A NEW APPROACH IN TWO-DIMENSIONAL HEAVY-TAILED DISTRIBUTIONS

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Abstract. We consider a new approach in the definition of two-dimensional heavy-tailed distributions. Namely, we introduce the classes of two-dimensional long-tailed, of two-dimensional dominatedly varying and of two-dimensional consistently varying distributions. Next, we define the closure property with respect to two-dimensional convolution and to joint max-sum equivalence in order to study if they are satisfied by these classes. Further we examine the joint behavior of two random sums, under generalized tail asymptotic independence. Afterward we study the closure property under scalar product and two dimensional product convolution and by these results we extended our main result in the case of jointly randomly weighted sums. Our results contained some applications where we establish the asymptotic expression of the ruin probability in a two-dimensional discrete-time risk model.

Keywords: two-dimensional heavy-tailed distributions; closure with respect to convolution; joint max-sum equivalence; generalized tail asymptotic independence; ruin probability.

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1. Introduction

1.1. **Preliminaries.** The heavy-tailed distributions describe precisely complicated situations. One of most important application is related to the risk theory in actuarial science. Although several one-dimensional problems remain still open, the multidimensional case meets popularity from both theoretical and practical aspect. Especial, with respect to practical point of view, the modern insurance industry does not operate with a single portfolio.

On this line there are some recent papers, as for example [18], [21], [44]. On this direction, we introduce some two-dimensional distribution classes, with heavy tails, that are convenient to calculations and permit direct and consistent generalization of the one-dimensional concepts.

In subsection 1.2, we remind some basic definitions, for one-dimensional heavy-tailed distributions, for easy comparison with the two-dimensional ones. In section 2, we introduce the closure property with respect to the two-dimensional convolution and the two-dimensional max-sum equivalence. Next, we present some results on these classes of distributions. In section 3, we estimate the joint asymptotic behavior of two random sums, under a dependence structure that generalizes the tail asymptotic independence, and we establish an asymptotic expression for the ruin probabilities, in a discrete-time two-dimensional risk model without stochastic discount factors. Furthermore in section 5 we study the closure property of some of new classes with respect to scalar product, and in section 6 we extended some of our results in section 4, in the case wich we have a common discount factor for the two portfolios. Last but not least we limited ourselves in the non-negative case and we study the closure

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property of new classes with respect to product convolution in two dimensions, and some previous results are extended.

We denote by $\overline{F} := 1-F$ the distribution tail, hence $\overline{F}(x) = \mathbf{P}[X > x]$ and holds $\overline{F}(x) > 0$ for any $x \geq 0$, except it is referred differently. For two positive functions $f(x)$ and $g(x)$, the asymptotic relation $f(x) = o[g(x)]$, as $x \to \infty$ means

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0\,,$$

the asymptotic relation $f(x) = O[g(x)]$, as $x \to \infty$ holds if

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty\,.$$

and the asymptotic relation $f(x) \asymp g(x)$, as $x \to \infty$ if both $f(x) = O[g(x)]$ and $g(x) = O[f(x)]$. For a real number x, y, we denote $x^+ := \max\{x, 0\}$, $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$.

1.2. **Uni-variate heavy-tailed distributions.** The following properties are to be extended in two dimensions.

(1) For two random variables X_1, X_2 with distributions F_1, F_2 respectively, the distribution of the sum is defined by $F_{X_1+X_2}(x) = \mathbf{P}[X_1 + X_2 \leq x]$ with tail $\overline{F_{X_1+X_2}}(x) = \mathbf{P}[X_1 + X_2 > x]$. If X_1, X_2 are independent, we write $F_1 * F_2$ instead of $F_{X_1+X_2}$.
(2) We say that the random variables X_1, X_2 or their distributions F_1, F_2 are max-sum equivalent if $\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x)$, as $x \to \infty$.

Now we consider some classes of heavy-tailed distributions. We say that a distribution F is heavy-tailed, and we write $F \in \mathcal{K}$, if holds

$$\int_{-\infty}^{\infty} e^{\varepsilon\, x}\, F(dx) = \infty\,,$$

for any $\varepsilon > 0$. A large enough class of heavy-tailed distributions is the class of long tails, denoted by $\mathcal{L}$. We have $F \in \mathcal{L}$ if holds

$$\lim_{x\to\infty} \frac{\overline{F}(x-a)}{\overline{F}(x)} = 1\,,$$

for any (or equivalently for some) $a > 0$. It is well-known that if $F \in \mathcal{L}$, then there exists a function $a\;:\;[0,\,\infty) \longrightarrow [0,\,\infty)$, such that $a(x) \to \infty$, $\overline{F}(x \pm a(x)) \sim \overline{F}(x)$, as $x \to \infty$. This kind of function $a(x)$ is called insensitivity function for F, see further in [13] or [20].

A little smaller class than $\mathcal{L}$ is the class of subexponential distributions, introduced in [9]. We say that a distribution F with support the interval $[0,\,\infty)$ belongs to the class of subexponential distributions, symbolically $F \in \mathcal{S}$ if holds

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n\,, \tag{1.1}$$

for any $n \in \mathbb{N}$, where $\overline{F^{n*}}$ represents the n-th order convolution power for F. The class $\mathcal{S}$ has found several applications in the risk models, as for example in [29], [16], [19].

We say that the distribution F belongs to the class of the dominatedly varying distributions, symbolically $F \in \mathcal{D}$, if holds

$$\limsup_{x\to\infty} \frac{\overline{F}(b\,x)}{\overline{F}(x)} < \infty\,,$$

for some (or equivalently for all) $b \in (0,\,1)$. Is well known that $\mathcal{D} \cap \mathcal{L} = \mathcal{D} \cap \mathcal{S} \subset \mathcal{K}$, see [17, Th.1].

Further, a smaller class of heavy-tailed distributions represents the class of consistently varying distributions, symbolically $F \in \mathcal{C}$. We say that $F \in \mathcal{C}$, if holds

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(y\,x)}{\overline{F}(x)} = 1\,, \tag{1.2}$$

or equivalently

$$\lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(y\,x)}{\overline{F}(x)} = 1\,.$$

Finally, we say that a distribution F belongs to the class of regularly varying distributions, with index $\alpha > 0$, symbolically $\overline{F} \in \mathcal{R}_{-\alpha}$ if holds

$$\lim_{x\to\infty} \frac{\overline{F}(t\,x)}{\overline{F}(x)} = x^{-\alpha}\,,$$

for any $t > 0$.

For these classes we obtain the following inclusions (see [2])

$$\mathcal{R} := \bigcup_{\alpha>0} \mathcal{R}_{-\alpha} \subsetneq \mathcal{C} \subsetneq \mathcal{D} \cap \mathcal{L} \subsetneq \mathcal{S} \subsetneq \mathcal{L} \subsetneq \mathcal{K}\,.$$

We can find numerous classes of heavy-tailed distributions, however we mentioned the most popular in the literature. In this paper we extend into two dimensions the classes $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{L}$.

In [3] we find the following results.

Proposition 1.1. *If $F_1 \in \mathcal{D}$ and $F_2 \in \mathcal{D}$ are distributions with support the interval* $[0,\,\infty)$*, then $F_{X_1+X_2} \in \mathcal{D}$.*

In Proposition 1.1 we find that for non-negative random variables, the class $\mathcal{D}$ satisfies the closure property with respect to convolution. As was mentioned in [3], the class $\mathcal{D}$ does NOT satisfy the max-sum equivalence, as it follows from the fact that $\mathcal{D} \not\subset \mathcal{S}$ and $\mathcal{S} \not\subset \mathcal{D}$, thence the relation $\overline{F^{2*}}(x) \sim 2\,\overline{F}(x)$, as $x \to \infty$, does NOT hold for $F \in \mathcal{D} \setminus \mathcal{S}$. In opposite to the dominated variation, the class of the consistently varying distributions satisfy both these properties.

Proposition 1.2. *If $F_1 \in \mathcal{C}$ and $F_2 \in \mathcal{C}$ are distributions with support the interval* $[0,\,\infty)$*, then holds $F_1 * F_2 \in \mathcal{C}$ and $\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x)$, as $x \to \infty$.*

In Propositions 1.1 and 1.2, the random variables with distributions F_1, F_2, are arbitrarily dependent.

2. Two-dimensional heavy tails

The reason why the multivariate distributions have been so popular is their ability to describe better multidimensional processes. This happens because of the interdependence among the components of the random vectors, the affect significantly on the final outcome.

The first heavy-tailed distributions class that was extended to multidimensional frame is the regular variation. We say that the random vector $\mathbf{X} = (X_1, \ldots, X_d)$ represents a multivariate regularly varying vector with index α and measure ν, symbolically $\mathbf{X} \in MRV(\alpha, F, \nu)$ if holds

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \mathbf{P}\left[\frac{\mathbf{X}}{x} \in \mathbb{B}\right] = \nu(\mathbb{B})\,,$$

for any $\mathbb{B} \subset [0, \infty]^d \setminus \{\mathbf{0}\}$, with $F \in \mathcal{R}_{-\alpha}$ and the measure ν is homogeneous, namely holds $\nu(\lambda\,\mathbb{B}) = \lambda^{-\alpha}\,\nu(\mathbb{B})$, for any $\lambda > 0$.

The frame of multivariate regular variation was introduced in [12]. Under this definition, the multivariate regular variation was used in the study of several issues in multivariate risk models and in risk management, as for example in [26], [39], [44].

Although this kind of extension to multidimensional setup is well-established, it does not happen to other multidimensional distribution classes. Most of the extensions cover the multivariate subexponential distribution class and the multivariate long tailed distribution class.

Initially in [10] was introduced these two distribution classes, as essential extension of the multivariate regular variation, namely using vague convergence and point processes. Later, in [31] appear three different formulations for the multivariate subexponentiality and the multivariate long-tailedness. The formulations, that are close to our definitions, are given in classes $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{L}(\mathbb{R}^d)$. We say that the multivariate distribution F belongs to class $\mathcal{S}(\mathbb{R}^d)$, if holds

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(\mathbf{t}\,x)}{\overline{F}(\mathbf{t}\,x)} = 2\,,$$

for any $\mathbf{x} > \mathbf{0}$, with $\min_{1\leq i\leq d}\{t_i\} < \infty$, and that the multivariate distribution F belongs to class $\mathcal{L}(\mathbb{R}^d)$, if holds

$$\lim_{x\to\infty} \frac{\overline{F}(\mathbf{t}\,x - \mathbf{a})}{\overline{F}(\mathbf{t}\,x)} = 1\,,$$

for any $\mathbf{a} \in \mathbb{R}^d$ and for any $\mathbf{t} > \mathbf{0}$, with $\min_{1\leq i\leq d}\{t_i\} < \infty$.

This approach was used to study the asymptotic behavior of the tail of randomly stopped sum of random vectors, namely $\mathcal{S}_N = \sum_{i=1}^{N} \mathbf{X}_i$, where N is a discrete random variable with support $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the $\mathbf{X}_i$ are independent, identically distributed random vectors with multivariate distribution F. For applications of this class, see in [32].

Finally, another formulation of multivariate subexponential distributions was provided in [33], which represents the only approach with results for the ruin probability in a multivariate continuous-time risk model.

In the present paper we confine ourselves in the two dimensions and and we stay close to the formulation in [31], however we keep two important differences. At first, we follow a direct approach to the uni-variate distribution class definition. At second, in the case of $d = 2$ the formulation in [31] and in the definition of multivariate regular variation adopts the convention $\mathbf{F}(x,\, y) = \mathbf{P}[X \leq x,\, Y \leq y]$ and the distribution tail $1-\mathbf{F}(x,\, y)$, that means the distribution tail $\overline{\mathbf{F}}$ can include the event $\{X > x\} \cup \{Y > y\}$. We consider only the case, in which there exist excesses of both random variables $\{X > x\} \cap \{Y > y\}$, namely we define by $\overline{\mathbf{F}}_{\mathbf{1}}(x,\, y) = \mathbf{P}[X > x,\, Y > y]$, as the distribution tail of $\mathbf{F}$, with notation $\overline{\mathbf{F}}_{\mathbf{b}}(x,\, y) = \mathbf{P}[X > b_1\, x,\, Y > b_2\, y]$, for all $\mathbf{b} = (b_1,\, b_2) \in (0,\, \infty)^2$, see [23, sec. 7].

This choice of definition, is due to both the consistency with the univariate case and the easiness in asymptotic calculation of the tail of joint random sums as well. We intent that our approach becomes more consistent with the ruin of all portfolios, which represents the worst event which can happen for an insurance company with multiple businesses.

Next, we introduce the bi-variate heavy-tailed distribution class.

Definition 2.1. *We say that the random pair (X, Y) with marginal distributions F, G belongs to the bi-variate long-tailed distributions, symbolically $(F,\, G) \in \mathcal{L}^{(2)}$, if hold*

(1) *$F \in \mathcal{L}$ and $G \in \mathcal{L}$.*
(2) *It holds*

$$\lim_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_{\mathbf{1}}(x-a_1,\, y-a_2)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\, y)} = \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x-a_1,\, Y > y-a_2]}{\mathbf{P}[X > x,\, Y > y]} = 1\,,$$

for some, or equivalently for any, $\mathbf{a} = (a_1,\, a_2) > (0, 0)$, with a_1 not necessarily equal to a_2.

From Definition 2.1 we obtain that, if $(F,\, G) \in \mathcal{L}^{(2)}$, then for any $(A_1,\, A_2) > (0,\, 0)$ holds

$$\sup_{|a_1|<A_1,\,|a_2|<A_2} \left|\mathbf{P}[X > x-a_1,\, Y > y-a_2] - \mathbf{P}[X > x,\, Y > y]\right| = o\left(\mathbf{P}[X > x,\, Y > y]\right), \tag{2.1}$$

as $x \wedge y \to \infty$, which follows from the uniformity of the convergence

$$\lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x-a_1\,,\; Y > y-a_2]}{\mathbf{P}[X > x\,,\; Y > y]} = 1\,, \tag{2.2}$$

over the parallelogram $[-A_1\,,\; A_1] \times [-A_2\,,\; A_2]$. Definition 2.2 provides the insensitivity property in joint distributions, see the uni-variate analogue for example in [13] or in [20].

Definition 2.2. *Let $a(x) > 0$ for any $x > 0$ be a non-decreasing function. The joint distribution $\mathbf{F} = (F,\, G)$, with right endpoint $r_{\mathbf{F}} := (r_F,\, r_G) = (\infty,\, \infty)$, is called a-joint insensitivity, if*

$$\sup_{|a_1|\leq a(x),|a_2|\leq a(y)} \left|\mathbf{P}[X > x-a_1, Y > y-a_2] - \mathbf{P}[X > x, Y > y]\right| = o\left(\mathbf{P}[X > x, Y > y]\right) \tag{2.3}$$

as $x \wedge y \to \infty$, which follows from uniformity of (2.2), for any $|a_1| \leq a(x)$ and any $|a_2| \leq a(y)$.

Now we show that class $\mathcal{L}^{(2)}$ satisfies the a-joint insensitive property.

Lemma 2.1. *Let assume that $(F,\, G) \in \mathcal{L}^{(2)}$. Then there exists some function $a(x)$ such that $a(x) \to \infty$, as $x \to \infty$, and $(F,\, G)$ satisfies the a-joint insensitive property.*

Proof. For any integer $n \in \mathbb{N}$, from relation (2.1) we can choose a sequence $\{u_n\}$ such that the inequality

$$\sup_{|a_1|\leq n,\,|a_2|\leq n} \left|\mathbf{P}[X > x-a_1,\, Y > y-a_2] - \mathbf{P}[X > x,\, Y > y]\right| \leq \frac{\mathbf{P}[X > x,\, Y > y]}{n}\,,$$

holds for any $x \geq u_n$ and any $y \geq u_n$. Without loss of generality we consider that the sequence $\{u_n\}$ increases to infinity. We put $a(x) = a(y) = n$, for any $(x,\, y) \in (u_n,\, u_{n+1}]^2$. From the fact that $u_n \to \infty$, as $n \to \infty$, we obtain that $a(x) \to \infty$, as $x \to \infty$, and $a(y) \to \infty$, as $y \to \infty$.

Whence, form the construction of $a(\cdot)$ we conclude that

$$\sup_{|a_1|\leq a(x),\,|a_2|\leq a(y)} |\mathbf{P}[X > x - a_1,\, Y > y - a_2] - \mathbf{P}[X > x,\, Y > y]| \leq \frac{\mathbf{P}[X > x,\, Y > y]}{n},$$

for any $x > u_n$ and any $y > u_n$, which is the required result. □

Remark 2.1. *From the a-joint insensitivity does not follow necessarily the a insensitivity for the marginal distributions. Furthermore Lemma 2.1 asserts that*

$$\lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x \pm a(x)\,,\; Y > y \pm a(y)]}{\mathbf{P}[X > x\,,\; Y > y]} = 1\,.$$

Let see now two examples, that help either to understanding or to constructing of such bi-variate distributions. In first case is the simplest, as we construct $(X,\, Y) \in \mathcal{L}^{(2)}$ through the independence between X and Y.

Example 2.1. *Let X and Y be random variables with distributions $F \in \mathcal{L}$ and $G \in \mathcal{L}$ respectively. We assume that X and Y are independent, thence*

$$\begin{aligned}
\lim_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_{\mathbf{1}}(x - a_1,\, y - a_2)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\, y)} &= \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x - a_1,\, Y > y - a_2]}{\mathbf{P}[X > x,\, Y > y]} \\
&= \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x - a_1]}{\mathbf{P}[X > x]}\, \frac{\mathbf{P}[Y > y - a_2]}{\mathbf{P}[Y > y]} = 1\,.
\end{aligned}$$

Therefore $(F,\, G) \in \mathcal{L}^{(2)}$.

The next example makes sense, as it can not be reduced into uni-variate distributions. The following dependence structure can be found in [28]. We say that the random variables X and Y are strongly asymptotic independent (SAI) if hold $\mathbf{P}[X^- > x,\, Y > y] = O[F(-x)\,\overline{G}(y)]$, $\mathbf{P}[X > x,\, Y^- > y] = O[\overline{F}(x)\, G(-y)]$, as $x \wedge y \to \infty$, and there exists a constant $C > 0$ such that holds

$$\mathbf{P}[X > x,\, Y > y] \sim C\, \overline{F}(x)\, \overline{G}(y)\,, \tag{2.4}$$

as $x \wedge y \to \infty$.

If the X and Y are bounded from below, then (2.4) is enough to be SAI.

Example 2.2. *Let X and Y be random variables with strongly asymptotic independence, with some constant $C > 0$ and distributions $F \in \mathcal{L}$ and $G \in \mathcal{L}$ respectively. Then*

$$\begin{aligned}
\lim_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_{\mathbf{1}}(x - a_1,\, y - a_2)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\, y)} &= \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[X > x - a_1,\, Y > y - a_2]}{\mathbf{P}[X > x,\, Y > y]} \\
&= \lim_{x\wedge y\to\infty} \frac{C\, \overline{F}(x - a_1)\, \overline{G}(y - a_2)}{C\, \overline{F}(x)\, \overline{G}(y)} = 1\,.
\end{aligned}$$

Therefore $(F,\, G) \in \mathcal{L}^{(2)}$.

We can find several dependence structures that satisfy the $\mathcal{L}^{(2)}$ condition. However, we choose to pursue theoretical results.

Now we pass to the bi-variate subexponential distribution class $\mathcal{S}^{(2)}$.

Definition 2.3. *We say that the random pair (X, Y), with marginal distributions F and G respectively, belongs to the class of bi-variate subexponential distributions, symbolically $(F, G) \in \mathcal{S}^{(2)}$, if*

(1) *$F \in \mathcal{S}$ and $G \in \mathcal{S}$.*
(2) *It holds*

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X_1 + X_2 > x\,,\ Y_1 + Y_2 > y]}{\mathbf{P}[X > x\,,\ Y > y]} = 2^2\,, \tag{2.5}$$

where X_1 and X_2 are independent and identically distributed and Y_1 and Y_2 are independent and identically distributed, with distributions F and G respectively.

Remark 2.2. *In case of d-variate distribution relation* (2.5) *becomes*

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X_{1,1} + X_{1,2} > x_1\,,\ \ldots,\ X_{d,1} + X_{d,2} > x_d]}{\mathbf{P}[X_{1,1} > x_1\,,\ \ldots,\ X_{d,1} > x_d]} = 2^d\,. \tag{2.6}$$

From this definition we obtain easily that $\mathcal{S}^{(2)} \subsetneq \mathcal{L}^{(2)}$.

Now we come to the bi-variate dominatedly varying distribution class $\mathcal{D}^{(2)}$.

Definition 2.4. *We say that the random pair (X, Y), with marginal distributions F and G respectively, belongs to the class of bi-variate dominatedly varying distributions, symbolically $(F, G) \in \mathcal{D}^{(2)}$, if*

(1) *$F \in \mathcal{D}$ and $G \in \mathcal{D}$.*
(2) *It holds*

$$\limsup_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_{\mathbf{b}}(x,\,y)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\,y)} = \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X > b_1\,x,\, Y > b_2\,y]}{\mathbf{P}[X > x,\, Y > y]} < \infty\,, \tag{2.7}$$

for some, or equivalently for all $\mathbf{b} = (b_1,\, b_2) \in (0,\, 1)^2$, with b_1 not necessarily equal to b_2.

It is obvious that 2.7 is equivalently with:

$$\liminf_{(x,y)\to(\infty,\infty)} \frac{\overline{\mathbf{F}}_{\mathbf{b}}(x,\,y)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,y)} > 0$$

for some, or equivalently for all $\mathbf{b} = (b_1,\, b_2) \in (1,\, \infty)^2$, with b_1 not necessarily equal to b_2.

Remark 2.3. *In* [23] *was introduced the class $\mathcal{D}_n$ (for some $n \in \mathbb{N}$) of multivariate dominatedly varying random vectors. It is obvious that in case $n = 2$ our approach include this definition. Namely*

$$\mathcal{D}_2 \subset \mathcal{D}^{(2)}$$

Definition 2.5. *We say that the random pair (X, Y), with marginal distributions F and G respectively, belongs to the class of bi-variate consistently varying distributions, symbolically $(F, G) \in \mathcal{C}^{(2)}$, if*

(1) *$F \in \mathcal{C}$ and $G \in \mathcal{C}$.*

(2) *It holds*

$$\lim_{\mathbf{z}\uparrow\mathbf{1}} \limsup_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_{\mathbf{z}}(x,\, y)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\, y)} = 1\,,$$

or equivalently

$$\lim_{\mathbf{z}\downarrow\mathbf{1}} \liminf_{x\wedge y\to\infty} \frac{\overline{\mathbf{F}}_{\mathbf{z}}(x,\, y)}{\overline{\mathbf{F}}_{\mathbf{1}}(x,\, y)} = 1\,,$$

where $\mathbf{z} = (z_1,\, z_2)$*, and* $\mathbf{1} = (1,\, 1)$.

Examples 2.1 and 2.2 remain in tact in classes $\mathcal{D}^{(2)}$ and $\mathcal{C}^{(2)}$, hence they keep functioning in class $(\mathcal{D}\cap\mathcal{L})^{(2)} := \mathcal{D}^{(2)} \cap \mathcal{L}^{(2)}$.

3. Max-sum equivalence and closure properties with respect to convolution

In this section we present two definitions. In the first one, we define the closure property with respect to convolution in bi-variate distributions. In this case we formulate the main result, showing that the class $\mathcal{D}^{(2)}$ is closed but the class $\mathcal{L}^{(2)}$ is not. The second definition, given at the end of section, under concrete dependence structures, also presented later, is fulfilled with respect to classes $(\mathcal{D}\cap\mathcal{L})^{(2)}$ and $\mathcal{C}^{(2)}$.

Definition 3.1. *Let* $X_1,\, X_2,\, Y_1,\, Y_2$ *be random variables, with distributions* $F_1,\, F_2,\, G_1$*and* G_2 *respectively. If the following conditions are true*

(1) $F_1 \in \mathcal{B}$*,* $F_2 \in \mathcal{B}$*,* $G_1 \in \mathcal{B}$*,* $G_2 \in \mathcal{B}$ *and for any* $k,\, l \in \{1,\, 2\}$*, holds* $(F_k,\, G_l) \in \mathcal{B}^{(2)}$*,*
(2) *Holds* $(F_{X_1+X_2},\, G_{Y_1+Y_2}) \in \mathcal{B}^{(2)}$*,*

where $\mathcal{B}^{(2)}$ *is some bi-variate class, defined in section 2, then we say that the class* $\mathcal{B}^{(2)}$ *is closed with respect to convolution.*

We wonder whether the closure property from Definition 3.1 follows trivially, in the sense that it comes directly from the corresponding uni-variate case. Although, this is true for the condition (1) in Definition 3.1, for example if $(F_{X_1+X_2},\, G_{Y_1+Y_2}) \in \mathcal{L}^{(2)}$ and also $F_1 \in \mathcal{L}$, $G_1 \in \mathcal{L}$, $F_2 \in \mathcal{L}$, $G_2 \in \mathcal{L}$, then to obtain the $F_1 * F_2 \in \mathcal{L}$ and $G_1 * G_2 \in \mathcal{L}$, can be reduced to uni-variate convolution problem, but for the second condition (2) in Definition 3.1, the task is far from trivial. Let start, with a corollary, where we can see clearly that the class $\mathcal{L}^{(2)}$ is not closed with respect to convolution. Evermore, this set up shows that Definition 3.1 is NOT a simple extension of the closure property in uni-variate case, as the dependence conditions play crucial role.

Corollary 3.1. *Let* $X_1,\, X_2,\, Y_1,\, Y_2$ *be positive random variables, with distributions* $F_1,\, F_2,$ $G_1,\, G_2$ *from class* $\mathcal{L}$*, respectively. We assume that all the random variables are mutually independent except* X_1 *and* Y_1*, which are SAI, with some constant* $C > 0$*. Then we obtain* $(F_1 * F_2,\, G_1 * G_2) \notin \mathcal{L}^{(2)}$*. Therefore, class* $\mathcal{L}^{(2)}$ *is not closed with respect to convolution.*

Proof. We observe that $(F_k,\, G_l) \in \mathcal{L}^{(2)}$ for any $k,\, l \in \{1,\, 2\}$, which follows by Examples 2.1 and 2.2. For the first condition, because of independence of X_1 and X_2 and independence of Y_1 and Y_2, with distributions belonging to class $\mathcal{L}$, we find that $F_1 * F_2 \in \mathcal{L}$ and $G_1 * G_2 \in \mathcal{L}$, see for example in [13].

For the second condition, we examine if holds

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X_1+X_2>x-a_1,\, Y_1+Y_2>y-a_2]}{\mathbf{P}[X_1+X_2>x,\, Y_1+Y_2>y]} = 1\,,$$

for any $(a_1,\, a_2)\in\mathbb{R}^2$. In case $a_1>0$ and $a_2>0$, then

$$\begin{aligned}\mathbf{P}[X_1+X_2>x-a_1,\, Y_1+Y_2>y-a_2] &\geq \mathbf{P}[X_1>x-a_1,\, Y_1>y-a_2]\\ &\sim \mathbf{P}[X_1>x,\, Y_1>y]\sim C\cdot\mathbf{P}[X>x]\,\mathbf{P}[Y_1>y]\,,\end{aligned}$$

as $x\wedge y\to\infty$, which follows from the fact that the random variables are non-negative, $(F_k,\, G_l)\in\mathcal{L}^{(2)}$ for any $k,\, l\in\{1,\, 2\}$ and the assumption of SAI $X_1,\, Y_1$.

We also obtain

$$\mathbf{P}[X_1+X_2>x,\, Y_1+Y_2>y]\leq\mathbf{P}[X_1+X_2>x]\leq\mathbf{P}[X_1>x/2]+\mathbf{P}[X_2>x/2]\,,$$

therefore

$$\frac{\mathbf{P}[X_1+X_2>x-a,\, Y_1+Y_2>y-a]}{\mathbf{P}[X_1+X_2>x,\, Y_1+Y_2>y]}\gtrsim\frac{C\cdot\mathbf{P}[X>x]\,\mathbf{P}[Y_1>y]}{\mathbf{P}[X_1>x/2]+\mathbf{P}[X_2>x/2]}\,, \tag{3.1}$$

as $x\wedge y\to\infty$. We wonder, if the right hand side in equation (3.1) can take value greater than unity:

$$\frac{C\cdot\mathbf{P}[X>x]\,\mathbf{P}[Y_1>y]}{\mathbf{P}[X_1>x/2]+\mathbf{P}[X_2>x/2]}>1\,,$$

but this is equivalent to

$$C>\frac{\mathbf{P}[X_1>x/2]+\mathbf{P}[X_2>x/2]}{\mathbf{P}[X>x]\,\mathbf{P}[Y_1>y]}\,,$$

which can be written as

$$C>\frac{\mathbf{P}[X_2>x/2]}{\mathbf{P}[X>x]\,\mathbf{P}[Y_1>y]}>\frac{\mathbf{P}[X_2>x/2]}{\mathbf{P}[Y_1>y]}\geq\frac{\mathbf{P}[X_2>x]}{\mathbf{P}[Y_1>y]}\,.$$

If random variable Y_1 has heavier distribution tail than X_2, then the last fraction in last relation tends to zero, as $x\wedge y\to\infty$. Hence, $C>0$ that means it is possible to find such a combination, that the lower bound becomes greater than unity, namely it holds $(F_1*F_2,\, G_1*G_2)\notin\mathcal{L}^{(2)}$. □

Remark 3.1. *The previous example shows, that for class $\mathcal{B}^{(2)}$ in general the proof of closure property does NOT follows directly from the closure property in the corresponding uni-dimensional distributions. A crucial role is played by the fact that usually we do not know the dependence structure between X_1+X_2 and Y_1+Y_2, even when we know the dependence structure between particular components.*

Next we see that the class $\mathcal{D}^{(2)}$ is closed with respect to convolution (of arbitrarily dependent random vectors with arbitrarily non-negative dependent components).

Theorem 3.1. *Let non-negative random variables $X_1,\, X_2,\, Y_1,\, Y_2$ with distributions $F_1,\, F_2,$ G_1 and G_2 from class $\mathcal{D}$ respectively. We assume that $(X_k,\, Y_l)\in\mathcal{D}^{(2)}$ for any $k,\, l\in\{1,\, 2\}$, then $(F_{X_1+X_2},\, G_{Y_1+Y_2})\in\mathcal{D}^{(2)}$.*

Proof. At first, for the first condition of $\mathcal{D}^{(2)}$, we obtain $F_1 * F_2 \in \mathcal{D}$ and $G_1 * G_2 \in \mathcal{D}$, because of Proposition 1.1.

Taking into consideration that all the distributions have support the interval $[0, \infty)$, we find

$$\begin{aligned}\mathbf{P}[X_1 + X_2 > x,\, Y_1 + Y_2 > y] &\leq \mathbf{P}[X_1 > x/2,\, Y_1 + Y_2 > y] + \mathbf{P}[X_2 > x/2,\, Y_1 + Y_2 > y]\\ &\leq \mathbf{P}\left[X_1 > \frac{x}{2},\, Y_1 > \frac{y}{2}\right] + \mathbf{P}\left[X_2 > \frac{x}{2},\, Y_2 > \frac{y}{2}\right]\\ &\quad + \mathbf{P}\left[X_2 > \frac{x}{2},\, Y_1 > \frac{y}{2}\right] + \mathbf{P}\left[X_2 > \frac{x}{2},\, Y_2 > \frac{y}{2}\right],\end{aligned}$$

hence

$$\mathbf{P}[X_1 + X_2 > x,\, Y_1 + Y_2 > y] \leq \sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > \frac{x}{2},\, Y_l > \frac{y}{2}\right]. \tag{3.2}$$

From the other side

$$\begin{aligned}\mathbf{P}[X_1 + X_2 > x,\, Y_1 + Y_2 > y] &\geq \frac{1}{2}\left(\mathbf{P}[X_1 > x,\, Y_1 + Y_2 > y] + \mathbf{P}[X_2 > x,\, Y_1 + Y_2 > y]\right)\\ &\geq \frac{1}{4}\left(\mathbf{P}[X_1 > x,\, Y_1 > y] + \mathbf{P}[X_1 > x,\, Y_2 > y]\right.\\ &\quad \left.+ \mathbf{P}[X_2 > x,\, Y_1 > y] + \mathbf{P}[X_2 > x,\, Y_2 > y]\right),\end{aligned}$$

from where we obtain

$$\mathbf{P}[X_1 + X_2 > x,\, Y_1 + Y_2 > y] \geq \frac{1}{4}\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}[X_k > x,\, Y_l > y]. \tag{3.3}$$

Therefore by relations (3.2) and (3.3), due to $(X_k,\, Y_l) \in \mathcal{D}^{(2)}$ for any $k,\, l \in \{1,\, 2\}$, we find

$$\begin{aligned}&\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X_1 + X_2 > x/2,\, Y_1 + Y_2 > y/2]}{\mathbf{P}[X_1 + X_2 > x,\, Y_1 + Y_2 > y]}\\ &\leq 4 \limsup_{x\wedge y\to\infty} \frac{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}[X_k > x/2,\, Y_l > y/2]}{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}[X_k > x,\, Y_l > y]}\\ &\leq 4 \max_{k,\, l \in \{1,\, 2\}} \left\{\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}[X_k > x/2,\, Y_l > y/2]}{\mathbf{P}[X_k > x,\, Y_l > y]}\right\} < \infty.\end{aligned}$$

So we conclude $(F_{X_1+X_2},\, G_{Y_1+Y_2}) \in \mathcal{D}^{(2)}$. $\square$

Now we are ready to define the max-sum equivalence in two dimensions.

Definition 3.2. *Let $X_1,\, X_2,\, Y_1,\, Y_2$ be random variables. Then we say that they are jointly max-sum equivalent if*

$$\mathbf{P}[X_1 + X_2 > x,\, Y_1 + Y_2 > y] \sim \sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}[X_k > x,\, Y_l > y],$$

as $x \wedge y \to \infty$.

This kind of asymptotic relation is to be established for classes $(\mathcal{D} \cap \mathcal{L})^{(2)}$ and $\mathcal{C}^{(2)}$, under the assumption of non-negative support and some specific dependence structure.

4. Joint behavior of random sums

In one dimension, the following asymptotic relation attracted attention

$$\mathbf{P}\left[\sum_{i=1}^{n} X_i > x\right] \sim \sum_{i=1}^{n} \mathbf{P}[X_i > x]\,, \tag{4.1}$$

as $x \to \infty$. Therefore, we study the behavior of both, the maximum $\bigvee_{i=1}^{n} X_i$ and the maximum of sums

$$\bigvee_{i=1}^{n} S_i := \max_{1\leq k\leq n} \sum_{i=1}^{k} X_i\,,$$

for some distributions and correspondingly with some dependence structures to examine if holds

$$\mathbf{P}\left[\sum_{i=1}^{n} X_i > x\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x\right] \sim \sum_{i=1}^{n} \mathbf{P}\left[X_i > x\right]\,, \tag{4.2}$$

as $x \to \infty$. The relations (4.1) and (4.2) have been studied for example in [1], [14], [15], [30]. A similar interest has been appeared for weighted sums of the form

$$S_n(\Theta) = \sum_{i=1}^{n} \Theta_i\, X_i\,, \qquad T_n(\Delta) = \sum_{j=1}^{n} \Delta_j\, Y_j\,,$$

and for the circumstances when they satisfy relations (4.1) and (4.2), see for example [37], [38], [43], [45].

In this section we study the relation (4.2) in two dimensions. This can be achieved for the class $(\mathcal{D}\cap\mathcal{L})^{(2)}$ under generalized tail asymptotic dependence. Although the uni-variate random weighted sums are well studied, this is not true for the multivariate case.

Let mention some papers, involved in the asymptotic behavior of the joint probability

$$\mathbf{P}\left[\sum_{i=1}^{n} \Theta_i\, X_i > x,\ \sum_{j=1}^{n} \Delta_i\, Y_i > y\right]\,,$$

as for example [6], [28], [34], [35], [41].

We restrict ourselves at moment, in the study of non-weighted random sums of the following form

$$\mathbf{P}\left[\sum_{i=1}^{n} X_i > x,\ \sum_{j=1}^{n} Y_j > y\right]\,.$$

We note that in all these papers the dependence structure for the main variables X_i, Y_j is either of the form: $\{(X_i,\, Y_i)\,,\ i \in \mathbb{N}\}$ independent random vectors and there exists some dependence structure in each random pair, or there exists dependence among $X_1,\,\ldots,\,X_n$ and $Y_1,\,\ldots,\,Y_n$, but the X_i and Y_j are independent for any $i,\ j$. Using generalized tail asymptotic independence (GTAI), introduced in [22], both dependence structures are simultaneously permitted. GTAI is defined as follows. Let consider two sequences of random variables $\{X_n,\ n \in \mathbb{N}\}\,,\ \{Y_m,\ m \in \mathbb{N}\}$. We say that the random variables $X_1,\,\ldots,\,X_n,\,Y_1,\,\ldots,\,Y_m$ follow the generalized tail asymptotic independence, if

(1) It holds

$$\lim_{\min\{x_i,\,x_k,\,y_j\}\to\infty} \mathbf{P}[|X_i| > x_i \mid X_k > x_k,\, Y_j > y_j] = 0\,,$$

for any $1 \leq k \notin \{i,\, j\} \leq n$.

(2) It holds

$$\lim_{\min\{x_i,\,y_k,\,y_j\}\to\infty} \mathbf{P}[|Y_j| > y_j \mid X_i > x_i,\, Y_k > y_k] = 0\,,$$

for any $1 \leq k \notin \{i,\, j\} \leq m$.

The aim of this dependence structure is the modeling the dependence both, in each sequence of random variables and in the interdependence between the sequences. We have to notice that if the X_i and Y_j are independent for any $i,\, j$, then each sequence of random variables follows tail asymptotic dependence (TAI), see definition bellow, however in any other case the GTAI does not restrict each sequence to TAI, but in a more general form of dependence.

It is easy to find that GTAI contains the case when $X_1,\,\ldots,\,X_n$ are independent or when $Y_1,\,\ldots,\,Y_n$ are independent or both. Even more this dependence structure indicates that the probability to happen three extreme events, is negligible with respect to probability to happen two extreme events, one in each sequence.

In the most of our results we use the TAI dependence structure as an extra assumption wich characterised the dependence of the terms of each sequence. This dependence structure was introduced by [15]. We say that they are Tail Asymptotic Independent, symbolically TAI, and in some works named as strong quasi-asymptotically independent, if for any pair $i,\, j = 1,\,\ldots,\,n$, with $i \neq j$ holds the limit

$$\lim_{x_i \wedge x_j \to\infty} \mathbf{P}[|X_i| > x_i \mid X_j > x_j] = 0\,.$$

Next result provides an asymptotic relation for the maximum of two sequences of random variables under the GTAI, WITHOUT to impose some assumption for the distributions of $X_1,\,\ldots,\,X_n,\,Y_1,\,\ldots,\,Y_m$, (except the infinite right point).

Theorem 4.1. *If $X_1,\,\ldots,\,X_n$ are random variables with distributions $F_1,\,\ldots,\,F_n$ respectively and $Y_1,\,\ldots,\,Y_m$ are random variables with distributions $G_1,\,\ldots,\,G_m$ and $X_1,\,\ldots,\,X_n$, $Y_1,\,\ldots,\,Y_m$ are GTAI then holds*

$$\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x\,,\ \bigvee_{j=1}^{m} Y_j > y\right] \sim \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x\,,\ Y_j > y\right]\,, \tag{4.3}$$

as $x \wedge y \to \infty$.

Proof. For $x > 0$, $y > 0$ holds

$$\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x\,,\ \bigvee_{j=1}^{m} Y_j > y\right] \leq \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x\,,\ Y_j > y\right]\,. \tag{4.4}$$

Further for the lower bound we use Bonferroni inequality

$$\begin{aligned}
&\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x\,,\ \bigvee_{j=1}^{m} Y_j > y\right]\\
&\geq \sum_{i=1}^{n} \mathbf{P}\left[X_i > x\,,\ \bigvee_{j=1}^{m} Y_j > y\right] - \sum\sum\nolimits_{l\neq i=1}^{n} \mathbf{P}\left[X_i > x\,,\ X_l > x\,,\ \bigvee_{j=1}^{m} Y_j > y\right]\\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x\,,\ Y_j > y\right] - \sum_{i=1}^{n}\sum\sum\nolimits_{k\neq j=1}^{m} \mathbf{P}\left[X_i > x\,,\ Y_j > y\,,\ Y_k > y\right]\\
&-\sum\sum\nolimits_{l\neq i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x\,,\ X_l > x\,,\ Y_j > y\right]\\
&=: I_1(x,\, y) - I_2(x,\, y) - I_3(x,\, y)\,.
\end{aligned}$$

For $I_2(x,\, y)$ we obtain

$$\begin{aligned}
I_2(x,\, y) &= \sum_{i=1}^{n}\sum\sum\nolimits_{k\neq j=1}^{m} \mathbf{P}\left[X_i > x\,,\ Y_j > y\,,\ Y_n > y\right]\\
&= \sum_{i=1}^{n}\sum\sum\nolimits_{k\neq j=1}^{m} \mathbf{P}\left[Y_k > y \mid X_i > x\,,\ Y_j > y\right]\,\mathbf{P}\left[X_i > x\,,\ Y_j > y\right]\\
&= o\left(\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x\,,\ Y_j > y\right]\right) = o(I_1(x,\, y))\,,
\end{aligned}$$

as $x \wedge y \to \infty$, where in the last step we use the GTAI property. In a similar way we can find

$$I_3(x,\, y) = o(I_1(x,\, y))\,,$$

as $x \wedge y \to \infty$. Hence we conclude

$$\mathbf{P}\left[\bigvee_{i=1}^{n} X_i > x\,,\ \bigvee_{j=1}^{m} Y_j > y\right] \gtrsim \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{P}\left[X_i > x\,,\ Y_j > y\right]\,, \tag{4.5}$$

as $x \wedge y \to \infty$. Now, from relations (4.4) and (4.5) we have the result. □

Before next theorem, we need some preliminary lemmas. Next lemma provides an important property of the GTAI structure, presenting itself as closure property with respect to sum.

Lemma 4.1. *If $X_1,\, \ldots,\, X_n,\, Y_1,\, \ldots,\, Y_m$ follow the generalized tail asymptotic independence (GTAI), then holds*

$$\lim_{\min\{x_I,\, x_k,\, y_j\}\to\infty} \mathbf{P}\left[\left|\sum_{i\in I} X_i\right| > x_I \;\middle|\; X_k > x_k\,,\ Y_j > y_j\right] = 0\,, \tag{4.6}$$

for $I \subsetneq \{1,\,\ldots,\,n\}$ *and* $k \in \{1,\,\ldots,\,n\} \setminus I$. *Similarly holds*

$$\lim_{\min\{x_i,\,y_k,\,y_J\}\to\infty} \mathbf{P}\left[\left|\sum_{j\in J} Y_j\right| > y_J \;\Big|\; Y_k > y_k\,,\; X_i > x_i\right] = 0\,, \tag{4.7}$$

for $J \subsetneq \{1,\,\ldots,\,m\}$ *and* $k \in \{1,\,\ldots,\,m\} \setminus J$.

Proof. It is enough to show relation (4.6) as relation (4.7) follows by similar way. Indeed, we observe that

$$\lim_{\min\{x_I,\,x_k,\,y_j\}\to\infty} \mathbf{P}\left[\left|\sum_{i\in I} X_i\right| > x_I \;\Big|\; X_k > x_k\,,\; Y_j > y_j\right]$$
$$\leq \lim_{\min\{x_I,\,x_k,\,y_j\}\to\infty} \mathbf{P}\left[|X_i| > \frac{x_I}{n} \;\Big|\; X_k > x_k\,,\; Y_j > y_j\right] = 0\,,$$

where the last step follows from GTAI property. □

From here and after we study only the case $n = m$. In the next lemma, we find the lower asymptotic bound of the joint tail of the random sums

$$S_n := \sum_{k=1}^{n} X_k\,, \qquad T_n := \sum_{l=1}^{n} Y_l\,,$$

when the summands follow distributions with long tails and the $\mathcal{L}^{(2)}$ property is true for any pair of the summands distribution. A similar result, for the uni-dimensional case, can be found in [15], where the dependence structure is TAI (tail asymptotic independence). In the next result we find generalization to two dimensions and furthermore the GTAI assumption, represents a wider dependence structure than the TAI for each one of the two sequences separately. Next, we introduce the notations

$$S_{n,k} := S_n - X_k\,, \qquad T_{n,l} := T_n - Y_l\,,$$

for some $k \in \{1,\,\ldots,\,n\}$ and some $l \in \{1,\,\ldots,\,n\}$. In what follows, we can choose

$$a := \min\left\{\bigwedge_{i=1}^{n} a_{F_i}\,,\; \bigwedge_{j=1}^{n} a_{G_j}\right\}\,, \tag{4.8}$$

namely the minimum of all the insensitivity functions, that means that the function $a(x)$ is insensitive for all the distributions $F_1,\,\ldots,\,F_n,\,G_1,\,\ldots,\,G_n$ of the corresponding random variables $X_1,\,\ldots,\,X_n,\,Y_1,\,\ldots,\,Y_n$.

Lemma 4.2. *Let* $X_1,\,\ldots,\,X_n,\,Y_1,\,\ldots,\,Y_n$ *be random variables with distributions* $F_1,\,\ldots,\,F_n$, $G_1,\,\ldots,\,G_n$ *from class* $\mathcal{L}$ *respectively. We also assume that* $X_1,\,\ldots,\,X_n,\,Y_1,\,\ldots,\,Y_n$ *satisfy the GTAI property and holds*

$$(X_k,\,Y_l) \in \mathcal{L}^{(2)}\,,$$

for any $k,\,l \in \{1,\,\ldots,\,n\}$. *Then holds*

$$\mathbf{P}\left[S_n > x\,,\; T_n > y\right] \gtrsim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]\,,$$

as $x \wedge y \to \infty$.

Proof. We choose as $a(x)$ a function with jointly insensitivity property for any random pair $(X_k,\, Y_l)$ for any $k,\, l \in \{1,\, \ldots,\, n\}$. A possible choice of this function is

$$a := \min_{1\leq k,\, l\leq n} a_{k,\,l}\,,$$

where $a_{k,l}$ is the jointly insensitivity function of the random pair $(X_k,\, Y_l)$ for any $k,\, l \in \{1,\, \ldots,\, n\}$. Next, we apply twice inequality Bonferroni to obtain

$$\begin{aligned}
&\mathbf{P}\left[S_n > x\,,\ T_n > y\right] \geq \mathbf{P}\left[S_n > x\,,\ T_n > y\,,\ \bigvee_{k=1}^{n} X_k > x + a(x)\,,\ \bigvee_{l=1}^{n} Y_l > y + a(y)\right] \\
&\geq \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[S_n > x\,,\ T_n > y\,,\ X_k > x + a(x)\,,\ Y_l > y + a(y)\right] \qquad (4.9) \\
&-\sum \sum_{1\leq k<i\leq n} \sum_{l=1}^{n} \mathbf{P}\left[X_i > x + a(x)\,,\ X_k > x + a(x)\,,\ Y_l > y + a(y)\right] \\
&-\sum_{k=1}^{n} \sum \sum_{1\leq l<i\leq n} \mathbf{P}\left[X_k > x + a(x)\,,\ Y_l > y + a(y)\,,\ Y_i > y + a(y)\right] \\
&=: \sum_{i=1}^{3} J_i(x,\, y)\,.
\end{aligned}$$

Now for $J_2(x,\, y)$ we find

$$\begin{aligned}
&\mathbf{P}\left[X_i > x + a(x)\,,\ X_k > x + a(x)\,,\ Y_l > y + a(y)\right] \\
&= \mathbf{P}\left[X_i > x + a(x) \mid X_k > x + a(x)\,,\ Y_l > y + a(y)\right]\, \mathbf{P}\left[X_k > x + a(x)\,,\ Y_l > y + a(y)\right] \\
&= o(\mathbf{P}\left[X_k > x\,,\ Y_l > y\right])\,,
\end{aligned}$$

as $x \wedge y \to \infty$, that follows from GTAI property, $\mathcal{L}^{(2)}$ membership and the definition of function $a(\cdot)$. Thence

$$J_2(x,\, y) = o(\mathbf{P}\left[X_k > x\,,\ Y_l > y\right])\,, \tag{4.10}$$

as $x \wedge y \to \infty$. Similarly, due to symmetry, we have

$$J_3(x,\, y) = o(\mathbf{P}\left[X_k > x\,,\ Y_l > y\right])\,, \tag{4.11}$$

as $x \wedge y \to \infty$. $\square$

The next result shows that in the non-negative part of class $(\mathcal{D}\cap\mathcal{L})^{(2)}$ the property of jointly max-sum equivalence as also under an extra assamption the closure property with respect to convolution are satisfied, as soon as the GTAI holds.

Lemma 4.3. *Let $X_1,\, X_2,\, Y_1,\, Y_2$ be non-negative random variables, with the following distributions $F_1,\, F_2,\, G_1,\, G_2$ from class $\mathcal{D}\cap\mathcal{L}$ respectively. Further we assume that the random variables $X_1,\, X_2,\, Y_1,\, Y_2$ satisfy the GTAI and*

$$(X_k,\, Y_l) \in (\mathcal{D}\cap\mathcal{L})^{(2)}\,,$$

for any $k,\, l \in \{1,\, 2\}$ *properties. Then hold*

$$\mathbf{P}\left[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y\right] \sim \sum_{k=1}^{2} \sum_{l=1}^{2} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]\,, \tag{4.12}$$

as $x \wedge y \to \infty$*. If further* X_1, X_2 *are TAI and* Y_1, Y_2 *are TAI then:*

$$(X_1 + X_2,\, Y_1 + Y_2) \in (\mathcal{D} \cap \mathcal{L})^{(2)}\,,$$

Proof. We start with relation (4.12). At first we choose the function $a(x)$ as before in the proof of Lemma 4.2. For the lower bound we obtain

$$\begin{aligned}
&\mathbf{P}[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y] \geq \mathbf{P}[X_1 > x\,,\; Y_1 + Y_2 > y] + \mathbf{P}[X_2 > x\,,\; Y_1 + Y_2 > y] \\
&-\mathbf{P}[X_1 > x\,,\; X_2 > x\,,\; Y_1 + Y_2 > y] \geq \mathbf{P}[X_1 > x\,,\; Y_1 > y] + \mathbf{P}[X_1 > x\,,\; Y_2 > y] \\
&-\mathbf{P}[X_1 > x\,,\; Y_1 > y\,,\; Y_2 > y] + \mathbf{P}[X_2 > x\,,\; Y_1 > y] + \mathbf{P}[X_2 > x\,,\; Y_2 > y] \\
&-\mathbf{P}[X_2 > x\,,\; Y_1 > y\,,\; Y_2 > y] - \mathbf{P}[X_1 > x\,,\; X_2 > x\,,\; Y_1 > y - a(y)] \\
&-\mathbf{P}[X_1 > x\,,\; X_2 > x\,,\; Y_2 > y - a(y)] + \mathbf{P}[X_1 > x\,,\; X_2 > x\,,\; Y_1 > y/2\,,\; Y_2 > a(y)] \\
&+\mathbf{P}[X_1 > x\,,\; X_2 > x\,,\; Y_1 > a(y)\,,\; Y_2 > y/2] = I_1(x,\, y) + I_2(x,\, y) - I_3(x,\, y) \\
&+I_4(x,\, y) + I_5(x,\, y) - I_6(x,\, y) - I_7(x,\, y) - I_8(x,\, y) - I_9(x,\, y) - I_{10}(x,\, y)\,.
\end{aligned}$$

Here we can see the following equality

$$I_1(x,\, y) + I_2(x,\, y) + I_4(x,\, y) + I_5(x,\, y) = \sum_{k=1}^{2} \sum_{l=1}^{2} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]\,. \tag{4.13}$$

and further we estimate the third term as follows

$$\begin{aligned}
I_3(x,\, y) &= \mathbf{P}[X_1 > x\,,\; Y_1 > y\,,\; Y_2 > y] \\
&= \mathbf{P}[Y_1 > y \mid X_1 > x\,,\; Y_2 > y]\,\mathbf{P}[X_1 > x\,,\; Y_2 > y] \\
&= o(\mathbf{P}\left[X_1 > x\,,\; Y_2 > y\right])\,,
\end{aligned}$$

as $x \wedge y \to \infty$, where the last step follows by the GTAI property. Similarly, for the $I_6(x,\, y)$, and using in $I_7(x,\, y)$ and $I_8(x,\, y)$ additionally the a-joint insensitivity property of $\mathcal{L}^{(2)}$, we can easily obtain

$$I_3(x,\, y) = I_6(x,\, y) = I_7(x,\, y) = I_8(x,\, y) = o(\mathbf{P}\left[X_1 > x\,,\; Y_2 > y\right])\,, \tag{4.14}$$

as $x \wedge y \to \infty$.

Finally, for the last two terms, we see the following asymptotic relations. For the first term $I_9(x,\, y)$ is true that

$$I_9(x,\, y) \leq \mathbf{P}\left[X_1 > x\,,\; Y_1 > y\,,\; Y_2 > \frac{y}{2}\right] = o(\mathbf{P}\left[X_1 > x\,,\; Y_2 > y\right])\,,$$

as $x \wedge y \to \infty$, by GTAI property and the membership in $(\mathcal{D} \cap \mathcal{L})^{(2)}$, hence we find the relation

$$I_9(x,\, y) = o(\mathbf{P}\left[X_1 > x\,,\; Y_2 > y\right])\,, \tag{4.15}$$

as $x\wedge y \to \infty$. By the symmetry between the two terms, easily we have the following relation

$$I_{10}(x,\, y) = o(\mathbf{P}\left[X_1 > x\,,\; Y_2 > y\right])\,, \tag{4.16}$$

as $x \wedge y \to \infty$. Therefore by relations (4.13), (4.14), (4.15) and (4.16), we conclude the asymptotic inequality

$$\mathbf{P}[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y] \gtrsim \sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]\,, \tag{4.17}$$

as $x \wedge y \to \infty$, which provides the lower asymptotic bound.

Let us examine now the upper asymptotic bound

$$\begin{aligned}
&\mathbf{P}[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y] \leq \mathbf{P}\left[X_1 > x - a(x)\,,\; Y_1 + Y_2 > y\right]\\
&+\mathbf{P}\left[X_2 > x - a(x)\,,\; Y_1 + Y_2 > y\right] + \mathbf{P}\left[X_1 > \frac{x}{2}\,,\; X_2 > a(x)\,,\; Y_1 + Y_2 > y\right]\\
&+\mathbf{P}\left[X_1 > a(x)\,,\; X_2 > \frac{x}{2}\,,\; Y_1 + Y_2 > y\right] \leq \mathbf{P}\left[X_1 > x - a(x)\,,\; Y_1 > y - a(y)\right]\\
&+\mathbf{P}\left[X_1 > x - a(x)\,,\; Y_2 > y - a(y)\right] + \mathbf{P}\left[X_1 > x - a(x)\,,\; Y_1 > a(y)\,,\; Y_2 > \frac{y}{2}\right]\\
&+\mathbf{P}\left[X_1 > x - a(x)\,,\; Y_1 > \frac{y}{2}\,,\; Y_2 > a(y)\right] + \mathbf{P}\left[X_2 > x - a(x)\,,\; Y_1 > y - a(y)\right]\\
&+\mathbf{P}\left[X_2 > x - a(x)\,,\; Y_2 > y - a(y)\right] + \mathbf{P}\left[X_2 > x - a(x)\,,\; Y_1 > a(y)\,,\; Y_2 > \frac{y}{2}\right]\\
&+\mathbf{P}\left[X_2 > x - a(x)\,,\; Y_1 > \frac{y}{2}\,,\; Y_2 > a(y)\right] + \mathbf{P}\left[X_1 > a(x)\,,\; X_2 > \frac{x}{2}\,,\; Y_1 > y - a(y)\right]\\
&+\mathbf{P}\left[X_1 > a(x)\,,\; X_2 > \frac{x}{2}\,,\; Y_2 > y - a(y)\right]\\
&+\mathbf{P}\left[X_1 > a(x)\,,\; X_2 > \frac{x}{2}\,,\; Y_1 > a(y)\,,\; Y_2 > \frac{y}{2}\right]\\
&+\mathbf{P}\left[X_1 > a(x)\,,\; X_2 > \frac{x}{2}\,,\; Y_1 > \frac{y}{2}\,,\; Y_2 > a(y)\right]\\
&+\mathbf{P}\left[X_1 > \frac{x}{2}\,,\; X_2 > a(x)\,,\; Y_1 > y - a(y)\right] + \mathbf{P}\left[X_1 > \frac{x}{2}\,,\; X_2 > a(x)\,,\; Y_2 > y - a(y)\right]\\
&+\mathbf{P}\left[X_1 > \frac{x}{2}\,,\; X_2 > a(x)\,,\; Y_1 > a(y)\,,\; Y_2 > \frac{y}{2}\right]\\
&+\mathbf{P}\left[X_1 > \frac{x}{2}\,,\; X_2 > a(x)\,,\; Y_1 > \frac{y}{2}\,,\; Y_2 > a(y)\right] =: \sum_{i=1}^{16} I_i(x,\, y)\,.
\end{aligned} \tag{4.18}$$

Taking into account the property $\mathcal{L}^{(2)}$ and the definition of function $a(x)$ we find the asymptotic expressions for

$$I_1(x,\, y) \sim \mathbf{P}\left[X_1 > x\,, Y_1 > y\right]\,, \qquad I_2(x,\, y) \sim \mathbf{P}\left[X_1 > x\,, Y_2 > y\right]\,,$$

$$I_5(x,\, y) \sim \mathbf{P}\left[X_2 > x\,, Y_1 > y\right]\,, \qquad I_6(x,\, y) \sim \mathbf{P}\left[X_2 > x\,, Y_2 > y\right]\,,$$

as $x \wedge y \to \infty$. Hence

$$I_1(x,\,y)+I_2(x,\,y)+I_5(x,\,y)+I_6(x,\,y)\sim \sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x\,,\;Y_l>y\right]\,, \tag{4.19}$$

as $x \wedge y \to \infty$.

Next, we follow a similar approach for $I_3(x,\,y)$, $I_4(x,\,y)$, $I_7(x,\,y)$, $I_8(x,\,y)$, $I_9(x,\,y)$ and $I_{10}(x,\,y)$. Now we see

$$\begin{aligned} I_3(x,\,y) &\sim \mathbf{P}\left[X_1>x\,,\;Y_1>a(y)\,,\;Y_2>\frac{y}{2}\right] \\ &= \mathbf{P}\left[Y_1>a(y)\;\Big|\;X_1>x\,,\;Y_2>\frac{y}{2}\right]\mathbf{P}\left[X_1>x\,,\;Y_2>\frac{y}{2}\right]=o\left(\mathbf{P}\left[X_2>x\,,Y_2>y\right]\right)\,, \end{aligned}$$

as $x \wedge y \to \infty$, that follows because of properties $(\mathcal{D}\cap\mathcal{L})^{(2)}$ and GTAI.

In similar way we find $I_4(x,\,y)=o(\mathbf{P}\left[X_1>x\,,Y_2>y\right])$, $I_7(x,\,y)=o(\mathbf{P}\left[X_2>x\,,Y_2>y\right])$, $I_8(x,\,y)=o(\mathbf{P}\left[X_2>x\,,Y_2>y\right])$, $I_9(x,\,y)=o(\mathbf{P}\left[X_2>x\,,Y_1>y\right])$ and finally $I_{10}(x,\,y)=o(\mathbf{P}\left[X_1>x\,,Y_2>y\right])$, as $x \wedge y \to \infty$. Hence

$$I_3(x,\,y)+I_4(x,\,y)+\sum_{i=7}^{10}I_i(x,\,y)=o\left(\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x\,,\;Y_l>y\right]\right)\,, \tag{4.20}$$

as $x \wedge y \to \infty$.

The $I_{11}(x,\,y)$, $I_{12}(x,\,y)$, $I_{13}(x,\,y)$, $I_{14}(x,\,y)$, $I_{15}(x,\,y)$, $I_{16}(x,\,y)$, can be handled also similarly

$$\begin{aligned} I_{11}(x,\,y) &\leq \mathbf{P}\left[X_2>\frac{x}{2}\,,\;Y_1>a(y)\,,\;Y_2>\frac{y}{2}\right] \\ &= \mathbf{P}\left[Y_1>a(y)\mid X_2>\frac{x}{2}\,,\;Y_2>\frac{y}{2}\right]\mathbf{P}\left[X_2>\frac{x}{2}\,,Y_2>\frac{y}{2}\right]\,, \end{aligned}$$

or equivalently $I_{11}(x,\,y)=o(\mathbf{P}\left[X_2>x\,,Y_2>y\right])$, as $x \wedge y \to \infty$, which follows because of properties $(\mathcal{D}\cap\mathcal{L})^{(2)}$ and GTAI. Similarly we find $I_{1j}(x,\,y)=o(\mathbf{P}\left[X_k>x\,,Y_l>y\right])$, for some $k,\,l\in\{1,\,2\}$ and for any $j\in\{1,\,\ldots,\,6\}$. Therefore we obtain

$$I_{1j}(x,\,y)=o\left(\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x\,,\;Y_l>y\right]\right)\,, \tag{4.21}$$

as $x \wedge y \to \infty$, for any $j\in\{1,\,\ldots,\,6\}$.

From (4.19), (4.20) and (4.21), in combination with (4.18) we find that

$$\mathbf{P}[X_1+X_2>x\,,\;Y_1+Y_2>y]\lesssim\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k>x\,,\;Y_l>y\right]$$

as $x \wedge y \to \infty$, which in combination with (4.17) leads to (4.12).

Now we check the validity of relation $(X_1 + X_2,\, Y_1 + Y_2) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$. At first, by (4.12) we obtain

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_1 + X_2 > b_1\, x\,,\; Y_1 + Y_2 > b_2\, y\right]}{\mathbf{P}\left[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y\right]}$$

$$= \limsup_{x\wedge y\to\infty} \frac{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > b_1\, x\,,\; Y_l > b_2\, y\right]}{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]}$$

$$\leq \max_{k,\,l\in\{1,\,2\}} \left\{ \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_k > b_1\, x\,,\; Y_l > b_2\, y\right]}{\mathbf{P}\left[X_k > x\,,\; Y_l > y\right]} \right\} < \infty\,,$$

for any $\mathbf{b} = (b_1,\, b_2) \in (0,\, 1)^2$, this means that, we have the one of two conditions of the closure property with respect to $\mathcal{D}^{(2)}$.

Next, we check the closure property with respect to $\mathcal{L}^{(2)}$. From (4.12) we obtain

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_1 + X_2 > x - a_1\,,\; Y_1 + Y_2 > y - a_2\right]}{\mathbf{P}\left[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y\right]}$$

$$= \limsup_{x\wedge y\to\infty} \frac{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > x - a_1\,,\; Y_l > y - a_2\right]}{\sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]}\,,$$

for any $\mathbf{a} = (a_1,\, a_2) \in \mathbb{R}^2$, and therefore

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_1 + X_2 > x - a_1\,,\; Y_1 + Y_2 > y - a_2\right]}{\mathbf{P}\left[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y\right]}$$

$$\leq \max_{k,\,l\in\{1,\,2\}} \left\{ \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_k > x - a_1\,,\; Y_l > y - a_2\right]}{\mathbf{P}\left[X_k > x\,,\; Y_l > y\right]} \right\} = 1\,,$$

and

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_1 + X_2 > x - a_1\,,\; Y_1 + Y_2 > y - a_2\right]}{\mathbf{P}\left[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y\right]}$$

$$\geq \min_{k,\,l\in\{1,\,2\}} \left\{ \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_k > x - a_1\,,\; Y_l > y - a_2\right]}{\mathbf{P}\left[X_k > x\,,\; Y_l > y\right]} \right\} = 1\,,$$

that means, we have the one of two conditions of the closure property with respect to $\mathcal{L}^{(2)}$ true. So by the extra assumption of TAI between X_1, X_2 and Y_1, Y_2 by Lemma 4.1 of [15] we have that $X_1 + X_2 \in \mathcal{D} \cap \mathcal{L}$ and $Y_1 + Y_2 \in \mathcal{D} \cap \mathcal{L}$, as a result we conclude

$$(X_1 + X_2,\, Y_1 + Y_2) \in (\mathcal{D} \cap \mathcal{L})^{(2)}\,. \quad \square$$

Here we provide a corollary, following from Lemma 4.3, where we establish the closure property with respect to $\mathcal{C}^{(2)}$ and the jointly max-sum equivalence, under condition GTAI.

Corollary 4.1. *Let $X_1,\, X_2,\, Y_1,\, Y_2$ be non-negative random variables, with the distributions $F_1,\, F_2,\, G_1,\, G_2$ from class $\mathcal{C}$ respectively and they satisfy the GTAI condition. If hold $(X_k,\, Y_l) \in \mathcal{C}^{(2)}$, for any $k,\, l \in \{1,\, 2\}$, then*

$$\mathbf{P}\left[X_1 + X_2 > x\,,\; Y_1 + Y_2 > y\right] \sim \sum_{k=1}^{2}\sum_{l=1}^{2} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]\,, \tag{4.22}$$

as $x\wedge y \to \infty$. If further X_1, X_2 are TAI and Y_1, Y_2 are TAI then holds $(X_1+X_2\,,\ Y_1+Y_2) \in \mathcal{C}^{(2)}$.

Proof. Relation (4.22) follows from the fact that $\mathcal{C}^{(2)} \subsetneq (\mathcal{D}\cap\mathcal{L})^{(2)}$ and by application of Lemma 4.3.

Next, we check the closure property with respect to convolution. From (4.22) we obtain

$$\mathbf{P}[X_1+X_2 > d_1\,x\,,\ Y_1+Y_2 > d_2\,y] \sim \sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k > d_1\,x\,,\ Y_l > d_2\,y\right]\,,$$

as $x\wedge y\to\infty$, for any $\mathbf{d}=(d_1,\,d_2)\in(0,\,1)^2$. Hence

$$\begin{aligned}
&\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_1+X_2 > d_1\,x\,,\ Y_1+Y_2 > d_2\,y\right]}{\mathbf{P}\left[X_1+X_2 > x\,,\ Y_1+Y_2 > y\right]}\\
&\qquad = \limsup_{x\wedge y\to\infty}\frac{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k > d_1\,x\,,\ Y_l > d_2\,y\right]}{\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbf{P}\left[X_k > x\,,\ Y_l > y\right]}\\
&\qquad \leq \limsup_{x\wedge y\to\infty}\max_{k,\,l\in\{1,\,2\}}\left\{\frac{\mathbf{P}\left[X_k > d_1\,x\,,\ Y_l > d_2\,y\right]}{\mathbf{P}\left[X_k > x\,,\ Y_l > y\right]}\right\}\,,
\end{aligned}$$

Whence, because of the definition of $\mathcal{C}^{(2)}$ we get

$$\begin{aligned}
1 &\leq \liminf_{\mathbf{d}\uparrow\mathbf{1}}\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_1+X_2 > d_1\,x\,,\ Y_1+Y_2 > d_2\,y\right]}{\mathbf{P}\left[X_1+X_2 > x\,,\ Y_1+Y_2 > y\right]}\\
&\leq \limsup_{\mathbf{d}\uparrow\mathbf{1}}\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_1+X_2 > d_1\,x\,,\ Y_1+Y_2 > d_2\,y\right]}{\mathbf{P}\left[X_1+X_2 > x\,,\ Y_1+Y_2 > y\right]}\\
&\leq \limsup_{\mathbf{d}\uparrow\mathbf{1}}\limsup_{x\wedge y\to\infty}\max_{k,\,l\in\{1,\,2\}}\left(\frac{\mathbf{P}\left[X_k > d_1\,x\,,\ Y_l > d_2\,y\right]}{\mathbf{P}\left[X_k > x\,,\ Y_l > y\right]}\right)\\
&\leq \max_{k,\,l\in\{1,\,2\}}\left(\limsup_{\mathbf{d}\uparrow\mathbf{1}}\limsup_{x\wedge y\to\infty}\frac{\mathbf{P}\left[X_k > d_1\,x\,,\ Y_l > d_2\,y\right]}{\mathbf{P}\left[X_k > x\,,\ Y_l > y\right]}\right) = 1\,,
\end{aligned}$$

this means that the one of two conditions of closedness under convolution holds. By the assumptions of TAI in each sequence, and by $\mathcal{C}\subsetneq\mathcal{D}\cap\mathcal{L}$, we use Lemma 4.1 of [15] and we take that:

$$\begin{aligned}
1 &\leq \liminf_{d_1\uparrow 1}\limsup_{x\to\infty}\frac{\mathbf{P}\left[X_1+X_2 > d_1\,x\right]}{\mathbf{P}\left[X_1+X_2 > x\right]}\\
&\leq \limsup_{d_1\uparrow 1}\limsup_{x\to\infty}\frac{\mathbf{P}\left[X_1+X_2 > d_1\,x\right]}{\mathbf{P}\left[X_1+X_2 > x\right]} = \limsup_{d_1\uparrow 1}\limsup_{x\to\infty}\frac{\mathbf{P}\left[X_1 > d_1\,x\right]+\mathbf{P}\left[X_1 > d_1\,x\right]}{\mathbf{P}\left[X_1 > x\right]+\mathbf{P}\left[X_1 > x\right]}\\
&\leq \limsup_{d_1\uparrow 1}\limsup_{x\to\infty}\max_{k\in\{1,\,2\}}\left(\frac{\mathbf{P}\left[X_k > d_1\,x\right]}{\mathbf{P}\left[X_k > x\right]}\right)\\
&\leq \max_{k\in\{1,\,2\}}\left(\limsup_{d_1\uparrow 1}\limsup_{x\to\infty}\frac{\mathbf{P}\left[X_k > d_1\,x\right]}{\mathbf{P}\left[X_k > x\right]}\right) = 1\,,
\end{aligned}$$

which gives that $(X_1 + X_2) \in \mathcal{C}$. With the same argument we have : $(Y_1 + Y_2) \in \mathcal{C}$. That means $(X_1 + X_2\,,\ Y_1 + Y_2) \in \mathcal{C}^{(2)}$. $\square$

Now we can give the main result, where we find an analogue to relation (4.2) in two dimensions.

Theorem 4.2. *Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be random variables with the following distributions $F_1, \ldots, F_n\, G_1, \ldots, G_n$ from class $\mathcal{D} \cap \mathcal{L}$ respectively and they satisfy the GTAI condition, with $(X_k, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$, for any $k, l \in \{1, \ldots, n\}$. If further $X_1, \ldots, X_n$ are TAI and $Y_1, \ldots, Y_n$ are TAI, then*

$$\begin{aligned}\mathbf{P}\left[\sum_{k=1}^{n} X_k > x\,,\ \sum_{l=1}^{n} Y_l > y\right] &\sim \mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x\,,\ \bigvee_{j=1}^{n} T_j > y\right] \\ &\sim \mathbf{P}\left[\bigvee_{k=1}^{n} X_k > x\,,\ \bigvee_{l=1}^{n} Y_l > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x\,,\ Y_l > y\right],\end{aligned}$$

as $x \wedge y \to \infty$.

Proof. By Lemma 4.2 we find

$$\mathbf{P}\left[\sum_{k=1}^{n} X_k > x\,,\ \sum_{l=1}^{n} Y_l > y\right] \gtrsim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x\,,\ Y_l > y\right]\,,$$

as $x \wedge y \to \infty$. Because of closure property of $(\mathcal{D} \cap \mathcal{L})^{(2)}$ with respect to convolution in the positive part, under GTAI condition, we can apply Lemma 4.3 and employing induction, we find

$$\begin{aligned}\mathbf{P}\left[\sum_{k=1}^{n} X_k > x\,,\ \sum_{l=1}^{n} Y_l > y\right] &\leq \mathbf{P}\left[\sum_{k=1}^{n} X_k^{+} > x\,,\ \sum_{l=1}^{n} Y_l^{+} > y\right] \\ &\sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x\,,\ Y_l > y\right]\,,\end{aligned}$$

as $x \wedge y \to \infty$. Whence, taking into consideration Theorem 4.1 we find

$$\mathbf{P}\left[\sum_{k=1}^{n} X_k > x,\ \sum_{l=1}^{n} Y_l > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x,\ Y_l > y\right] \sim \mathbf{P}\left[\bigvee_{k=1}^{n} X_k > x,\ \bigvee_{l=1}^{n} Y_l > y\right],$$

as $x \wedge y \to \infty$. Finally, due to the inequality

$$\begin{aligned}\mathbf{P}\left[\sum_{k=1}^{n} X_k > x\,,\ \sum_{l=1}^{n} Y_l > y\right] &\leq \mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x\,,\ \bigvee_{j=1}^{n} T_j > y\right] \\ &\leq \mathbf{P}\left[\sum_{k=1}^{n} X_k^{+} > x\,,\ \sum_{l=1}^{n} Y_l^{+} > y\right]\,,\end{aligned}$$

we get the asymptotic relation

$$\mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x\,,\ \bigvee_{j=1}^{n} T_j > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x\,,\ Y_l > y\right]\,,$$

as $x \wedge y \to \infty$. $\square$

Recently more and more researchers study two dimensional risk models, we refer to the reader [7], [8], [18] among many others. For

$$U_1(k,\,x) := x - \sum_{i=1}^{k} X_i\,, \qquad U_2(k,\,y) := y - \sum_{j=1}^{k} Y_j\,,$$

for $1 \leq k \leq n$, we define now two ruin times,

$$T_{max} := \inf\left\{1 \leq k \leq n \;:\; U_1(k,\,x) \wedge U_2(k,\,y) < 0\right\}\,,$$

that denotes the first moment when both portfolios are found with negative surplus, and for each portfolio we define:

$$T_1(x) := \inf\left\{1 \leq k \leq n : U_1(k,\,x) < 0 | U_1(0,\,x) = x\right\}\,,$$

$$T_2(y) := \inf\left\{1 \leq k \leq n : U_2(k,\,y) < 0 | U_2(0,\,y) = y\right\}\,,$$

as a result the second type of ruin type is:

$$T_{and} := \max\left\{T_1(x), T_2(y)\right\}\,,$$

that corresponds to the first moment, when both portfolios have been with negative surplus, but not necessarily simultaneously. Hence we define the ruin probabilities as

$$\psi_{max}(x,\,y,\,n) = \mathbf{P}[T_{max} \leq n]\,, \qquad \psi_{and}(x,\,y,\,n) = \mathbf{P}[T_{and} \leq n]\,, \tag{4.23}$$

for any $n \in \mathbb{N}$ and $x,\,y > 0$. From (4.23) we easily find out that

$$\psi_{and}(x,\,y,\,n) = \mathbf{P}\left[\bigvee_{i=1}^{n} S_i > x\,,\; \bigvee_{i=1}^{n} T_i > y\right]\,.$$

Thence by Theorem 4.2 follows the next result.

Corollary 4.2. *Under conditions of Theorem 4.2 we obtain*

$$\psi_{and}(x,\,y,\,n) \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]\,, \tag{4.24}$$

as $x \wedge y \to \infty$.

Remark 4.1. *From relation (4.24) and the definitions for T_{max} and T_{and} we can easily observe that $\psi_{max}(x,\,y,\,n) \leq \psi_{and}(x,\,y,\,n)$, for any $x,\,y > 0$ and any $n \in \mathbb{N}$. Thus, for $\psi_{max}(x,\,y,\,n)$ we find the asymptotic upper bound*

$$\psi_{max}(x,\,y,\,n) \lesssim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[X_k > x\,,\; Y_l > y\right]\,,$$

as $x \wedge y \to \infty$, for any $n \in \mathbb{N}$.

5. Scalar product

Now we examine the closure property of scalar product in $\mathcal{L}^{(2)}$, $\mathcal{D}^{(2)}$, and their intersection. Later we check the same for random sums in two dimensions.

The scalar product has the following tail

$$\overline{\mathbf{H}}(x,\, y) := \mathbf{P}[\Theta\, X > x,\, \Theta\, Y > y]\,. \tag{5.1}$$

Here we set Θ to be a non-negative random variable with distribution B, such that $B(0-) = 0$ and $B(0) < 1$. These products in relation (5.1) have many applications in actuarial mathematics, in risk management, and stochastic fields. Next, we use an assumption from [23].

Assumption 5.1. *There exist a function $b\,:\,[0,\,\infty) \longrightarrow (0,\,\infty)$, such that*

(1) $b(x) \to \infty\,,$ *as* $x \to \infty$.
(2) $b(x) = o(x)\,,$ *as* $x \to \infty$.
(3) $\overline{B}[b(x \wedge y)] = o\left(\mathbf{P}\left[\Theta\, X > x\,,\ \Theta\, Y > y\right]\right) =: o\left[\overline{\mathbf{H}}(x,\, y)\right]\,,$ *as* $x \wedge y \to \infty$.

Remark 5.1. *From Assumption 5.1 we conclude*

$$\frac{\overline{B}[b(x)]}{\mathbf{P}[\Theta\, X > x]} \leq \frac{\overline{B}[b(x)]}{\mathbf{P}[\Theta\, X > x\,,\ \Theta\, Y > y]} \longrightarrow 0\,,$$

as $x \to \infty$, and from parts (1) and (2), in combination with [36, Lem. 3.2], *it follows*

$$\overline{B}(c\, x) = o\left(\mathbf{P}\left[\Theta\, X > x\right]\right)\,, \tag{5.2}$$

as $x \to \infty$, for any $c > 0$, and with similar manipulation we find

$$\overline{B}(c\, y) = o\left(\mathbf{P}\left[\Theta\, Y > y\right]\right)\,, \tag{5.3}$$

as $y \to \infty$.

Now we study the closedness of class $\mathcal{D}^{(2)}$ under the scalar product.

Theorem 5.1. *Let $(X,\, Y)$ be random vector and Θ be random variable, with tail distribution $\overline{\mathbf{F_1}}(x\, y) = \mathbf{P}[X > x,\, Y > xy]$ and B respectively, and assume $B(0-) = 0, B(0) < 1$. If Θ and $(X,\, Y)$ are independent, Assumption 5.1 holds and $(F, G) \in \mathcal{D}^{(2)}$, then*

$$\mathbf{H}(x,\, y) \in \mathcal{D}^{(2)}\,.$$

Proof. Initially, we get from $F,\, G \in \mathcal{D}$ and by [11, Th. 3.3 (i)] that the products $\Theta\, X$ and $\Theta\, Y$ follow distributions from $\mathcal{D}$. From Assumption 5.1 we obtain that for any $\mathbf{b} \in (0,\, 1)^n$

$$\begin{aligned}
\limsup_{x\wedge y \to \infty} \frac{\overline{\mathbf{H}}_{\mathbf{b}}(x,\, y)}{\overline{\mathbf{H}}(x,\, y)} &= \limsup_{x\wedge y \to \infty} \frac{\mathbf{P}[\Theta\, X > b_1\, x,\ \Theta\, Y > b_2\, y]}{\mathbf{P}[\Theta\, X > x\,,\ \Theta\, Y > y]} \qquad (5.4)\\
&= \limsup_{x\wedge y \to \infty} \frac{\left(\int_0^{b(x\wedge y)} + \int_{b(x\wedge y)}^{\infty}\right) \mathbf{P}\left[X > \dfrac{b_1\, x}{s}\,,\ Y > \dfrac{b_2\, y}{s}\right]\, B(ds)}{\mathbf{P}[\Theta\, X > x\,,\ \Theta\, Y > y]}\\
&=: \limsup_{x\wedge y \to \infty} \frac{I_1 + I_2}{\mathbf{P}[\Theta\, X > x, \Theta\, Y > y]}\,.
\end{aligned}$$

Further we calculate

$$I_2 \leq \int_{b(x\wedge y)}^{\infty} B(ds) = \overline{B}[b(x\wedge y)] = o\left[\overline{\mathbf{H}}(x,\, y)\right]\,,$$

as $x \wedge y \to \infty$, due to Assumption 5.1. Hence, taking into account also relation (5.4) we find

$$\begin{aligned}\limsup_{x\wedge y\to\infty} \frac{\overline{\mathbf{H}}_{\mathbf{b}}(x,\, y)}{\overline{\mathbf{H}}(x,\, y)} &\leq \limsup_{x\wedge y\to\infty} \frac{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{b_1\,x}{s}\,,\; Y > \frac{b_2\,y}{s}\right]\, B(ds)}{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x}{s}\,,\; Y > \frac{y}{s}\right]\, B(ds)} \\ &\leq \limsup_{x\wedge y\to\infty} \sup_{0<s\leq b(x\wedge y)} \frac{\mathbf{P}\left[X > b_1\,x/s,\; Y > b_2\,y/s\right]}{\mathbf{P}\left[X_1 > x/s,\; Y > y/s\right]} \\ &\leq \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X > b_1\,x,\; Y > b_2\,y\right]}{\mathbf{P}\left[X > x,\; Y > y\right]} < \infty\,,\end{aligned}$$

where in the last step we used the condition $(F, G) \in \mathcal{D}^{(2)}$. So we conclude

$$\mathbf{H}(x,\, y) \in \mathcal{D}^{(2)}\,. \quad \square$$

Now we provide an analogue for class $\mathcal{L}^{(2)}$.

Theorem 5.2. *Let $(X,\, Y)$ be a random vector and θ be a non-negative random variable, with distributions $\mathbf{F}$, B respectively, under condition $B(0) < 1$. If $(X,\, Y)$, Θ are independent, Assumption 5.1 holds, and $(F,\, G) \in \mathcal{L}^{(2)}$, then*

$$\mathbf{H}(x,\, y) \in \mathcal{L}^{(2)}\,.$$

Proof. From the fact that $(X,\, Y)$ is independent of Θ, $F,\, G \in \mathcal{L}$ and relations (5.2) and (5.3), using [11, Th 2.2 (iii)], we find that distributions of $\Theta\, X$ and $\Theta\, Y$ belong to $\mathcal{L}$. Let $\mathbf{a} = (a_1,\, a_2) \in (0,\, \infty)^2$. Then we easily obtain

$$\liminf_{x\wedge y\to\infty} \frac{\overline{\mathbf{H}}(x-a_1,\, y-a_2)}{\overline{\mathbf{H}}(x,\, y)} = \lim_{x\wedge y\to\infty} \frac{\mathbf{P}[\Theta\, X > x-a_1,\; \Theta\, Y > y-a_2]}{\mathbf{P}[\Theta\, X > x,\; \Theta\, Y > y]} \geq 1\,. \tag{5.5}$$

Next, we show the opposite asymptotic inequality. Using Assumption 5.1 we obtain

$$\begin{aligned}&\limsup_{x\wedge y\to\infty} \frac{\overline{\mathbf{H}}(x-a_1,\, y-a_2)}{\overline{\mathbf{H}}(x,\, y)} \qquad\qquad (5.6)\\ &= \lim_{x\wedge y\to\infty} \frac{1}{\overline{\mathbf{H}}(x,\, y)} \left(\int_0^{b(x\wedge y)} + \int_{b(x\wedge y)}^{\infty}\right) \mathbf{P}\left[X > \frac{x-a_1}{s},\; Y > \frac{y-a_2}{s}\right] B(ds) \\ &=: \lim_{x\wedge y\to\infty} \frac{I_1(x,\, y) + I_2(x,\, y)}{\overline{\mathbf{H}}(x,\, y)}\,.\end{aligned}$$

Thence, by Assumption 5.1 we find

$$I_2(x,\, y) = \int_{b(x\wedge y)}^{\infty} \mathbf{P}\left[X > \frac{x-a_1}{s}\,,\; Y > \frac{y-a_2}{s}\right] B(ds) \leq \overline{B}[b(x\wedge y)] = o\left[\overline{\mathbf{H}}(x,\, y)\right]\,,$$

hence,

$$\frac{I_2(x,\, y)}{\overline{\mathbf{H}}(x,\, y)} = o(1)\,,$$

as $x \wedge y \to \infty$. As a consequence, taking into account also (5.6) we get

$$\begin{aligned}
\limsup_{x\wedge y\to\infty} \frac{\overline{\mathbf{H}}_{\mathbf{1}}(x-a_1,\, y-a_2)}{\overline{\mathbf{H}}(x,\, y)} &= \lim_{x\wedge y\to\infty} \int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x-a_1}{s},\; Y > \frac{y-a_2}{s}\right] \frac{B(ds)}{\overline{\mathbf{H}}(x,\, y)} \\
&\leq \lim_{x\wedge y\to\infty} \frac{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x-a_1}{s},\; Y > \frac{y-a_2}{s}\right] B(ds)}{\int_0^{b(x\wedge y)} \mathbf{P}\left[X > \frac{x}{s},\; Y > \frac{y}{s}\right] B(ds)} \\
&\leq \lim_{x\wedge y\to\infty} \sup_{0<s\leq b(x\wedge y)} \frac{\mathbf{P}\left[X > \frac{x-a_1}{s},\; Y > \frac{y-a_2}{s}\right]}{\mathbf{P}\left[X > \frac{x}{s},\; Y > \frac{y}{s}\right]} \\
&= \lim_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X > x-a_1,\; Y > y-a_2\right]}{\mathbf{P}\left[X > x\,,\; Y > y\right]} = 1\,.
\end{aligned}$$

where in the last step we consider the fact that $(F, G) \in \mathcal{L}^{(2)}$. So we have

$$\limsup_{x\wedge y\to\infty} \frac{\overline{\mathbf{H}}(x-a_1,\, y-a_2)}{\overline{\mathbf{H}}(x,\, y)} \leq 1\,. \tag{5.7}$$

From relations (5.5) and (5.7) we conclude $\mathbf{H}(x,\, y) \in \mathcal{L}^{(2)}$. □

The next statement stems from a combination of previous results.

Corollary 5.1. *Let $(X,\, Y)$ be a random vector and Θ be a non-negative random variable with distributions (F, G), B respectively, under condition $B(0) < 1$. If $(X,\, Y)$ and Θ are independent, with $(X,\, Y) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ and satisfy the Assumption 5.1, then $\mathbf{H}(x,\, y) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$.*

Proof. This follows directly from Theorem 5.1 and Theorem 5.2. □

6. Randomly weighted sums

In this section we extend Theorem 4.2 into weighted sums. The first kind of weighted sums takes the form

$$S_n^{\Theta} = \sum_{k=1}^{n} \Theta\, X_k\,, \qquad T_n^{\Theta} = \sum_{l=1}^{n} \Theta\, Y_l\,.$$

These quantities have the same discount factor Θ, hence the $(X_k,\, Y_l)$, for $k,\, l = 1,\, \ldots,\, n$, are the losses or gains of the two lines of business during the k-th period. If the $(x,\, y)$ represents the two initial capitals respectively, then the ruin probability in this model comes in the form

$$\psi_{and}(x,\, y,\, n) := \mathbf{P}\left[\bigvee_{i=1}^{n} S_i^{\Theta} > x\,,\; \bigvee_{j=1}^{n} T_j^{\Theta} > y\right]\,. \tag{6.1}$$

The ruin probability plays a significant role in risk theory. For example we refer to, [27], [42] and [7], [19] for discrete-time or continuous-time models respectively.

The next result is based on Theorem 4.2 and Corollary 5.1. We have to notice that there exists the asymptotic behavior of the ruin probability in (6.1) as well.

Corollary 6.1. *Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be random variables with the following distributions $F_1, \ldots, F_n, G_1, \ldots, G_n$ respectively form class $\mathcal{D} \cap \mathcal{L}$ and they satisfy the GTAI dependence structure. We assume that Θ represents a non-negative upper-bounded random variable, it is independent of $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and hold Assumption 5.1 and $(X_k, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$, for $k, l = 1, \ldots, n$. If further $X_1, \ldots, X_n$ are TAI and $Y_1, \ldots, Y_n$ are TAI then the following asymptotic relation is true*

$$\begin{aligned}\mathbf{P}\left[S_n^{\Theta} > x\,,\ T_n^{\Theta} > y\right] \ &\sim\ \mathbf{P}\left[\bigvee_{i=1}^{n} S_i^{\Theta} > x\,,\ \bigvee_{j=1}^{n} T_j^{\Theta} > y\right] \sim \mathbf{P}\left[\bigvee_{k=1}^{n} \Theta\, X_k > x\,,\ \bigvee_{l=1}^{n} \Theta\, Y_l > y\right] \\ &\sim\ \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[\Theta\, X_k > x\,,\ \Theta\, Y_l > y\right]\,, \qquad (6.2)\end{aligned}$$

as $x \wedge y \to \infty$.

Proof. We start from [22, Lem. 3.1] and because of the upper-bound of Θ, we obtain that the products $\Theta\, X_1, \ldots, \Theta\, X_n, \Theta\, Y_1, \ldots, \Theta\, Y_n$ are GTAI. Now we can apply [11, Th. 2.2(iii), Th.3.3(iii)], to find $\Theta\, X_k \in \mathcal{D} \cap \mathcal{L}$, and $\Theta\, Y_l \in \mathcal{D} \cap \mathcal{L}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Because of the closedness of class $\mathcal{D}$ we using Theorem 2.2 of [25] $\Theta_1\, X_1, \ldots, \Theta_n\, X_n$ are TAI and $\Delta_1\, Y_1, \ldots, \Delta_n\, Y_n$ are TAI.

Next, because of Assumption 5.1, to obtain $(\Theta\, X_k, \Theta\, Y_l) \in \mathcal{D}^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$, thence applying Theorem 5.2 we find $(\Theta\, X_k, \Theta\, Y_l) \in \mathcal{L}^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Therefore the $(\Theta\, X_k, \Theta\, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ and the $\Theta\, X_1, \ldots, \Theta\, X_n$, $\Theta\, Y_1, \ldots, \Theta\, Y_n$ are GTAI. Finally, applying Theorem 4.2, we conclude (6.2). □

Now we need some preliminary results. Several times before proving that the convolution product satisfies $H \in \mathcal{B}$, with $\mathcal{B}$ some distribution class, we need to prove that $H_\varepsilon(x) := \mathbf{P}[(\Theta \vee \varepsilon)\, X \leq x]$ belongs to this class $\mathcal{B}$ for any $\varepsilon > 0$. Following the approach in [11] we show that for some constant $\delta > 0$, if $H_\varepsilon \in \mathcal{L}^{(2)}$, for any $\varepsilon \in (0, \delta)$, then $H \in \mathcal{L}^{(2)}$. However the next results, deserves theoretical attention by its own merit.

From here until to the end of paper we assume that X, Y are non negative random variables.

Lemma 6.1. *If for some constant vector $\delta = (\delta_1, \delta_2) > (0, 0)$, holds the inclusion*

$$((\Theta \vee \varepsilon_1)\, X,\ (\Delta \vee \varepsilon_2)\, Y) \in \mathcal{L}^{(2)}\,,$$

for any $\varepsilon_1 \in (0, \delta_1)$ (with X, Y, Θ, Δ non-negative random variables) and for any $\varepsilon_2 \in (0, \delta_2)$, then we conclude that $(\Theta\, X, \Delta\, Y) \in \mathcal{L}^{(2)}$.

Proof. Keeping in mind that $((\Theta \vee \varepsilon_1)\, X, (\Delta \vee \varepsilon_2)\, Y) \in \mathcal{L}^{(2)}$, we start by [11, th. 2.2(i)] to establish that due to

$$(\Theta \vee \varepsilon_1)\, X \in \mathcal{L}\,, \qquad (\Delta \vee \varepsilon_2)\, Y \in \mathcal{L}\,,$$

we get $\Theta\, X \in \mathcal{L}$, and $\Delta\, Y \in \mathcal{L}$. Next we check the property of class $\mathcal{L}^{(2)}$. Let $(a_1,\, a_2) > (0,\, 0)$, then

$$\lim_{x\wedge y\to\infty} \frac{\mathbf{P}\,[\Theta\, X > x - a_1\,,\ \Delta\, Y > y - a_2]}{\mathbf{P}\,[\Theta\, X > x\,,\ \Delta\, Y > y]} \geq 1\,, \tag{6.3}$$

Next, for any $(\varepsilon_1,\, \varepsilon_2) > (0,\, 0)$, we find

$$\begin{aligned}
&\mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x\,,\ (\Delta \vee \varepsilon_2)\, Y > y] \geq \mathbf{P}[\Theta\, X > x\,,\ (\Delta \vee \varepsilon_2)\, Y > y] \\
&\quad \geq \mathbf{P}[\Theta\, X > x\,,\ \Theta > \varepsilon_1\,,\ (\Delta \vee \varepsilon_2)\, Y > y] \\
&\quad = \mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x\,,\ (\Delta \vee \varepsilon_2)\, Y > y] - \mathbf{P}[\Theta \leq \varepsilon_1]\, \mathbf{P}[X\, \varepsilon_1 > x,\, (\Delta \vee \varepsilon_2)\, Y > y] \\
&\quad \geq \mathbf{P}[\Theta > \varepsilon_1]\mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x,\, (\Delta \vee \varepsilon_2)\, Y > y]\,,
\end{aligned}$$

hence we conclude

$$\mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x\,,\ (\Delta \vee \varepsilon_2)\, Y > y] \geq \mathbf{P}[\Theta > \varepsilon_1]\mathbf{P}[(\Theta \vee \varepsilon_1)\, X > x\,,\ (\Delta \vee \varepsilon_2)\, Y > y]\,. \tag{6.4}$$

Therefore, using (6.4) and due to properties of $\mathcal{L}^{(2)}$, for $((\Theta \vee \varepsilon_1)\, X,\, (\Delta \vee \varepsilon_2)\, Y)$ we obtain

$$\begin{aligned}
&\lim_{x\wedge y\to\infty} \frac{\mathbf{P}\,[\Theta\, X > x - a_1\,,\ \Delta\, Y > y - a_2]}{\mathbf{P}\,[\Theta\, X > x\,,\ \Delta\, Y > y]} \\
&\qquad \leq \lim_{x\wedge y\to\infty} \frac{\mathbf{P}\,[(\Theta \vee \varepsilon_1)\, X > x - a_1\,,\ (\Delta \vee \varepsilon_2)\, Y > y - a_2]}{\mathbf{P}[\Theta > \varepsilon_1]\, \mathbf{P}\,[(\Theta \vee \varepsilon_1)\, X > x\,,\ (\Delta \vee \varepsilon_2)\, Y > y]} = \frac{1}{\mathbf{P}[\Theta > \varepsilon_1]}\,,
\end{aligned}$$

and leaving ε_1 to tend to zero we get

$$\lim_{x\wedge y\to\infty} \frac{\mathbf{P}\,[\Theta\, X > x - a_1\,,\ \Delta\, Y > y - a_2]}{\mathbf{P}\,[\Theta\, X > x\,,\ \Delta\, Y > y]} \leq 1\,,$$

whence from (1.1) and (1.2) we reach to $(\Theta\, X,\, \Delta\, Y) \in \mathcal{L}^{(2)}$. $\square$

Lemma 6.2. *Let X and Y be non-negative random variables, with $(X,\, Y) \in \mathcal{L}^{(2)}$ and Θ and Δ be non-negative, non-degenerated to zero random variables, independent of $(X,\, Y)$. We assume that*

$$\begin{aligned}
\mathbf{P}\,[\Theta > x] &= o(\mathbf{P}\,[\Theta\, X > c_1\, x\,,\ \Delta\, Y > c_2\, y])\,, \\
\mathbf{P}\,[\Delta > y] &= o(\mathbf{P}\,[\Theta\, X > c_1\, x\,,\ \Delta\, Y > c_2\, y])\,,
\end{aligned} \tag{6.5}$$

as $x \to \infty$, $y \to \infty$, for any $c_1,\, c_2 > 0$. Then $(\Theta\, X,\, \Delta\, Y) \in \mathcal{L}^{(2)}$.

Proof. From (6.5) we obtain

$$\frac{\mathbf{P}\,[\Theta > x]}{\mathbf{P}\,[\Theta\, X > c_1\, x]} \leq \frac{\mathbf{P}\,[\Theta > x]}{\mathbf{P}\,[\Theta\, X > c_1\, x\,,\ \Delta\, Y > c_2\, y]} \longrightarrow 0\,,$$

as $x \wedge y \to \infty$, and similarly we find $\mathbf{P}\,[\Delta > y] = o(\mathbf{P}\,[\Delta\, Y > c_2\, y])$, as $x \wedge y \to \infty$, for any $c_1,\, c_2 > 0$. Hence, by [11, Th. 2.2] we find $\Theta\, X \in \mathcal{L}$ and $\Delta\, Y \in \mathcal{L}$. Next, we show $(\Theta\, X,\, \Delta\, Y) \in \mathcal{L}^{(2)}$. Indeed, from Lemma 6.1 we see that it is enough to show this for any $\Theta \geq \varepsilon_1$ and $\Delta \geq \varepsilon_2$ almost surely for any $\varepsilon_1,\, \varepsilon_2 > 0$. Let consider some $a_1,\, a_2 > 0$ and some $k_1,\, k_2,\, k > 0$, such that for a large enough $x_0 > 0$ holds

$$\mathbf{P}\left[X > x - \frac{a_1}{\varepsilon_1}\,,\ Y > y - \frac{a_2}{\varepsilon_2}\right] \leq (1 + k)\, \mathbf{P}\,[X > x\,,\ Y > y]\,, \tag{6.6}$$

for any $x,\, y > x_0$ and

$$\mathbf{P}\left[X > x - \frac{a_1}{\varepsilon_1}\right] \leq (1+k_1)\,\mathbf{P}\,[X > x]\;, \quad \mathbf{P}\left[Y > y - \frac{a_2}{\varepsilon_2}\right] \leq (1+k_2)\,\mathbf{P}\,[Y > y]\;, \tag{6.7}$$

for any $x > x_0$ and $y > x_0$, respectively. Then we have

$$\begin{aligned}
&\mathbf{P}\,[\Theta\, X > x - a_1\,,\; \Delta\, Y > y - a_2] \qquad (6.8)\\
&= \left(\int_{\varepsilon_1}^{x/x_0} + \int_{x/x_0}^{\infty}\right)\left(\int_{\varepsilon_2}^{y/x_0} + \int_{y/x_0}^{\infty}\right)\mathbf{P}\left[X > \frac{x-a_1}{s}\,,\; Y > \frac{y-a_2}{t}\right]\mathbf{P}[\Theta \in ds,\; \Delta \in dt]\\
&\qquad\qquad =: I_1(x,\,y) + I_2(x,\,y) + I_3(x,\,y) + I_4(x,\,y)\,,
\end{aligned}$$

where we find

$$I_4(x,\,y) = \int_{x/x_0}^{\infty}\int_{y/x_0}^{\infty}\mathbf{P}\left[X > \frac{x-a_1}{s}\,,\; Y > \frac{y-a_2}{t}\right]\mathbf{P}[\Theta \in ds,\; \Delta \in dt]\,,$$

that gives

$$I_4(x,\,y) \leq \mathbf{P}\left[\Theta \geq \frac{x}{x_0}\,,\; \Delta \geq \frac{y}{x_0}\right]\,. \tag{6.9}$$

Now we estimate $I_1(x,\,y)$

$$\begin{aligned}
I_1(x,\,y) &= \int_{\varepsilon_1}^{x/x_0}\int_{\varepsilon_2}^{y/x_0}\mathbf{P}\left[X > \frac{x-a_1}{s}\,,\; Y > \frac{y-a_2}{t}\right]\mathbf{P}[\Theta \in ds,\; \Delta \in dt]\\
&\leq \int_{\varepsilon_1}^{x/x_0}\int_{\varepsilon_2}^{y/x_0}\mathbf{P}\left[X > \frac{x}{s} - \frac{a_1}{\varepsilon_1}\,,\; Y > \frac{y}{t} - \frac{a_2}{\varepsilon_2}\right]\mathbf{P}[\Theta \in ds,\; \Delta \in dt]\\
&\leq \int_{\varepsilon_1}^{x/x_0}\int_{\varepsilon_2}^{y/x_0}\mathbf{P}\left[X > x_0 - \frac{a_1}{\varepsilon_1}\,,\; Y > x_0 - \frac{a_2}{\varepsilon_2}\right]\mathbf{P}[\Theta \in ds,\; \Delta \in dt]\\
&\leq (1+k)\,\mathbf{P}\,[\Theta\, X > x\,,\; \Delta\, Y > y]\;,
\end{aligned}$$

thus we get $I_1(x,\,y) \leq (1+k)\,\mathbf{P}\,[\Theta\, X > x\,,\; \Delta\, Y > y]$, which follows from (6.6).

Next we consider $I_2(x,\,y)$

$$\begin{aligned}
I_2(x,\,y) &= \int_{\varepsilon_1}^{x/x_0}\int_{y/x_0}^{\infty}\mathbf{P}\left[X > \frac{x-a_1}{s}\,,\; Y > \frac{y-a_2}{t}\right]\mathbf{P}[\Theta \in ds,\; \Delta \in dt]\\
&\leq \int_{\varepsilon_1}^{x/x_0}\mathbf{P}\left[X > \frac{x}{s} - \frac{a_1}{\varepsilon_1}\right]\mathbf{P}\left[\Theta \in ds,\; \Delta > \frac{y}{x_0}\right]\\
&\leq \int_{\varepsilon_1}^{x/x_0}\mathbf{P}\left[X > x_0 - \frac{a_1}{\varepsilon_1}\right]\mathbf{P}\left[\Theta \in ds,\; \Delta > \frac{y}{x_0}\right]\\
&\leq (1+k_1)\,\mathbf{P}\left[\Theta\, X > x\,,\; \Delta > \frac{y}{x_0}\right] \leq (1+k_1)\,\mathbf{P}\left[\Delta > \frac{y}{x_0}\right]\,,
\end{aligned}$$

that means $I_2(x,\,y) \leq (1+k_1)\,\mathbf{P}\,[\Delta > y/x_0]$, where in the pre-last step we use the first relation in (6.7). For $I_3(x,\,y)$ we use the second relation in (6.7) and due to symmetry we find

$$
\begin{aligned}
I_3(x,\,y) &= \int_{x/x_0}^{\infty}\int_{\varepsilon_2}^{\infty} \mathbf{P}\left[X > \frac{x-a_1}{s}\,,\ Y > \frac{y-a_2}{t}\right] \mathbf{P}[\Theta \in ds,\ \Delta \in dt] \\
&\leq (1+k_2)\,\mathbf{P}\left[\Theta > \frac{x}{x_0}\right]\,, \qquad (6.10)
\end{aligned}
$$

therefore putting the estimation from (6.9)-(6.10) into (6.8) we conclude

$$
\begin{aligned}
&\mathbf{P}\left[\Theta\, X > x - a_1\,,\ \Delta\, Y > y - a_2\right] \leq \mathbf{P}\left[\Theta > \frac{x}{x_0}\,,\ \Delta > \frac{y}{x_0}\right] \\
&\qquad +(1+k)\,\mathbf{P}\left[\Theta\, X > x\,,\ \Delta\, Y > y\right] + (1+k_2)\,\mathbf{P}\left[\Theta > \frac{x}{x_0}\right] + (1+k_1)\,\mathbf{P}\left[\Delta > \frac{y}{x_0}\right]\,,
\end{aligned}
$$

Now, because of (6.5) and the relation

$$
\frac{\mathbf{P}\left[\Theta > x\,,\ \Delta > y\right]}{\mathbf{P}\left[\Theta\, X > x\,,\ \Delta\, Y > y\right]} \leq \frac{\mathbf{P}\left[\Theta > x\right]}{\mathbf{P}\left[\Theta\, X > x\,,\ \Delta\, Y > y\right]} \longrightarrow 0\,,
$$

as $x \wedge y \to \infty$, we find

$$
\lim_{x\wedge y \to \infty} \frac{\mathbf{P}\left[\Theta\, X > x - a_1\,,\ \Delta\, Y > y - a_2\right]}{\mathbf{P}\left[\Theta\, X > x\,,\ \Delta\, Y > y\right]} \leq 1 + k\,,
$$

which in combination of the arbitrary choice of k and relation (6.3) we have $(\Theta\, X,\, \Delta\, Y) \in \mathcal{L}^{(2)}$. $\square$

Next, we consider a two-dimensional risk model on discrete-time, where the vector $(X_k,\, Y_k)$ represents losses in two lines of business at the k-th period, while the $(\Theta_k,\, \Delta_k)$ represent the discount factors of these two lines of business respectively. In this risk model we study only the aggregate claims, and we accept that the $\Theta_1,\, \ldots,\, \Theta_n,\, \Delta_1,\, \ldots,\, \Delta_m$ are independent of claims $X_1,\, \ldots,\, X_n,\, Y_1,\, \ldots,\, Y_m$. For further reading on risk models with dependence among the discount factors and main claims see in [4], [5], [40], but only in one dimension. Namely we have the sums:

$$
S_n(\Theta) := \sum_{k=1}^{n} \Theta_k X_k\,, \qquad T_n(\Delta) := \sum_{l=1}^{n} \Delta_l Y_l\,.
$$

Assumption 6.1. *There exist constants $0 < \xi_k \leq \delta_k$ such that hold $\xi_k \leq \Theta_k \leq \delta_k$ almost surely, for any $k = 1,\, \ldots,\, n$ and there exist constants $0 < \gamma_l \leq \zeta_l$ such that hold $\gamma_l \leq \Delta_l \leq \zeta_l$ almost surely, for any $l = 1,\, \ldots,\, n$.*

Theorem 6.1. *Let $X_1,\, \ldots,\, X_n,\, Y_1,\, \ldots,\, Y_n$ be random variables with the following distributions $F_1,\, \ldots,\, F_n,\, G_1,\, \ldots,\, G_n$ respectively form class $\mathcal{D} \cap \mathcal{L}$ and they satisfy the GTAI dependence structure, with $(X_k,\, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1,\, \ldots,\, n$ and $l = 1,\, \ldots,\, n$. We suppose that the random discount factors $\Theta_1,\, \ldots,\, \Theta_n,\, \Delta_1,\, \ldots,\, \Delta_n$ satisfy Assumption 6.1 and are independent of $X_1,\, \ldots,\, X_n,\, Y_1,\, \ldots,\, Y_n$. Then the products $\Theta_1\, X_1,\, \ldots,\, \Theta_n\, X_n$, $\Delta_1\, Y_1,\, \ldots,\, \Delta_n\, Y_n$, are GTAI with $(\Theta_k\, X_k,\, \Delta_l\, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ and if further $X_1,\, \ldots,\, X_n$ are*

TAI and $Y_1, \ldots, Y_n$ *are TAI then holds the asymptotic relations*

$$\mathbf{P}\left[S_n(\Theta) > x\,,\ T_n(\Delta) > y\right] \sim \mathbf{P}\left[\bigvee_{i=1}^{n} S_i(\Theta) > x\,,\ \bigvee_{j=1}^{n} T_j(\Delta) > y\right] \tag{6.11}$$

$$\sim \mathbf{P}\left[\bigvee_{k=1}^{n} \Theta_k\,X_k > x\,,\ \bigvee_{l=1}^{n} \Delta_l\,Y_l > y\right] \sim \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{P}\left[\Theta\,X_k > x\,,\ \Delta\,Y_l > y\right]\,,$$

as $x \wedge y \to \infty$.

Proof. Taking into account the upper bound for discount factors $\Theta_1, \ldots, \Theta_n, \Delta_1, \ldots, \Delta_n$ and their independence from $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, we apply [22, Lem. 1] to find that the products $\Theta_1\,X_1, \ldots, \Theta_n\,X_n, \Delta_1\,Y_1, \ldots, \Delta_n\,Y_n$ are GTAI. Now by [11, Th. 3.3(i)] we get $\Theta_k\,X_k \in \mathcal{D}$ and $\Delta_l\,Y_l \in \mathcal{D}$, for any $k = 1, \ldots, n$ and for any $l = 1, \ldots, n$. As a result by class $\mathcal{D}$ we using Theorem 2.2 of [25] $\Theta_1\,X_1, \ldots, \Theta_n\,X_n$ are TAI and $\Delta_1\,Y_1, \ldots, \Delta_n\,Y_n$ are TAI.

Next, we check if $(\Theta_k\,X_k,\, \Delta_l\,Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Let $\mathbf{b} = (b_1,\, b_2) \in (0,\,1)^2$, then

$$\limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[\Theta_k\,X_k > b_1\,x,\ \Delta_l\,Y_l > b_2\,y\right]}{\mathbf{P}\left[\Theta_k\,X_k > x,\ \Delta_l\,Y_l > y\right]} \le \limsup_{x\wedge y\to\infty} \frac{\mathbf{P}\left[X_k > b_1\dfrac{x}{\delta_k},\ Y_l > b_2\dfrac{y}{\zeta_l}\right]}{\mathbf{P}\left[X_k > \dfrac{x}{\xi_k}\,,\ Y_l > \dfrac{y}{\gamma_l}\right]} < \infty,$$

which follows from the inequalities

$$\frac{b_1}{\delta_k} < \frac{1}{\xi_k}\,, \qquad \frac{b_2}{\zeta_l} < \frac{1}{\gamma_l}\,,$$

and the membership $(X_k,\, Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Thence we find the relation $(\Theta_k\,X_k,\, \Delta_l\,Y_l) \in \mathcal{D}^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$.

Now, noticing that relations (6.5) are satisfied because of Assumption 6.1, we obtain directly from Lemma 6.2 the inclusion $(\Theta_k\,X_k,\, \Delta_l\,Y_l) \in \mathcal{L}^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$. Hence $(\Theta_k\,X_k,\, \Delta_l\,Y_l) \in (\mathcal{D} \cap \mathcal{L})^{(2)}$ for any $k = 1, \ldots, n$ and $l = 1, \ldots, n$ and by application of Theorem 4.2 for the products we conclude relation (6.11). ☐

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References

[1] Asmussen, S., Foss, S., Korshunov, D. (2003) Asymptotics for Sums of Random Variables with Local Subexponential Behaviour. *J. Theor. Probab.*, **16**, 489–518.

[2] Bingham. N.H., Goldie, C.M., Teugels, J.L. (1987) *Regular Variation* Cambridge University Press, Cambridge .

[3] Cai, J., Tang, Q. (2004) On max-sum equivalence and convolution closure of heavy-tailed distributions and their applications. *J. Appl. Probab.* **41**, 117–130.

[4] Chen, Y. (2011) The finite-time ruin probabilities with dependent insurance and financial risks. *J. Appl. Probab.* **48**, 1035–1048.

[5] Chen, Y. (2017) Interplay of subexponential and dependent insurance and financial risks. *Insur. Math. Econom.* **77**, 78–83.

[6] Chen, Y., Yang, Y. (2019) Bivariate regular variation among randomly weighted sums in general insurance. *Eur. Actuar. J.* **9**, 301–322.

[7] Cheng, D. (2021) Uniform asymptotics for the finite-time ruin probability of a generalized bidimensional risk model with Brownian perturbations. *Stochastics* **93**, Vol 1. 56–71.

[8] Cheng, D., Yu, C. (2019) Uniform asymptotics for the ruin probabilities in a bidimensional renewal risk model with strongly subexponential claims. *Stochastics* **91**, Vol 1. 643–656.

[9] Chistyakov, V.P. (1964) A theorem on sums of independent positive random variables and its applications to branching random processes. *Theory Probab. Appl.*, **9**, 640–648.

[10] Cline, D.B.H., Resnick, S. (1992) Multivariate subexponential distributions. *Stoch. Process. Appl.*, **42**, no.1, 49–72.

[11] Cline, D.B.H., Samorodnitsky, G. (1994) Subexponentiality of the product of independent random variables. *Stoch. Process. Appl.*, **49**, 75–98.

[12] De Haan, L., Resnick, S. (1982) On the observation closet to the origin. *Stoch. Process. Appl.*, **11**, no. 3, 301–308.

[13] Foss, S., Korshunov, D., Zachary, S. (2013) *An Introduction to Heavy-Tailed and Subexponential Distributions* Springer, New York, 2nd ed.

[14] Geluk, J., Ng, K.W. (2006) Tail behavior of negatively associated heavy-tailed sums. *J. Appl. Probab.*, **43**, no. 2, 587–593.

[15] Geluk, J., Tang, Q. (2009) Asymptotic tail probabilities of sums of dependent subexponential random variables. *J. Theor. Probab.*, **22**, 871–882.

[16] Geng, B., Liu, Z., Wang, S. (2023) A Kesten-type inequality for randomly weighted sums of dependent subexponential random variables with application to risk theory. *Lith. Math. J.* **63**, 81–91.

[17] Goldie, C.M. (1978) Subexponential distributions and dominated variation tails *J. Appl. Probab.*, **15**, 440–442.

[18] Hu, Z., Jiang, B. (2013) On joint ruin probabilities of a two-dimensional risk model with constant interest rate. *J. Appl. Probab.*, **50**, no. 2, 309–322.

[19] Ji, X., Wang, B., Yan, J., Cheng, D. (2023) Asymptotic estimates for finite-time ruin probabilities in a generalized dependent bidimensional risk model with CMC simulations. *J. Industr. Manag. Optim.*, **19**, 2140–2155.

[20] Konstantinides, D.G. (2018) *Risk Theory. A Heavy Tail Approach* World Scientific, New Jersey.

[21] Konstantinides, D.G., Li, J. (2016) Asymptotic ruin probabilities for a multidimensional renewal risk model with multivariate regularly varying claims. *Insur. Math. and Econom.*, **69**, 38–44.

[22] Konstantinides, D.G., Passalidis, C.D. (2024(a)) Background risk model in presence of heavy tails under dependence. *Preprint.*

[23] Konstantinides, D.G., Passalidis, C.D. (2024(b)) Closure properties and heavy tails: random vectors in the presence of dependence *Preprint, arXiv:2402.09041.*

[24] Leipus, R., Šiaulys, J., Konstantinides, D.G. (2023) *Closure Properties for Heavy-Tailed and Related Distributions: An Overview.* Springer Nature, Cham.

[25] Li, J. (2013) On pairwise quasi-asymptotically independent random variables and their applications. *Stat. Prob. Lett.*, **83**, 2081–2087.

[26] Li, J. (2016) Uniform asymptotics for a multi-dimensional time-dependent risk model with multivariate regularly varying claims and stochastic return. *Insur. Math. Econom.*, **71**, 195–204.

[27] Li, J., Tang, Q. (2015) Interplay of insurance and financial risks in a discrete time model with strongly regular variation. *Bernoulli* 21, no.3, 1800–1823

[28] Li, J. (2018) On the joint tail behavior of randomly weighted sums of heavy-tailed random variables. *J. Mutlivar. Anal.*, **164**, 40–53.

[29] Li, J., Tang, Q., Wu, R. (2010) Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model. *Adv. Appl. Probab.*, **42**, no. 4, 1126-1146.

[30] Ng, K.W., Tang, Q., Yang, H. (2002) Maxima of sums of heavy-tailed random variables. *ASTIN bull.*, **32**, no.1, 43–55.

[31] Omey, E. (2006) Subexponential distribution functions in R^d. *J. Math. Sci.*, **138**, no.1, 5434–5449.

[32] Omey, E., Mallor, F., Santos, J. (2006) Multivariate subexponential distribution and random sums of random vectors. *Adv. Appl. Probab.*, **38**, 1028–1046.

[33] Samorodnitsky, G., Sun, J. (2016) Multivariate subexponential distributions and their applications. *Extremes*, **19**, no. 2, 171–196.
[34] Shen, X., Du, K. (2023) Uniform approximation for the tail behavior of bidimensional randomly weighted sums. *Methodol. Comput. Appl. Probab.*, **25**, no. 26.
[35] Shen, X., Ge, M., Fu, K.A. (2018) Approximation of the tail probabilities for bidimensional randomly weighted sums with dependent components. *Probab. Eng. Inf. Sci.*, **34**, no. 1, 112–130.
[36] Tang, Q. (2006) The subexponentiality of products revisited. *Extremes*, **9**, 231–241.
[37] Tang, Q., Yuan, Z. (2014) Randomly weighted sums of subexponential random variables with application to capital allocation. *Extremes*, **17**, 467–493.
[38] Tang, Q., Tsitsiashvili, G. (2003) Randomly weighted sums of subexponential random variables with application to ruin theory. *Extremes*, **6**, 171–188.
[39] Tang, Q., Yang, Y. (2019) Interplay of insurance and financial risks in a stochastic environment. *Scand. Actuar. J.*, no. 5, 432–451.
[40] Yang, H., Gao, W., Li, J. (2016) Asymptotic ruin probabilities for a discrete-time risk model with dependent insurance and financial risks *Scand. Actuar. J.*, **1**, 1–17.
[41] Yang, Y., Chen, S., Yuen, C. (2024) Asymptotic for the joint probability of bidimensional randomly stopped sums with applications to insurance. *Sci. China Math.*, **67**, 163–186.
[42] Yang, Y., Konstantinides, K. (2015) Asymptotic for ruin probabilities in a discrete-time risk model with dependent financial and insurance risks. *Scand. Actuar. J.*, **8**, 641–659.
[43] Yang, Y., Leipus, R., Šiaulys, J. (2012) Tail probability of randomly weighted sums of subexponential random variables under a dependence structure. *Stat. Probab. Lett.*, **82**, 1727–1736.
[44] Yang, Y., Su, Q. (2023) Asymptotic behavior of ruin probabilities in a multidimensional risk model with investment and multivariate regularly varying claims. *J. Math. Anal. Appl.*, **525**, 127319.
[45] Zhang, Y., Shen, X., Weng, C. (2009) Approximation of the tail probability of randomly weighted sums and applications. *Stoch. Process. Appl.*, **119**, 655–675.

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