant Riemannian metric. The master phase space is obtained by replacing the cotangent bundle by the so-called Heisenberg double [62], which as a manifold is provided by the complexification $G^{\mathbb{C}}$ of the group G. This phase space carries a symplectic structure for which a generalization of the conjugation action of G on T^*G represents Poisson–Lie symmetry with respect to the standard multiplicative Poisson structure on G [36]. There exist also Hamiltonians on the Heisenberg double that generate 'free motion' in the sense that their flows project on the geodesic lines on G.

The free motion modeled on T^*G yields a degenerate integrable system[1], and its reductions by the conjugation action of G inherit the integrability properties on generic symplectic leaves of the quotient space T^*G/G [53]. The reduced systems are spin Sutherland models built on 'collective spin variables' belonging to the reduction of the dual space $\mathcal{G}^*$ of the Lie algebra of G (or a coadjoint orbit therein) with respect to the maximal torus $G_0 < G$, at the zero value of the moment map. (For the spin Sutherland Hamiltonian, see equation (6.35).) In the Poisson–Lie setting, the space of collective spin variables becomes a similar reduction of the dual Poisson–Lie group (or a dressing orbit therein), which is the Lie group $G^* = B$ whose Lie algebra $\mathcal{B}$ appears in the Manin triple [10, 63] displayed in equation (2.2). The Poisson–Lie analogues of the spin Sutherland models were first introduced in [19], where Marsden–Weinstein type reductions of the Heisenberg double were studied employing the shifting trick of symplectic reduction [50]. This means that the phase space was initially extended by a dressing orbit of G in B, and then the reduction was defined by setting the relevant B-valued moment map to the identity value. The resulting systems were further investigated in [21] using Poisson reduction, i.e., by directly taking the quotient of the phase space by the action of the symmetry group G. Via restriction to symplectic leaves after reduction, the two methods give the same models. The first method is better suited for describing the reduced symplectic form, while the second one leads more directly to the reduced Poisson algebra.

In [19, 21] we collected heuristic arguments in favour of the degenerate integrability of the reduced systems that descend from the master systems of free motion supported by the Heisenberg doubles, but have not obtained a full proof. The main achievement of this paper is that we will establish in a mathematically exact manner the degenerate integrability of the reduced systems after restriction to a dense open subset of the Poisson quotient. This subset corresponds to the principal orbit type with respect to the G-action on the Heisenberg double. After restriction to a smaller dense open subset, degenerate integrability on the generic symplectic leaves will be proved as well. Our proof of degenerate

[1] Degenerate integrable systems are also called superintegrable, the notion as we use it is defined in Section 2.2.

integrability was motivated by ideas that we learned from papers by Reshetikhin [53, 54]. It also relies on techniques introduced in our joint work with Fairon [16] and on the recent note [22] dealing with reduced integrability on T^*G/G.

Besides the main achievement, the analyses of [19, 21] will be further developed in several other respects as well. For example, we shall present two useful, alternative descriptions of the reduced Poisson brackets. In the first one 'particle positions' and the Lax matrix are used as variables. The explicit formula given by equation (5.10) contains the dynamical r-matrix $\mathcal{R}(Q)$ (5.1) depending on the former. The second formulation (given by equation (6.21)) relies on variables that may be interpreted as particle positions, their canonical conjugates and collective spin degrees of freedom. These 'decoupled variables' make it possible to view the reduced systems as Ruijsenaars–Schneider type deformations of the standard spin Sutherland models, and we shall present a detailed elaboration of the relevant 'scaling limit'.

1.1. **Organization and results.** In the next section, we collect the necessary background material concerning Lie theory, degenerate integrability and Poisson–Lie symmetry. In Section 3, we first give a careful presentation of the Heisenberg double, describing its Poisson structure in terms of three distinct sets of variables; each have their own advantages as it turns out subsequently. Then, we expose the master system of free motion and explain its degenerate integrability. The core of the paper is Section 4, where we define the reduction of the master system and demonstrate the integrability properties of the resulting reduced system. Our main new results, Theorem 4.6 with Corollary 4.7, and Theorem 4.9 can be found in this section. Section 5.1 contains the derivation of the dynamical r-matrix form of the reduced Poisson brackets. The result is given by Theorem 5.2, which can be considered as an improvement of a previous result found in [21]. In Section 5.2, we describe the reduced Hamiltonian vector fields and present a quadrature leading to their integral curves. Here, we employ the partial gauge fixing associated with the gauge slice $\mathbb{M}_0$ (5.4), which covers a dense open subset of the reduced phase space. Then, in Section 6.1 we exhibit canonical conjugates of the position coordinates and a 'collective spin variable' whereby the reduced Poisson bracket takes the 'decoupled form' displayed in Theorem 6.4. This result appeared implicitly already in [19], and explicitly in the $G = \mathrm{U}(n)$ case in the paper [26]. In Section 6.2, we utilize the decoupled variables to explain how our reduced systems are connected to the standard spin Sutherland models in the so-called scaling limit characterized by Propositions 6.7 and 6.8. These propositions strengthen and make more precise previous results of [19]. In the final section, we offer a brief summary and an outlook towards open problems. There are also three appendices developing technical issues. Appendix A illustrates how a Poisson–Lie moment map generates a G-action, Appendix B explains a connection with the paper [19], and in Appendix C the previous derivation [23] of the spinless trigonometric Ruijsenaars–Schneider model [59] is recovered from the formalism used in the present work.

The exposition of the material that follows is detailed and mostly self-contained, with the intention to facilitate further studies of the subject.

2. Background material

Here, we first summarize a few Lie theoretic facts for later use. More details can be found in [21] and in the textbooks [12, 37, 60]. Then, we review the notion of degenerate integrability, and recall crucial features of Poisson–Lie groups and their actions.

2.1. **Lie theoretic preparations.** Let $\mathcal{G}^{\mathbb{C}}$ be a complex simple Lie algebra with Killing form $\langle -, - \rangle$. The choice of a Cartan subalgebra $\mathcal{G}_0^{\mathbb{C}} < \mathcal{G}^{\mathbb{C}}$ and a system of positive roots leads to the triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_-^{\mathbb{C}} + \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_+^{\mathbb{C}}. \tag{2.1}$$

Then, the 'realification' $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ of $\mathcal{G}^{\mathbb{C}}$ (i.e. $\mathcal{G}^{\mathbb{C}}$ viewed as a real Lie algebra) can be written as the vector space direct sum of two subalgebras

$$\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B}, \tag{2.2}$$

where $\mathcal{G}$ is a compact simple Lie algebra containing the maximal Abelian subalgebra $\mathcal{G}_0 < \mathcal{G}$ for which

$$\mathcal{G}_0^{\mathbb{C}} = \mathcal{G}_0 + \mathrm{i}\mathcal{G}_0, \tag{2.3}$$

and

$$\mathcal{B} := \mathrm{i}\mathcal{G}_0 + \mathcal{G}_+^{\mathbb{C}} \tag{2.4}$$

is a 'Borel' subalgebra. We shall also employ the vector space decompositions

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_\perp^{\mathbb{C}} \quad \text{with} \quad \mathcal{G}_\perp^{\mathbb{C}} := \mathcal{G}_-^{\mathbb{C}} + \mathcal{G}_+^{\mathbb{C}}, \tag{2.5}$$

and

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_\perp, \quad \mathcal{B} = \mathcal{B}_0 + \mathcal{B}_+ \tag{2.6}$$

with

$$\mathcal{G}_\perp = \mathcal{G} \cap \mathcal{G}_\perp^{\mathbb{C}}, \quad \mathcal{B}_0 = \mathrm{i}\mathcal{G}_0, \quad \mathcal{B}_+ = \mathcal{G}_+^{\mathbb{C}}. \tag{2.7}$$

Referring to (2.1) and (2.5), we may write any $X \in \mathcal{G}_\mathbb{R}^\mathbb{C}$ as

$$X = X_- + X_0 + X_+ = X_0 + X_\perp, \tag{2.8}$$

or by using (2.2) as

$$X = X_\mathcal{G} + X_\mathcal{B}, \tag{2.9}$$

and can also further decompose $X_\mathcal{G} \in \mathcal{G}$ and $X_\mathcal{B} \in \mathcal{B}$ according to (2.6).

Let us equip $\mathcal{G}_\mathbb{R}^\mathbb{C}$ with the invariant, nondegenerate, symmetric bilinear form

$$\langle X, Y\rangle_\mathbb{I} := \Im\langle X, Y\rangle, \qquad \forall X, Y \in \mathcal{G}^\mathbb{C}. \tag{2.10}$$

The decomposition (2.2) is a well known example of a Manin triple [10, 63], meaning that $\mathcal{G}$ and $\mathcal{B}$ are isotropic subalgebras of $\mathcal{G}_\mathbb{R}^\mathbb{C}$. Consequently, the bilinear form gives rise to the following identifications of linear dual spaces:

$$\mathcal{G}^* = \mathcal{B}, \quad \mathcal{B}^* = \mathcal{G}, \quad (\mathcal{B}_+)^* = \mathcal{G}_\perp, \quad (\mathcal{B}_0)^* = \mathcal{G}_0, \quad (\mathcal{G}_0)^* = \mathcal{B}_0. \tag{2.11}$$

For the simplest series of examples $\mathcal{G}^\mathbb{C} = \mathrm{sl}(n, \mathbb{C})$, $\mathcal{G}_0^\mathbb{C}$ is the standard Cartan subalgebra of traceless diagonal matrices, $\mathcal{G} = \mathrm{su}(n)$, $\mathcal{B}$ consists of the upper triangular elements of $\mathrm{sl}(n, \mathbb{C})$ with real diagonal entries, and the Killing form $\langle X, Y\rangle$ is a multiple of $\mathrm{tr}(XY)$ by a positive constant.

Let $G_\mathbb{R}^\mathbb{C}$ be a connected and simply connected real Lie group whose Lie algebra is $\mathcal{G}_\mathbb{R}^\mathbb{C}$, and denote G and B its connected subgroups associated with the Lie subalgebras $\mathcal{G}$ and $\mathcal{B}$. These subgroups are simply connected and G is compact. We have the connected subgroup $G_0^\mathbb{C} < G_\mathbb{R}^\mathbb{C}$ corresponding to $\mathcal{G}_0^\mathbb{C}$, as well as the subgroups $G_0 < G$, $B_0 < B$, $B_+ < B$ associated with $\mathcal{G}_0$ and the subalgebras $\mathcal{B}_0$ and $\mathcal{B}_+$ of $\mathcal{B}$.

The real vector space $\mathcal{G}_\mathbb{R}^\mathbb{C}$ can be presented as the direct sum

$$\mathcal{G}_\mathbb{R}^\mathbb{C} = \mathcal{G} + \mathrm{i}\mathcal{G}, \tag{2.12}$$

and we let θ denote the corresponding complex conjugation,

$$\theta(Y_1 + \mathrm{i}Y_2) := Y_1 - \mathrm{i}Y_2 \quad \text{for} \quad Y_1, Y_2 \in \mathcal{G}. \tag{2.13}$$

This is an involutive automorphism of the real Lie algebra $\mathcal{G}_\mathbb{R}^\mathbb{C}$, which lifts to an involutive automorphism Θ of the group $G_\mathbb{R}^\mathbb{C}$. They are known as infinitesimal and global Cartan involutions, respectively. It is customary to denote

$$Z^\dagger := -\theta(Z), \qquad K^\dagger = \Theta(K^{-1}), \qquad \forall Z \in \mathcal{G}_\mathbb{R}^\mathbb{C},\ \forall K \in G_\mathbb{R}^\mathbb{C}. \tag{2.14}$$

The maps $Z \mapsto Z^\dagger$ and $K \mapsto K^\dagger$ are antiautomorphisms. For the classical Lie groups one can choose the conventions in such a way that dagger coincides with the matrix adjoint [37].

The compact subgroup $G < G_\mathbb{R}^\mathbb{C}$ is the fixed point set of Θ. The closed submanifold

$$\mathfrak{P} := \exp(\mathrm{i}\mathcal{G}) \subset G_\mathbb{R}^\mathbb{C} \tag{2.15}$$

is diffeomorphic to $\mathrm{i}\mathcal{G}$ by the exponential map and is a connected component of the fixed point set of the antiautomorphism $K \mapsto K^\dagger$. The group B also admits global exponential parametrization, and the map

$$\nu : B \to \mathfrak{P}, \quad \nu(b) := bb^\dagger \tag{2.16}$$

is a diffeomorphism.

Next, we describe a chain of diffeomorphisms between the manifolds

$$M := G_\mathbb{R}^\mathbb{C}, \quad \mathfrak{M} := G \times B \quad \text{and} \quad \mathbb{M} := G \times \mathfrak{P}. \tag{2.17}$$

We start by recalling that every element $K \in M$ admits unique (Iwasawa) decompositions [37] into products of elements of G and B,

$$K = g_L b_R^{-1} = b_L g_R^{-1} \quad \text{with} \quad g_L, g_R \in G,\ b_L, b_R \in B. \tag{2.18}$$

These decompositions induce the (real-analytic) maps $\Xi_L, \Xi_R : M \to G$ and $\Lambda_L, \Lambda_R : M \to B$,

$$\Xi_L(K) := g_L, \quad \Xi_R(K) := g_R, \quad \Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R. \tag{2.19}$$

Besides the pairs (Ξ_L, Λ_R) and (Ξ_R, Λ_L), also the pair (Ξ_R, Λ_R) yields a diffeomorphism,

$$m_1 := (\Xi_R, \Lambda_R) : M \ \to \mathfrak{M}, \qquad m_1(K) = (g_R, b_R). \tag{2.20}$$

In addition to this, we need the diffeomorphism

$$m_2 : \mathfrak{M} \to \mathbb{M}, \qquad m_2(g, b) := (g, \nu(b)). \tag{2.21}$$

The map ν (2.16) intertwines the so-called dressing action of G on B with the obvious conjugation action of G on $\mathfrak{P}$. That is, we have

$$\mathrm{Dress}_\eta(b)(\mathrm{Dress}_\eta(b))^\dagger = \eta b b^\dagger \eta^{-1}, \qquad \forall \eta \in G,\, b \in B. \tag{2.22}$$

It follows that any element of B can be transformed into $B_0 = \exp(\mathrm{i}\mathcal{G}_0)$ by the dressing action. The relation (2.22) can be taken as the definition of the dressing action. More explicitly, one has

$$\mathrm{Dress}_\eta(b) = \Lambda_L(\eta b), \qquad \forall \eta \in G,\, b \in B, \tag{2.23}$$

and the corresponding infinitesimal action

$$\mathrm{dress}_Y(b) := \frac{d}{dt}\Big|_{t=0} \mathrm{Dress}_{e^{tY}}(b) = b(b^{-1}Yb)_{\mathcal{B}}, \quad \forall Y \in \mathcal{G}. \tag{2.24}$$

Remark 2.1. *The notation used on the right side of* (2.24) *'pretends' that our Lie groups are groups of matrices. Such symbolic matrix notations are adopted throughout the paper. If desired, one may rewrite all of the relevant equations in equivalent abstract form (which is often longer), or can employ faithful matrix representations.*

For a real function $\varphi \in C^\infty(B)$, we define the $\mathcal{G}$-valued left and right derivatives, $D\varphi$ and $D'\varphi$, by

$$\langle X, D\varphi(b)\rangle_{\mathbb{I}} + \langle X', D'\varphi(b)\rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \varphi(e^{tX} b e^{tX'}), \quad \forall b \in B,\, X, X' \in \mathcal{B}. \tag{2.25}$$

In general, these obey the relation $D\varphi(b) = (bD'\varphi(b)b^{-1})_{\mathcal{G}}$. If the function is invariant with respect to the dressing action, $\varphi \in C^\infty(B)^G$, then $\langle D'\varphi(b), (b^{-1}Yb)_{\mathcal{B}}\rangle_{\mathbb{I}} = 0$, $\forall Y \in \mathcal{G}$, and from this we get $D\varphi(b) = bD'\varphi(b)b^{-1}$. Equivalently, we have

$$(b^{-1}D\varphi(b)b)_{\mathcal{B}} = 0, \qquad \forall \varphi \in C^\infty(B)^G,\, b \in B. \tag{2.26}$$

By (2.24), this means that $D\varphi(b)$ belongs to the Lie algebra of the isotropy group $G_b < G$ of b with respect to the dressing action. Even more, this derivative belongs the center of the isotropy Lie algebra, because the derivative of an invariant function is equivariant:

$$D\varphi(\mathrm{Dress}_\eta(b)) = \eta D\varphi(b)\eta^{-1}, \qquad \forall \eta \in G,\, b \in B. \tag{2.27}$$

The isotropy subgroup G_b is generically a maximal torus of G, and the elements for which this holds constitute the dense open subset $B^{\mathrm{reg}} \subset B$. The derivatives of the invariant functions actually span the center of the isotropy Lie algebra at any $b \in B$. This can be seen, for example, with the aid of the natural isomorphisms

$$C^\infty(\mathcal{G}_0)^W \longleftrightarrow C^\infty(\mathcal{G})^G \longleftrightarrow C^\infty(\mathfrak{P})^G \longleftrightarrow C^\infty(B)^G, \tag{2.28}$$

where $C^\infty(\mathcal{G}_0)^W$ denotes the Weyl invariant smooth functions on $\mathcal{G}_0$. The isomorphisms are induced by the maps

$$\mathcal{G}_0 \xrightarrow{\iota} \mathcal{G} \xrightarrow{\exp_{\mathrm{i}}} \mathfrak{P} \xleftarrow{\nu} B, \tag{2.29}$$

where $\iota : \mathcal{G}_0 \to \mathcal{G}$ is the inclusion, $\exp_{\mathrm{i}}(X) := \exp(\mathrm{i}X)$, and ν is defined in (2.16). These maps also relate the dense open subsets $\mathcal{G}_0^{\mathrm{reg}}$, $\mathcal{G}^{\mathrm{reg}}$, $\mathfrak{P}^{\mathrm{reg}}$ and B^{reg}. It is well known that the gradients (with respect to the Killing form of $\mathcal{G}$) of the invariant functions on $\mathcal{G}$ span the center of the corresponding isotropy subalgebra. The dimension of the span of the derivatives of the invariant functions does not change under these maps, since the derivatives are equivalent to the ordinary exterior derivatives (for example, $D\varphi(b) \in \mathcal{G} \simeq \mathcal{B}^*$ encodes $d\varphi(b) \in T_b^*B$).

2.2. **Degenerate integrability.** The notion of degenerate integrability of Hamiltonian systems on symplectic manifolds is due to Nekhoroshev [46]. Degenerate integrable systems have more first integrals (constants of motion) than half the dimension of the phase space, which characterizes Liouville integrability. In the extreme case the trajectories are completely determined by fixing the constants of motion, as is exemplified by the classical Kepler problem that possesses 5 independent constants of motion. A closely related concept of non-Abelian or non-commutative integrability was introduced by Mischenko and Fomenko [44], and this is especially fitting for systems whose basic constants of motion form a finite-dimensional, non-Abelian Lie algebra.

A systematic exploration of natural quantum mechanical Hamiltonians with many conserved quantities having specific form was initiated by Fris *et al* in 1965 [30], and later this has become a very active research subject [45]. The systems studied in this field are nowadays called superintegrable, a term that apparently goes back to Wojciechowski [64]. The adjective *superintegrable* is now often used to characterize both quantum and classical mechanical systems [18, 45, 55, 56]. We prefer to stick to the original terminology of Nekhoroshev, which highlights the important feature that in comparison to classical Liouville integrability the dynamics takes place on lower dimensional submanifolds of the phase space (typically, degenerate tori when compact). The extensive literature on the subject of integrability (see e.g. [33, 38, 57]) contains several variants of the basic notions. The definition that we find the most convenient is presented below.

Definition 2.2. *Suppose that $\mathcal{M}$ is a symplectic manifold of dimension $2m$ with associated Poisson bracket $\{-,-\}$ and two distinguished subrings $\mathfrak{H}$ and $\mathfrak{F}$ of $C^\infty(\mathcal{M})$ satisfying the following conditions:*

(1) The ring $\mathfrak{H}$ has functional dimension r and $\mathfrak{F}$ has functional dimension s such that $r+s=\dim(\mathcal{M})$ and $r<m$.
(2) Both $\mathfrak{H}$ and $\mathfrak{F}$ form Poisson subalgebras of $C^\infty(\mathcal{M})$, satisfying $\mathfrak{H}\subset\mathfrak{F}$ and $\{\mathcal{F},\mathcal{H}\}=0$ for all $\mathcal{F}\in\mathfrak{F}$, $\mathcal{H}\in\mathfrak{H}$.
(3) The Hamiltonian vector fields of the elements of $\mathfrak{H}$ are complete.

Then, $(\mathcal{M},\{-,-\},\mathfrak{H},\mathfrak{F})$ is called a degenerate integrable system of rank r. The rings $\mathfrak{H}$ and $\mathfrak{F}$ are referred to as the ring of Hamiltonians and constants of motion, respectively.

Recall that the functional dimension of a ring $\mathfrak{R}$ of functions on a manifold $\mathcal{M}$ is d if the exterior derivatives of the elements of $\mathfrak{R}$ *generically*, that is on a dense open submanifold, span a d-dimensional subspace of the cotangent space. Condition (3) above on the completeness of the flows is superfluous if the joint level surfaces of the elements of $\mathfrak{F}$ are compact. Degenerate integrability of a single Hamiltonian $\mathcal{H}$ is understood to mean that there exist rings $\mathfrak{H}$ and $\mathfrak{F}$ with the above properties such that $\mathcal{H}\in\mathfrak{H}$. Observe that $\mathfrak{F}$ is either equal to or can be enlarged to the centralizer of $\mathfrak{H}$ in the Poisson algebra $(C^\infty(\mathcal{M}),\{-,-\})$. In the literature the definition is often formulated in terms of functions $f_1,\dots,f_r,f_{r+1},\dots,f_s$ so that they generate $\mathfrak{F}$ and the first r of them generate $\mathfrak{H}$. If the definition is modified by setting $r=s=m$ and $\mathfrak{H}=\mathfrak{F}$, then one obtains the notion of Liouville integrability.

The concepts of integrability can be extended to Poisson manifolds [38] beyond the symplectic class. In fact, we shall construct a series of examples that satisfy the requirements of the next definition.

Definition 2.3. *Consider a Poisson manifold $(\mathcal{M},\{-,-\})$ whose Poisson tensor has maximal rank $2m\le\dim(\mathcal{M})$ on a dense open subset. Then, $(\mathcal{M},\{-,-\},\mathfrak{H},\mathfrak{F})$ is called a degenerate integrable system of rank r if conditions (1), (2), (3) of Definition 2.2 hold, and the Hamiltonian vector fields of the elements of $\mathfrak{H}$ span an r-dimensional subspace of the tangent space over a dense open subset of $\mathcal{M}$.*

The integrable systems of Definition 2.2 are integrable in the sense of Definition 2.3, too, since in the symplectic case the condition on the span of the Hamiltonian vector fields of $\mathfrak{H}$ holds automatically. Liouville integrability in the Poisson case results by imposing $r=m$ instead of $r<m$ in the definition. In that case, our definition implies that $\mathfrak{F}$ is an Abelian Poisson algebra (in [38] this condition appears in the definition).

In a degenerate integrable system, the evolution equation associated with any $\mathcal{H}\in\mathfrak{H}$ can be integrated by quadrature (see, e.g., [33, 46, 57]). A description of 'action-angle and spectator' coordinates in the Poisson case can be found in [38]. Under further conditions, it can be shown [33] that degenerate integrable systems are integrable also in the Liouville sense. However, in general there is no canonical way to enlarge $\mathfrak{H}$ by elements of $\mathfrak{F}$ to obtain an Abelian Poisson algebra of the required functional dimension. This freedom can be used to manufacture very different Liouville integrable systems out of a given degenerate integrable system. For spin Calogero–Moser type systems, and their generalizations that we are interested in, $\mathfrak{H}$ is distinguished by its group theoretic origin [21, 53, 54].

2.3. **Poisson–Lie symmetry.**

Poisson–Lie groups are the quasi-classical analogues of quantum groups introduced by Drinfeld [10, 11]. Their role in classical integrable systems was pioneered by Semenov–Tian–Shansky [62], whose review [63] is highly recommended as a general reference.

By definition, a Poisson–Lie group is a pair $(G,\{-,-\}_G)$, where $\{-,-\}_G$ is a Poisson bracket on the smooth (or holomorphic etc) functions on the Lie group G such that the group product $G\times G\to G$ is a Poisson map. A Poisson action of $(G,\{-,-\}_G)$ on a Poisson manifold $(\mathcal{M},\{-,-\})$ is an action for which the action map $\mathcal{A}:G\times\mathcal{M}\to\mathcal{M}$ is Poisson. In these definitions, $G\times G$ and $G\times\mathcal{M}$ are equipped with the respective product Poisson structures. Take arbitrary points $g\in G$, $p\in\mathcal{M}$ and for any $F\in C^\infty(\mathcal{M})$ define $F_g\in C^\infty(\mathcal{M})$ and $F^p\in C^\infty(G)$ by

$$F_g(p)=F^p(g)=F(\mathcal{A}_g(p)) \tag{2.30}$$

using $\mathcal{A}_g(p):=\mathcal{A}(g,p)$. The Poisson property of the map $\mathcal{A}$ means that

$$\{F,H\}(\mathcal{A}_g(p))=\{F_g,H_g\}(p)+\{F^p,H^p\}_G(g),\qquad \forall F,H\in C^\infty(\mathcal{M}). \tag{2.31}$$

The Poisson tensor of every Poisson Lie group vanishes at the unit element $e \in G$. Thus, the linearization of the Poisson bracket $\{-,-\}_G$ yields [10, 63] a Lie bracket $[-,-]_*$ on the dual space $\mathcal{G}^* = T_e^* G$ of the Lie algebra $\mathcal{G} = T_e G$. For any $X \in \mathcal{G}$, let $X_{\mathcal{M}}$ denote the infinitesimal generator of the (left) G-action on $\mathcal{M}$, such that $X \mapsto X_{\mathcal{M}}$ is an antihomomorphism, and let $\mathcal{L}_{X_{\mathcal{M}}} F = dF(X_{\mathcal{M}})$ denote the derivative of the function $F \in C^\infty(\mathcal{M})$. Pick a basis $\{T_a\}$ of $\mathcal{G}$ with dual basis $\{T^a\}$ of $\mathcal{G}^*$, and define $\zeta_F \in C^\infty(\mathcal{M}, \mathcal{G}^*)$ by

$$\zeta_F := \sum_a T^a dF((T_a)_{\mathcal{M}}), \qquad \forall F \in C^\infty(\mathcal{M}). \tag{2.32}$$

Then, the Poisson property (2.31) implies the identity

$$\mathcal{L}_{X_{\mathcal{M}}}\{F,H\} - \{\mathcal{L}_{X_{\mathcal{M}}}F, H\} - \{F, \mathcal{L}_{X_{\mathcal{M}}}H\} - ([\zeta_F, \zeta_H]_*, X) = 0, \tag{2.33}$$

for all $X \in \mathcal{G}$, $F, H \in C^\infty(\mathcal{M})$, where in the last term the pairing between $\mathcal{G}^*$ and $\mathcal{G}$ is used. Indeed, (2.33) follows by putting $g = \exp(tX)$ in (2.31) and taking derivative with respect to $t \in \mathbb{R}$ at $t = 0$. It is also worth noting that

$$\zeta_F(p) = (d_G F^p)(e) \quad \text{and} \quad [\zeta_F(p), \zeta_H(p)]_* = (d_G\{F^p, H^p\}_G)(e), \tag{2.34}$$

where d_G denotes the exterior derivation of functions on G.

We assume that G is connected, and then (2.33) is equivalent to the Poisson property of the G-action. Two consequences of the identity (2.33) are important for us. First, if both F and H are G-invariant, then so is their Poisson bracket, i.e., $C^\infty(\mathcal{M})^G$ is closed under the Poisson bracket. The statement holds because $\mathcal{L}_{X_{\mathcal{M}}}\{F,H\} = 0$ in this case. Second, if $F \in C^\infty(\mathcal{M})$ is arbitrary and $H \in C^\infty(\mathcal{M})^G$, then (2.33) becomes

$$\mathcal{L}_{X_{\mathcal{M}}}\{F,H\} - \{\mathcal{L}_{X_{\mathcal{M}}}F, H\} = 0. \tag{2.35}$$

Defining the Hamiltonian vector field V_H by $\{F,H\} =: \mathcal{L}_{V_H}(F)$, the identity means that

$$[X_{\mathcal{M}}, V_H] = 0, \qquad \forall X \in \mathcal{G},\ H \in C^\infty(\mathcal{M})^G. \tag{2.36}$$

This entails that the corresponding flows, denoted φ_τ^X and φ_t^H, commute

$$\varphi_\tau^X \circ \varphi_t^H = \varphi_t^H \circ \varphi_\tau^X. \tag{2.37}$$

Since G is supposed to be connected, this in turn implies that the Hamiltonian flow φ_t^H is G-equivariant. In favourable circumstances, e.g. if the group G is compact, one may identify $C^\infty(\mathcal{M})^G$ with the ring of smooth functions on the quotient space $\mathcal{M}^{\mathrm{red}} := \mathcal{M}/G$. In this way, $C^\infty(\mathcal{M}^{\mathrm{red}})$ becomes a Poisson algebra, and the Hamiltonian flows generated by its elements are the projections of the flows φ_t^H living upstairs. The process of descending to the quotient space $\mathcal{M}^{\mathrm{red}}$ is known as Poisson reduction, or Hamiltonian reduction if a G-invariant Hamiltonian is also specified. It should be noted that the quotient space $\mathcal{M}^{\mathrm{red}}$ is usually not a smooth Poisson manifold, but a so-called stratified Poisson space [50, 61].

In the theory of Poisson actions of $(G, \{-,-\}_G)$ the G^*-valued Poisson–Lie moment map plays an important role [39, 40]. Here, G^* is the dual Poisson–Lie group [10, 39, 63], whose Lie algebra is $(\mathcal{G}^*, [-,-]_*)$ mentioned above and the linearization of the Poisson bracket on G^* reproduces the Lie algebra of G. The precise notion of the moment map will be recalled in Section 4 focusing on the groups our interest. The Poisson–Lie moment map can be used for finding Poisson subspaces of $\mathcal{M}^{\mathrm{red}}$ quite in the same way as for the standard $\mathcal{G}^*$-valued moment map [50]. For compact semisimple Lie groups, there is a direct link between ordinary Hamiltonian G-actions and their Poisson–Lie analogues. One can be converted into the other by means of a modification of the symplectic form, without changing the reduced structure [1].

3. Integrable master system on the Heisenberg double

In the first subsection we give a terse overview of the Poisson geometry of the standard Heisenberg double of the compact Lie group G. The second subsection is devoted to the description of a degenerate integrable system on this phase space.

3.1. **Three models of the Heisenberg double.** We recall [62, 63] that the group manifold $M = G_{\mathbb{R}}^{\mathbb{C}}$ carries the following two Poisson brackets:

$$\{\Phi_1, \Phi_2\}_\pm := \langle \nabla\Phi_1, \rho\nabla\Phi_2\rangle_{\mathbb{I}} \pm \langle \nabla'\Phi_1, \rho\nabla'\Phi_2\rangle_{\mathbb{I}}, \quad \forall \Phi_1, \Phi_2 \in C^\infty(M). \tag{3.1}$$

Here, $\rho := \frac{1}{2}(\pi_{\mathcal{G}} - \pi_{\mathcal{B}})$ with $\pi_{\mathcal{G}}$ and $\pi_{\mathcal{B}}$ denoting the projections from $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ onto $\mathcal{G}$ and $\mathcal{B}$, which correspond to the direct sum in (2.2). For any real function $\Phi \in C^\infty(M)$, the $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-valued left and right derivatives $\nabla\Phi$ and $\nabla'\Phi$ are defined by

$$\langle X, \nabla\Phi(K)\rangle_{\mathbb{I}} + \langle X', \nabla'\Phi(K)\rangle_{\mathbb{I}} := \frac{d}{dt}\bigg|_{t=0} \Phi(e^{tX} K e^{tX'}), \quad \forall K \in M,\ X, X' \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}. \tag{3.2}$$

The minus bracket makes M into a Poisson–Lie group, of which G and B are Poisson–Lie subgroups, i.e., (embedded) Lie subgroups and Poisson submanifolds. Their *inherited Poisson brackets* take the form

$$\{\chi_1, \chi_2\}_G(g) = -\langle D'\chi_1(g), g^{-1}(D\chi_2(g))g\rangle_{\mathbb{I}}, \tag{3.3}$$

and

$$\{\varphi_1, \varphi_2\}_B(b) = \langle D'\varphi_1(b), b^{-1}(D\varphi_2(b))b\rangle_{\mathbb{I}}. \tag{3.4}$$

The derivatives are $\mathcal{B}$-valued for $\chi_i \in C^\infty(G)$ and $\mathcal{G}$-valued for $\varphi_i \in C^\infty(B)$. Concretely, we use the definition (2.25) for any $\varphi \in C^\infty(B)$, and

$$\langle Y, D\chi(g)\rangle_{\mathbb{I}} + \langle Y', D'\chi(g)\rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \chi(e^{tY} g e^{tY'}), \quad \forall g \in G,\, Y, Y' \in \mathcal{G}, \tag{3.5}$$

for any $\chi \in C^\infty(G)$. The Poisson manifolds $(M, \{-,-\}_-)$ and $(M, \{-,-\}_+)$ are known, respectively, as the Drinfeld double and the Heisenberg double associated with the standard Poisson structures of B and G. The Poisson bracket $\{-,-\}_+$ is nondegenerate. The corresponding symplectic form was found in [2], but we shall not use its formula here. It is also known that the maps

$$(\Lambda_L, \Lambda_R) : M \to B \times B \quad \text{and} \quad (\Xi_L, \Xi_R) : M \to G \times G \tag{3.6}$$

are Poisson maps with respect to $(M, \{-,-\}_+)$ and the direct product Poisson structures on the targets obtained from $(B, \{-,-\}_B)$ and from $(G, \{-,-\}_G)$, respectively.

Below we focus on the Heisenberg double $(M, \{-,-\}_+)$, and transfer its Poisson structure to $\mathfrak{M}$ and $\mathbb{M}$ (2.17) by the diffeomorphisms m_1 (2.20) and m_2 (2.21). As was proved in [21], the usage of m_1 results in the Poisson bracket $\{-,-\}_{\mathfrak{M}}$ on $\mathfrak{M}$ having the following explicit form:

$$\{f, h\}_{\mathfrak{M}}(g, b) = \left\langle D_2'f, b^{-1}(D_2h)b\right\rangle_{\mathbb{I}} - \left\langle D_1'f, g^{-1}(D_1h)g\right\rangle_{\mathbb{I}} + \langle D_1f, D_2h\rangle_{\mathbb{I}} - \langle D_1h, D_2f\rangle_{\mathbb{I}} \tag{3.7}$$

for functions $f, h \in C^\infty(\mathfrak{M})$. The derivatives on the right-hand side are taken at $(g, b) \in G \times B$, with respect to the first and second variable, according to the definitions (3.5) and (2.25), respectively. In particular, D_1f is $\mathcal{B}$-valued and D_2f is $\mathcal{G}$-valued.

For any real function $\mathcal{F} \in C^\infty(\mathfrak{P})$, define its $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$*-valued derivative* $\mathcal{DF}$ as follows:

$$\langle X, \mathcal{DF}(L)\rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \mathcal{F}(e^{tX} L e^{tX^\dagger}), \qquad \forall X \in \mathcal{B}, \tag{3.8}$$

with $X^\dagger$ given by (2.14), and

$$\langle Y, \mathcal{DF}(L)\rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \mathcal{F}(e^{tY} L e^{-tY}), \qquad \forall Y \in \mathcal{G}. \tag{3.9}$$

Referring to (2.9), the first equation determines $(\mathcal{DF}(L))_{\mathcal{G}}$ and the second one $(\mathcal{DF}(L))_{\mathcal{B}}$. (These two equations could be 'unified' since $-Y = Y^\dagger$ for $Y \in \mathcal{G}$, but we prefer to display them separately.) Because the natural action of B on $\mathfrak{P}$ is transitive[2], all information about $\mathcal{DF}$ is contained in the $\mathcal{G}$-component. This is clear from the next lemma, too.

Lemma 3.1. *Let $\mathcal{F} \in C^\infty(\mathfrak{P})$ and $\varphi \in C^\infty(B)$ connected by the diffeomeorphism ν (2.16), i.e.,*

$$\mathcal{F}(bb^\dagger) = \varphi(b), \qquad \forall b \in B. \tag{3.10}$$

Then their derivatives satisfy

$$(\mathcal{DF}(bb^\dagger))_{\mathcal{G}} = D\varphi(b) \equiv (bD'\varphi(b)b^{-1})_{\mathcal{G}}, \qquad (\mathcal{DF}(bb^\dagger))_{\mathcal{B}} = (bD'\varphi(b)b^{-1})_{\mathcal{B}}, \tag{3.11}$$

and consequently

$$\mathcal{DF}(bb^\dagger) = bD'\varphi(b)b^{-1}. \tag{3.12}$$

Proof. Take $L = bb^\dagger$ and consider the curve e^{tX} for $X \in \mathcal{B}$. Since $\nu(e^{tX}b) = e^{tX}Le^{tX^\dagger}$, we obtain

$$\langle X, bD'\varphi(b)b^{-1}\rangle_{\mathbb{I}} = \langle X, D\varphi(b)\rangle_{\mathbb{I}} = \frac{d}{dt}\Big|_{t=0} \varphi(e^{tX}b) = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(e^{tX}Le^{tX^\dagger}) = \langle X, \mathcal{DF}(L)\rangle_{\mathbb{I}}, \tag{3.13}$$

which implies the first equality in (3.11). Next, take any $Y \in \mathcal{G}$ and consider the curve $\mathrm{Dress}_{e^{tY}}(b)$. Using (2.24) we get

$$\frac{d}{dt}\Big|_{t=0} \varphi(\mathrm{Dress}_{e^{tY}}(b)) = \langle D'\varphi(b), (b^{-1}Yb)_{\mathcal{B}}\rangle_{\mathbb{I}} = \langle bD'\varphi(b)b^{-1}, Y\rangle_{\mathbb{I}}. \tag{3.14}$$

[2]The corresponding action map is $B \times \mathfrak{P} \in (b, L) \mapsto bLb^\dagger \in \mathfrak{P}$.

Due to the identity $\nu(\mathrm{Dress}_{e^{tY}}(b)) = e^{tY}Le^{-tY}$, this is equal to

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{F}(e^{tY}Le^{-tY}) = \langle(\mathcal{D}\mathcal{F}(L))_{\mathcal{B}}, Y\rangle_{\mathbb{I}}. \tag{3.15}$$

Consequently, the second equality in (3.11) holds, too. $\square$

We use the map ν (2.16) to transfer the Poisson bracket $\{-,-\}_B$ (3.4) from B to $\mathfrak{P}$. With the aid of Lemma 3.1, this leads to

$$\{\mathcal{F},\mathcal{H}\}_{\mathfrak{P}}(L) = \langle(\mathcal{D}\mathcal{F}(L))_{\mathcal{B}}, (\mathcal{D}\mathcal{H}(L))_{\mathcal{G}}\rangle_{\mathbb{I}} = -\langle(\mathcal{D}\mathcal{F}(L))_{\mathcal{G}}, (\mathcal{D}\mathcal{H}(L))_{\mathcal{B}}\rangle_{\mathbb{I}}, \quad \forall \mathcal{F},\mathcal{H} \in C^\infty(\mathfrak{P}). \tag{3.16}$$

The Hamiltonian vector field generated by $\mathcal{H} \in C^\infty(\mathfrak{P})$ on $\mathfrak{P}$ yields the evolution equation

$$\dot{L} = [(\mathcal{D}\mathcal{H}(L))_{\mathcal{G}}, L]. \tag{3.17}$$

The identical vanishing of the right-hand side characterizes the elements of the center of the Poisson bracket (3.16). The elements of the center are constant along all Hamiltonian flows, and this gives their equivalent characterization: $(\mathcal{D}\mathcal{H}(L))_{\mathcal{B}} = 0$. This holds if and only $\mathcal{H}$ is G-invariant, that is, for $\mathcal{H} \in C^\infty(\mathfrak{P})^G$. As one can verify, the derivative of every invariant function is equivariant,

$$\mathcal{D}\mathcal{H}(\eta L\eta^{-1}) = \eta\mathcal{D}\mathcal{H}(L)\eta^{-1}, \qquad \forall \eta \in G,\ L \in \mathfrak{P},\ \mathcal{H} \in C^\infty(\mathfrak{P})^G. \tag{3.18}$$

Similarly to the relation between B and $\mathfrak{P}$, we can transfer the Poisson bracket (3.7) from $\mathfrak{M}$ to $\mathbb{M}$ (2.17) via the map m_2 (2.21). To display the result, for $\mathcal{F} \in C^\infty(\mathbb{M})$ let $\mathcal{D}_1\mathcal{F}(g,L) \in \mathcal{B}$ and $\mathcal{D}_1'\mathcal{F}(g,L) \in \mathcal{B}$ denote the usual derivatives with respect to the first variable, and $\mathcal{D}_2\mathcal{F}(g,L) \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ denote the derivative with respect to the second variable defined according to equations (3.8) and (3.9).

Proposition 3.2. *Via the map m_2 (2.21), the formula (3.7) of the Poisson bracket on $\mathfrak{M}$ is equivalent to*

$$\{\mathcal{F},\mathcal{H}\}_{\mathbb{M}}(g,L) = \langle\mathcal{D}_2\mathcal{F}, (\mathcal{D}_2\mathcal{H})_{\mathcal{G}}\rangle_{\mathbb{I}} - \left\langle g\mathcal{D}_1'\mathcal{F}g^{-1}, \mathcal{D}_1\mathcal{H}\right\rangle_{\mathbb{I}} + \langle\mathcal{D}_1\mathcal{F}, \mathcal{D}_2\mathcal{H}\rangle_{\mathbb{I}} - \langle\mathcal{D}_1\mathcal{H}, \mathcal{D}_2\mathcal{F}\rangle_{\mathbb{I}}, \tag{3.19}$$

where the derivatives of $\mathcal{F},\mathcal{H} \in C^\infty(\mathbb{M})$ are evaluated at $(g,L) \in \mathbb{M} = G \times \mathfrak{P}$.

Proof. We simply substitute the following relations into (3.7):

$$D_1 f(g,b) = \mathcal{D}_1\mathcal{F}(g,L), \quad bD_2' f(g,b)b^{-1} = \mathcal{D}_2\mathcal{F}(g,L), \quad D_2 f(g,b) = (\mathcal{D}_2\mathcal{F}(g,L))_{\mathcal{G}}. \tag{3.20}$$

Using also the corresponding relations for $h = \mathcal{H} \circ m_2$, we get (3.19) from (3.7). Note that $\langle\mathcal{D}_1\mathcal{F}, \mathcal{D}_2\mathcal{H}\rangle_{\mathbb{I}} = \langle\mathcal{D}_1\mathcal{F}, (\mathcal{D}_2\mathcal{H})_{\mathcal{G}}\rangle_{\mathbb{I}}$, because $\mathcal{D}_1\mathcal{F}(g,L) \in \mathcal{B}$. $\square$

Remark 3.3. *For the alert reader, a word on 'tricky signs' is in order. Using the identifications $\mathcal{B}^* = \mathcal{G}$ and $\mathcal{G}^* = \mathcal{B}$ in (2.11), the linearization of the Poisson bracket (3.4) on B gives the Lie bracket on the subalgebra $\mathcal{G} < \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$, and the linearization of the opposite of the Poisson bracket on G gives the Lie bracket on $\mathcal{B} < \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$. In standard terminology, this means that $(G, (-1)\{-,-\}_G)$ and $(B, \{-,-\}_B)$ form a pair of mutually dual Poisson Lie groups. The dual group of $(G, \{-,-\}_G)$ is obtained from $(B, \{-,-\}_B)$ by keeping the Poisson structure on the manifold B but replacing the group product by its opposite, defined by $b_1 \star b_2 = b_2 b_1$, which also changes the corresponding Lie bracket on $\mathcal{B}$ to its opposite.*

3.2. **The integrable master system of free motion.** By the projection

$$\pi_2 : \mathbb{M} \to \mathfrak{P}, \quad \pi_2(g,L) = L, \tag{3.21}$$

we can pull-back the elements of $C^\infty(\mathfrak{P})^G$ to $\mathbb{M}$ (2.17). Since π_2 is a Poisson map, this yields Poisson commuting Hamiltonians on $\mathbb{M}$. We next describe the flows and the constants of motion for these Hamiltonians.

Proposition 3.4. *Let $\mathcal{H} = \pi_2^*(\phi)$ for a function $\phi \in C^\infty(\mathfrak{P})^G$ and pick an initial value $(g(0), L(0)) \in \mathbb{M}$. The corresponding integral curve of the Hamiltonian vector field of $\mathcal{H}$, defined by means of $\{-,-\}_{\mathbb{M}}$ (3.19), is provided by*

$$(g(t), L(t)) = (\exp(t\mathcal{D}\phi(L(0)))\, g(0), L(0)). \tag{3.22}$$

The map $\Psi : \mathbb{M} \to \mathfrak{P} \times \mathfrak{P}$ defined by

$$\Psi(g,L) := (\tilde{L}, L) \quad \textit{with} \quad \tilde{L} := g^{-1}Lg \tag{3.23}$$

is constant along the integral curves (3.22). The map Ψ is Poisson with respect to (3.19) and the direct product Poisson structure on $\mathfrak{P} \times \mathfrak{P}$ obtained from $\{-,-\}_{\mathfrak{P}}$ (3.16) on the second $\mathfrak{P}$ factor and its opposite (multiple by -1) on the first $\mathfrak{P}$ factor.[3]

[3] When we wish to emphasize its Poisson structure, we denote this space as $\mathfrak{P}_- \times \mathfrak{P}$.

Proof. For the Hamiltonian $\mathcal{H}$ at hand, $\mathcal{D}_1\mathcal{H}(g,L)=0$ and $\mathcal{D}_2\mathcal{H}(g,L)=\mathcal{D}\phi(L)\in\mathcal{G}$. Therefore we have

$$\{\mathcal{F},\mathcal{H}\}_{\mathbb{M}}(g,L)=\langle\mathcal{D}_1\mathcal{F}(g,L),\mathcal{D}\phi(L)\rangle_{\mathbb{I}},\quad \forall\mathcal{F}\in C^\infty(\mathbb{M}). \tag{3.24}$$

Hence, $\mathcal{H}$ generates the evolution equation

$$\dot g=(\mathcal{D}\phi(L))g,\qquad \dot L=0, \tag{3.25}$$

which is solved by (3.22). The statement that $\tilde L$ is constant along the flows is then verified by using that $[L,\mathcal{D}\phi(L)]=0$ for all $\phi\in C^\infty(\mathfrak{P})^G$.

To see the Poisson map property of Ψ, consider the diffeomorphism

$$m:=m_2\circ m_1: M\to\mathbb{M}, \tag{3.26}$$

based on (2.20) and (2.21). From (2.18), we have

$$g_R^{-1}b_R=b_L^{-1}g_L,\qquad \forall K\in M. \tag{3.27}$$

Using also (2.16), this implies the equality

$$\Psi\circ m=\left(\nu\circ(\Lambda_L)^{-1},\nu\circ\Lambda_R\right). \tag{3.28}$$

The Poisson property of Ψ is then a consequence of the facts that $(\Lambda_L,\Lambda_R): M\to B\times B$ is a Poisson map, where $B\times B$ carries the product Poisson structure with $\{-,-\}_B$ on the two copies, and that taking the inverse is an anti-Poisson map on every Poisson–Lie group. □

It is worth noting that the map Ψ is not surjective and its image

$$\mathfrak{C}:=\Psi(\mathbb{M})\subset\mathfrak{P}\times\mathfrak{P} \tag{3.29}$$

is *not* a smooth manifold in any natural way. However, the dense subset $\mathfrak{C}_{\mathrm{reg}}\subset\mathfrak{C}$, given by

$$\mathfrak{C}_{\mathrm{reg}}:=\Psi(\pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}})),\qquad \pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}})=G\times\mathfrak{P}^{\mathrm{reg}}, \tag{3.30}$$

is an embedded submanifold of $\mathfrak{P}^{\mathrm{reg}}\times\mathfrak{P}^{\mathrm{reg}}$ of co-dimension r. (Recall that $\mathfrak{P}^{\mathrm{reg}}$ contains those elements of $\mathfrak{P}$ whose isotropy groups in G are maximal tori.) In fact, $\mathfrak{C}_{\mathrm{reg}}$ is also a Poisson submanifold of $\mathfrak{P}_-^{\mathrm{reg}}\times\mathfrak{P}^{\mathrm{reg}}$ since it can presented as the intersection of $\mathfrak{P}^{\mathrm{reg}}\times\mathfrak{P}^{\mathrm{reg}}$ with the joint zero set of Casimir functions $F_i\in C^\infty(\mathfrak{P}_-\times\mathfrak{P})$ of the form

$$F_i(\mathcal{L}_1,\mathcal{L}_2)=C_i(\mathcal{L}_1)-C_i(\mathcal{L}_2),\quad \forall(\mathcal{L}_1,\mathcal{L}_2)\in\mathfrak{P}\times\mathfrak{P}, \tag{3.31}$$

where the differentials of the functions $C_i\in C^\infty(\mathfrak{P})^G$ $(i=1,\dots,r)$ span an r-dimensional subspace of the cotangent space of at every point of $\mathfrak{P}^{\mathrm{reg}}$. For example, one may obtain the C_i out of independent invariant polynomials on $\mathcal{G}$ using the exponential parametrization, $\mathfrak{P}=\exp(\mathrm{i}\mathcal{G})$.

Next, we verify an important consequence of Proposition 3.4.

Corollary 3.5. *The two subrings of $C^\infty(\mathbb{M})$ defined by*

$$\mathfrak{H}:=\pi_2^*\left(C^\infty(\mathfrak{P})^G\right)\quad \text{and}\quad \mathfrak{F}:=\Psi^*\left(C^\infty(\mathfrak{P}_-\times\mathfrak{P})\right) \tag{3.32}$$

engender a degenerate integrable system on the symplectic Poisson manifold $(\mathbb{M},\{-,-\}_{\mathbb{M}})$. The rank of this integrable system is equal to the rank $r=\dim(\mathcal{G}_0)$ of the Lie algebra $\mathcal{G}$.

Proof. The elements of $\mathfrak{F}$ are constant along the flows of the elements of $\mathfrak{H}$, because Ψ is constant along those flows. Since Ψ is a Poisson map, $\mathfrak{F}\subset C^\infty(\mathbb{M})$ is a Poisson subalgebra, and we only have to establish the functional dimensions of $\mathfrak{H}$ and $\mathfrak{F}$. To this end, let us denote $r:=\dim(\mathcal{G}_0)$, and note that the exterior derivatives of the elements of $C^\infty(\mathfrak{P})^G$ span an r-dimensional space at every point $L\in\mathfrak{P}^{\mathrm{reg}}$. Thus, the same is true for their π_2 pullbacks, at every point of $\pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}})$, which is a dense open submanifold of $\mathbb{M}$. Hence, the functional dimension of $\mathfrak{H}$ is r.

One can verify by an easy inspection that the derivative $D\Psi$ has constant rank, equal to $\dim(\mathbb{M})-r$, at every point of $G\times\mathfrak{P}^{\mathrm{reg}}$. As a result, the transpose $(D\Psi)^*$ satisfies

$$\dim\left(\mathrm{Im}\left(D\Psi(g,L)^*\right)\right)=\dim(\mathbb{M})-r,\qquad \forall(g,L)\in G\times\mathfrak{P}^{\mathrm{reg}}. \tag{3.33}$$

This implies immediately that $\mathfrak{F}$ has functional dimension $\dim(\mathbb{M})-r$. If $\mu_2:\mathfrak{P}\times\mathfrak{P}$ is the projection onto the second factor, then $\pi_2=\mu_2\circ\Psi$ (with π_2 in (3.21)). Thus, $\mathfrak{H}\subset\mathfrak{F}$, which completes the proof. □

Remark 3.6. *We refer to the system of Corollary 3.5 as the* integrable master system *of free motion on the Heisenberg double. The presentation of the Heisenberg double via its model $\mathbb{M}=G\times\exp(\mathrm{i}\mathcal{G})$ highlights the close analogy with the standard degenerate integrable system on the cotangent bundle $T^*G\simeq G\times\mathcal{G}$ associated with the invariant functions $C^\infty(\mathcal{G})^G$. Of course, we can transfer the master system to the models M and $\mathfrak{M}$ (2.17) by means of the diffeomorhisms $m: M\to\mathbb{M}$ and $m_2:\mathfrak{M}\to\mathbb{M}$. We shall make use of all three models in what follows.*

4. Hamiltonian reduction of the master system

We consider the reduction of the master system relying on an action of G on the Heisenberg double. The degenerate integrability of the reduced system will be proved after restriction to a dense open subset of the reduced phase space. It will be convenient to start with the model M in the first subsection, but utilize the model $\mathbb{M}$ for the proof of reduced integrability given in the second subsection.

4.1. **Actions of G and Hamiltonian reduction.** We begin by recalling the concept of Poisson–Lie moment map for G-actions [39, 40, 63], adapted to our case. Let $(\mathcal{M}, \{-,-\})$ be a Poisson manifold and $\Lambda : \mathcal{M} \to B$ a Poisson map with respect to the Poisson bracket $\{-,-\}_B$ (3.4) on the target space. Such a map can be used to generate an (infinitesimal) Poisson action of the group $(G, \{-,-\}_G)$ (3.3) on $\mathcal{M}$. This works by associating a vector field $X_{\mathcal{M}}$ on $\mathcal{M}$, i.e. a derivation of $C^\infty(\mathcal{M})$, to every $X \in \mathcal{G}$ via the following formula:

$$df(X_{\mathcal{M}}) := -\langle X, \{\Lambda, f\}_{\mathcal{M}} \Lambda^{-1} \rangle_{\mathbb{I}}, \qquad \forall f \in C^\infty(\mathcal{M}). \tag{4.1}$$

To explain the meaning of this formula, note that, for any $x_0 \in \mathcal{M}$, we have

$$\{\Lambda, f\}_{\mathcal{M}}(x_0) := \left.\frac{d}{dt}\right|_{t=0} \Lambda(x(t)), \tag{4.2}$$

where $x(t)$ is the integral curve of the Hamiltonian vector field of f satisfying $x(0) = x_0$. This yields an element of $T_{\Lambda(x_0)}B$, which is then translated into $\mathcal{B} = T_e B$ via right multiplication by $\Lambda(x_0)^{-1}$. According to the general theory [39, 40], the so-obtained map $\mathcal{G} \ni X \mapsto X_{\mathcal{M}} \in \mathrm{Vect}(M)$ is an antihomomorphism satisfying the key relation (2.33). If this $\mathcal{G}$-action can be integrated to an action of G, then it is automatically a Poisson action. The map Λ is called the Poisson–Lie moment map for the pertinent action.[4] It enjoys G-equivariance with respect to the dressing action (2.23) of G on B and the action that it generates on $\mathcal{M}$. The construction can be applied in two ways. Either one starts with a Poisson action and searches for a corresponding (equivariant) moment map, or one starts with a Poisson map Λ and looks for a G-action that integrates the vector fields $X_{\mathcal{M}}$. Since G is connected and simply connected, a G-action results from the infinitesimal action whenever the vector fields $X_{\mathcal{M}}$ are complete. As an example, one may check that the dressing action (2.23) is a Poisson action of $(G, \{-,-\}_G)$ on $(B, \{-,-\}_B)$ with the identity map from B to B being the moment map.

The Heisenberg double $(M, \{-,-\}_+)$ supports three natural Poisson maps into the Poisson–Lie group $(B, \{-,-\}_B)$. These serve as moment maps generating corresponding Poisson actions of $(G, \{-,-\}_G)$ on M. In fact, Λ_L and Λ_R (2.19) are the moment maps for the G-actions given by left and right multiplications of the elements of $M = G^{\mathbb{R}}_{\mathbb{C}}$ by the elements of the subgroup G, and their pointwise product

$$\Lambda := \Lambda_L \Lambda_R : M \to B \tag{4.3}$$

is the moment map for the so-called quasi-adjoint action of G on M. As was shown by Klimčík [36], the moment map (4.3) generates a global G-action whose action map is given explicitly by

$$\mathcal{A}^M : G \times M \to M, \qquad \mathcal{A}^M_\eta(K) := \mathcal{A}^M(\eta, K) = \eta K \Xi_R(\eta \Lambda_L(K)). \tag{4.4}$$

The map Λ is equivariant with respect to this G-action on M and the dressing action on B. Since the center $Z(G)$ of G is contained in the center of $G^{\mathbb{C}}_{\mathbb{R}}$, we obtain the equality $\Xi_R(\eta\Lambda_L(K)) = \eta^{-1}$ for $\eta \in Z(G)$. By using this, it is easily seen that $Z(G)$ acts trivially, and the action of G descends to an effective action of adjoint group $\bar{G} := G/Z(G)$ on M. Since we have a Poisson action, i.e. $\mathcal{A}^M$ is a Poisson map with respect to the product of the Poisson structures on G and on M, the ring of G-invariant functions, $C^\infty(M)^G$, is closed under the Poisson bracket on M. The same is true for the functions of the moment map; and the Poisson algebras

$$(C^\infty(M)^G, \{-,-\}_+) \quad \text{and} \quad (\Lambda^* C^\infty(B), \{-,-\}_+) \tag{4.5}$$

are the centralizers of each other in $(M, \{-,-\}_+)$. If one defines the reduced phase space by

$$M^{\mathrm{red}} := M/G, \tag{4.6}$$

then the identification $C^\infty(M^{\mathrm{red}}) \equiv C^\infty(M)^G$ equips $C^\infty(M^{\mathrm{red}})$ with a Poisson bracket. In this way, one obtains the reduced Poisson space $(M^{\mathrm{red}}, \{-,-\}_+^{\mathrm{red}})$. The pullbacks by Λ of the dressing invariant functions on B engender Casimir functions on the reduced phase space, because

$$\Lambda^*(C^\infty(B)^G) \subset C^\infty(M)^G. \tag{4.7}$$

[4] The equivalence of our moment map condition (4.1) to the original one introduced by Lu [39, 40] follows from Remark 3.3, since the right-invariant Maurer–Cartan 1-form $(db)b^{-1}$ on the group B, which features in (4.1), becomes the left-invariant Maurer–Cartan form $b^{-1} \star db$ on B equipped with the opposite multiplication, $b_1 \star b_2 = b_2 b_1$. Agreement between (4.1) with its counterpart (5.20) in [63] is seen by additionally noting that $X_{\mathcal{M}}$ corresponds to $-\widehat{X}$ used in [63].

The joint level surfaces of these Casimir functions are unions of symplectic leaves. The reduced phase space M^{red} is not a smooth manifold, but is a disjoint union of smooth strata. However, like in the smooth case, the Hamiltonian vector fields of the smooth functions on M^{red} can still be obtained as projections of the Hamiltonian vector fields of the corresponding elements of $C^\infty(M)^G$.

We are mostly interested in the 'big cell' of M^{red} that results by restriction to the principal orbit type [12, 43, 50] for the G-action. The principal orbits fill the dense open submanifold

$$M_* := \{x \in M \mid G_x = Z(G)\}, \tag{4.8}$$

where G_x denotes the isotropy group of the point x. Three important features of the restriction to M_* are as follows. First,

$$M_*^{\text{red}} := M_*/G \tag{4.9}$$

is a smooth manifold, and is a connected, dense open subset of M^{red}. Second, the restriction of the moment map to M_* is a submersion, i.e., its derivative $D\Lambda(x) : T_x M_* \to T_{\Lambda(x)} B$ is surjective at every $x \in M_*$. Third, M_* is invariant with respect to the Hamiltonian flows of all the elements of $C^\infty(M)^G$. These statements are immediate consequences of well known general results. For example, the third property is a consequence of the fact that the flow φ_t^H of any invariant function $H \in C^\infty(M)^G$ is equivariant,

$$\mathcal{A}_\eta^M \circ \varphi_t^H = \varphi_t^H \circ \mathcal{A}_\eta^M, \qquad \forall \eta \in G, \tag{4.10}$$

which implies that the isotropy group $G_{\varphi_t^H(x)}$ is constant in t for every $x \in M$.

In order to transfer the action $\mathcal{A}^M$ (4.4) to the alternative models $\mathfrak{M}$ and $\mathbb{M}$ of the Heisenberg double (2.17), we use the relations (2.18) and (3.27) that lead to the identities

$$\Lambda_L(K) = \Lambda_L(b_L g_R^{-1}) = \Lambda_R(b_R^{-1} g_R)^{-1} = \Lambda_L(g_R^{-1} b_R)^{-1} =: \beta_L(g_R, b_R). \tag{4.11}$$

The last equality is the definition of the map $\beta_L : G \times B \to B$, which can also be written as

$$\beta_L(g, b) = (\text{Dress}_{g^{-1}}(b))^{-1}, \qquad \forall (g, b) \in G \times B. \tag{4.12}$$

Combining this with the diffeomorphisms m_1 (2.20) and m_2 (2.21), we see that the Poisson action $\mathcal{A}^M$ acquires the following form in terms of the models $\mathfrak{M}$ and $\mathbb{M}$

$$\mathcal{A}_\eta^{\mathfrak{M}}(g, b) = \left(\Xi_R(\eta\beta_L(g,b))^{-1} g \Xi_R(\eta\beta_L(g,b)), \text{Dress}_{\Xi_R(\eta\beta_L(g,b))^{-1}}(b)\right), \tag{4.13}$$

and

$$\mathcal{A}_\eta^{\mathbb{M}}(g, L) = \left(\tilde\eta g \tilde\eta^{-1}, \tilde\eta L \tilde\eta^{-1}\right), \quad \text{with} \quad \tilde\eta = \Xi_R(\eta\beta_L(g, \nu^{-1}(L)))^{-1}, \tag{4.14}$$

where we applied the inverse of the diffeomorphism ν (2.16). For any fixed (g, b), the map

$$\eta \mapsto \Xi_R(\eta\beta_L(g,b))^{-1} \tag{4.15}$$

yields a diffeomorphism of G. Consequently, the Poisson actions (4.13) and (4.14) are orbit equivalent (have the same orbits) as the simpler G-actions given on $\mathfrak{M}$ and on $\mathbb{M}$ by the formulae

$$A_\eta^{\mathfrak{M}}(g, b) := (\eta g \eta^{-1}, \text{Dress}_\eta(b)), \qquad A_\eta^{\mathbb{M}}(g, L) := (\eta g \eta^{-1}, \eta L \eta^{-1}), \tag{4.16}$$

for all $\eta \in G$, $(g, b) \in \mathfrak{M}$ and $(g, L) \in \mathbb{M}$. These simpler actions are *not* Poisson action of G, but for taking the quotients of the respective model phase spaces they can be used in the same way as their parent Poisson actions.

In Proposition 3.4, we introduced the Poisson space $\mathfrak{P}_- \times \mathfrak{P}$, which is $\mathfrak{P} \times \mathfrak{P}$ equipped with the Poisson bracket $(-1)\{-,-\}_{\mathfrak{P}} \times \{-,-\}_{\mathfrak{P}}$ with (3.16). We may write the elements $(\mathcal{L}_1, \mathcal{L}_2) \in \mathfrak{P}_- \times \mathfrak{P}$ in the form

$$(\mathcal{L}_1, \mathcal{L}_2) = (\nu(b_1^{-1}), \nu(b_2)) \quad \text{with} \quad (b_1, b_2) \in B \times B, \tag{4.17}$$

and then we obtain a Poisson map $\hat\Lambda : \mathfrak{P}_- \times \mathfrak{P} \to B$ by the definition

$$\hat\Lambda : (\mathcal{L}_1, \mathcal{L}_2) \mapsto b_1 b_2. \tag{4.18}$$

As a moment map, $\hat\Lambda$ generates a Poisson action of G. The action map $\hat{\mathcal{A}} : G \times (\mathfrak{P}_- \times \mathfrak{P}) \to \mathfrak{P}_- \times \mathfrak{P}$ operates for $\eta \in G$ by

$$\hat{\mathcal{A}}_\eta : (\mathcal{L}_1, \mathcal{L}_2) \mapsto \left(\Xi_R(\eta b_1)^{-1} \mathcal{L}_1 \Xi_R(\eta b_1), \Xi_R(\eta b_1)^{-1} \mathcal{L}_2 \Xi_R(\eta b_1)\right), \tag{4.19}$$

using $b_1 = (\nu^{-1}(\mathcal{L}_1))^{-1}$. It is an instructive exercise to verify this statement, which we do in Appendix A. The Poisson action (4.19) possesses the same orbits as the alternative G-action having the action map

$$\hat A_\eta : (\mathcal{L}_1, \mathcal{L}_2) \mapsto (\eta \mathcal{L}_1 \eta^{-1}, \eta \mathcal{L}_2 \eta^{-1}). \tag{4.20}$$

The Poisson map $\Psi : \mathbb{M} \to \mathfrak{P}_- \times \mathfrak{P}$ (3.23) relates the relevant moment maps according to

$$\Lambda \circ m^{-1} = \hat\Lambda \circ \Psi, \tag{4.21}$$

where we used Λ in (4.3) and $m : M \to \mathbb{M}$ in (3.26). It also satisfies the equivariance properties

$$\Psi \circ \mathcal{A}^{\mathbb{M}}_\eta = \hat{\mathcal{A}}_\eta \circ \Psi \quad \text{and} \quad \Psi \circ A^{\mathbb{M}}_\eta = \hat{A}_\eta \circ \Psi, \quad \forall \eta \in G. \tag{4.22}$$

Our construction implies that

$$C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G \subset C^\infty(\mathfrak{P}_- \times \mathfrak{P}) \tag{4.23}$$

is a Poisson subalgebra, and this entails that

$$\mathfrak{F}^G := \Psi^*(C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G) \subset C^\infty(\mathbb{M})^G \tag{4.24}$$

is also a Poisson subalgebra.

Remark 4.1. *Let us take the opportunity to clarify a potentially confusing point that occurs in our earlier paper [21]. Namely, it was verified in Appendix C of [21] that the identity $A^{\mathbb{M}}_\eta \circ \varphi^H_t = \varphi^H_t \circ A^{\mathbb{M}}_\eta$ does not hold for certain G-invariant Hamiltonians on $\mathbb{M}$. This is not surprising since $A^{\mathbb{M}}_\eta$ (4.16) is not a Poisson action (in fact, the failure of the identity proves this), and the analogous identity holds if one uses the original action $\mathcal{A}^{\mathbb{M}}_\eta$ (4.14).*

4.2. **Reduced integrability.** Now we use the model $\mathbb{M}$ of the Heisenberg double (2.17), whereby M_* (4.8) and $M^{\rm red}_*$ (4.9) get replaced by $\mathbb{M}_*$ and $\mathbb{M}^{\rm red}_*$, respectively. That is, with the map m (3.26), we have

$$\mathbb{M}_* = m(M_*), \qquad \mathbb{M}^{\rm red}_* = \mathbb{M}_*/G. \tag{4.25}$$

In Section 3.2, we described the master integrable system $(\mathbb{M}, \{-,-\}_{\mathbb{M}}, \mathfrak{H}, \mathfrak{F})$, whose Hamiltonians and constants of motion (3.32) were constructed relying on the Poisson map Ψ (3.23). Eventually, we shall demonstrate that the quadruple $(\mathbb{M}, \{-,-\}_{\mathbb{M}}, \mathfrak{H}, \mathfrak{F}^G)$, with $\mathfrak{F}^G$ in (4.24), engenders a degenerate integrable system on the Poisson manifold $\mathbb{M}^{\rm red}_*$. However, it will be advantageous to first deal with a restriction of the reduced system on a certain dense open subset of $\mathbb{M}^{\rm red}_*$, which will be found to satisfy stronger conditions than those required by Definition 2.3.

Let $\mathit{\Lambda} : \mathbb{M} \to B$ be the moment map Λ (4.3) transferred to $\mathbb{M}$ and $\mathit{\Lambda}_*$ its restriction to $\mathbb{M}_*$, i.e.,

$$\mathit{\Lambda} = \Lambda \circ m^{-1}, \qquad \mathit{\Lambda}_* = \mathit{\Lambda}_{|\mathbb{M}_*}. \tag{4.26}$$

Lemma 4.2. *The inverse image $\mathit{\Lambda}_*^{-1}(B^{\rm reg})$ is a dense open subset of $\mathbb{M}_*$. Then,*

$$\mathit{\Lambda}_*^{-1}(B^{\rm reg})/G \subset \mathbb{M}^{\rm red}_* \tag{4.27}$$

is a dense open subset, which consists of symplectic leaves of co-dimension $r = \dim(\mathcal{G}_0)$.

Proof. The map $\mathit{\Lambda}_*$ is continuous. Since the action of $G/Z(G)$ is free on $\mathbb{M}_*$, $\mathit{\Lambda}_* : \mathbb{M}_* \to B$ is a submersion, and thus it is also an open map. The inverse image of a dense set under an open map is dense, and the inverse image of an open set under a continuous map is open. Therefore, $\mathit{\Lambda}_*^{-1}(B^{\rm reg}) \subset \mathbb{M}_*$ is dense and open.

Let $(\mathbb{M}^{\rm red}_*, \{-,-\}^{\rm red}_*)$ denote the reduced Poisson manifold obtained by taking the quotient of $\mathbb{M}_*$ by the action of G. The general reduction theory [40, 50, 63] says that the symplectic leaves of this Poisson manifold are the connected components of the sets of the form

$$\mathit{\Lambda}_*^{-1}(\mathcal{O}_B)/G, \tag{4.28}$$

where $\mathcal{O}_B \subset B$ is a dressing orbit contained in $\mathit{\Lambda}_*(\mathbb{M}_*)$. The dressing orbits of maximal dimension are those that lie in $B^{\rm reg}$, and their co-dimension is r.

The symplectic leaves (4.28) can also be identified as the connected components of the level surfaces of the Casimir functions on $\mathbb{M}^{\rm red}_*$ that arise from $\mathit{\Lambda}^*(C^\infty(B)^G)$ restricted on $\mathbb{M}_*$. By using that both $\mathit{\Lambda}_*$ and the projection $\pi : \mathbb{M}_* \to \mathbb{M}^{\rm red}_*$ are submersions, it is easily seen that the differentials of the Casimir functions span an r-dimensional space at every point of $\mathit{\Lambda}_*^{-1}(B^{\rm reg})/G$. Hence, the dressing orbits lying in $\mathit{\Lambda}_*(\mathbb{M}_*) \cap B^{\rm reg}$ yield symplectic leaves of co-dimension r, which are the leaves of maximal dimension. □

Remark 4.3. *It is known [19] that $\mathit{\Lambda} : \mathbb{M} \to B$ is a surjective map. Because $\mathit{\Lambda}$ is continuous and its restriction $\mathit{\Lambda}_*$ (4.26) is an open map, we see that*

$$B^{\rm reg} \cap \mathit{\Lambda}_*(\mathbb{M}_*) \subset B \tag{4.29}$$

is dense and open. We suspect that $B^{\rm reg}$ is contained in $\mathit{\Lambda}_*(\mathbb{M}_*)$, but have not proved this.

We previously introduced the 'space of constants of motion' $\mathfrak{C}$ (3.29) and its dense subset $\mathfrak{C}_{\rm reg}$ (3.30), which is a Poisson submanifold of $\mathfrak{P}^{\rm reg}_- \times \mathfrak{P}^{\rm reg}$. Explicitly, $\mathfrak{C}_{\rm reg}$ consists of the pairs $(\mathcal{L}_1, \mathcal{L}_2) \in \mathfrak{P}^{\rm reg} \times \mathfrak{P}^{\rm reg}$ for which $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to the same G-orbit:

$$\mathfrak{C}_{\rm reg} := \{(g^{-1}Lg, L) \mid L \in \mathfrak{P}^{\rm reg},\, g \in G\}. \tag{4.30}$$

The group G acts by componentwise conjugations on $\mathfrak{P}^{\mathrm{reg}} \times \mathfrak{P}^{\mathrm{reg}}$, and this restricts to the submanifold $\mathfrak{C}_{\mathrm{reg}}$. Now we introduce $\mathfrak{C}_* \subset \mathfrak{C}_{\mathrm{reg}}$, which by definition is the dense open subset given by the principal orbit type for the G-action on $\mathfrak{C}_{\mathrm{reg}}$. It is easily seen that

$$\mathfrak{C}_* = \{(g^{-1}Lg, L) \in \mathfrak{C}_{\mathrm{reg}} \mid G_{(g^{-1}Lg,L)} = Z(G)\}, \tag{4.31}$$

where $G_{(g^{-1}Lg,L)}$ is the isotropy group. We observe that $C^\infty(\mathfrak{C}_*)^G$ gives rise to a Poisson structure on the smooth manifold

$$\mathfrak{C}_*^{\mathrm{red}} := \mathfrak{C}_*/G. \tag{4.32}$$

Then, we define the following subset of $\mathbb{M}$:

$$\mathbb{M}_{**} := \Psi^{-1}(\mathfrak{C}_*). \tag{4.33}$$

It is clear from the G-equivariance of Ψ that $\mathbb{M}_{**}$ is mapped to itself by the G-action. After some preparation, our goal is to show that the restriction of the master system of free motion on $\mathbb{M}_{**}$ descends to a degenerate integrable system on the corresponding quotient space.

Lemma 4.4. *The inverse image $\mathbb{M}_{**}$ (4.33) is a dense open subset of $\mathbb{M}_* \cap \pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}})$.*

Proof. The G-equivariance of the map Ψ (3.23) implies that $G_x < G_{\Psi(x)}$ holds for all $x \in \mathbb{M}$. It follows that $G_x = Z(G)$ for all $x \in \mathbb{M}_{**}$, i.e.,

$$\mathbb{M}_{**} \subset \mathbb{M}_* \cap \pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}}). \tag{4.34}$$

We know from the proof of Corollary 3.5 that the restricted map

$$\Psi : \pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}}) \to \mathfrak{C}_{\mathrm{reg}} \tag{4.35}$$

is a surjective submersion. In particular, it is both continuous and open, and therefore the inverse image of the dense open subset $\mathfrak{C}_* \subset \mathfrak{C}_{\mathrm{reg}}$ enjoys the claimed property. □

Remark 4.5. *It is an easy consequence of what we proved that in the chain*

$$\mathbb{M}_{**} \subset \mathbb{M}_* \subset \mathbb{M} \tag{4.36}$$

every subset is dense and open in the one that contains it, including $\mathbb{M}_{**} \subset \mathbb{M}$. Now we explain a few properties of these sets, including that $\mathbb{M}_{**} \subset \mathbb{M}_*$ is a proper subset.

First, by choosing both g and L to be the unit element of $G_{\mathbb{R}}^{\mathbb{C}}$, we see that $\mathbb{M}_*$ is a proper subset of $\mathbb{M}$. Next, let us recall that the center $Z(G)$ is the intersection of all maximal tori of G, and for a fixed maximal torus G_0 one can find (see e.g. [42]) another one, G_0', such that

$$G_0 \cap G_0' = Z(G). \tag{4.37}$$

Clearly, one can choose $(\tilde{L}, L) = \Psi(g, L)$ in such way that the isotropy subgroups with respect to the conjugation action of G on $\mathfrak{P}$ are $G_L = G_0$ and $G_{\tilde{L}} = G_0'$. Then, the isotropy group $G_{(\tilde{L},L)}$ with respect to the diagonal conjugation action of G is $Z(G)$. Consequently, $(\tilde{L}, L) \in \mathfrak{C}_*$ and $(g, L) \in \mathbb{M}_{**}$.

For concreteness, consider $G = \mathrm{SU}(n)$ and choose a pair (g, L), where L is a diagonal matrix with distinct positive eigenvalues and g is a multiple of the matrix of cyclic permutation, i.e.,

$$g = C(E_{n,1} + \sum_{i=1}^{n-1} E_{i,i+1}), \tag{4.38}$$

with a constant C such that $\det(g) = 1$. In this case, one can verify that $G_{(g,L)} = Z(G)$, while $G_{\Psi(g,L)}$ is the standard maximal torus of G. This implies that $(g, L) \in \mathbb{M}_* \setminus \mathbb{M}_{**}$. We can generalize this example to other groups by picking a regular L whose isotropy group is the maximal torus G_0, and taking $g \in G$ to be a representative of a Coxeter element of the Weyl group with respect to G_0. By using that [42] the fixed point set of the action of the Coxeter element on G_0 is the center $Z(G)$, it is easy to verify that for this choice $G_{\Psi(g,L)} = Z(G)$ and $(g, L) \in \mathbb{M}_* \setminus \mathbb{M}_{**}$.

Consider an arbitrary pair (g_0', L_0) for which $g_0' \in G_0'$ an $L_0 \in \exp(\mathrm{i}\mathcal{G}_0)$ are regular elements, i.e., their isotropy groups in G are G_0' and G_0, respectively, which are subject to (4.37). The G-orbit through (g_0', L_0) belongs to $\mathbb{M}_*$, and from this one sees that

$$G^{\mathrm{reg}} \subset \pi_1(\mathbb{M}_*) \quad \text{and} \quad \mathfrak{P}^{\mathrm{reg}} \subset \pi_2(\mathbb{M}_*), \tag{4.39}$$

since the regular elements of G (and $\mathfrak{P}$) are those elements whose isotropy subgroups under the conjugation action of G are maximal tori, and all maximal tori are conjugate to each other. We also note that $\pi_2(\mathbb{M}_{**}) = \mathfrak{P}^{\mathrm{reg}}$, but it is not clear at present if in the relations (4.39) one has equalities or not.

The Abelian Poisson algebra $\mathfrak{H}$ (3.32) gives rise to the reduced Abelian Poisson algebra $\mathfrak{H}_{\rm red}$ of Hamiltonians defined on $\mathbb{M}^{\rm red} = \mathbb{M}/G$. Similarly, $\mathfrak{F}^G$ (4.24) descends to a Poisson algebra of functions on $\mathbb{M}^{\rm red}$, which we denote $\mathfrak{F}_{\rm red}$ and call *the Poisson algebra of reduced constants of motion.* Resulting from (4.36), we have the chain of dense open subsets

$$\mathbb{M}^{\rm red}_{**} \subset \mathbb{M}^{\rm red}_{*} \subset \mathbb{M}^{\rm red}, \tag{4.40}$$

and we let $\mathfrak{H}^{**}_{\rm red}$ and $\mathfrak{F}^{**}_{\rm red}$ denote the restrictions of $\mathfrak{H}_{\rm red}$ and $\mathfrak{F}_{\rm red}$ on $\mathbb{M}^{\rm red}_{**}$, respectively. Analogously, we denote by $\mathfrak{H}^{*}_{\rm red}$ and $\mathfrak{F}^{*}_{\rm red}$ the corresponding restrictions on $\mathbb{M}^{\rm red}_{*}$. These Poisson algebras enjoy the natural isomorphisms

$$\mathfrak{H}^{**}_{\rm red} \simeq \mathfrak{H}_{|\mathbb{M}_{**}}, \quad \mathfrak{F}^{**}_{\rm red} \simeq \mathfrak{F}^G{}_{|\mathbb{M}_{**}} \quad \text{and} \quad \mathfrak{H}^{*}_{\rm red} \simeq \mathfrak{H}_{|\mathbb{M}_{*}}, \quad \mathfrak{F}^{*}_{\rm red} \simeq \mathfrak{F}^G{}_{|\mathbb{M}_{*}}. \tag{4.41}$$

Since $\mathbb{M}_{**}$ was defined by placing constraints on the constants of motion, the flows of all Hamiltonians $\mathcal{H} \in \mathfrak{H}$ (3.32) preserve this dense open submanifold of $\mathbb{M}_*$. Taking into account that $\mathbb{M}_{**}$ is also preserved by the G-action, we can consider the simultaneous reduction of the pertinent Hamiltonian systems after restriction on $\mathbb{M}_{**}$. This leads to the *'restricted reduced system'* denoted

$$(\mathbb{M}^{\rm red}_{**}, \{-,-\}^{\rm red}_{**}, \mathfrak{H}^{**}_{\rm red}), \tag{4.42}$$

where the Poisson structure on $\mathbb{M}^{\rm red}_{**} = \mathbb{M}_{**}/G$ results from the identification

$$(C^\infty(\mathbb{M}^{\rm red}_{**}), \{-,-\}^{\rm red}_{**}) \simeq (C^\infty(\mathbb{M}_{**})^G, \{-,-\}_{\mathbb{M}_{**}}). \tag{4.43}$$

The commutative diagram of maps displayed in Figure 1 will be utilized for proving Theorem 4.6 below, which represents our first main result. Here, p_1 and p_2 are the canonical projections, $\psi := \Psi_{|\mathbb{M}_{**}}$ and

$$\psi_{\rm red} : \mathbb{M}^{\rm red}_{**} \to \mathfrak{C}^{\rm red}_{*} = \mathfrak{C}_*/G \tag{4.44}$$

is induced by the G-equivariance of the map ψ. All these maps are *smooth, surjective submersions* and are *Poisson maps.* In fact, p_1 and p_2 are projections of principal fiber bundles, with structure group $G/Z(G)$. Since the map $\Psi : \pi_2^{-1}(\mathfrak{P}^{\rm reg}) \to \mathfrak{C}_{\rm reg}$ is a surjective submersion, this property is inherited by its restriction ψ. Then, one sees by tracing the diagram that $\psi_{\rm red}$ is also a smooth submersion. The Poisson property of the maps follows immediately from the definitions. Of course, the Poisson structure on $\mathfrak{C}^{\rm red}_{*}$ is defined by the isomorphism

$$C^\infty(\mathfrak{C}^{\rm red}_{*}) \simeq C^\infty(\mathfrak{C}_*)^G. \tag{4.45}$$

Below, we shall first use the Poisson algebra

$$\mathfrak{F}^\sharp_{\rm red} := \psi^*_{\rm red}(C^\infty(\mathfrak{C}^{\rm red}_{*})). \tag{4.46}$$

Its relation to $\mathfrak{F}^{**}_{\rm red}$ (4.41) will be clarified later (see Lemma 4.8).

$$\begin{tikzcd}
\mathbb{M}_{**} \arrow[r, "\psi"] \arrow[d, "p_1"'] & \mathfrak{C}_* \arrow[d, "p_2"] \\
\mathbb{M}^{\rm red}_{**} \arrow[r, "\psi_{\rm red}"] & \mathfrak{C}^{\rm red}_{*}
\end{tikzcd}$$

FIGURE 1. The sets and maps used in the proof of Theorem 4.6. All sets are smooth Poisson manifolds and all maps are smooth Poisson submersions. $\mathfrak{C}_*$ is the subset of principal orbit type for the G-action on $\mathfrak{C}_{\rm reg} \subset \mathfrak{C}$ (4.30). The map ψ is the restriction of Ψ (3.23) to $\mathbb{M}_{**} = \Psi^{-1}(\mathfrak{C}_*)$, p_1 and p_2 are projections of principal fibre bundles.

The $r = 1$ case, i.e. the case of $G = {\rm SU}(2)$, is excluded in the subsequent theorem, since in that case the reduced system is 'only' Liouville integrable. The proof given below is similar to the proof of an analogous statement[5] concerning Poisson reduction of the cotangent bundle T^*G.

[5]Incidentally, in our work we first considered the Heisenberg double; the reduced integrability for the cotangent bundle was presented in [22] in order to expound the ideas in a simpler context.

Theorem 4.6. *Suppose that $r = \dim(\mathcal{G}_0) \neq 1$. Then, the restricted reduced system (4.42) is a degenerate integrable system of rank r with constants of motion provided by the ring of functions $\mathfrak{F}^\sharp_{\rm red}$ (4.46), that is, the quadruple $(\mathbb{M}^{\rm red}_{**}, \{-,-\}^{\rm red}_{**}, \mathfrak{H}^{**}_{\rm red}, \mathfrak{F}^\sharp_{\rm red})$ satisfies the stipulations of Definition 2.3. The reduced Hamiltonian vector fields associated with $\mathfrak{H}^{**}_{\rm red}$ span an r-dimensional subspace of the tangent space at every point of $\mathbb{M}^{\rm red}_{**}$, and the differentials of the elements of $\mathfrak{F}^\sharp_{\rm red}$ span a co-dimension r subspace of the cotangent space.*

Proof. By the definition of the reduction, every element $\mathcal{H}_{\rm red} \in \mathfrak{H}^{**}_{\rm red}$ obeys the relation

$$\mathcal{H}_{\rm red} \circ p_1 = \mathcal{H}|_{\mathbb{M}_{**}} \quad \text{with some} \quad \mathcal{H} \in \mathfrak{H} = \pi_2^* \left(C^\infty(\mathfrak{P})^G \right), \tag{4.47}$$

and every integral curve $y(t)$ of $\mathcal{H}_{\rm red}$ in $\mathbb{M}^{\rm red}_{**}$ can be presented as the projection $p_1(x(t))$ of an integral curve $x(t)$ of $\mathcal{H}$ in $\mathbb{M}_{**}$. Since the map ψ is constant along $x(t)$, we see from Figure 1 that $\psi_{\rm red}$ is constant along $y(t)$. This implies immediately that the elements $\mathfrak{F}^\sharp_{\rm red}$ (4.46) are constants of motion for every $\mathcal{H}_{\rm red} \in \mathfrak{H}^{**}_{\rm red}$. The fact that $\psi_{\rm red}$ is a Poisson map entails that $\mathfrak{F}^\sharp_{\rm red}$ forms a Poisson subalgebra of $C^\infty(\mathbb{M}^{\rm red}_{**})$. Recalling that $\psi_{\rm red}$ is a submersion, we obtain that the dimension of the span of the differentials of the elements of $\mathfrak{F}^\sharp_{\rm red}$ is equal to $\dim(\mathfrak{C}^{\rm red}_*)$ at every point of $\mathbb{M}^{\rm red}_{**}$. Because $G/Z(G)$ acts freely on $\mathfrak{C}_*$ and on $\mathbb{M}_{**}$, we have

$$\dim(\mathfrak{C}^{\rm red}_*) = \dim(G) - r \quad \text{and} \quad \dim(\mathbb{M}^{\rm red}_{**}) = \dim(G), \tag{4.48}$$

which confirms the statement concerning the differentials of the elements of $\mathfrak{F}^\sharp_{\rm red}$.

Next, we verify the claim about the dimension of the span of the reduced Hamiltonian vector fields. To do this, we pick an arbitrary point $y := p_1(x) \in \mathbb{M}^{\rm red}_{**}$, with some $x = (g, L) \in \mathbb{M}_{**}$. We know that the values of the reduced Hamiltonian vector fields at y result by applying the tangent map $Dp_1(x)$ to the values of the original Hamiltonian vector field at x. It follows from (3.22) that the latter are the tangent vectors of the form

$$(Vg, 0) \in T_x\mathbb{M}_{**} = T_gG \oplus T_L\mathfrak{P}, \tag{4.49}$$

where $V \in \mathcal{G}$ is given by the $\mathcal{G}$-valued derivative of some function $\phi \in C^\infty(\mathfrak{P})^G$ at $L \in \mathfrak{P}$. Since L is a regular element, these derivatives span a maximal Abelian subalgebra of $\mathcal{G}$, and thus the linear space of the tangent vectors (4.49) is r-dimensional. We have to verify that this linear space has zero intersection with $\mathrm{Ker}(Dp_1(x))$, consisting of the elements

$$([W, g], [W, L]) \in T_x\mathbb{M}_{**}, \qquad \forall\, W \in \mathcal{G}. \tag{4.50}$$

Now, if two tangent vectors having the respective forms (4.49) and (4.50) coincide, then so do their images in $T_{(\tilde{L},L)}\mathfrak{C}_*$ obtained by the map $D\psi(x)$. But the image of the tangent vector in (4.49) is zero, while the image of the one in (4.50) is

$$([W, \tilde{L}], [W, L]) \in T_{\psi(x)}\mathfrak{C}_* \subset T_{\tilde{L}}\mathfrak{P} \oplus T_L\mathfrak{P}, \quad \text{with} \quad \tilde{L} = g^{-1}Lg. \tag{4.51}$$

Since $G/Z(G)$ acts freely $\mathfrak{C}_*$, the vector in (4.51) vanishes only for $W = 0$, and then the vector (4.50) also vanishes. In conclusion, the $Dp_1(x)$ image of the tangent vectors in (4.49) has dimension r.

The differentials of the elements of $\mathfrak{H}^{**}_{\rm red}$ span an r-dimensional subspace of $T^*_y\mathbb{M}^{\rm red}_{**}$ at every $y \in \mathbb{M}^{\rm red}_{**}$ since their Hamiltonian vector fields span an r-dimensional subspace of $T_y\mathbb{M}^{\rm red}_{**}$. That is, the functional dimensions of $\mathfrak{H}^{**}_{\rm red}$ and $\mathfrak{H}$ are the same. It is obvious that $\mathfrak{H}^{**}_{\rm red}$ is contained in $\mathfrak{F}^\sharp_{\rm red}$.

Lemma 4.2 implies that a dense open subset of $\mathbb{M}^{\rm red}_{**}$ is filled by symplectic leaves of maximal dimension, which have co-dimension r. If $r \neq 1$, then we have

$$r < \frac{1}{2}\left(\dim(G) - r\right) = \frac{1}{2}(\dim(\mathbb{M}^{\rm red}_{**}) - r). \tag{4.52}$$

This means that r is strictly smaller than half the maximal dimension of the symplectic leaves in $\mathbb{M}^{\rm red}_{**}$, and thus our restricted reduced system satisfies all conditions of Definition 2.3.

In the excluded $r = 1$ case, that is for $G = \mathrm{SU}(2)$, the reduced system is Liouville integrable, but there is no room for degenerate integrability in this case. $\square$

Next, we prove an important consequence of Theorem 4.6.

Corollary 4.7. *The restriction of the system $(\mathbb{M}^{\rm red}_{**}, \{-,-\}^{\rm red}_{**}, \mathfrak{H}^{**}_{\rm red}, \mathfrak{F}^\sharp_{\rm red})$ of Theorem 4.6 to any symplectic leaf of $\mathbb{M}^{\rm red}_{**}$ of co-dimension r (where $r > 1$) is a degenerate integrable system in the sense of Definition 2.2.*

Proof. Choose r functions $C_1, \dots, C_r \in C^\infty(\mathfrak{P})^G$ that are functionally independent at every point of $\mathfrak{P}^{\mathrm{reg}}$. The restrictions of the functions $C_i \circ \Lambda_*$ on $\mathbb{M}_{**}$ descend to Casimir functions $\mathcal{C}_i \in C^\infty(\mathbb{M}^{\mathrm{red}}_{**})$. These functions belong to $\mathfrak{F}^\sharp_{\mathrm{red}}$, and any symplectic leaf of co-dimension r in $\mathbb{M}_{**}$ is (a connected component of) a joint level surface thereof. Now, we consider a symplectic leaf S of co-dimension r and fix an arbitrary point $y \in S$. Then, we can select additional $(\dim(\mathfrak{C}^{\mathrm{red}}_*) - r)$ elements of $\mathfrak{F}^\sharp_{\mathrm{red}}$, say f_a, so that

$$\mathcal{C}_i, f_a, \qquad i = 1, \dots, r, \; a = 1, \dots, \dim(\mathfrak{C}^{\mathrm{red}}_*) - r \tag{4.53}$$

are functionally independent at y. We can also find further r functions, say $z_1, \dots, z_r$, so that the functions

$$\mathcal{C}_i, f_a, z_i \tag{4.54}$$

yield local coordinates on an open set $U \subset \mathbb{M}^{\mathrm{red}}_{**}$, containing y. It follows that the restriction of the functions f_a, z_i to the level surface S of the Casimirs gives local coordinates on S around $y \in S$. Consequently, the differentials of the restrictions on S of the elements of $\mathfrak{F}^\sharp_{\mathrm{red}}$ span a $\dim(S) - r$ dimensional subspace of the cotangent space $T^*_y S$ at every $y \in S$. This is all what we needed to prove, since we know from Theorem 4.6 that the Hamiltonian vector fields of $\mathfrak{H}^{**}_{\mathrm{red}}$ span an r-dimensional space at every $y \in \mathbb{M}^{\mathrm{red}}_{**}$. They are tangent to the every symplectic leaf, and give the Hamiltonian vector fields of $\mathfrak{H}^{**}_{\mathrm{red}}$ restricted onto the leaf. □

Incidentally, for any generating set C_i $(i = 1, \dots, r)$ of $C^\infty(\mathfrak{P})^G$, the functions $\mathcal{H}_i = C_i \circ \pi_2 \in \mathfrak{H}$ are independent at every point of $\pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}}) \subset \mathbb{M}$, and their restrictions on $\mathbb{M}_*$ descend to functions $\mathcal{H}^{\mathrm{red}}_i \in C^\infty(\mathbb{M}^{\mathrm{red}}_*)$, which are independent at every point of $(\mathbb{M}_* \cap \pi_2^{-1}(\mathfrak{P}^{\mathrm{reg}}))/G$.

Finally, we wish to prove the reduced integrability on the 'big cell' $\mathbb{M}^{\mathrm{red}}_*$ of the reduced phase space. We begin by recalling that $\mathfrak{C}_{\mathrm{reg}}$ (4.30) is a regular (embedded) submanifold of $\mathfrak{P}^{\mathrm{reg}} \times \mathfrak{P}^{\mathrm{reg}}$ and $\mathfrak{C}_*$ is a dense open subset of $\mathfrak{C}_{\mathrm{reg}}$. It follows that $\mathfrak{C}_*$ is also a regular submanifold of $\mathfrak{P} \times \mathfrak{P}$. With the tautological embeddding

$$\iota : \mathfrak{C}_* \to \mathfrak{P} \times \mathfrak{P}, \tag{4.55}$$

we obtain

$$\iota^*(C^\infty(\mathfrak{P} \times \mathfrak{P})^G) \subset C^\infty(\mathfrak{C}_*)^G. \tag{4.56}$$

It is easy to see that this is a proper subset, since one can construct[6] smooth invariant functions on $\mathfrak{C}_{\mathrm{reg}}$ that blow up when approaching limit points of $\mathfrak{C}_{\mathrm{reg}}$ lying outside $\mathfrak{P}^{\mathrm{reg}} \times \mathfrak{P}^{\mathrm{reg}}$. The maps in Figure 1 give

$$p_1^*(\mathfrak{F}^{**}_{\mathrm{red}}) = \psi^*(\iota^*(C^\infty(\mathfrak{P} \times \mathfrak{P})^G)) \quad \text{and} \quad p_1^*(\mathfrak{F}^\sharp_{\mathrm{red}}) = \psi^*(C^\infty(\mathfrak{C}_*)^G), \tag{4.57}$$

where we used (4.41) and (4.46). Then, since p_1 and ψ are surjective submersions, we may conclude from (4.56) that

$$\mathfrak{F}^{**}_{\mathrm{red}} \subset \mathfrak{F}^\sharp_{\mathrm{red}} \tag{4.58}$$

is a proper subset. Nevertheless, the following crucial lemma holds.

Lemma 4.8. *At each point $y \in \mathbb{M}^{\mathrm{red}}_{**}$, the differentials of the elements of $\mathfrak{F}^{**}_{\mathrm{red}}$ (4.41) span the same linear subspace of the cotangent space $T^*_y \mathbb{M}^{\mathrm{red}}_{**}$ as do the differentials of the elements of $\mathfrak{F}^\sharp_{\mathrm{red}}$ (4.46).*

Proof. For any point $p \in \mathfrak{C}_*$, define the vector spaces

$$\mathcal{V}(p) := \mathrm{span}\{dF(p) \mid F \in C^\infty(\mathfrak{P} \times \mathfrak{P})^G\} < T^*_p(\mathfrak{P} \times \mathfrak{P}) \tag{4.59}$$

and

$$\mathcal{W}(p) := \mathrm{span}\{dK(p) \mid K \in C^\infty(\mathfrak{C}_*)^G\} < T^*_p \mathfrak{C}_*. \tag{4.60}$$

Relying on (4.57), we observe that the claim is equivalent to the equality

$$(D\iota(p))^*(\mathcal{V}(p)) = \mathcal{W}(p). \tag{4.61}$$

Because of (4.56), we have

$$(D\iota(p))^*(\mathcal{V}(p)) < \mathcal{W}(p). \tag{4.62}$$

Thus, it is enough to demonstrate that these vector spaces have the same dimension. To show this, we recall that for any smooth action of a compact Lie group on a connected manifold $\mathcal{M}$ the dimension of the span of the differentials of the smooth invariant functions at any $p \in \mathcal{M}$ belonging the principal orbit type is equal to the co-dimension of the orbit through p. (For a proof of this well known result, see the Appendix of [22].) In our case, this implies that

$$\dim(\mathcal{V}(p)) = \dim(\mathcal{W}(p)) + r, \tag{4.63}$$

[6] For example, for $\mathfrak{P} = \exp(\mathrm{i}\mathfrak{su}(n))$ take the function $1/f$, where $f(\tilde{L}, L) := \prod_{i=1}^{n-1}(\lambda_i - \lambda_{i+1})$ with the λ_i denoting the ordered eigenvalues of $L \in \mathfrak{P}_{\mathrm{reg}}$.

since $\dim(\mathcal{V}(p)) = \dim(G)$ and $\dim(\mathcal{W}(p)) = \dim(\mathfrak{C}_*^{\mathrm{red}})$. Now, we notice that the kernel of $(D\iota(p))^*$ is the span of the differentials $dF_i(p)$ $(i = 1, \dots, r)$ of the functions $F_i \in C^\infty(\mathfrak{P} \times \mathfrak{P})^G$ defined in (3.31). This follows since $p \in \mathfrak{C}_*$ has a coordinate neighbourhood $U_p \subset \mathfrak{P} \times \mathfrak{P}$ whose intersection with $\mathfrak{C}_*$ is the joint zero set of the functions F_i in U_p. Here, we used the description of $\mathfrak{C}_{\mathrm{reg}}$ outlined after (3.30) and that $\mathfrak{C}_* \subset \mathfrak{C}_{\mathrm{reg}}$ is an open subset. The kernel lies in $\mathcal{V}(p)$ and is of dimension r. By putting these arguments together, we find that

$$\dim((D\iota(p))^*(\mathcal{V}(p))) = \dim(\mathcal{W}(p)), \tag{4.64}$$

which implies the claim of the lemma. □

Now we are ready to state the principal new result of the present paper.

Theorem 4.9. *Suppose that $r = \mathrm{rank}(G) > 1$ and consider the restriction of the master system of free motion (described in Section 3.2) on the dense, open submanifold $\mathbb{M}_* \subset \mathbb{M}$ of principal orbit type with respect to the G-action (4.16). Then, this system descends to the degenerate integrable system $(\mathbb{M}_*^{\mathrm{red}}, \{-,-\}_{\mathbb{M}_*^{\mathrm{red}}}, \mathfrak{H}_{\mathrm{red}}^*, \mathfrak{F}_{\mathrm{red}}^*)$ on the Poisson manifold $\mathbb{M}_*^{\mathrm{red}} = \mathbb{M}_*/G$, where the Poisson subalgebras $\mathfrak{H}_{\mathrm{red}}^*$ and $\mathfrak{F}_{\mathrm{red}}^*$ of $C^\infty(\mathbb{M}_*^{\mathrm{red}}) \simeq C^\infty(\mathbb{M}_*)^G$ arise from the restrictions of $\mathfrak{H}$ (3.32) and $\mathfrak{F}^G$ (4.24) on $\mathbb{M}_* \subset \mathbb{M}$, respectively.*

Proof. The statement follows by combining Theorem 4.6 with Lemma 4.8. Indeed, $\mathfrak{H}_{\mathrm{red}}^*$ and $\mathfrak{F}_{\mathrm{red}}^*$ satisfy the conditions of Definition 2.3 on $\mathbb{M}_*^{\mathrm{red}}$ because of the properties of their restrictions $\mathfrak{H}_{\mathrm{red}}^{**}$ and $\mathfrak{F}_{\mathrm{red}}^{**}$ on the dense open subset $\mathbb{M}_{**}^{\mathrm{red}} \subset \mathbb{M}_*^{\mathrm{red}}$. In particular, Theorem 4.6 and Lemma 4.8 imply that the differentials of the elements of $\mathfrak{F}_{\mathrm{red}}^*$ span a co-dimension r subspace of the cotangent space at every point of $\mathbb{M}_{**}^{\mathrm{red}}$. □

Remark 4.10. *The full reduced phase space $\mathbb{M}^{\mathrm{red}} = \mathbb{M}/G$ is not a smooth manifold, but it still carries the Poisson algebra of smooth functions descending from $C^\infty(\mathbb{M})^G$. Moreover (see [50, 61]), $\mathbb{M}^{\mathrm{red}}$ can be decomposed into a disjoint union of symplectic leaves, each of which inherits an Abelian Poisson algebra from $\mathfrak{H}$ (3.32) and a Poisson algebra of constants of motion from $\mathfrak{F}^G$ (4.24). We conjecture that the reduced system is integrable on every such symplectic leaf. It is worth noting that the derivation [23] of the trigonometric Ruijsenaars–Schneider model by Hamiltonian reduction of the Heisenberg double of $\mathrm{SU}(n)$ (see also Appendic C) provides examples in which $\mathbb{M}_*^{\mathrm{red}}$ contains symplectic leaves of dimension $2(n-1)$, smaller than the dimension of the generic symplectic leaves if $n \neq 2$, and the reduced system on these leaves is 'only' Liouville integrable.*

5. Dynamical r-matrix formulation of the reduced system

Here, we first derive an explicit formula for the reduced Poisson bracket based on a convenient partial gauge fixing. The 'gauge slice' $\mathbb{M}_0$ (5.4) intersects every G-orbit contained in the dense open submanifold $\pi_1^{-1}(G^{\mathrm{reg}})$ of $\mathbb{M}$, where $\pi_1 : \mathbb{M} \to G$ is the projection onto the first factor of $\mathbb{M} = G \times \mathfrak{P}$. The formula (5.10) characterizes the reduced Poisson bracket since every invariant function $\mathcal{F} \in C^\infty(\mathbb{M})^G$ can be recovered from its restriction $\bar{\mathcal{F}}$ on $\mathbb{M}_0$. Then, we describe the reduced dynamics induced on the gauge slice.

Consider the set of elements, $G_0^{\mathrm{reg}} = G_0 \cap G^{\mathrm{reg}}$, whose centralizer in G is G_0. Since $G_0 < \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$, the adjoint action of the elements of G_0 is well-defined on $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$. As is easy to see, for any $Q \in G_0^{\mathrm{reg}}$ the linear operator $(\mathrm{Ad}_Q - \mathrm{id}) \in \mathrm{End}(\mathcal{G}_{\mathbb{R}}^{\mathbb{C}})$ is invertible on $\mathcal{G}_\perp^{\mathbb{C}} = \mathcal{G}_>^{\mathbb{C}} + \mathcal{G}_<^{\mathbb{C}}$ (2.1). Thus, one can define the linear operator $\mathcal{R}(Q) \in \mathrm{End}(\mathcal{G}_{\mathbb{R}}^{\mathbb{C}})$ by

$$\mathcal{R}(Q)(X) := \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id}) \circ (\mathrm{Ad}_Q - \mathrm{id})^{-1}_{|\mathcal{G}_\perp^{\mathbb{C}}}(X_\perp), \qquad \forall Q \in G_0^{\mathrm{reg}},\ \forall X = (X_0 + X_\perp) \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}, \tag{5.1}$$

where $X_0 \in \mathcal{G}_0^{\mathbb{C}}$ and $X_\perp \in \mathcal{G}_\perp^{\mathbb{C}}$ according to (2.5). One can check that $\mathcal{R}(Q)$ maps $\mathcal{B}$ to $\mathcal{B}$ and $\mathcal{G}$ to $\mathcal{G}$, and (writing $\mathcal{R}(Q)X := \mathcal{R}(Q)(X)$) it enjoys the identities

$$\langle \mathcal{R}(Q)X, Y\rangle_{\mathbb{I}} = -\langle X, \mathcal{R}(Q)Y\rangle_{\mathbb{I}}, \qquad \forall X, Y \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}, \tag{5.2}$$

and

$$\langle \mathcal{R}(Q)X, Y\rangle_{\mathbb{I}} = \langle \mathcal{R}(Q)X_{\mathcal{G}}, Y_{\mathcal{B}}\rangle_{\mathbb{I}} - \langle X_{\mathcal{B}}, \mathcal{R}(Q)Y_{\mathcal{G}}\rangle_{\mathbb{I}}, \qquad \forall X, Y \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}. \tag{5.3}$$

With the aid of the exponential parametrization of Q and restriction to a linear operator on $\mathcal{G}$, $\mathcal{R}(Q)$ yields the standard trigonometric solution of the modified classical dynamical Yang–Baxter equation [15] for the pair $\mathcal{G}_0 \subset \mathcal{G}$. This dynamical r-matrix features in Theorem 5.2 below.

5.1. **Reduced Poisson brackets.** Let us introduce

$$\mathbb{M}_0 := \{(Q,L) \in \mathbb{M} \mid Q \in G_0^{\mathrm{reg}}\}. \tag{5.4}$$

The G-orbits that intersect $\mathbb{M}_0$ fill the dense, open, G-invariant submanifold

$$\pi_1^{-1}(G^{\mathrm{reg}}) = G^{\mathrm{reg}} \times \mathfrak{P} \subset \mathbb{M}. \tag{5.5}$$

The intersection of $\mathbb{M}_0$ with a G-orbit is an orbit of the normalizer

$$\mathfrak{N} := N_G(G_0) = \{\eta \in G \mid \eta G_0 \eta^{-1} = G_0\}, \tag{5.6}$$

which acts on $\mathbb{M}_0$. Here, we are referring to the action of the group elements $\eta \in \mathfrak{N} < G$ determined by equation (4.16). Colloquially, we may call $\mathfrak{N}$ with this action the 'residual gauge group'. Then, as is easily seen, the restriction of functions gives rise to the following isomorphism:

$$C^\infty(G^{\mathrm{reg}} \times \mathfrak{P})^G \longleftrightarrow C^\infty(\mathbb{M}_0)^{\mathfrak{N}}. \tag{5.7}$$

The next definition relies on this isomorphism.

Definition 5.1. *Let $\bar{\mathcal{F}}, \bar{\mathcal{H}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$ be the restrictions of $\mathcal{F}, \mathcal{H} \in C^\infty(G^{\mathrm{reg}} \times \mathfrak{P})^G$. Then, we define*

$$\{\bar{\mathcal{F}}, \bar{\mathcal{H}}\}^{\mathrm{red}}_{\mathbb{M}_0}(Q,L) := \{\mathcal{F}, \mathcal{H}\}_{\mathbb{M}}(Q,L), \qquad \forall (Q,L) \in \mathbb{M}_0. \tag{5.8}$$

On the right-hand side of (5.8) the restriction of $\{-,-\}_{\mathbb{M}}$ (3.19) to the open submanifold (5.5) is used. The ring of functions $C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$ becomes a Poisson algebra when equipped with the 'reduced Poisson bracket' $\{-,-\}^{\mathrm{red}}_{\mathbb{M}_0}$. We shall express $\{-,-\}^{\mathrm{red}}_{\mathbb{M}_0}$ in terms of the derivatives $\mathcal{D}_1\bar{\mathcal{F}}(Q,L) \in \mathcal{B}_0$ and $\mathcal{D}_2\bar{\mathcal{F}}(Q,L) \in \mathcal{G}^{\mathbb{C}}_{\mathbb{R}}$, where $\mathcal{D}_1\bar{\mathcal{F}}(Q,L)$ is defined by

$$\langle Y_0, \mathcal{D}_1\bar{\mathcal{F}}(Q,L)\rangle_{\mathbb{I}} = \frac{d}{dt}\Big|_{t=0} \bar{\mathcal{F}}(e^{tY_0}Q, L), \quad \forall Y_0 \in \mathcal{G}_0, \tag{5.9}$$

and the derivative $\mathcal{D}_2\bar{\mathcal{F}}$ with respect to the second variable is determined according to (3.8) and (3.9).

Theorem 5.2. *The definition (5.8) and equation (3.19) imply the following formula:*

$$\{\bar{\mathcal{F}}, \bar{\mathcal{H}}\}^{\mathrm{red}}_{\mathbb{M}_0}(Q,L) = \langle \mathcal{D}_1\bar{\mathcal{F}}, \mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}} - \langle \mathcal{D}_1\bar{\mathcal{H}}, \mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}} + \langle \mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}}, (\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{B}}\rangle_{\mathbb{I}} - \langle (\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{B}}, \mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{G}}\rangle_{\mathbb{I}}. \tag{5.10}$$

Here, the derivatives $\mathcal{D}_1\bar{\mathcal{F}} \in \mathcal{B}_0$ and $\mathcal{D}_2\bar{\mathcal{F}} \in \mathcal{G}^{\mathbb{C}}_{\mathbb{R}}$ are taken at (Q,L), and $\mathcal{R}(Q)$ is given by (5.1).

Proof. Let $\bar{\mathcal{F}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$ be the restriction of $\mathcal{F} \in C^\infty(G^{\mathrm{reg}} \times \mathfrak{P})^G$. To begin, note that

$$(\mathcal{D}_1\mathcal{F}(Q,L))_{\mathcal{B}_0} = \mathcal{D}_1\bar{\mathcal{F}}(Q,L) \quad \text{and} \quad \mathcal{D}_2\mathcal{F}(Q,L) = \mathcal{D}_2\bar{\mathcal{F}}(Q,L), \quad \forall (Q,L) \in \mathbb{M}_0. \tag{5.11}$$

Then, we take the derivative at $t=0$ of

$$\mathcal{F}(e^{tY}Qe^{-tY}, e^{tY}Le^{-tY}) = \mathcal{F}(Q,L), \quad \forall Y \in \mathcal{G}, \tag{5.12}$$

and from this obtain

$$\mathcal{D}_1'\mathcal{F}(Q,L) - \mathcal{D}_1\mathcal{F}(Q,L) = (\mathcal{D}_2\mathcal{F}(Q,L))_{\mathcal{B}}, \tag{5.13}$$

which implies

$$\left(\mathrm{Ad}_{Q^{-1}} - \mathrm{id}\right)\mathcal{D}_1\mathcal{F}(Q,L) = (\mathcal{D}_2\bar{\mathcal{F}}(Q,L))_{\mathcal{B}}. \tag{5.14}$$

This in turn entails that

$$(\mathcal{D}_2\mathcal{F}(Q,L))_{\mathcal{B}_0} = 0 \quad \text{and} \quad (\mathcal{D}_1\mathcal{F}(Q,L))_{\mathcal{B}_+} = -(\frac{1}{2}\mathrm{id} + \mathcal{R}(Q))(\mathcal{D}_2\bar{\mathcal{F}}(Q,L))_{\mathcal{B}}, \tag{5.15}$$

where the subscripts refer to the decomposition of $\mathcal{B}$ in (2.6), Thus, at $(Q,L) \in \mathbb{M}_0$, we expressed the derivatives of $\mathcal{F}$ in terms of the derivatives of $\bar{\mathcal{F}}$. It remains to insert these expressions, and their counterparts for $\mathcal{H}$, into the right-hand side of (5.8) given by (3.19).

Regarding the third term of (3.19), we have $\big\langle Q\mathcal{D}_1'\mathcal{F}(Q,L)Q^{-1},\mathcal{D}_1\mathcal{H}(Q,L)\big\rangle_{\mathbb{I}}=0$, since $\mathcal{B}$ is stable under Ad_Q. We now use (5.11) and (5.15) to write the last two terms of (3.19), at (Q,L), as follows:

$$\begin{aligned}\langle\mathcal{D}_1\mathcal{F},\mathcal{D}_2\mathcal{H}\rangle_{\mathbb{I}}-\langle\mathcal{D}_1\mathcal{H},\mathcal{D}_2\mathcal{F}\rangle_{\mathbb{I}}=&\langle\mathcal{D}_1\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}}-\langle\mathcal{D}_1\bar{\mathcal{H}},\mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}}\\&-\frac{1}{2}\langle(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{B}},(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}}\rangle_{\mathbb{I}}+\frac{1}{2}\langle(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{B}},(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{G}}\rangle_{\mathbb{I}}\\&-\langle\mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{B}},(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}}\rangle_{\mathbb{I}}+\langle\mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{B}},(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{G}}\rangle_{\mathbb{I}}\\=&\langle\mathcal{D}_1\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}}-\langle\mathcal{D}_1\bar{\mathcal{H}},\mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}}\\&-\frac{1}{2}\langle\mathcal{D}_2\bar{\mathcal{F}},(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}}\rangle_{\mathbb{I}}+\frac{1}{2}\langle\mathcal{D}_2\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}-(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}}\rangle_{\mathbb{I}}\\&+\langle(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{B}},\mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}}\rangle_{\mathbb{I}}+\langle(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{G}},\mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{B}}\rangle_{\mathbb{I}}\\=&\langle\mathcal{D}_1\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}}-\langle\mathcal{D}_1\bar{\mathcal{H}},\mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}}\\&-\langle\mathcal{D}_2\bar{\mathcal{F}},(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}}\rangle_{\mathbb{I}}+\langle\mathcal{D}_2\bar{\mathcal{F}},\mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{H}})\rangle_{\mathbb{I}}+\frac{1}{2}\langle\mathcal{D}_2\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}}.\end{aligned}\tag{5.16}$$

By adding also the first term of (3.19), we get

$$\{\bar{\mathcal{F}},\bar{\mathcal{H}}\}^{\mathrm{red}}_{\mathbb{M}_0}=\langle\mathcal{D}_1\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}}-\langle\mathcal{D}_1\bar{\mathcal{H}},\mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}}+\langle\mathcal{D}_2\bar{\mathcal{F}},\mathcal{R}(Q)(\mathcal{D}_2\bar{\mathcal{H}})\rangle_{\mathbb{I}}+\frac{1}{2}\langle\mathcal{D}_2\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}}.\tag{5.17}$$

Since the L.H.S. is antisymmetric with respect to the exchange of $\bar{\mathcal{F}}$ and $\bar{\mathcal{H}}$, and the sum of the first three terms of the R.H.S. is also antisymmetric, we must have $\langle\mathcal{D}_2\bar{\mathcal{F}},\mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}}=0$ (as is confirmed by (3.16)), and the the claim (5.10) follows on account of the identity (5.3). □

Observe that the first term of (5.10) contains only $(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}_0}$ since $(\mathcal{D}_1\bar{\mathcal{F}})_1\in\mathcal{B}_0$. The third term depends only on $(\mathcal{D}_2\bar{\mathcal{H}})_{\mathcal{G}_\perp}$ and on $(\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{B}_+}$, because $\mathcal{R}(Q)$ vanishes on $\mathcal{G}_0$. Here, we refer to the decompositions in (2.6). Of course, similar remarks hold for the second and fourth terms.

There are two alternative ways of dealing with the residual $\mathfrak{N}$ 'gauge freedom' that remains after the restriction to $\mathbb{M}_0$ (5.4). The first is based on the fact that G_0 is a normal subgroup of $\mathfrak{N}$, with the factor group being the Weyl group

$$W_G=\mathfrak{N}/G_0.\tag{5.18}$$

This leads to the isomorphisms

$$\pi_1^{-1}(G^{\mathrm{reg}})/G\simeq\mathbb{M}_0/\mathfrak{N}\simeq(\mathbb{M}_0/G_0)/W_G,\tag{5.19}$$

i.e., one may first take the quotient of $\mathbb{M}_0$ by G_0 and then divide by the Weyl group. We shall proceed in this way in the description of a specific example in Appendix C. The second possibility is to take into account that G_0^{reg} is disconnected, and its connected components are permuted by the Weyl group. Therefore, one may restrict to a connected component, a so-called Weyl alcove in G_0^{reg}, and then there remains only the residual G_0 gauge symmetry. Denoting a fixed Weyl alcove by $\check{G}_0^{\mathrm{reg}}$, one introduces the new gauge slice

$$\check{\mathbb{M}}_0:=\{(Q,L)\in\mathbb{M}\mid Q\in\check{G}_0^{\mathrm{reg}}\},\tag{5.20}$$

which induces the isomorphism

$$C^\infty(\mathbb{M}_0)^{\mathfrak{N}}\longleftrightarrow C^\infty(\check{\mathbb{M}}_0)^{G_0}.\tag{5.21}$$

On the other hand, since $\mathbb{M}_0$ (5.4) is disconnected and its connected components are permuted by W_G, it is clear that the expression (5.10) defines a Poisson algebra structure on the larger ring $C^\infty(\mathbb{M}_0)^{G_0}$, too. In fact, the Poisson bracket on $C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$ represents a Poisson structure on the quotient space $\mathbb{M}_0/\mathfrak{N}$, and this lifts to its W_G covering space $\mathbb{M}_0/G_0$, giving rise to a Poisson bracket on $C^\infty(\mathbb{M}_0)^{G_0}$.

5.2. **Reduced dynamics on the gauge slice $\mathbb{M}_0$.** The Hamiltonian vector fields associated with the commuting Hamiltonians from $\mathfrak{H}$ (3.32) are projectable on the reduced phase space $\mathbb{M}/G$, and the projected vector fields are Hamiltonian with respect to the reduced Poisson structure. Restricting to the dense open submanifold of $\pi_1^{-1}(G^{\mathrm{reg}})\subset\mathbb{M}$, we describe the vector fields on $\mathbb{M}_0$ (5.4) whose projections on $\mathbb{M}_0/\mathfrak{N}$ coincide with the reduced Hamiltonian vector fields on $\pi_1^{-1}(G^{\mathrm{reg}})/G$. Then, we present a construction of the corresponding integral curves on $\mathbb{M}_0$.

Below, we use the terms reduced Hamiltonian vector fields and reduced dynamics on $\mathbb{M}_0$. This is a slight abuse of terminology, since the true reduced dynamics lives on $\mathbb{M}/G$, of which $\mathbb{M}_0/\mathfrak{N}$ is a dense open subset.

A vector field V on $\mathbb{M}_0=G_0^{\mathrm{reg}}\times\mathfrak{P}$ can be presented as

$$V(Q,L)=(V^1(Q,L),V^2(Q,L))\quad\text{with}\quad V^1(Q,L)\in T_QG_0,\ V^2(Q,L)=T_L\mathfrak{P}.\tag{5.22}$$

Let us consider a Hamiltonian $\bar{\mathcal{H}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$ which is the restriction of $\mathcal{H} = \pi_2^*\phi \in \mathfrak{H}$. This means that

$$\bar{\mathcal{H}}(Q,L) = \phi(L) \quad \text{where} \quad \phi \in C^\infty(\mathfrak{P})^G. \tag{5.23}$$

For the derivatives of $\bar{\mathcal{H}}$, we have

$$\mathcal{D}_1\bar{\mathcal{H}}(Q,L) = 0 \quad \text{and} \quad \mathcal{D}_2\bar{\mathcal{H}}(Q,L) = \mathcal{D}\phi(L) \in \mathcal{G}. \tag{5.24}$$

See equations (3.8) and (3.9) for the definition of $\mathcal{D}\phi$. Now, using (5.10), we associate with $\bar{\mathcal{H}}$ the 'reduced Hamiltonian vector field' $V_{\bar{\mathcal{H}}}$ on $\mathbb{M}_0$ by imposing the following condition:

$$\{\bar{\mathcal{F}}, \bar{\mathcal{H}}\}^{\text{red}}_{\mathbb{M}_0} = V_{\bar{\mathcal{H}}}[\bar{\mathcal{F}}], \qquad \forall \bar{\mathcal{F}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}, \tag{5.25}$$

where $V_{\bar{\mathcal{H}}}[\bar{\mathcal{F}}]$ denotes the derivative of $\bar{\mathcal{F}}$ along the vector field. There is an ambiguity in $V_{\bar{\mathcal{H}}}$ because of the invariance property of $\bar{\mathcal{F}}$. It is convenient to require (5.25) for all $\bar{\mathcal{F}} \in C^\infty(\mathbb{M}_0)^{G_0}$, and then the residual ambiguity in $V_{\bar{\mathcal{H}}}$ is the addition of an arbitrary vector field that is tangent to the G_0-orbits, representing an infinitesimal G_0 gauge transformation.

Proposition 5.3. *The 'reduced Hamiltonian vector field' $V_{\bar{\mathcal{H}}}$ defined above has the following form:*

$$V^1_{\bar{\mathcal{H}}}(Q,L) = \mathcal{D}\phi(L)_0 Q, \qquad V^2_{\bar{\mathcal{H}}}(Q,L) = [\mathcal{R}(Q)\mathcal{D}\phi(L), L], \tag{5.26}$$

up to the ambiguity of adding an arbitrary vector field tangent to the G_0-orbits in $\mathbb{M}_0$.

Proof. On account of (5.10), the equality (5.25) can be spelled out as

$$\langle \mathcal{D}_1\bar{\mathcal{F}}(Q,L), \mathcal{D}\phi(L)_0\rangle_{\mathbb{I}} + \langle \mathcal{D}_2\bar{\mathcal{F}}(Q,L)_{\mathcal{B}}, \mathcal{R}(Q)\mathcal{D}\phi(L)\rangle_{\mathbb{I}} = V_{\bar{\mathcal{H}}}[\bar{\mathcal{F}}](Q,L). \tag{5.27}$$

The definition of the derivatives $\mathcal{D}_i\bar{\mathcal{F}}$ implies that (5.27) is equivalent to the claimed formula (5.26), up to the ambiguity of $V_{\bar{\mathcal{H}}}$ discussed above. □

The vector field $V_{\bar{\mathcal{H}}}$ can also be derived by first taking the original Hamiltonian vector field of $\mathcal{H} \in \mathfrak{H}$ on $\mathbb{M}$, then restricting it to $\mathbb{M}_0$, and adding a vector field tangent to the G-orbits in a such a way that the result is tangent to $\mathbb{M}_0$. The original Hamiltonian vector field is provided by (3.25), and one can verify that

$$V^1_{\bar{\mathcal{H}}}(Q,L) = \mathcal{D}\phi(L)Q + [\mathcal{R}(Q)\mathcal{D}\phi(L) - \frac{1}{2}\mathcal{D}\phi(L), Q], \quad V^2_{\bar{\mathcal{H}}}(Q,L) = [\mathcal{R}(Q)\mathcal{D}\phi(L) - \frac{1}{2}\mathcal{D}\phi(L), L], \tag{5.28}$$

where the Lie brackets define an element of $T_{(Q,L)}\mathbb{M}$ that is tangent to the G-orbit through $(Q,L) \in \mathbb{M}_0$. Note that $[\mathcal{D}\phi(L), L] = 0$ because of the G-invariance of ϕ, and also $[\mathcal{D}\phi(L)_0, Q] = 0$.

Next, we present a quadrature for finding the integral curves of the vector fields (5.26), which govern the dynamics induced on the gauge slice $\mathbb{M}_0$ (5.4).

Proposition 5.4. *For a fixed function $\phi \in C^\infty(\mathfrak{P})^G$ and a point $(Q_0, L_0) \in \mathbb{M}_0$, let $\eta_1(t)$ be a G-valued smooth function on an interval $(-\epsilon, \epsilon) \subset \mathbb{R}$ such that $\eta_1(0) = e$ and*

$$Q(t) := \eta_1(t)\exp(t\mathcal{D}\phi(L_0))Q_0\eta_1(t)^{-1} \in G_0^{\text{reg}}, \quad \forall t \in (-\epsilon, \epsilon). \tag{5.29}$$

Furthermore, let $\eta_0(t) \in G_0$ with $\eta_0(0) = e$ be the unique solution of the differential equation

$$\dot{\eta}_0(t)\eta_0(t)^{-1} = -\frac{1}{2}\mathcal{D}\phi(\eta_1(t)L_0\eta_1(t)^{-1})_0 - (\dot{\eta}_1(t)\eta_1(t)^{-1})_0. \tag{5.30}$$

Then, $Q(t)$ and $L(t) := \eta_0(t)\eta_1(t)L_0\eta_1(t)^{-1}\eta_0(t)^{-1}$, defined on the interval $(-\epsilon, \epsilon)$, gives an integral curve of the vector field $V_{\bar{\mathcal{H}}}$ (5.26), with initial value (Q_0, L_0).

Proof. After setting $L_1(t) := \eta_1(t)L_0\eta_1(t)^{-1}$, a simple calculation gives $\dot{Q}(t) = \mathcal{D}\phi(L_1(t))_0 Q(t)$ and

$$\dot{L}_1(t) = [\mathcal{R}(Q(t))\mathcal{D}\phi(L_1(t)) + (\dot{\eta}_1(t)\eta_1(t)^{-1})_0 + \frac{1}{2}\mathcal{D}\phi(L_1(t))_0, L_1(t)]. \tag{5.31}$$

To get this, one uses that $\eta_1(t)\mathcal{D}\phi(L_0)\eta_1(t)^{-1} = \mathcal{D}\phi(L_1(t))$ and that $\dot{Q}(t)Q(t)^{-1} \in \mathcal{G}_0$ implies

$$\left(\dot{\eta}_1(t)\eta_1(t)^{-1}\right)_\perp = \left(\mathcal{R}(Q(t)) - \frac{1}{2}\right)\mathcal{D}\phi(L_1(t))_\perp. \tag{5.32}$$

Conjugation by $\eta_0(t)$ does not change $Q(t)$, and the result follows since $\mathcal{D}\phi(L(t)) = \eta_0(t)\mathcal{D}\phi(L_1(t))\eta_0(t)^{-1}$. □

The first step of the above quadrature is the construction of $\eta_1(t)$ in (5.29), which is a purely (linear) algebraic problem. The subsequent construction of $\eta_0(t)$ requires the calculation of an integral,

$$\eta_0(t) = \exp\left(-\int_0^t \left(\dot{\eta}_1(\tau)\eta_1(\tau)^{-1} + \frac{1}{2}\mathcal{D}\phi(L_1(\tau))\right)_0 d\tau\right). \tag{5.33}$$

This second step can actually be omitted, since the conjugation by $\eta_0(t)$ does not affect the eventual projection of the integral curve on the reduced phase space.

Remark 5.5. *The Hamiltonian vector field of any $\mathcal{H} \in \mathfrak{H}$ on $\mathbb{M}$, and consequently also its projection on the full reduced phase space $\mathbb{M}/G$, is complete. However, the vector field $V_{\bar{\mathcal{H}}}$ (5.26) is not complete on $\mathbb{M}_0$ in general. This is a consequence of the fact that not all unreduced integral curves (3.22) starting in $G^{\text{reg}} \times \mathfrak{P}$ stay in this dense open submanifold.*

6. Decoupled variables and the scaling limit

Our first goal here is to recast the reduced Poisson bracket (5.10) in terms of new variables consisting of canonically conjugate pairs q_i, p_i $(i = 1, \dots, r)$ and a 'decoupled spin variable' λ. This is described by Theorem 6.4, which generalizes a similar result presented in [26] for the $G = \mathrm{U}(n)$ case. At an intermediate stage, we shall also recover the form of the reduced Poisson bracket given in [21]. Then, based on Theorem 6.4, we shall elaborate the connection between our reduced systems and the well known spin Sutherland models obtained by reduction from the cotangent bundles T^*G.

6.1. Canonically conjugate pairs and 'spin' variables.

Let us begin by noting that in terms of the alternative model $\mathfrak{M}$ (2.17) of the Heisenberg double the gauge slice $\mathbb{M}_0$ (5.4) turns into

$$\mathfrak{M}_0 := G_0^{\text{reg}} \times B. \tag{6.1}$$

The connection between $\mathfrak{M}_0$ and $\mathbb{M}_0$ is given by the diffeomorphism $\bar{m}_2$, which is the restriction of the map m_2 (2.21),

$$\bar{m}_2 : \mathfrak{M}_0 \to \mathbb{M}_0, \qquad \bar{m}_2(Q, b) = (Q, bb^\dagger). \tag{6.2}$$

The map $\bar{m}_2$ is $\mathfrak{N}$-equivariant with respect to the restrictions of the actions (4.16), and therefore it induces an isomorphism

$$C^\infty(\mathbb{M}_0)^{\mathfrak{N}} \longleftrightarrow C^\infty(\mathfrak{M}_0)^{\mathfrak{N}}. \tag{6.3}$$

Proposition 6.1. *For functions $\bar{f}, \bar{h} \in C^\infty(\mathfrak{M}_0)^{\mathfrak{N}}$, the isomorphism (6.3) converts (5.10) into the following equivalent formula of the reduced Poisson bracket:*

$$\{\bar{f}, \bar{h}\}^{\text{red}}_{\mathfrak{M}_0}(Q, b) = \langle D_1\bar{f}, D_2\bar{h}\rangle_{\mathbb{I}} - \langle D_1\bar{h}, D_2\bar{f}\rangle_{\mathbb{I}} + \langle \mathcal{R}(Q)(bD_2'\bar{h}b^{-1}), bD_2'\bar{f}b^{-1}\rangle_{\mathbb{I}}. \tag{6.4}$$

Here, the derivatives are evaluated at (Q, b), with $D_1\bar{f} \in \mathcal{B}_0$ defined similarly to(5.9) and $D_2\bar{f}, D_2'\bar{f} \in \mathcal{G}$ defined by applying (2.25) to the second variable, and $\mathcal{R}(Q)$ is given by (5.1).

Proof. For functions $\bar{\mathcal{F}}$ and $\bar{f}$ related by $\bar{f} = \bar{\mathcal{F}} \circ \bar{m}_2$, one has

$$\mathcal{D}_1\bar{\mathcal{F}} = D_1\bar{f} \quad \text{and} \quad \mathcal{D}_2\bar{\mathcal{F}} = bD_2'\bar{f}b^{-1}, \tag{6.5}$$

at the corresponding arguments, similarly to (3.12). Furthermore, $D_2\bar{f} = (bD_2'\bar{f}b^{-1})_{\mathcal{G}} = (\mathcal{D}_2\bar{\mathcal{F}})_{\mathcal{G}}$. The formula (6.4) is obtained from (5.10) by direct substitution of these relations, and their counterparts for $\bar{\mathcal{H}} = \bar{h} \circ \bar{m}_2$. More precisely, we also used the identity (5.3). □

The formula (6.4) was obtained previously in [21] starting from the Poisson bracket (3.19) on the model $\mathfrak{M}$ (2.17) of the Heisenberg double. Its equivalence with the claim of Theorem 5.2 provides a good consistency check on our considerations. According to the discussion at the end of Section 5.1, the formula (6.4) yields a Poisson bracket also on $C^\infty(\mathfrak{M}_0)^{G_0}$.

Now, our goal is to recast the Poisson bracket (6.4) on $C^\infty(\mathfrak{M}_0)^{G_0}$ in terms of convenient new variables. To do this, we shall use that any $b \in B$ can be decomposed uniquely as

$$b = e^p b_+ \quad \text{with} \quad p \in \mathcal{B}_0,\, b_+ \in B_+, \tag{6.6}$$

and that the subgroups $B_0 = \exp(\mathcal{B}_0)$ and $B_+ = \exp(\mathcal{B}_+)$ of B admit global exponential parametrization. Our construction relies on the map

$$\zeta : \mathfrak{M}_0 = G_0^{\text{reg}} \times B \to G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+ \tag{6.7}$$

defined by the formula

$$\zeta : (Q, e^p b_+) \mapsto (Q, p, \lambda) \quad \text{with} \quad \lambda := b_+^{-1}Q^{-1}b_+Q. \tag{6.8}$$

We remark that λ can also be written as $\lambda = b^{-1}Q^{-1}bQ$. The definition (6.8) comes from [19], where a 'spin variable' (denoted there S_+) given by the same formula as our λ, but restricted to the intersection of an arbitrary dressing orbit of G with B_+, was used. The geometric origin of the definition is expounded in Appendix B.

Lemma 6.2. *The map ζ (6.7) is a diffeomorphism. It is equivariant with respect to the G_0-actions for which $\eta_0 \in G_0$ sends (Q, b) to $(Q, \eta_0 b \eta_0^{-1})$ and (Q, p, λ) to $(Q, p, \eta_0 \lambda \eta_0^{-1})$. Consequently, ζ induces an isomorphism*

$$C^\infty(\mathfrak{M}_0^{\mathrm{reg}})^{G_0} \longleftrightarrow C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}. \tag{6.9}$$

Proof. The properties of the map ζ were analyzed in Section 5.1 of [19], from which the statement follows. (Incidentally, this paper contains an explicit formula for the inverse of ζ in the $G = \mathrm{SU}(n)$ case.) See also the proof of Proposition 6.8 below. $\square$

Let $D_Q F$, $d_p F$ and $D_\lambda F$, $D'_\lambda F$ denote the derivatives of any real function $F \in C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)$ with respect to its three variables, respectively. We have $D_Q F \in \mathcal{B}_0$, $d_p F \in \mathcal{G}_0$ and $D_\lambda F, D'_\lambda F \in \mathcal{G}_\perp$. Here, $D_Q F$ and $d_p F$ are defined by

$$\langle Y_0, D_Q F(Q,p,\lambda)\rangle_{\mathbb{I}} + \langle X_0, D_p F(Q,p,\lambda)\rangle_{\mathbb{I}} = \frac{d}{dt}\Big|_{t=0} F(Qe^{tY_0}, p + tX_0, \lambda), \quad \forall X_0 \in \mathcal{B}_0,\ Y_0 \in \mathcal{G}_0, \tag{6.10}$$

using that $Qe^{tY_0} \in G_0^{\mathrm{reg}}$ for small t. Recalling the decompositions (2.2) and (2.6), $\mathcal{G}_\perp$ is taken as the model of the dual space of $\mathcal{B}_+$. Then, $D_\lambda F$ and $D'_\lambda F$ are defined by

$$\langle X, D_\lambda F(Q,p,\lambda)\rangle_{\mathbb{I}} + \langle X', D'_\lambda F(Q,p,\lambda)\rangle_{\mathbb{I}} = \frac{d}{dt}\Big|_{t=0} F(Q, p, e^{tX}\lambda e^{tX'}), \quad \forall X, X' \in \mathcal{B}_+. \tag{6.11}$$

Lemma 6.3. *Consider two functions $\bar f \in C^\infty(\mathfrak{M}_0)$ and $F \in C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)$ related by $\bar f = F \circ \zeta$ with the diffeomorphism (6.8). Then, their derivatives are connected according to*

$$D'_2 \bar f = d_p F + Q D'_\lambda F Q - (\lambda D'_\lambda F \lambda^{-1})_{\mathcal{G}} \tag{6.12}$$

and

$$D_1 \bar f = D_Q F - \left(bQD'_\lambda F Q^{-1} b^{-1}\right)_{\mathcal{B}_0}, \tag{6.13}$$

where the derivatives on the left and right sides are taken at (Q, b) and at $(Q, p, \lambda) = \zeta(Q, b)$, respectively. The subscripts $\mathcal{G}$ and $\mathcal{B}_0$ refer to the decompositions (2.2) and (2.6).

Proof. Taking any $X \in \mathcal{B}_+$, and inspecting the derivatives along the curve $(Q, b\exp(tX))$ in $\mathfrak{M}_0$ and its ζ-image in $G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+$, we get

$$\langle X, D'_2 \bar f(Q,b)\rangle_{\mathbb{I}} = \langle X, QD'_\lambda F(Q,p,\lambda)Q^{-1} - (\lambda D'_\lambda F(Q,p,\lambda)\lambda^{-1})_{\mathcal{G}_\perp}\rangle_{\mathbb{I}}. \tag{6.14}$$

Inspection of the derivatives along the curve $(Q, b\exp(tX_0))$ in $\mathfrak{M}_0$, with $X_0 \in \mathcal{B}_0$, gives

$$\langle X_0, D'_2 \bar f(Q,b)\rangle_{\mathbb{I}} = \langle X_0, d_p F(Q,p,\lambda) - (\lambda D'_\lambda F(Q,p,\lambda)\lambda^{-1})_{\mathcal{G}_0}\rangle_{\mathbb{I}}. \tag{6.15}$$

Together, these imply the equality (6.12). To derive (6.13), we use the identity

$$\bar f(Qe^{tY_0}, b) = F(Qe^{tY_0}, p, \lambda Q^{-1} b^{-1} e^{-tY_0} b Q e^{tY_0}), \quad \forall Y_0 \in \mathcal{G}_0, \tag{6.16}$$

and note that

$$\frac{d}{dt}\Big|_{t=0} \left(Q^{-1}b^{-1}e^{-tY_0}bQe^{tY_0}\right) = \left(Y_0 - Q^{-1}b^{-1}Y_0 bQ\right) \in \mathcal{B}_+. \tag{6.17}$$

Consequently, at the arguments related by ζ, we get

$$\langle Y_0, D_1 \bar f\rangle_{\mathbb{I}} = \langle Y_0, D_Q F\rangle_{\mathbb{I}} + \langle Y_0 - Q^{-1}b^{-1}Y_0 bQ, D'_\lambda F\rangle_{\mathbb{I}} = \langle Y_0, D_Q F - \left(bQD'_\lambda F Q^{-1}b^{-1}\right)_{\mathcal{B}_0}\rangle_{\mathbb{I}}, \tag{6.18}$$

which completes the proof. $\square$

Since any two functions $F, H \in C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ are related to unique functions $\bar f, \bar h \in C^\infty(\mathfrak{M}_0)^{G_0}$ by

$$F \circ \zeta = \bar f, \quad H \circ \zeta = \bar h, \tag{6.19}$$

we can define $\{F, H\}_0^{\mathrm{red}} \in C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ by the requirement

$$\{F, H\}_0^{\mathrm{red}} \circ \zeta := \{\bar f, \bar h\}_{\mathfrak{M}_0}^{\mathrm{red}}. \tag{6.20}$$

Theorem 6.4. *In terms of the new variables introduced via the map ζ (6.8), using the definition (6.20), the reduced Poisson bracket (6.4) acquires the 'decoupled form'*

$$\{F,H\}_0^{\mathrm{red}}(Q,p,\lambda) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}}, \tag{6.21}$$

where the derivatives of $F, H \in C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ are taken at (Q,p,λ). The functions of the form $F(Q,p,\lambda) = \varphi(\lambda)$ with $\varphi \in C^\infty(B)^G$ are in the center of this Poisson bracket.

Proof. The formula (6.21) follows from the direct substitution of the relations of Lemma 6.3 into (6.4). The required tedious manipulations leading from (6.4) to (6.21) are omitted, since they essentially coincide with the calculation presented in [26], where the $G = \mathrm{U}(n)$ analogue of the formula was derived.

If F depends only on λ and is the restriction of $\varphi \in C^\infty(B)^G$, then we have

$$D'_\lambda F = D'\varphi(\lambda) - X_0 \quad \text{with} \quad X_0 = (D'\varphi(\lambda))_0. \tag{6.22}$$

In this case, since $\lambda D'\varphi(\lambda)\lambda^{-1} \in \mathcal{G}$ by (2.26), we can write

$$\{F,H\}_0^{\mathrm{red}}(Q,p,\lambda) = \langle X_0 - \lambda X_0 \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}}. \tag{6.23}$$

Notice that $Y := X_0 - \lambda X_0 \lambda^{-1}$ belongs to $\mathcal{B}_+$ and

$$\left.\frac{d}{dt}\right|_{t=0} e^{tY}\lambda = \left.\frac{d}{dt}\right|_{t=0} e^{tX_0}\lambda e^{-tX_0}. \tag{6.24}$$

This identity and the fact that H is G_0-invariant imply

$$\langle Y, D_\lambda H\rangle_{\mathbb{I}} = \left.\frac{d}{dt}\right|_{t=0} H(Q,p,e^{tY}\lambda) = \left.\frac{d}{dt}\right|_{t=0} H(Q,p,e^{tX_0}\lambda e^{-tX_0}) = 0, \tag{6.25}$$

which completes the proof. Incidentally, an alternative proof can be obtained starting from the reduced symplectic form derived in [19]. □

Remark 6.5. *As it was discussed around (5.20), the variable Q may be restricted to a Weyl alcove $\check{G}_0^{\mathrm{reg}}$. By using (5.21) and the relevant restriction of the map ζ (6.7), we obtain the isomorphism*

$$C^\infty(\mathfrak{M}_0)^{\mathfrak{N}} \longleftrightarrow C^\infty(\check{\mathfrak{M}}_0)^{G_0} \longleftrightarrow C^\infty(\check{G}_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{G_0} \tag{6.26}$$

with $\check{\mathfrak{M}}_0 = \check{G}_0^{\mathrm{reg}} \times B$. This may be a preferred way to proceed, since we do not have an explicit formula for the action of the group $\mathfrak{N}$ on $G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+$. Such an action is determined by transferring the action (4.16) of $\mathfrak{N} < G$ on $\mathfrak{M}_0$ via the diffeomorphism ζ, but its explicit form is not easy to find. According to Theorem 6.4, the additional restriction of the variable λ to the intersection of B_+ with a dressing orbit can be achieved by fixing Casimir functions. In fact, this leads to a Poisson subspace of $(\check{G}_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)/G_0$ and the Poisson bracket on this subspace corresponds to the reduced symplectic form exhibited in [19].

Remark 6.6. *With very small modifications, all results of the paper are valid for non-Abelian reductive Lie groups, too, and the simply connectedness of G is also not essential. In the paper [20] we dealt with the important example for which*

$$G = \mathrm{U}(n), \quad G^{\mathbb{C}} = \mathrm{GL}(n,\mathbb{C}), \quad B = \mathrm{B}(n), \tag{6.27}$$

where $\mathrm{B}(n)$ consists of those upper triangular elements of $\mathrm{GL}(n,\mathbb{C})$ whose diagonal entries are real, positive numbers. The bilinear form on the real Lie algebra $\mathrm{gl}(n,\mathbb{C})$ can be taken to be

$$\langle X, Y\rangle_{\mathbb{I}} = \Im\mathrm{tr}(XY), \quad \forall X, Y \in \mathrm{gl}(n,\mathbb{C}). \tag{6.28}$$

In this case $\mathcal{G} = \mathfrak{u}(n)$, and $\mathfrak{P} = \exp(\mathrm{i}\mathfrak{u}(n))$ is the set of positive definite, Hermitian matrices, which is an open *subset of the vector space $\mathrm{i}\mathfrak{u}(n)$ of Hermitian matrices. The vector spaces $\mathrm{i}\mathfrak{u}(n)$ and $\mathfrak{u}(n)$ are in duality with respect to bilinear form (6.28). Thus, for any function $\mathcal{F} \in C^\infty(\mathfrak{P})$, one can define its $\mathfrak{u}(n)$-valued differential $d\mathcal{F}$ by the requirement*

$$\langle Z, d\mathcal{F}(L)\rangle_{\mathbb{I}} = \left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(L + tZ), \qquad \forall Z \in \mathrm{i}\mathfrak{u}(n), \tag{6.29}$$

since $(L+tZ)$ belongs to $\mathfrak{P}$ for sufficiently small t. Relating $\mathcal{F} \in C^\infty(\mathfrak{P})$ to $\varphi \in C^\infty(B)$ by

$$\mathcal{F}(bb^\dagger) = \varphi(b), \qquad \forall b \in \mathrm{B}(n), \tag{6.30}$$

one can verify the following identity:

$$2Ld\mathcal{F}(L) = bD'\varphi(b)b^{-1} \qquad \textit{for} \quad L = bb^\dagger. \tag{6.31}$$

By applying the counterpart of this identity to related functions defined on $\mathbb{M}$ and $\mathfrak{M}$ (2.17), and on $\mathbb{M}_0$ (5.4) and $\mathfrak{M}_0$ (6.1), the formulas (3.7) and (6.4) are converted into those derived in [20], which served as the starting in the joint paper with Marshall [26].

6.2. **Connection with spin Sutherland models.** It is easily seen from the formula (3.3) that $G_0 < G$ is a Poisson–Lie subgroup on which the Poisson bracket vanishes. This entails that the restriction of the dressing action of G on B yields a classical Hamiltonian action of G_0. This action operates by conjugations,

$$\mathrm{Dress}_{\eta_0}(S) = \eta_0 S \eta_0^{-1}, \qquad \forall \eta_0 \in G_0,\, S \in B, \tag{6.32}$$

and is generated by the classical moment map $S \mapsto \log S_0 \in \mathcal{B}_0 \simeq \mathcal{G}_0^*$, which is defined by applying the decomposition $S = S_0\lambda$ with $S_0 \in B_0$ and $\lambda \in B_+$. A particular reduction of the Poisson space $(B, \{-,-\}_B)$ is obtained by setting this moment map to zero. Identifying the smooth functions on B_+/G_0 with $C^\infty(B_+)^{G_0}$, the third term of the Poisson bracket (6.21) represents precisely the reduced Poisson structure arising from $(B, \{-,-\}_B)$ in this way.

The Poisson–Lie group $(B, \{-,-\}_B)$ is a nonlinear analogue of $(\mathcal{G}^*, \{-,-\}_{\mathcal{G}^*})$ equipped with the linear Lie–Poisson bracket. Using the pairing (2.10), $\mathcal{B}$ can be taken as the model of the dual space $\mathcal{G}^*$, and its reduction with respect to G_0 at the zero value of the moment map gives a Poisson structure on $C^\infty(\mathcal{B}_+)^{G_0}$. This is a building block of a linear counterpart of the Poisson bracket (6.21). Denoting the elements of $G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+$ by (Q,p,X), the Poisson bracket at issue has the form

$$\{f,h\}_{\mathrm{lin}}(Q,p,X) = \langle D_Q f, d_p h\rangle_{\mathbb{I}} - \langle D_Q h, d_p f\rangle_{\mathbb{I}} + \langle X, [d_X f, d_X h]\rangle_{\mathbb{I}}, \tag{6.33}$$

where the derivatives of $f,h \in C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+)^{G_0}$ are taken at (Q,p,X), and $d_X f \in \mathcal{G}_\perp \simeq (\mathcal{B}_+)^*$ denotes the differential of f with respect to its third variable.

We recall (see e.g. [53]) that the Poisson algebra

$$(C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+)^{G_0}, \{-,-\}_{\mathrm{lin}}) \tag{6.34}$$

encodes the reduced Poisson structure obtained by taking the quotient of the cotangent bundle T^*G with respect to the conjugation action of G. More precisely, this is true for the dense open subset $T^*G^{\mathrm{reg}}/G \subset T^*G/G$, after further restricting the variable Q to a Weyl alcove $\check{G}_0^{\mathrm{reg}} \subset G_0^{\mathrm{reg}}$ (or considering only those functions that are invariant with respect to the normalizer $\mathfrak{N}$ (5.6)). The cotangent bundle T^*G carries the degenerate integrable system whose main Hamiltonian is the kinetic energy corresponding to the bi-ivariant Riemannian metric on G. The reduction of the kinetic energy yields the spin Sutherland Hamiltonian [14, 53], represented by the following element of the Poisson algebra (6.34):

$$H_{\mathrm{spin-Suth}}(e^{\mathrm{i}q},p,X) = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\Phi^+}\frac{1}{|\alpha|^2}\frac{|X_\alpha|^2}{\sin^2(\alpha(q)/2)} \quad \text{with} \quad X = \sum_{\alpha\in\Phi^+} X_\alpha E_\alpha \in \mathcal{B}_+. \tag{6.35}$$

Here, Φ^+ denotes the set of positive roots of $\mathcal{G}^{\mathbb{C}}$, so that $\mathcal{B}_+$ is the complex span of the root vectors E_α for $\alpha \in \Phi^+$. We employ a Weyl–Chevalley basis [60] of $\mathcal{G}^{\mathbb{C}}$, for which $E_{-\alpha} = -\theta(E_\alpha)$ with the Cartan involution θ (2.13), and $\langle E_\alpha, E_{-\alpha}\rangle = 2/|\alpha|^2$ holds. It is worth stressing that in (6.35) $\mathcal{B}$ is taken as the model of $\mathcal{G}^*$.

The next result clarifies the connection between the Poisson algebras of the spin Sutherland models and our models.

Proposition 6.7. *For any real $\epsilon \neq 0$, let us define the G_0-equivariant diffeomorphism*

$$\mu_\epsilon : G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+ \to G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+, \quad \mu_\epsilon : (Q,p,X) \mapsto (Q, \epsilon p, \exp(\epsilon X)). \tag{6.36}$$

Then, the 'linear Poisson structure' (6.33) is related to the nonlinear one (6.21) according to the formula

$$\{f,h\}_{\mathrm{lin}} = \lim_{\epsilon\to 0} \epsilon \{f\circ\mu_\epsilon^{-1}, h\circ\mu_\epsilon^{-1}\}_0^{\mathrm{red}} \circ \mu_\epsilon. \tag{6.37}$$

Proof. To keep the formulae short, let us focus on functions that do not depend on Q and p. Choosing a basis $\{T^a\}$ of $\mathcal{G}_\perp$, we may use the components

$$X^a = \langle X, T^a\rangle_{\mathbb{I}} \quad \text{and} \quad \sigma^a := \langle \sigma, T^a\rangle_{\mathbb{I}} \quad \text{with} \quad \lambda = e^\sigma \in B_+ \tag{6.38}$$

as coordinates on the respective spaces $\mathcal{B}_+$ and B_+. Both formulae (6.21) and (6.33) define bi-derivations (bivector fields), which can be used to calculate the brackets of arbitrary smooth functions (not only G_0-invariant ones). Adopting the usual summation convention, we can write

$$\{f,h\}_{\mathrm{lin}} = \Pi_{\mathrm{lin}}^{a,c}\partial_a f \partial_c h, \qquad \Pi_{\mathrm{lin}}^{a,c}(X) = \langle X, [T^a, T^c]\rangle_{\mathbb{I}}, \tag{6.39}$$

where ∂_a denotes partial derivative with respect to the coordinate X^a. On the other hand, we obtain

$$\{F,H\}_0^{\mathrm{red}} = \Pi_0^{a,c}\partial_a F \partial_c H, \qquad \Pi_0^{a,c}(\sigma) = \Pi_{\mathrm{lin}}^{a,c}(\sigma) + \mathcal{P}^{a,c}(\sigma), \tag{6.40}$$

where $\mathcal{P}^{a,c}(\sigma)$ is a polynomial in the components of σ, whose lowest order terms are quadratic. Here, the partial derivatives are with respect to the coordinates σ^a. The formula (6.40) follows from (6.21) by means of a simple calculation of

$$\Pi_0^{a,c}(\sigma) := \langle \lambda D'_\lambda \sigma^a \lambda^{-1}, D_\lambda \sigma^c \rangle_{\mathbb{I}}. \tag{6.41}$$

The point is to notice from (6.11) that the derivatives of the coordinate functions σ^a have the form

$$D_\lambda \sigma^a = T^a + \mathcal{P}^a(\sigma), \quad D'_\lambda \sigma^a = T^a + \mathcal{P}'^a(\sigma), \tag{6.42}$$

with certain $\mathcal{G}_\perp$-valued functions $\mathcal{P}^a(\sigma)$ and $\mathcal{P}'^a(\sigma)$ whose components are multivariable polynomials in the coordinate functions, *without constant terms*. By using the chain rule, and noting that the partial derivatives of the components of μ_ϵ^{-1} give ϵ^{-1}-times the unit matrix, (6.40) leads to

$$\left(\epsilon\{f \circ \mu_\epsilon^{-1}, h \circ \mu_\epsilon^{-1}\}_0^{\mathrm{red}} \circ \mu_\epsilon\right)(X) = \frac{1}{\epsilon}\left(\Pi_{\mathrm{lin}}^{a,c}(\epsilon X) + \mathcal{P}^{a,c}(\epsilon X)\right)(\partial_a f)(X)(\partial_c h)(X). \tag{6.43}$$

This implies the claim (6.37) for functions f, h that depend only on $X \in \mathcal{B}_+$. The possible dependence on Q and p is taken into account effortlessly. □

In view of Proposition 6.7, we say that the 'linear structure' (6.33) is the *scaling limit* of the nonlinear one (6.21). Notice that in (6.37) the bracket $\{-,-\}_0^{\mathrm{red}}$ has also been rescaled by ϵ. We put 'linear Poisson structure' in quotation marks, since we are dealing with Poisson brackets of G_0-invariant functions, and no linear function of $X \in \mathcal{B}_+$ is G_0-invariant.

Finally, we explain how the spin Sutherland Hamiltonian (6.35) can be recovered from specific Hamiltonians of our reduced system obtained from the Heisenberg double. For this purpose, we take an arbitrary finite dimensional irreducible representation $\rho : G^{\mathbb{C}} \to \mathrm{SL}(V)$, and introduce an inner product on the complex vector space V in such a way that we have,

$$\rho(K^\dagger) = \rho(K)^\dagger, \qquad \forall K \in G^{\mathbb{C}}, \tag{6.44}$$

where $K^\dagger$ is defined in (2.14) and $\rho(K)^\dagger$ denotes adjoint with respect to the inner product. This ensures that G and $\mathfrak{P}$ are represented by unitary and by positive Hermitian operators, respectively. Then, the character of the representation gives the element $\phi^\rho \in C^\infty(\mathfrak{P})^G$,

$$\phi^\rho(L) := \mathrm{tr}_\rho(L) := c_\rho \mathrm{tr}\rho(L), \qquad \forall L \in \mathfrak{P}. \tag{6.45}$$

Here, c_ρ is a (positive) normalization constant chosen in such a way that the trace taken in the representation reproduces the Killing form, i.e.,

$$\langle X, Y \rangle = c_\rho \mathrm{tr}(\rho(X)\rho(Y)), \qquad \forall X, Y \in \mathcal{G}^{\mathbb{C}}, \tag{6.46}$$

where the Lie algebra representation is also denoted by ρ.

Now, let us express L in terms of the decoupled variables (Q, p, λ) introduced in equation (6.8), with $\lambda \in B_+$ written as $\lambda = \exp(\sigma)$. This yields the Hamiltonian

$$H^\rho(Q, p, \sigma) := \mathrm{tr}_\rho(L(Q, p, \sigma)) \quad \text{with} \quad L(Q, p, \sigma) = e^p b_+(Q, \sigma) b_+(Q, \sigma)^\dagger e^p, \tag{6.47}$$

where $b_+(Q, \sigma)$ is determined by the relation

$$b_+^{-1} Q^{-1} b_+ Q = e^\sigma. \tag{6.48}$$

Proposition 6.8. *The spin Sutherland Hamiltonian* (6.35) *is the scaling limit of* H^ρ (6.47) *as follows:*

$$H_{\mathrm{spin-Suth}} = \lim_{\epsilon \to 0} \frac{1}{4\epsilon^2}(H^\rho \circ \mu_\epsilon - c_\rho \mathrm{dim}_\rho). \tag{6.49}$$

Here, we use the map $\mu_\epsilon : (Q, p, X) \mapsto (Q, \epsilon p, \epsilon X)$*, which is just* (6.36) *written in terms of the exponential parametrization of* B_+*.*

Proof. The proof is based on calculations that appeared in [19] (without the interpretation as a scaling limit). Let us employ the parametrizations

$$b_+ = \exp(\beta), \quad \lambda = \exp(\sigma) \quad \text{with} \quad \beta = \sum_{\alpha \in \Phi^+} \beta_\alpha E_\alpha, \quad \sigma = \sum_{\alpha \in \Phi^+} \sigma_\alpha E_\alpha \tag{6.50}$$

and spell out the relation (6.48) as

$$\exp(-\beta + Q^{-1}\beta Q + \frac{1}{2}[Q^{-1}\beta Q, \beta] + \cdots) = \exp(\sigma), \tag{6.51}$$

which results from the Baker–Campbell–Hausdorff formula. The dots denote higher 'commutators', of which there appear only finitely many, for $\mathcal{B}_+$ is nilpotent. Since the exponential map from $\mathcal{B}_+$ to B_+

is a diffeomorphism, one may use (6.51) to establish the form of the dependence of β on σ and $Q = e^{\mathrm{i}q}$. With the aid of induction according to the height of the roots, one finds [19] that

$$\beta_\alpha = \frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} + \Gamma_\alpha(e^{\mathrm{i}q}, \sigma), \tag{6.52}$$

where Γ_α has the form

$$\Gamma_\alpha = \sum_{k\geq 2} \sum_{\varphi_1,\dots,\varphi_k} f_{\varphi_1,\dots,\varphi_k}(e^{\mathrm{i}q})\sigma_{\varphi_1}\dots\sigma_{\varphi_k}. \tag{6.53}$$

The sum is taken over the unordered collections $\varphi_1, \dots, \varphi_k$ of positive roots, with possible repetitions, such that $\alpha = \varphi_1 + \dots + \varphi_k$. The coefficients $f_{\varphi_1,\dots,\varphi_k}$ are rational functions in $e^{\mathrm{i}\alpha_1(q)}, \dots, e^{\mathrm{i}\alpha_r(q)}$, where $\alpha_1, \dots, \alpha_r$ are the simple roots. By substituting (6.52) of β_α into (6.47), and expanding $\rho(b_+) = \exp(\rho(\beta))$, one obtains the formula

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = c_\rho \mathrm{tr}\left(e^{2\rho(p)}\left(\mathbf{1}_V + \frac{1}{4}\sum_{\alpha\in\Phi^+}\frac{|\sigma_\alpha|^2\rho(E_\alpha)\rho(E_{-\alpha})}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma,\sigma^*)\right)\right). \tag{6.54}$$

Here, $\mathrm{o}_2(\sigma, \sigma^*)$ stands for a finite number of terms that have total degree 3 and higher in the components of σ and their complex conjugates. These terms depend also on $Q = \exp(\mathrm{i}q)$, and $\mathbf{1}_V$ denotes the identity operator on the representation space V. To get (6.54), we used that $\mathrm{tr}\left(\rho(E_\alpha)\rho(E_{-\gamma})e^{2\rho(p)}\right) = 0$ unless $\gamma = \alpha$. Then, by expanding $e^{2\rho(p)}$ as well and noting that $\mathrm{tr}(\rho(p)) = 0$ because $\mathcal{G}^{\mathbb{C}}$ is simple, the claim (6.49) follows from (6.54). □

Remark 6.9. *The standard spin Sutherland Lax matrix can be recovered as the scaling limit of our Lax matrix $L(Q, p, \sigma)$ (6.47). It can be shown that the Hamiltonians H^ρ (6.47) corresponding to the r fundamental highest weight representations of $G^{\mathbb{C}}$ are functionally independent on a dense open subset. Motivated by the presence of the factor e^{2p} in (6.54) and the relation (6.49), H^ρ (6.47) may be called a Hamiltonian of spin Ruijsenaars–Schneider type. We explain in Appendix C that a special case of these Hamiltonians for $G = \mathrm{SU}(n)$ reproduces the standard (spinless) trigonometric Ruijsenaars–Schneider Hamiltonian [59] on a symplectic leaf of the reduced phase space.*

7. Discussion

In this work we continued our previous investigations [19, 21] of Poisson–Lie analogues of spin Sutherland models. We solved an important open question regarding the integrability of these models, and further developed various aspects of the earlier results.

Reduced integrability was argued in [19, 21] by exhibiting a large set of constants of motion, but the precise counting and other technical details were missing. Our principal new results are given by Theorem 4.6 with Corollary 4.7 and Theorem 4.9. Theorem 4.9 states the degenerate integrability of our models on the Poisson manifold $\mathbb{M}_*^{\mathrm{red}}$, which is the smooth component of the reduced phase space corresponding to the principal orbit type for the underlying group action on the Heisenberg double. Theorem 4.6 establishes even stronger properties of the restricted reduced system on the dense open subset $\mathbb{M}_{**}^{\mathrm{red}} \subset \mathbb{M}_*^{\mathrm{red}}$ associated with the subset $\mathfrak{C}_*$ (4.31) of principal orbit type in the space of unreduced constants of motion. Corollary 4.7 deals with the generic symplectic leaves in $\mathbb{M}_{**}^{\mathrm{red}}$.

In addition to the thorough analysis of reduced integrability, we also presented a novel description of the reduced Poisson brackets. This is given by Theorem 5.2, which was derived utilizing the model $\mathbb{M} = G \times \mathfrak{P}$ (2.15) of the Heisenberg double developed in this paper. Then, in Theorem 6.4, we expressed the reduced Poisson brackets in terms of canonically conjugate pairs and decoupled spin variables, and subsequently used this to deepen the previously found [19] connection between our models and the standard spin Sutherland models. The latter models are recovered via the scaling limit characterized by Propositions 6.7 and 6.8.

Turning to open problems, we wish to stress that further work is required to clarify the integrability properties of the restrictions of the reduced systems on arbitrary symplectic leaves of the full reduced phase space. This is true concerning both the spin Sutherland models and their Poisson–Lie deformations. Other challenging problems concern the quantization and the construction of elliptic counterparts of our trigonometric systems. It is well known that the spin Sutherland models can be quantized by combining harmonic analysis on the underlying Lie groups with quantum Hamiltonian reduction [14, 28], and it should be possible to generalize this to our systems.

Throughout the paper, we strove for a careful exposition of the nontrivial technical issues in the hope that the resulting text may serve as a useful starting point for future studies of open problems of the subject. The auxiliary material of the appendices is included having the same goal in mind.

Data availability statement. No new data were created or analysed in this study.

Acknowledgement. I wish to thank Maxime Fairon for useful remarks on the manuscript. This work was supported in part by the NKFIH research grant K134946. The publication was also supported by the University of Szeged Open Access Fund, under Grant Number 6814.

Appendix A. The Poisson action on $\mathfrak{P}_- \times \mathfrak{P}$

Here, we sketch the derivation of the Poisson action (4.19) of G on $\mathfrak{P}_- \times \mathfrak{P}$. We proceed by first deriving an equivalent action on $B \times B$, which we then transfer to $\mathfrak{P}_- \times \mathfrak{P}$ by means of the Poisson diffeomorphism $\mu : B \times B \to \mathfrak{P}_- \times \mathfrak{P}$ given (with ν in (2.16)) by

$$\mu : (b_1, b_2) \mapsto (\nu(b_1^{-1}), \nu(b_2)). \tag{A.1}$$

Our reasoning illustrates how one may find the action starting from a Poisson–Lie moment map.

To begin, we note from (3.4) that the Hamiltonian vector field, V_F, associated with $F \in C^\infty(B \times B)$ by means of the product Poisson structure reads

$$V_F(b_1, b_2) = \left(b_1(b_1^{-1} D_1 F(b_1, b_2) b_1)_{\mathcal{B}}, b_2(b_2^{-1} D_2 F(b_1, b_2) b_2)_{\mathcal{B}}\right), \tag{A.2}$$

where $D_1 F$ and $D_2 F$ are the derivatives with respect to the first and second variable, respectively. Next, we define the Poisson map $J : B \times B \to B$ by $J(b_1, b_2) := b_1 b_2$, and from $\{F, J\} = -V_F[J]$ find

$$\langle X, \{F, J\}_{B\times B}(p) J(p)^{-1}\rangle_{\mathbb{I}} = \langle (b_1^{-1} X b_1)_{\mathcal{B}}, D_1' F(p)\rangle_{\mathbb{I}} + \langle (b_2^{-1}(b_1^{-1} X b_1)_{\mathcal{G}} b_2)_{\mathcal{B}}, D_2' F(p)\rangle_{\mathbb{I}}, \tag{A.3}$$

at any $p = (b_1, b_2) \in B \times B$, for any $X \in \mathcal{G}$. This means that the vector field $X_{B\times B}$ generated by the moment map J has the form

$$X_{B\times B}(b_1, b_2) = \left(\mathrm{dress}_X(b_1), \mathrm{dress}_{(b_1^{-1} X b_1)_{\mathcal{G}}}(b_2)\right). \tag{A.4}$$

We claim that this is the infinitesimal form of the G-action on $B \times B$ defined by the maps

$$\mathcal{A}_\eta(b_1, b_2) = \left(\mathrm{Dress}_\eta(b_1), \mathrm{Dress}_{\Xi_R(\eta b_1)^{-1}}(b_2)\right), \qquad \forall \eta \in G. \tag{A.5}$$

The action property $\mathcal{A}_{\eta_1} \circ \mathcal{A}_{\eta_2} = \mathcal{A}_{\eta_1 \eta_2}$ is proved by using that $\mathrm{Dress}_{\eta_1} \circ \mathrm{Dress}_{\eta_2} = \mathrm{Dress}_{\eta_1 \eta_2}$ and that

$$\Xi_R(\eta_1 \eta_2 b_1)^{-1} = \Xi_R(\eta_1 \mathrm{Dress}_{\eta_2}(b_1))^{-1} \Xi_R(\eta_2 b_1)^{-1}. \tag{A.6}$$

The last equality is verified by substituting

$$\Xi_R(\eta_1 \eta_2 b_1)^{-1} = (\mathrm{Dress}_{\eta_1 \eta_2}(b_1))^{-1} \eta_1 \eta_2 b_2, \tag{A.7}$$

and similarly rewriting the two factors on the right side of (A.6). Having verified that (A.5) is a G-action, it remains to ascertain that

$$X_{B\times B}(b_1, b_2) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}_{e^{tX}}(b_1, b_2). \tag{A.8}$$

The first component of this equality is obvious from (2.24), and second one is seen from

$$\left.\frac{d}{dt}\right|_{t=0} \Xi_R(e^{tX} b_1)^{-1} = \left.\frac{d}{dt}\right|_{t=0} \left((\mathrm{Dress}_{e^{tX}}(b_1))^{-1} e^{tX} b_1\right) = -(b_1^{-1} X b_1)_{\mathcal{B}} + b_1^{-1} X b_1 = (b_1^{-1} X b_1)_{\mathcal{G}}. \tag{A.9}$$

The final step is to convert the action (A.5) on $B \times B$ into the action $\hat{\mathcal{A}}_\eta$ (4.19) on $\mathfrak{P}_- \times \mathfrak{P}$ by means of the map μ (A.1). The desired result, $\hat{\mathcal{A}}_\eta \circ \mu = \mu \circ \mathcal{A}_\eta$, follows immediately from the identity

$$(\mathrm{Dress}_\eta(b_1))^{-1} = \mathrm{Dress}_{\Xi_R(\eta b_1)^{-1}}(b_1^{-1}), \tag{A.10}$$

because ν (2.16) intertwines the dressing action (2.23) on B and the conjugation action on $\mathfrak{P}$. The identity (A.10) itself is obtained by applying Λ_L (2.19) to both sides of the equality

$$(\mathrm{Dress}_\eta(b_1))^{-1} = \Xi_R(\eta b_1)^{-1} b_1^{-1} \eta^{-1}. \tag{A.11}$$

The moment maps $\hat{\Lambda}$ (4.18) and J above are related by $\hat{\Lambda} \circ \mu = J$, and thus we have indeed established the form of the Poisson action on $\mathfrak{P}_- \times \mathfrak{P}$ generated by $\hat{\Lambda}$.

Incidentally, the formula (4.4) of the quasi-adjoint action can be verified following a train of thoughts similar to the one presented in this appendix.

Appendix B. Poisson reduction via the shifting trick

We now explain the origin of the defining equation (6.8) of the 'spin variable' λ by utilizing the so-called shifting trick of Hamiltonian reduction [50]. In the context of Marsden–Weinstein type reductions, the shifting trick means that one first extends the phase space by a coadjoint orbit or dressing orbit, and then reduces the extended phase space at the trivial moment map value. Under mild conditions, the outcome is equivalent to the result of the corresponding 'point reductions' based on taking a moment map value from the 'opposite' orbit.

In our case, we may start with the extended Heisenberg double

$$M_{\text{ext}} := M \times B = \{(K, S) \mid K \in M,\, S \in B\}, \tag{B.1}$$

and equip it with the direct product Poisson structure $\{-,-\}_{\text{ext}}$ built from $\{-,-\}_+$ (3.1) on M and $\{-,-\}_B$ (3.4) on B. This extended phase space carries the extended moment map $\Lambda_{\text{ext}} : M_{\text{ext}} \to B$,

$$\Lambda_{\text{ext}}(K, S) := \Lambda(K)S = \Lambda_L(K)\Lambda_R(K)S, \tag{B.2}$$

which generates a Poisson action of $(G, \{-,-\}_G)$ (3.3) on M_{ext}. Then, we reduce the extended phase space by imposing the moment map constraint

$$\Lambda_{\text{ext}}(K, S) = e. \tag{B.3}$$

By using that on the 'constraint surface' $S = \Lambda(K)^{-1}$, one arrives at the identification

$$\Lambda_{\text{ext}}^{-1}(e)/G \simeq M/G. \tag{B.4}$$

Moreover, for every G-invariant function on $\Lambda_{\text{ext}}^{-1}(e)$ one can define a G-invariant function on M_{ext} in such a way that the extended function does not depend on S. In this way, one obtains the identification

$$C^\infty(\Lambda_{\text{ext}}^{-1}(e))^G \simeq C^\infty(M)^G. \tag{B.5}$$

Essentially because (B.3) represents first class constraints in Dirac's sense [32], the identification (B.5) holds at the level of reduced Poisson algebras as well.

On the other hand, coming to the crux, we may introduce a convenient partial gauge fixing in the moment map constraint surface (B.3) by imposing the condition that $\Xi_R(K) \in G_0$. Then $K \in M$ can be presented as

$$K = (Q^{-1}b^{-1}Q)Q^{-1} = Q^{-1}b^{-1} \quad \text{with} \quad Q = \Xi_R(K) \in G_0,\ b = \Lambda_R(K) \in B. \tag{B.6}$$

On this 'gauge slice', applying the parametrization $b = e^p b_+$ (with $p \in \mathcal{B}_0, b_+ \in B_+$), we get

$$\Lambda(K) = Q^{-1}b^{-1}Qb = Q^{-1}b_+^{-1}Qb_+. \tag{B.7}$$

Consequently, the moment map condition (B.3) becomes

$$b_+^{-1}Q^{-1}b_+Q = S. \tag{B.8}$$

This relation enforces that $S \in B_+$, and after re-naming S to λ we recognize the formula (B.8) as equation (6.8) that we started with in Section 6.1. By imposing the additional condition $Q \in G^{\text{reg}}$, one may ensure that the residual gauge transformations of the partial gauge fixing (B.6) are associated with the normalizer $\mathfrak{N}$ of G_0 (which means that η in (4.4) is restricted so that $\Xi_R(\eta\Lambda_L(K))$ belongs to $\mathfrak{N}$).

The shifting trick was applied in [19] working in the symplectic framework, by restricting the variable S in (B.1) to a dressing orbit of G in B throughout the procedure.

Appendix C. Derivation of the trigonometric RS model

It is known [23] that the standard (real) trigonometric Ruijsenaars–Schneider (RS) model [59] can be derived by a specific Marsden–Weinstein type reduction of the Heisenberg double of the unitary group $\mathrm{U}(n)$. The goal of this appendix is to explain how this result can be recovered in the framework of the present paper. Here, we take $G := \mathrm{SU}(n)$, and obtain the model in the 'center of mass frame'. In this case the group B consists of the upper triangular matrices in $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ having positive diagonal entries. The diagonal elements of the matrices in $B_+ < B$ are equal to 1. The crucial point is that we restrict the variable λ (6.8) to a minimal dressing orbit, of dimension $2(n-1)$, which leads to a symplectic leaf in the reduced phase space. There exists a one parameter family of such orbits, and their parameter will appear as the coupling constant of the RS model.

The minimal dressing orbits at issue admit representatives of the form

$$\Delta(x) := \exp\left(\mathrm{diag}((n-1)x/2, -x/2, \cdots, -x/2)\right), \quad \text{for} \quad x \in \mathbb{R}^*, \tag{C.1}$$

where the eigenvalue $e^{-x/2}$ of $\Delta(x)$ has multiplicity $(n-1)$. Let $\mathcal{O}(x)$ denote the dressing orbit through $\Delta(x)$. The only redundancy of these representatives occurs for $n = 2$, in which case $\Delta(x)$ and $\Delta(-x)$ lie on the same orbit. Another representative of the orbit $\mathcal{O}(x)$ is the upper-triangular matrix $\nu(x) \in B_+$, whose diagonal entries are equal to 1 and

$$\nu(x)_{jk} = (1 - e^{-x}) \exp((k-j)x/2), \qquad \forall j < k. \tag{C.2}$$

This matrix satisfies the equality

$$\nu(-x) = \nu(x)^{-1}. \tag{C.3}$$

More importantly, as was shown in [23], one has

$$\mathcal{O}(x) \cap B_+ = \{T\nu(x)T^{-1} \mid T \in G_0\}, \tag{C.4}$$

where G_0 is the standard maximal torus of $G = \mathrm{SU}(n)$. Thus, $(\mathcal{O}(x) \cap B_+)/G_0$ consists of a single point.

The defining equation (6.8) entails that λ belongs to the subgroup $B_+ < B$, and we know from Theorem 6.4 that λ can be restricted to the intersection of B_+ with any dressing orbit. Now we choose to restrict it to the orbit $\mathcal{O}(-x)$ with a fixed $x \in \mathbb{R}^*$. On account of the relations (C.3) and (C.4), we then obtain a complete fixing of the residual G_0 'gauge freedom' by imposing the condition

$$\lambda = b_+^{-1} Q^{-1} b_+ Q = \nu(x)^{-1}. \tag{C.5}$$

The paper [23] analysed the symplectic reduction of the Heisenberg double based on the moment map constraint

$$\Lambda(K) = \nu(x) \tag{C.6}$$

with Λ in (4.3). After diagonalizing $g_R = \Xi_R(K)$ (2.18), i.e., by setting $g_R = Q = \mathrm{diag}(Q_1, \dots, Q_n) \in G_0$, $K \in \mathrm{SL}(n, \mathbb{C})$ takes the form

$$K = Q^{-1} b^{-1}, \quad \text{with some} \quad b = e^p b_+, \tag{C.7}$$

where $p \in \mathcal{B}_0$ and $b_+ \in B_+$. Then the constraint (C.6) leads precisely to equation (C.5). It was proved in [23] that (C.5) implies that $Q \in G_0^{\mathrm{reg}}$ and b_+ in (C.5) can be expressed in terms of $Q \in G_0^{\mathrm{reg}}$ as follows:

$$(b_+)_{kl} = Q_k \bar{Q}_l \prod_{m=1}^{l-k} \frac{e^{-\frac{x}{2}} \bar{Q}_k - e^{\frac{x}{2}} \bar{Q}_{k+m-1}}{\bar{Q}_k - \bar{Q}_{k+m}}, \quad 1 \le k < l \le n, \tag{C.8}$$

where $\bar{Q}_k = Q_k^{-1}$. Of course, $(b_+)_{kk} = 1$ and the matrix elements below the diagonal are zero. (The correspondence between our notations and those in [23] is explained in the subsequent Remark C.1.)

We have taken the quotient by the G_0-symmetry, but there still remains a residual $S_n = \mathfrak{N}/G_0$ redundancy in our description. Consequently, the variables Q and p parametrize an S_n covering space of a Poisson subspace of the full reduced phase space. Since λ became a constant by the gauge fixing, it follows from (6.21) that the Poisson bracket on this covering space is given by the formula

$$\{F, H\}^{\mathrm{red}}(Q, p) = \langle D_Q F, d_p H \rangle_{\mathbb{I}} - \langle D_Q H, d_p F \rangle_{\mathbb{I}}, \tag{C.9}$$

which corresponds to the symplectic form

$$\omega_{\mathrm{red}} = \Im \mathrm{tr}(dp \wedge Q^{-1} dQ). \tag{C.10}$$

The elements of S_n permute the n diagonal entries of the matrix Q. However, a careful analysis [24] shows that their action on $p = \mathrm{diag}(p_1, \dots, p_n)$ has a complicated form, and what are permuted in the obvious manner are the entries of the traceless diagonal matrix ϑ given by the following formula:

$$\vartheta_k = 2p_k - \frac{1}{2} \sum_{m<k} \ln \left[1 + \frac{\sinh^2(x/2)}{\sin^2(q_k - q_m)} \right] + \frac{1}{2} \sum_{m>k} \ln \left[1 + \frac{\sinh^2(x/2)}{\sin^2(q_k - q_m)} \right], \quad k = 1, \dots, n. \tag{C.11}$$

Here, we use the parametrization

$$Q = \exp(2\mathrm{i}q) \quad \text{with} \quad q = \mathrm{diag}(q_1, \dots, q_n), \quad \mathrm{tr}(q) = 0. \tag{C.12}$$

Equation (C.11) yields a canonical transformation, since in terms of q and ϑ one has

$$\omega_{\mathrm{red}} = \mathrm{tr}(d\vartheta \wedge dq). \tag{C.13}$$

This means that Q and ϑ are the natural variables on $T^* G_0^{\mathrm{reg}}$.

We are also reducing the 'free Hamiltonians' given by the dressing invariant functions of $b_R = b = e^p b_+$. The main Hamiltonian of the reduced system is

$$H_{\mathrm{RS}} = \frac{1}{2}(H_{+,\mathrm{RS}} + H_{-,\mathrm{RS}}), \quad \text{with} \quad H_{+,\mathrm{RS}} = \mathrm{tr}(bb^\dagger), \quad H_{-,\mathrm{RS}} = \mathrm{tr}(bb^\dagger)^{-1}. \tag{C.14}$$

By using (C.8) and the canonical transformation (C.11) to express p in terms of q and ϑ, one finds

$$H_{\pm,\mathrm{RS}}(q,\vartheta)=\sum_{k=1}^{n}e^{\pm\vartheta_k}\prod_{m\neq k}\left[1+\frac{\sinh^2(x/2)}{\sin^2(q_k-q_m)}\right]^{\frac{1}{2}}, \tag{C.15}$$

and $H_{\mathrm{RS}}(q,\vartheta)$ is just the standard trigonometric Ruijsenaars–Schneider Hamiltonian introduced in [59]. In conclusion, we have shown that the reduction of the Heisenberg double gives the trigonometric RS system on a symplectic leaf of the reduced phase space, which is symplectomorphic to $(T^*G_0^{\mathrm{reg}})/S_n$. It is worth noting that this system is Liouville integrable, but is not superintegrable [58].

Remark C.1. *It follows from the results of* [23] *that the isotropy group of the elements* $\Lambda^{-1}(\mathcal{O}(x))$ *is the center of* $G=\mathrm{SU}(n)$*, i.e.,* $\Lambda^{-1}(\mathcal{O}(x))$ *is a subset of* M_* (4.8)*. In that paper the variables* b_L *and* g_R *constituting* $K=b_L g_R^{-1}$ (2.18) *were used, while here we mostly worked with* $b\equiv b_R$ *and* $g\equiv g_R$*. After bringing* g_R *into* G_0*, the relation of the variables becomes* $b_L=Q^{-1}b^{-1}Q$*. In* [23] b_L *was parametrized as* $b_L=\mathcal{N}a$ *with* $\mathcal{N}\in B_+$ *and* $a\in B_0$*. These are related to our variables* b_+ *and* p *by* $\mathcal{N}^{-1}=Q^{-1}b_+Q$ *and* $a=e^{-p}$*. Finally, for readers interested in a detailed comparison with* [23, 24]*, we also note that the notations* T_k *and* $\zeta_k=\ln a_k$ *in* [23] *correspond to* Q_k *and* $-p_k$ *as used in the present paper; and what is denoted by* p_k *in* [23] *corresponds to* ϑ_k (C.11)*.*

References

[1] A.Yu. Alekseev, *On Poisson actions of compact Lie groups on symplectic manifolds.* J. Diff. Geom. **45** (1997) 241-256; arXiv:dg-ga/9602001

[2] A.Yu. Alekseev and A.Z. Malkin, *Symplectic structures associated to Lie–Poisson groups.* Commun. Math. Phys. **162** (1994) 147-173; arXiv:hep-th/9303038

[3] A. Alekseev, Y. Kosmann-Schwarzbach and E. Meinrenken, *Quasi-Poisson manifolds.* Canad. J. Math. **54** (2002) 3-29; arXiv:math/0006168

[4] A. Alekseev, A. Malkin and E. Meinrenken, *Lie group valued moment maps.* J. Differential Geom. **48** (1998) 445-495; arXiv:dg-ga/9707021

[5] S. Arthamonov and N. Reshetikhin, *Superintegrable systems on moduli spaces of flat connections.* Commun. Math. Phys. **386** (2021) 1337-1381; arXiv:1909.08682

[6] G. Arutyunov, Elements of Classical and Quantum Integrable Systems, Springer, 2019

[7] G.E. Arutyunov, S.A. Frolov and P.B. Medvedev, *Elliptic Ruijsenaars-Schneider model from the cotangent bundle over the two-dimensional current group.* J. Math. Phys. **38** (1997) 5682-5689; arXiv:hep-th/9608013

[8] G. Arutyunov and E. Olivucci, *Hyperbolic spin Ruijsenaars–Schneider model from Poisson reduction.* Proc. Steklov Inst. Math. **309** (2020) 31-45; arXiv:1906.02619

[9] O. Chalykh and M. Fairon, *On the Hamiltonian formulation of the trigonometric spin Ruijsenaars–Schneider system.* Lett. Math. Phys. **110** (2020) 2893-2940; arXiv:1811.08727

[10] V. Chari and A.N. Pressley, A Guide to Quantum Groups, Cambridge University Press, 1995

[11] V.G. Drinfel'd, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations.* Soviet Math. Dokl. **27** (1983) 68-71

[12] J.J. Duistermaat and J.A.C. Kolk, Lie Groups, Springer, 2000

[13] P. Etingof, Calogero–Moser Systems and Representation Theory, European Mathematical Society, 2007

[14] P.I. Etingof, I.B. Frenkel and A.A. Kirillov Jr., *Spherical functions on affine Lie groups.* Duke Math. J. **80** (1995) 59-90; arXiv:hep-th/9407047

[15] P. Etingof and A. Varchenko, *Geometry and classification of solutions of the classical dynamical Yang–Baxter equation.* Commun. Math. Phys. **192** (1998) 77-120; arXiv:q-alg/9703040

[16] M. Fairon and L. Fehér, *Integrable multi-Hamiltonian systems from reduction of an extended quasi-Poisson double of* U(n). Ann. Henri Poincaré **24** (2023) 3461-3529; preptrint arXiv:2302.14392

[17] M. Fairon, L. Fehér and I. Marshall, *Trigonometric real form of the spin RS model of Krichever and Zabrodin.* Ann. Henri Poincaré **22** (2021) 615–675; arXiv:2007.08388

[18] F. Fasso, *Superintegrable Hamiltonian systems: geometry and perturbations.* Acta Appl. Math. **87** (2005) 93-121

[19] L. Fehér, *Poisson–Lie analogues of spin Sutherland models.* Nucl. Phys. B **949** (2019) Paper No. 114807; arXiv:1809.01529

[20] L. Fehér, *Reduction of a bi-Hamiltonian hierarchy on* $T^*\mathrm{U}(n)$ *to spin Ruijsenaars–Sutherland models.* Lett. Math. Phys. **110** (2020) 1057-1079; arXiv:1908.02467

[21] L. Fehér, *Poisson reductions of master integrable systems on doubles of compact Lie groups.* Ann. Henri Poincaré **24** (2023) 1823-1876; arXiv:2208.03728

[22] L. Fehér, Notes on the degenerate integrability of reduced systems obtained from the master systems of free motion on cotangent bundles of compact Lie groups, preprint arXiv:2309.16245

[23] L. Fehér and C. Klimčík, *Poisson–Lie generalization of the Kazhdan-Kostant-Sternberg reduction.* Lett. Math. Phys. **87** (2009) 125-138; arXiv:0809.1509

[24] L. Fehér and C. Klimčík, *Poisson–Lie interpretation of trigonometric Ruijsenaars duality.* Commun. Math. Phys. **301** (2011) 55-104; arXiv:0906.4198

[25] L. Fehér and C. Klimčík, *Self-duality of the compactified Ruijsenaars–Schneider system from quasi-Hamiltonian reduction.* Nucl. Phys. B **860** (2012) 464-515; arXiv:1101.1759

[26] L. Fehér and I. Marshall, *On the bi-Hamiltonian structure of the trigonometric spin Ruijsenaars–Sutherland hierarchy.* Geometric Methods in Physics XXXVIII, eds. P. Kielanowski et al, Birkhauser, pp. 75-87, 2020; arXiv:2007.09658

[27] L. Fehér and B.G. Pusztai, *A class of Calogero type reductions of free motion on a simple Lie group.* Lett. Math. Phys. **79** (2007) 263-277; arXiv:math-ph/0609085

[28] L. Fehér and B.G. Pusztai, *Twisted spin Sutherland models from quantum Hamiltonian reduction.* J. Phys. A: Math. Theor. **41** (2008) Paper No. 194009; arXiv:0711.4015

[29] V.V. Fock and A.A. Rosly, *Poisson structure on moduli of flat connections on Riemann surfaces and the r-matrix.* Moscow Seminar in Mathematical Physics, AMS Transl. Ser. 2, Vol. 191, pp. 67-86, 1999; arXiv:math/9802054

[30] I. Fris, V. Mandrosov, Ya.A. Smorodinsky, M. Uhlir M and P. Winternitz, *On higher symmetries in quantum mechanics.* Phys. Lett. **16** (1965) 354–356

[31] A. Gorsky and N. Nekrasov, *Relativistic Calogero-Moser model as gauged WZW theory.* Nucl. Phys. B **436** (1995) 582-608; arXiv:hep-th/9401017

[32] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, 1992

[33] B. Jovanovic, *Symmetries and integrability.* Publ. Institut Math. **49** (2008) 1-36; arXiv:0812.4398

[34] D. Kazhdan, B. Kostant and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type.* Comm. Pure Appl. Math. **31** (1978) 481-507

[35] S. Kharchev, A. Levin, M. Olshanetsky and A. Zotov, *Quasi-compact Higgs bundles and Calogero–Sutherland systems with two types spins.* J. Math. Phys. **59** (2018) Paper No. 103509; arXiv:1712.08851

[36] C. Klimčík, *On moment maps associated to a twisted Heisenberg double.* Rev. Math. Phys. **18** (2006) 781-821; arXiv:math-ph/0602048

[37] A.W. Knapp, Lie Groups Beyond an Introduction, Birkhäuser, 1996

[38] C. Laurent-Gengoux, E. Miranda and P. Vanhaecke, *Action-angle coordinates for integrable systems on Poisson manifolds.* Int. Math. Res. Not. **2011** 1839-1869; arXiv:0805.1679 [math.SG]

[39] J-H. Lu, *Multiplicative and Affine Poisson Structures on Lie Groups*, PhD thesis, Berkeley, 1990

[40] J-H. Lu, *Momentum mappings and reduction of Poisson actions.* Symplectic Geometry, Groupoids, and Integrable Systems, Springer, pp. 209-226, 1991

[41] I. Marshall, *A new model in the Calogero-Ruijsenaars family.* Commun. Math. Phys. **338** (2015) 563-587; arXiv:1311.4641

[42] E. Meinrenken, *Verlinde formulas for nonsimply conected groups.* Lie Groups, Geometry, and Representation Theory, Progress in Mathematics (Birkhäuser) **326** (2018) 381-417; arXiv:1706.04045

[43] P.W. Michor, Topics in Differential Geometry, American Mathematical Society, 2008

[44] A.S. Mischenko and A.T. Fomenko, *Generalized Liouville method for integrating Hamiltonian systems.* Funct. Anal. Appl. **12** (1978) 113-125

[45] W. Miller Jr, S. Post and P. Winternitz, *Classical and quantum superintegrability with applications.* J. Phys. A **46** (2013) Paper No. 423001; arXiv:1309.2694

[46] N.N. Nekhoroshev, *Action-angle variables and their generalizations.* Trans. Moscow Math. Soc. **26** (1972) 180-197

[47] N. Nekrasov, *Infinite-dimensional algebras, many-body systems and gauge theories.* Moscow Seminar in Mathematical Physics, AMS Transl. Ser. 2, Vol. 191, Amer. Math. Soc., pp. 263-299, 1999

[48] A. Oblomkov, *Double affine Hecke algebras and Calogero-Moser spaces.* Represent. Theory **8** (2004) 243-266; arXiv:math/0303190

[49] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras.* Phys. Rept. **71** (1981) 313-400

[50] J.-P. Ortega and T. Ratiu, Momentum Maps and Hamiltonian Reduction, Birkhäuser, 2004

[51] A.M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkhäuser, 1990

[52] A.P. Polychronakos, *Physics and mathematics of Calogero particles.* J. Phys. A: Math. Gen. **39** (2006) 12793-12845; arXiv:hep-th/0607033

[53] N. Reshetikhin, *Degenerate integrability of spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems.* Lett. Math. Phys. **63** (2003) 55-71; arXiv:math/0202245

[54] N. Reshetikhin, *Degenerately integrable systems.* J. Math. Sci. **213** (2016) 769-785; arXiv:1509.00730

[55] N. Reshetikhin, *Spin Calogero–Moser models on symmetric spaces.* Integrability, Quantization, and Geometry. I. Integrable Systems, Proc. Sympos. Pure Math., Vol. 103.1, Amer. Math. Soc., pp. 377-402, 2021; arXiv:1903.03685

[56] N. Reshetikhin, *Periodic and open classical spin Calogero–Moser chains.* preprint arXiv:2302.14281

[57] G. Rudolph and M. Schmidt, Differential Geometry and Mathematical Physics. Part I. Manifolds, Lie Groups and Hamiltonian Systems, Springer, 2013

[58] S.N.M. Ruijsenaars, *Action-angle maps and scattering theory for some finite-dimensional integrable systems III. Sutherland type systems and their duals.* Publ. RIMS **31** (1995) 247-353

[59] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons.* Ann. Phys. **170** (1986) 370-405

[60] H. Samelson, Notes on Lie Algebras, third corrected edition, Springer, 1990

[61] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction.* Ann. of Math. **134** (1991) 375-422

[62] M.A. Semenov-Tian-Shansky, *Dressing transformations and Poisson group actions.* Publ. RIMS **21** (1985) 1237-1260

[63] M.A. Semenov-Tian-Shansky, *Integrable systems: an r-matrix approach.* Kyoto preprint RIMS-1650, 2008; kurims.kyoto-u.ac.jp/preprint/file/RIMS1650.pdf

[64] S. Wojciechowski, *An integrable marriage of the Euler equations with the Calogero–Moser system.* Phys. Lett. A **111** (1985) 101-103