

Parameterized Vertex Integrity Revisited

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Abstract

Vertex integrity is a graph parameter that measures the connectivity of a graph. Informally, its meaning is that a graph has small vertex integrity if it has a small separator whose removal disconnects the graph into connected components which are themselves also small. Graphs with low vertex integrity are extremely structured; this renders many hard problems tractable and has recently attracted interest in this notion from the parameterized complexity community. In this paper we revisit the NP-complete problem of computing the vertex integrity of a given graph from the point of view of structural parameterizations. We present a number of new results, which also answer some recently posed open questions from the literature. Specifically: We show that unweighted vertex integrity is $W[1]$ -hard parameterized by treedepth; we show that the problem remains $W[1]$ -hard if we parameterize by feedback edge set size (via a reduction from a BIN PACKING variant which may be of independent interest); and complementing this we show that the problem is FPT by max-leaf number. Furthermore, for weighted vertex integrity, we show that the problem admits a single-exponential FPT algorithm parameterized by vertex cover or by modular width, the latter result improving upon a previous algorithm which required weights to be polynomially bounded.

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1 Introduction

The *vertex integrity* of a graph is a vulnerability measure indicating how easy it is to break down the graph into small pieces. More precisely, the vertex integrity $vi(G)$ of a graph G is defined as $vi(G) = \min_{S \subseteq V(G)} \{|S| + \max_{D \in cc(G-S)} |D|\}$, that is, to calculate the vertex integrity of a graph we must find a separator that minimizes the size of the separator itself plus the size of the largest remaining connected component. Intuitively, a graph has low vertex integrity not only when it contains a small separator, but more strongly when it contains a small separator such that its removal leaves a collection of small connected components.

Vertex integrity was first introduced more than thirty years ago by Barefoot et al. [1], but has recently received particular attention from the parameterized complexity community, because it can be considered as a very natural structural parameter: when a graph has vertex

integrity k , large classes of NP-hard problems admit FPT¹ algorithms with running times of the form $f(k)n^{\mathcal{O}(1)}$ [28]. Note that vertex integrity has a clear relationship with other, well-known structural parameters: it is more restrictive than treedepth, pathwidth, and treewidth (all these parameters are upper-bounded by vertex integrity) but more general than vertex cover (a graph of vertex cover k has vertex integrity at most $k+1$). “Price of generality” questions, where one seeks to discover for a given problem the most general parameter for which an FPT algorithm is possible, are a central topic in structural parameterized complexity, and vertex integrity therefore plays a role as a natural stepping stone in the hierarchy of standard parameters [4, 16, 17, 21, 23, 28].

The investigation of the parameterized complexity aspects of vertex integrity is, therefore, an active field of research, but it is important to remember that a prerequisite for any such parameter to be useful is that it should be tractable to calculate the parameter itself (before we try to use it to solve other problems). Since, unsurprisingly, computing the vertex integrity exactly is NP-complete [7], in this paper we focus on this problem from the point of view of parameterized complexity. We consider both the unweighted, as well as a natural weighted variant of the problem. Formally, we want to solve the following:

UNWEIGHTED (WEIGHTED) VERTEX INTEGRITY

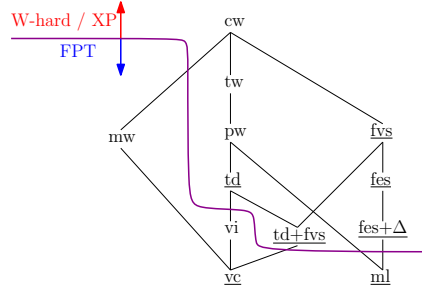
Instance: A graph G (with binary vertex weights $w : V(G) \rightarrow \mathbb{Z}^+$), an integer k .
Goal: Determine whether $\text{vi}(G) \leq k$ ($\text{wvi}(G) \leq k$).

The point of view we adopt is that of structural parameterized complexity, where vertex integrity is the target problem we are trying to solve, and not necessarily the parameter. Instead, we parameterize by standard structural width measures, such as variations of treewidth. The questions we would like to address are of several forms:

1. For which structural parameters is it FPT to compute the vertex integrity?
2. For which such parameters is it possible to obtain an FPT algorithm with single-exponential complexity?
3. For which parameters can the weighted version of the problem be handled as well as the unweighted version?

To put these questions in context, we recall some facts from the state of the art. When the parameter k is the vertex integrity itself, Fellows and Stueckle show an $\mathcal{O}(k^{3k}n)$ -time algorithm for UNWEIGHTED VERTEX INTEGRITY [14], and later Drange et al. proposed an $\mathcal{O}(k^{k+1}n)$ -time algorithm even for WEIGHTED VERTEX INTEGRITY [10], so this problem is FPT. More recently, Gima et al. [22] took up the study of vertex integrity in the same structurally parameterized spirit as the one we adopt here and presented numerous results which already give some answers to the questions we posed above. In particular, for the first question they showed that UNWEIGHTED VERTEX INTEGRITY is W[1]-hard by pathwidth (and hence by treewidth); for the second question they showed that the problem admits a single-exponential algorithm for parameter modular-width; and for the third question they showed that the problem is (weakly) NP-hard on sub-divided stars, which rules out FPT algorithms for most structural parameters.

¹ We assume the reader is familiar with the basics of parameterized complexity, as given e.g. in [8]. We give precise definitions of all parameters in the next section.



■ **Figure 1** The parameterized complexity of UNWEIGHTED VERTEX INTEGRITY, with the underlined parameters indicating our results. A connection between two parameters implies that the one above generalizes the one below; that is, the one below is lower-bounded by a function of the one above. All of our FPT algorithms have single-exponential parametric dependence, while the ones for vc and mw extend to the weighted case as well.

Our results. Although the results of [22] are rather comprehensive, they leave open several important questions about the complexity of vertex integrity. In this paper we resolve the questions explicitly left open by [22] and go on to present several other results that further clarify the picture for vertex integrity. In particular, our results are as follows (see also Figure 1):

The first question we tackle is an explicit open problem from [22]: is UNWEIGHTED VERTEX INTEGRITY FPT parameterized by treedepth? This is a very natural question, because treedepth is the most well-known parameter that sits between pathwidth, where the problem is $W[1]$ -hard by [22], and vertex integrity itself, where the problem is FPT. We resolve this question via a reduction from BOUNDED DEGREE VERTEX DELETION, showing that UNWEIGHTED VERTEX INTEGRITY is $W[1]$ -hard for treedepth (Theorem 2).

A second explicitly posed open question of [22] is the complexity of UNWEIGHTED VERTEX INTEGRITY for parameter feedback vertex set. Taking a closer look at our reduction from BOUNDED DEGREE VERTEX DELETION, which is known to be $W[1]$ -hard for this parameter, we observe that it also settles this question, showing that UNWEIGHTED VERTEX INTEGRITY is also hard. However, in this case we are motivated to dig a little deeper and consider a parameter, feedback edge set, which is a natural restriction of feedback vertex set and typically makes most problems FPT. Our second result is to show that UNWEIGHTED VERTEX INTEGRITY is in fact $W[1]$ -hard even when parameterized by feedback edge set and the maximum degree of the input graph (Theorem 7). We achieve this via a reduction from UNARY BIN PACKING parameterized by the number of bins, which is $W[1]$ -hard [24]. An aspect of our reduction which may be of independent interest is that we use a variant of UNARY BIN PACKING where we are given a choice of only two possible bins per item (we observe that the reduction of [24] applies to this variant).

We complement these mostly negative results with a fixed-parameter tractability result for a more restrictive parameter: we show that UNWEIGHTED VERTEX INTEGRITY is FPT by max-leaf number (Theorem 10) indeed by a single-exponential FPT algorithm. Note that when a graph has bounded max-leaf number, then it has bounded degree and bounded feedback edge set number, therefore this parameterization is a special case of the one considered in Theorem 7. Hence, this positive result closely complements the problem's hardness in the more general case.

Moving on, we consider the parameterization by modular width, and take a second look at the $2^{\mathcal{O}(mw)} n^{\mathcal{O}(1)}$ algorithm provided by [22], which is able to handle the weighted case of

the problem, but only for polynomially-bounded weights. Resolving another open problem posed by [22], we show how to extend their algorithm to handle the general case of weights encoded in binary (Theorem 13).

Finally, we ask the question of whether a single-exponential FPT algorithm is possible for parameters other than max-leaf and modular width. We answer this affirmatively for vertex cover, even in the weighted case (Theorem 16), obtaining a faster and simpler algorithm for the unweighted case (Theorem 14).

Related work. The concept of vertex integrity is natural enough that it has appeared in many slight variations under different names in the literature. We mention in particular, the fracture number [11], which is the minimum k such that it is possible to delete k vertices from a graph so that all remaining components have size at most k , and the starwidth [31], which is the minimum width of a tree decomposition that is a star. Both of these are easily seen to be at most a constant factor away from vertex integrity. Similarly, the safe set number [3, 15] seeks a separator such that every component of the separator is only connected to smaller components. These concepts are so natural that sometimes they are used as parameters without an explicit name, for example [5] uses the parameter “size of a deletion set to a collection of components of bounded size”. As observed by [22], despite these similarities, sometimes computing these parameters can have different complexity, especially when weights are allowed. Another closely related computational problem, that we also use, is the COMPONENT ORDER CONNECTIVITY [10] problem, where we are given explicit distinct bounds on the size of the separator sought and the allowed size of the remaining components.

2 Preliminaries

Throughout the paper we use standard graph notation [9] and assume familiarity with the basic notions of parameterized complexity [8]. All graphs considered are undirected without loops. Given a graph G and $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S , while $G - S$ denotes $G[V(G) \setminus S]$. If we are additionally given a weight function $w : V(G) \rightarrow \mathbb{Z}^+$, $w(S)$ denotes the sum of the weights of the vertices of S , i.e. $w(S) = \sum_{s \in S} w(s)$. For $x, y \in \mathbb{Z}$, let $[x, y] = \{z \in \mathbb{Z} : x \leq z \leq y\}$, while $[x] = [1, x]$. For a set of integers $S \subseteq \mathbb{Z}^+$, let $\Sigma(S)$ denote the sum of its elements, i.e. $\Sigma(S) = \sum_{s \in S} s$, while $\binom{S}{c}$ denotes the set of subsets of S of size c , i.e. $\binom{S}{c} = \{S' \subseteq S : |S'| = c\}$.

2.1 Vertex Integrity

For a vertex-weighted graph G with $w : V(G) \rightarrow \mathbb{Z}^+$, we define its *weighted vertex integrity*, denoted by $\text{wvi}(G)$, as

$$\text{wvi}(G) = \min_{S \subseteq V(G)} \left\{ w(S) + \max_{D \in \text{cc}(G-S)} w(D) \right\},$$

where $\text{cc}(G - S)$ is the set of connected components of $G - S$. A set S such that $w(S) + \max_{D \in \text{cc}(G-S)} w(D) \leq k$ is called a $\text{wvi}(k)$ -set. The *vertex integrity* of an unweighted graph G , denoted by $\text{vi}(G)$, is defined in an analogous way, by setting $w(v) = 1$ for all $v \in V(G)$. In that case, $S \subseteq V(G)$ is a $\text{vi}(k)$ -set if $|S| + \max_{D \in \text{cc}(G-S)} |D| \leq k$.

A vertex $v \in S$ is called *redundant* if at most one connected component of $G - S$ contains neighbors of v . A set $S \subseteq V(G)$ is *irredundant* if S contains no redundant vertex. Thanks

to Proposition 1, it suffices to only search for irredundant $\text{wvi}(k)$ -sets when solving VERTEX INTEGRITY.

► **Proposition 1** ([10, 22]). *A graph with a $\text{wvi}(k)$ -set has an irredundant $\text{wvi}(k)$ -set.*

2.2 Graph Parameters

We use several standard graph parameters, so we recall here their definitions and known relations between them. A graph G has *feedback vertex* (respectively *edge*) *set* k if there exists a set of k vertices (respectively edges) such that removing them from G destroys all cycles. We use $\text{fvs}(G)$ and $\text{fes}(G)$ to denote these parameters. Note that even though computing $\text{fvs}(G)$ is NP-complete [25], in all connected graphs with m edges and n vertices $\text{fes}(G) = m - n + 1$. The *vertex cover* of a graph G , denoted by $\text{vc}(G)$, is the size of the smallest set whose removal destroys all edges. The *treedepth* of a graph G can be defined recursively as follows: $\text{td}(K_1) = 1$; if G is disconnected $\text{td}(G)$ is equal to the maximum of the treedepth of its connected components; otherwise $\text{td}(G) = \min_{v \in V(G)} \text{td}(G - v) + 1$. The *max-leaf number* of a graph G , denoted by $\text{ml}(G)$, is the maximum number of leaves of any spanning tree of G .

A module of a graph $G = (V, E)$ is a set of vertices $M \subseteq V$ such that for all $x \in V \setminus M$ we have that x is either adjacent to all vertices of M or to none. The *modular width* of a graph $G = (V, E)$ ([18, 19]) is the smallest integer k such that, either $|V| \leq k$, or V can be partitioned into at most $k' \leq k$ sets $V_1, \dots, V_{k'}$, with the following two properties: (i) for all $i \in [k']$, V_i is a module of G , (ii) for all $i \in [k']$, $G[V_i]$ has modular width at most k .

Let us also briefly explain the relations depicted in Figure 1. Clearly, for all G , we have $\text{fvs}(G) \leq \text{fes}(G)$, because we can remove from the graph one endpoint of each edge of the feedback edge set. It is known that if a graph has $\text{ml}(G) = k$, then G contains at most $\mathcal{O}(k)$ vertices of degree 3 or more (Lemma 8), and clearly such a graph has maximum degree at most k . Since vertices of degree at most 2 are irrelevant for fes , we conclude that the parameterization by ml is more restrictive than that for $\text{fes} + \Delta$. It is also not hard to see that for all G , $\text{td}(G) \leq \text{vi}(G) \leq \text{vc}(G) + 1$. Note also that even though vc can be seen as a parameter more restrictive than mw , when a graph has vertex cover k , the best we can say is that its modular width is at most $2^k + k$ [27]. As a result, the algorithm of Theorem 13 does not imply a single-exponential FPT algorithm for parameter vc (but does suffice to show that the problem is FPT). We also note that the reductions of Theorem 2 (for td) and Theorem 7 (for $\text{fes} + \Delta$) are complementary and cannot be subsumed by a single reduction. The reason for this is that if in a graph we bound simultaneously the treedepth and the maximum degree, then we actually bound the size of the graph (rendering all problems FPT).

3 Treedepth

Our main result in this section is the following theorem, resolving a question of [22]. We obtain it via a parameter-preserving reduction from BOUNDED DEGREE VERTEX DELETION, which is known to be $W[1]$ -hard parameterized by treedepth plus feedback vertex set [20].

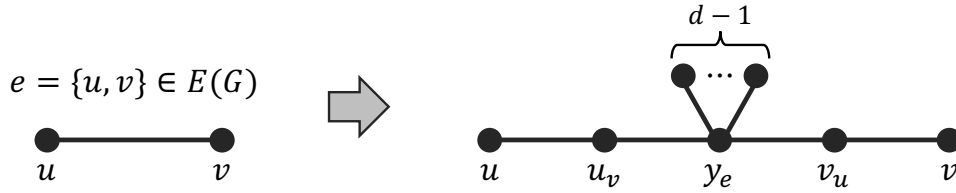
► **Theorem 2.** *UNWEIGHTED VERTEX INTEGRITY is $W[1]$ -hard parameterized by $\text{td} + \text{fvs}$. Moreover, it cannot be solved in time $f(\text{td})n^{o(\text{td})}$ under the ETH.*

Proof. First we define the closely related COMPONENT ORDER CONNECTIVITY problem: given a graph G as well as integers ℓ and p , we want to determine whether there exists $S \subseteq V(G)$ such that $|S| \leq p$ and all components of $G - S$ have size at most ℓ . We will

proceed in two steps: we first reduce BOUNDED DEGREE VERTEX DELETION to COMPONENT ORDER CONNECTIVITY, and then employ the reduction of [22] that reduces the latter to UNWEIGHTED VERTEX INTEGRITY. Notice that [22, Lemma 4.4] creates an equivalent instance of UNWEIGHTED VERTEX INTEGRITY by solely adding disjoint stars and leaves in the vertices of the initial graph, therefore it suffices to prove the statement for COMPONENT ORDER CONNECTIVITY instead.

We give a parameterized reduction from BOUNDED DEGREE VERTEX DELETION, which is $W[1]$ -hard by treedepth plus feedback vertex set number [20] and cannot be solved in time $f(\text{td})n^{o(\text{td})}$ under the ETH [29]. In BOUNDED DEGREE VERTEX DELETION we are given a graph $G = (V, E)$ and two integers k and d , and we are asked to determine whether there exists $S \subseteq V$ of size $|S| \leq k$ such that the maximum degree of $G - S$ is at most d . In the following, let $n = |V(G)|$ and $m = |E(G)|$.

Given an instance (G, k, d) of BOUNDED DEGREE VERTEX DELETION, we construct an equivalent instance (G', ℓ, p) of COMPONENT ORDER CONNECTIVITY. We construct G' from G as follows: We subdivide every edge $e = \{u, v\} \in E(G)$ three times, thus replacing it with a path on vertices u, u_v, y_e, v_u , and v , where $T_e = \{u_v, y_e, v_u\}$. Next, we attach $d - 1$ leaves to y_e (see Figure 2). This concludes the construction of G' . Notice that the subdivision of the edges three times and the attachment of pendant vertices does not change the feedback vertex set number, while the treedepth is only increased by an additive constant. Thus, it holds that $\text{fvs}(G') = \text{fvs}(G)$ and $\text{td}(G') = \text{td}(G) + \mathcal{O}(1)$.



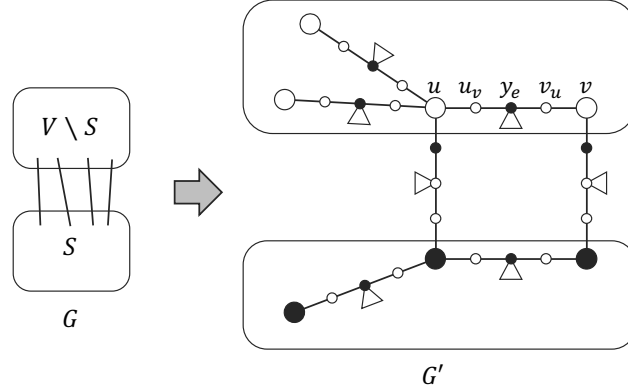
■ **Figure 2** Edge gadget for edge $e = \{u, v\} \in E(G)$.

In the following, we show that (G, k, d) is a yes-instance of BOUNDED DEGREE VERTEX DELETION if and only if (G', ℓ, p) is a yes-instance of COMPONENT ORDER CONNECTIVITY, where $\ell = d + 1$ and $p = k + m$.

For the forward direction, let S be a set of vertices of size at most k such that the maximum degree of $G - S$ is at most d . We will construct a set $S' \subseteq V(G')$ such that $|S'| \leq p$ and every connected component of $G' - S'$ has size at most ℓ . Initially set $S' = S$. Then, add one vertex to S' per edge $e = \{u, v\} \in E(G)$ as follows. If $u, v \in S$ or $u, v \notin S$, we add y_e to S' . Otherwise, if $u \in S$ and $v \notin S$, we add v_u to S' ; symmetrically, if $u \notin S$ and $v \in S$, we add u_v instead (see Figure 3). Notice that $|S'| = |S| + m \leq k + m = p$, therefore it suffices to show that the size of each connected component of $G' - S'$ is at most $\ell = d + 1$.

Consider a connected component D of $G' - S'$. Assume that D does not contain any vertices of $V \setminus S$. If D is a leaf it holds that $|D| \leq d + 1$. Alternatively, D is a subgraph of the graph induced by u_v (or v_u), y_e , and its attached leaves, for some $e = \{u, v\} \in E(G)$, in which case $|D| \leq d + 1$. Now assume that D contains $u \in V \setminus S$. Notice that u is the only vertex of $V \setminus S$ present in D , since $S' \cap T_e \neq \emptyset$ for all $e \in E(G)$. Moreover, let $N(u) \setminus S = \{u_i : i \in [q]\}$ denote its neighbors in $G - S$, where $q \leq d$ since the maximum degree of $G - S$ is at most d . In that case, it follows that D consists of u , as well as the vertices u_{u_i} for all $i \in [q]$. Consequently, $|D| = q + 1 \leq d + 1$.

For the converse direction, assume there exists $S' \subseteq V(G')$ such that $|S'| \leq p = k + m$



■ **Figure 3** Black vertices belong to S' .

and $|D| \leq \ell = d + 1$, for all connected components $D \in \text{cc}(G' - S')$. Assume that S' does not contain any leaves; if it does, substitute them with their single neighbor. Moreover, $S' \cap T_e \neq \emptyset$ for all $e \in E(G)$, since otherwise $G' - S'$ has a component of size at least $d + 2 > \ell$, which is a contradiction. Assume without loss of generality that $|S' \cap T_e| = 1$, for all $e = \{u, v\} \in E(G)$; if that is not the case, there is always a vertex of $\{u_v, v_u\}$, say u_v , such that $u_v \in S'$ and $S' \cap \{y_e, v_u\} \neq \emptyset$, in which case one may consider the deletion set $(S' \cup \{u\}) \setminus \{u_v\}$ instead (the argument is symmetric in case $v_u \in S'$).

Let $S = S' \cap V$, where $|S| \leq k$. We will prove that $G - S$ has maximum degree at most d . Let D_u denote the connected component of $G' - S'$ that contains $u \in V \setminus S'$; in fact this is the only vertex of $V \setminus S'$ present in D_u , since $S' \cap T_e \neq \emptyset$ for all $e \in E(G)$. Notice that for all $e = \{u, v\} \in E(G)$ where $u, v \notin S'$, it holds that $y_e \in S'$: if that were not the case, then either D_u or D_v contains at least $d + 2 > \ell$ vertices, due to $\{u, u_v, y_e\}$ or $\{v, v_u, y_e\}$ and the leaves of y_e respectively. For $u \in V \setminus S$, let $N(u) \setminus S = \{u_i : i \in [q]\}$, for some integer q , denote its neighbors in $G - S$, where $e_i = \{u, u_i\} \in E(G)$ for $i \in [q]$. It suffices to show that $q \leq d$. Assume that this is not the case, i.e. $q > d$. Then, since $S' \cap T_{e_i} = \{y_{e_i}\}$ for $i \in [q]$, it follows that D_u contains vertices u and u_{u_i} , therefore $|D_u| \geq q + 1 > d + 1 = \ell$, which is a contradiction. Consequently, $|N(u) \setminus S| \leq d$ for all $u \in V \setminus S$, i.e. $G - S$ has maximum degree d . ◀

4 Feedback Edge Set plus Maximum Degree

In this section we prove that VERTEX INTEGRITY is W[1]-hard parameterized by $\text{fes} + \Delta$. Since our reduction is significantly more involved than the one of Theorem 2, we proceed in several steps. We start from an instance of UNARY BIN PACKING where the parameter is the number of bins and consider a variant where we are also supplied in the input, for each item, a choice of two possible bins to place it. We first observe that the reduction of [24] shows that this variant is also W[1]-hard. We then reduce this to a *semi-weighted* version of VERTEX INTEGRITY, where placing a vertex in the separator always costs 1, but vertices have weights which they contribute to their components if they are not part of the separator, and where we are prescribed the size of the separator to use (this is called the COMPONENT ORDER CONNECTIVITY problem). Subsequently, we show how to remove the weights and the prescription on the separator size to obtain hardness for VERTEX INTEGRITY.

4.1 Preliminary Tools

Unary Bin Packing. Given a set $S = \{s_1, \dots, s_n\}$ of integers in unary (i.e. $s_i = \mathcal{O}(n^c)$ for some constant c), as well as $k \in \mathbb{Z}^+$, UNARY BIN PACKING asks whether we can partition S into k subsets $S_1, \dots, S_k$, such that $\Sigma(S_i) = \Sigma(S)/k$, for all $i \in [k]$. This problem is well known to be $W[1]$ -hard parameterized by the number of bins k [24]. We formally define a restricted version where every item is allowed to choose between *exactly two* bins, and by delving deeper into the proof of [24] we observe that an analogous hardness result follows.

RESTRICTED UNARY BIN PACKING

- Instance:** A set $S = \{s_1, \dots, s_n\}$ of integers in unary, $k \in \mathbb{Z}^+$, as well as a function $f : S \rightarrow \binom{[k]}{2}$.
- Goal:** Determine whether we can partition S into k subsets $S_1, \dots, S_k$, such that for all $i \in [k]$ it holds that (i) $\Sigma(S_i) = \Sigma(S)/k$, and (ii) $\forall s \in S_i, i \in f(s)$.

► **Theorem 3.** $(\star)$ RESTRICTED UNARY BIN PACKING is $W[1]$ -hard parameterized by the number of bins.

Semi-weighted problems. In this section we study semi-weighted versions of COMPONENT ORDER CONNECTIVITY and VERTEX INTEGRITY, which we first formally define. Then, we prove that the first can be reduced to the latter, while retaining the size of the minimum feedback edge set and the maximum degree.

SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY

- Instance:** A vertex-weighted graph $G = (V, E, w)$, as well as integers $\ell, p \in \mathbb{Z}^+$.
- Goal:** Determine whether there exists $S \subseteq V$ of size $|S| \leq p$, such that $w(D) \leq \ell$ for all $D \in \text{cc}(G - S)$.

SEMI-WEIGHTED VERTEX INTEGRITY

- Instance:** A vertex-weighted graph $G = (V, E, w)$, as well as an integer $\ell \in \mathbb{Z}^+$.
- Goal:** Determine whether there exists $S \subseteq V$ such that $|S| + w(D) \leq \ell$ for all $D \in \text{cc}(G - S)$.

► **Theorem 4.** $(\star)$ SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY parameterized by $\text{fes} + \Delta$ is *fpt-reducible* to SEMI-WEIGHTED VERTEX INTEGRITY parameterized by $\text{fes} + \Delta$.

4.2 Hardness Result

Using the results of Section 4.1, we proceed to proving the main theorem of this section. To this end, we present a reduction from RESTRICTED UNARY BIN PACKING to SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY such that for the produced graph G it holds that $\text{fes}(G) + \Delta(G) \leq f(k)$, for some function f and k denoting the number of bins of the RESTRICTED UNARY BIN PACKING instance.

We first provide a sketch of our reduction. For every bin of the RESTRICTED UNARY BIN PACKING instance, we introduce a clique of $\mathcal{O}(k)$ heavy vertices, and then connect any pair of such cliques via two paths. The weights are set in such a way that an optimal solution will only delete vertices from said paths. In order to construct a path for a pair of bins, we compute the set of all subset sums of the items that can be placed in these two bins, and introduce a vertex of medium weight per such subset sum. Moreover, every such vertex

corresponding to subset sum s is preceded by exactly s vertices of weight 1. An optimal solution will cut the path in such a way that the number of vertices of weight 1 will be partitioned between the two bins, encoding the subset sum of the elements that are placed on each bin. The second path that we introduce has balancing purposes, allowing us to exactly count the number of vertices of medium weight that every connected component will end up with.

► **Theorem 5.** $(\star)$ SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY is $W[1]$ -hard parameterized by $\text{fes} + \Delta$.

By Theorems 4 and 5, the hardness of SEMI-WEIGHTED VERTEX INTEGRITY follows.

► **Theorem 6.** SEMI-WEIGHTED VERTEX INTEGRITY is $W[1]$ -hard parameterized by $\text{fes} + \Delta$.

Moreover, we can easily reduce an instance (G, w, k) of SEMI-WEIGHTED VERTEX INTEGRITY to an instance (G', k) of UNWEIGHTED VERTEX INTEGRITY by attaching a path on $w(v) - 1$ vertices to each vertex v (we assume that $w(v) \leq k$, otherwise v belongs to the deletion set). Thus, $\text{fes}(G') = \text{fes}(G)$ and $\Delta(G') = \Delta(G) + 1$, and due to Theorem 6 the main result of this section follows.

► **Theorem 7.** UNWEIGHTED VERTEX INTEGRITY is $W[1]$ -hard parameterized by $\text{fes} + \Delta$.

5 Max-Leaf Number

In this section, we consider UNWEIGHTED VERTEX INTEGRITY parameterized by the max-leaf number. For a connected graph G we denote by $\text{ml}(G)$ the maximum number of leaves of any spanning tree of G . This is a well-studied but very restricted parameter [12, 13, 27]. In particular, it is known that if a graph G has $\text{ml}(G) \leq k$, then in fact G is a subdivision of a graph on $\mathcal{O}(k)$ vertices [26]. We are motivated to study this parameter because in a sense it lies close to the intractability boundary established in Section 4. Observe that if a graph is a sub-division of a graph on k vertices, then it has maximum degree at most k and feedback edge set at most k^2 ; however, graphs of small feedback edge set and small degree do not necessarily have small max-leaf number (consider a long path where we attach a leaf to each vertex). Interestingly, the graphs we construct in Section 4.2 *do* have small max-leaf number, if we consider semi-weighted instances. However, adding the necessary simple gadgets in order to simulate weights increases the max-leaf number of the graphs of our reduction. It is thus a natural question whether this is necessary. In this section, we show that indeed this is inevitable, as VERTEX INTEGRITY is FPT parameterized by ml .

We start with a high-level overview of our approach. As mentioned, we will rely on the result of Kleitman and West [26] who showed that if a graph $G = (V, E)$ has $\text{ml}(G) \leq k$, then there exists a set X of size $|X| = \mathcal{O}(k)$ such that all vertices of $V \setminus X$ have degree at most 2. Our main tool is a lemma (Lemma 9) which allows us to “rotate” solutions: whenever we have a cycle in our graph, we can, roughly speaking, exchange every vertex of S in the cycle with the next vertex, until we reach a point where our solution removes strictly more vertices of X . We therefore guess the largest intersection of an optimal separator with X , and can now assume that in every remaining cycle, the separator S is not using any vertices. This allows us to simplify the graph in a way that removes all cycles and reduces the case to a tree, which is polynomial-time solvable.

Let us now give more details. We first recall the result of [26].

► **Lemma 8.** $(\star)$ In any graph G , the set X of vertices of degree at least 3 has size at most $|X| \leq 12\text{ml}(G) + 32$.

We will solve COMPONENT ORDER CONNECTIVITY: for a given ℓ we want to calculate the minimum number of vertices p such that there exists a separator S of size at most p with all components of $G - S$ having size at most ℓ . To obtain an algorithm for UNWEIGHTED VERTEX INTEGRITY, we will try all possible values of ℓ and select the solution which minimizes $\ell + p$.

Our main lemma is now the following:

► **Lemma 9.** *(*) Let $G = (V, E)$ be a graph and X be a set of vertices such that all vertices of $V \setminus X$ have degree at most 2 in G . For all positive integers ℓ, p , if there exists a separator S of size at most p such that all components of $G - S$ have size at most ℓ , then there exists such a separator S that also satisfies the following property: for every cycle C of G with $C \cap X \neq \emptyset$ we either have $C \cap S = \emptyset$ or $C \cap X \cap S \neq \emptyset$.*

We are now ready to state the main result of this section.

► **Theorem 10.** *(*) UNWEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(\text{ml})} n^{\mathcal{O}(1)}$.*

6 Modular Width

In this section we revisit an algorithm of [22] establishing that WEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(\text{mw})} n^{\mathcal{O}(1)}$, on graphs of modular width mw , but only if weights are polynomially bounded in n (or equivalently, if weights are given in unary). It was left as an explicit open problem in [22] whether this algorithm can be extended to the case where weights are given in binary and can therefore be exponential in n . We resolve this problem positively, by showing how the algorithm of [22] can be modified to work also in this case, without a large increase in its complexity.

The high-level idea of the algorithm of [22] is to perform dynamic programming to solve the related WEIGHTED COMPONENT ORDER CONNECTIVITY problem. In this problem we are given a target component weight ℓ and a deletion budget p and are asked if it is possible to delete from the graph a set of vertices with total weight at most p so that the maximum weight of any remaining component is at most ℓ . Using this algorithm as a black box, we can then solve WEIGHTED VERTEX INTEGRITY by iterating over all possible values of ℓ , between 1 and the target vertex integrity. If vertex weights are polynomially bounded, this requires a polynomial number of iterations, giving the algorithm of [22]. However, if weights are given in binary, the target vertex integrity could be exponential in n , so in general, it does not appear possible to guess the weight of the heaviest component in an optimal solution.

Our contribution to the algorithm of [22] is to observe that for graphs of modular width mw the weight of the heaviest component may take at most $2^{\mathcal{O}(\text{mw})} n$ distinct possible values. Hence, for this parameter, guessing the weight of the heaviest component in an optimal solution can be done in FPT time. We can therefore plug in this result to the algorithm of [22] to obtain an algorithm for WEIGHTED VERTEX INTEGRITY with binary weights running in time $2^{\mathcal{O}(\text{mw})} n^{\mathcal{O}(1)}$.

Our observation is based on the following lemma.

► **Lemma 11.** *(*) Let $G = (V, E)$ be an instance of WEIGHTED VERTEX INTEGRITY. There exists an optimal solution using a separator S such that for all connected components D of $G - S$ and modules M of G we have one of the following: (i) $M \cap D = \emptyset$, (ii) $M \subseteq D$, or (iii) $D \subseteq M$.*

We also recall the algorithmic result of [22].

► **Theorem 12.** ([22]) *There exists an algorithm that takes as input a vertex-weighted graph $G = (V, E)$ and an integer ℓ and computes the minimum integer p such that there exists a separator S of G of weight at most p such that each component of $G - S$ has weight at most ℓ . The algorithm runs in time $2^{\mathcal{O}(\text{mw})}n^{\mathcal{O}(1)}$, where n is the size of the input.*

Putting Lemma 11 and Theorem 12 together we obtain the main result of this section.

► **Theorem 13.** (*) *There exists an algorithm that solves WEIGHTED VERTEX INTEGRITY in time $2^{\mathcal{O}(\text{mw})}n^{\mathcal{O}(1)}$, where mw is the modular width of the input graph, n is the size of the input, and weights are allowed to be written in binary.*

7 Vertex Cover Number

In this section, we design single-exponential algorithms for VERTEX INTEGRITY parameterized by vertex cover number. We suppose that a minimum vertex cover C of size vc is given since it can be computed in time $\mathcal{O}(1.2738^{\text{vc}} + \text{vc}n)$ [6]. We start by presenting an algorithm for UNWEIGHTED VERTEX INTEGRITY, before moving on to the weighted version of the problem.

► **Theorem 14.** (*) *UNWEIGHTED VERTEX INTEGRITY can be solved in time $5^{\text{vc}}n^{\mathcal{O}(1)}$.*

We now move on to the weighted case of the problem. It is clear that WEIGHTED VERTEX INTEGRITY is FPT parameterized by vertex cover, due to Theorem 13 and the relation between modular-width and vertex cover. However, this gives a double-exponential dependence on vc , as $\text{mw} \leq 2^{\text{vc}} + \text{vc}$ and there are some graphs for which this is essentially tight. We would like to obtain an algorithm that is as efficient as that of Theorem 14. The algorithm of Theorem 14, however, cannot be applied to the weighted case because the case of the branching where we place a vertex of the independent set in the separator is not guaranteed to make much progress (the vertex could have very small weight compared to our budget).

Before we proceed, it is worth thinking a bit about how this can be avoided. One way to obtain a faster FPT algorithm would be, rather than guessing only the intersection of the optimal separator S with the vertex cover C , to also guess how the vertices of $C \setminus S$ are partitioned into connected components in the optimal solution. This would immediately imply the decision for all vertices of the independent set: vertices with neighbors in two components must clearly belong to S , while the others cannot belong to S if S is irredundant. This algorithm would give a complexity of $\text{vc}^{\mathcal{O}(\text{vc})}n^{\mathcal{O}(1)}$, however, because the number of partitions of C is slightly super-exponential.

Let us sketch the high level idea of how we handle this. Our first step is, similarly to Theorem 13, to calculate the weight $w_{\max}$ of the most expensive connected component of the optimal solution. For this, there are at most $2^{\text{vc}} + n$ possibilities, because this component is either a single vertex, or it has a non-empty intersection with C . However, if we fix its intersection with C , then this fixes its intersection with the independent set: the component must contain (by irredundancy) exactly those vertices of the independent set all of whose neighbors in C are contained in the component. Having fixed a value of $w_{\max}$ we simply seek the best separator so that all components have weight at most $w_{\max}$. The reason we perform this guessing step is that this version of this problem is easier to decompose: if we have a disconnected graph, we simply calculate the best separator in each part and take the sum (this is not as clear for the initial version of VERTEX INTEGRITY).

Suppose then that we have fixed $w_{\max}$, how do we find the best partition of C into connected components? We apply a win/win argument: if the optimal partition has a

connected component that contains many (say, more than $vc/10$) vertices of C , we simply guess the intersection of the component with C and complete it with vertices from the independent set, as previously, while placing vertices with neighbors inside and outside the component in the separator. If the weight of the component is at most $w_{\max}$, we recurse in the remaining instance, which has vertex cover at most $9vc/10$. The complexity of this procedure works out as $T(vc) \leq 2^{vc} \cdot T(9vc/10) = 2^{\mathcal{O}(vc)}$.

What if the optimal partition of C only has components with few vertices of C ? In that case we observe that we do not need to compute the full partition (which would take time vc^{vc}), but it suffices to guess a good bipartition of C into two sets, of roughly the same size (say, both sets have size at least $2vc/5$), such that the two sets are a coarsening of the optimal partition. In other words, we compute two subsets of C , of roughly equal size, such that the intersection of each connected component with C is contained in one of the two sets. This is always possible in this case, because no connected component has a very large intersection with C . Now, all vertices of I which have neighbors on both sides of the bipartition of C must be placed in the separator. But once we do this, we have disconnected the instance into two independent instances, each of vertex cover at most $3vc/5$. The complexity of this procedure again works out as $T(vc) \leq 2^{vc} \cdot 2 \cdot T(3vc/5) = 2^{\mathcal{O}(vc)}$.

Let us now proceed to the technical details. To solve WEIGHTED VERTEX INTEGRITY, we first define the annotated and optimization version of the problem.

ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER

Instance: A vertex-weighted graph $G = (V, E, w)$, a vertex cover C of G , an integer $w_{\max}$.
Goal: Find a minimum weight irredundant wvi-set $S \subseteq V \setminus C$ such that $w(D) \leq w_{\max}$ for all $D \in \text{cc}(G - S)$. If there is no such S , report NO.

Then we give an algorithm that solves ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER.

► **Theorem 15.** $(\star)$ ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER can be solved in time $2^{\mathcal{O}(|C|)} n^{\mathcal{O}(1)}$.

► **Theorem 16.** $(\star)$ WEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(vc)} n^{\mathcal{O}(1)}$.

8 Conclusion

We have presented a number of new results on the parameterized complexity of computing vertex integrity. The main question that remains open is whether the slightly super-exponential $k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ algorithm, where k is the vertex integrity itself, can be improved to single-exponential. Although we have given such an algorithm for the more restricted parameter vertex cover, we conjecture that for vertex integrity the answer is negative. Complementing this question, it would be interesting to consider approximation algorithms for vertex integrity, whether trying to obtain FPT approximations in cases where the problem is W[1]-hard, or trying to obtain almost-optimal solutions via algorithms that run with a better parameter dependence. Again, a constant-factor or even $(1 + \varepsilon)$ -approximation running in time $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ would be the ideal goal. Do such algorithms exist or can they be ruled out under standard complexity assumptions?

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A

 Proofs for Section 4 (Feedback Edge Set plus Maximum Degree)

► **Theorem 3.** RESTRICTED UNARY BIN PACKING is $W[1]$ -hard parameterized by the number of bins.

Proof. The $W[1]$ -hardness of UNARY BIN PACKING parameterized by the number of bins is shown via an intermediate problem, called 10-UNARY VECTOR BIN PACKING [24]. In said problem, we are given n items $\mathcal{S} = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$, where every item is a 10-dimensional vector belonging to $\mathbb{N}^{10}$ encoded in unary. Additionally, we are given k “bin” vectors $\mathcal{B} = \{\mathbf{B}_1, \dots, \mathbf{B}_k\}$, and the question is whether we can partition the items into k sets $J_1, \dots, J_k$ such that $\sum_{\mathbf{s} \in J_i} \mathbf{s} \leq \mathbf{B}_i$, for all $i \in [k]$.

This problem is known to be $W[1]$ -hard parameterized by the number of bins, via an fpt-reduction from SUBGRAPH ISOMORPHISM parameterized by the number of edges of the sought subgraph. Fix $1 \leq i < j \leq k$ and notice that for the intended solution of said reduction ([24, Lemma 10]) it holds that:

- every item $\mathbf{s}_{i,j}(e)$ is either placed in bin $\mathbf{P}_{i,j}$ or bin $\mathbf{R}$,
- every item $\mathbf{t}_{i,i}(v)$ is either placed in bin $\mathbf{Q}_i$ or bin $\mathbf{R}$,
- every item $\mathbf{t}_{i,j}(v)$ and $\mathbf{t}_{j,i}(v)$ is either placed in bin $\mathbf{P}_{i,j}$ or bin $\mathbf{Q}_i$.

Consequently, the hardness result holds in the case when every item is allowed to be placed in one amongst two specified bins, called the 10-UNARY RESTRICTED VECTOR BIN PACKING problem.

As a second step, the authors reduce 10-UNARY VECTOR BIN PACKING to UNARY BIN PACKING ([24, Lemma 6]). Assume that $(\mathcal{S}, \mathcal{B}, f)$ denotes the initial instance of 10-UNARY RESTRICTED VECTOR BIN PACKING, where $f(\mathbf{s}) \in \binom{\mathcal{B}}{2}$ denotes the bins an item $\mathbf{s} \in \mathcal{S}$ may be placed into. To this end, the authors introduce a bin B_i per bin $\mathbf{B}_i$, an item s_i per item $\mathbf{s}_i$, as well as k additional items $t_1, \dots, t_k$, which are used to encode the capacity of bins $\mathbf{B}_i$ and are sufficiently large to guarantee that no two of them are placed in the same bin. It suffices to slightly modify said reduction, in order to show hardness for RESTRICTED UNARY BIN PACKING. In particular, we introduce an additional copy s'_i per item s_i , an additional copy t'_i per item t_i , as well as an additional copy B'_i per bin B_i . Next, if $f(\mathbf{s}) = \{\mathbf{B}_i, \mathbf{B}_j\}$, we set $f'(s) = \{B_i, B_j\}$ and $f'(s') = \{B'_i, B'_j\}$. Moreover, we set $f'(t_i) = f'(t'_i) = \{B_i, B'_i\}$. Since all items t_i and t'_i are placed in distinct bins, the correctness follows as in the proof of [24]. ◀

► **Theorem 4.** SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY parameterized by $\text{fes} + \Delta$ is fpt-reducible to SEMI-WEIGHTED VERTEX INTEGRITY parameterized by $\text{fes} + \Delta$.

Proof. We will closely follow the proof of [22, Lemma 4.4]. Let (G, w, ℓ, p) be an instance of SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY. Assume without loss of generality that for every vertex $v \in V(G)$ it holds that $w(v) \leq \ell$, since otherwise it necessarily belongs to the deletion set. We will construct an equivalent instance (G', w', k) of SEMI-WEIGHTED VERTEX INTEGRITY, where $k = \ell p + \ell + p$. Construct G' in the following way: Make a copy of G , where $w'(v) = w(v)$, for all $v \in V(G)$. Then, attach to every vertex v a leaf l_v , and set $w'(l_v) = p \cdot w(v)$. Finally, introduce an independent set $I = \{v_i : i \in [k+1]\}$, with every vertex of which having weight $w'(v_i) = k - p = \ell p + \ell$. This concludes the construction of the instance.

For the forward direction, assume there exists $S \subseteq V(G)$ such that $|S| \leq p$ and $w(D) \leq \ell$, for all connected components D of $G - S$. It suffices to prove that $w(D') \leq k - p = \ell p + \ell$, for all connected components D' of $G' - S$. If $D' \cap V(G) = \emptyset$, then D' contains (i) either a single vertex l_v of weight $w'(l_v) = p \cdot w(v) \leq p\ell$, (ii) or a single vertex v_i of weight $w'(v_i) = \ell p + \ell$,

and the statement holds. Alternatively, $D' \cap V(G)$ induces a connected component D of $G - S$, therefore $w(D') \leq w(D) + p \cdot w(D) = \ell + \ell p$.

For the converse direction, assume there exists $S' \subseteq V(G')$ such that $|S'| + w'(D') \leq k$, for all connected components D' of $G' - S'$. Assume without loss of generality that S' does not contain any vertex of degree 1; if that is the case, substitute it with its single neighbor and this set remains a valid solution. Notice that $|I| = k + 1 > |S'|$, and let $v_i \in I$ belong to $G' - S'$. In that case, it follows that $\max_{D' \in \text{cc}(G' - S')} w'(D') \geq w'(v_i) = k - p$, where $\text{cc}(G' - S')$ denotes the connected components of $G' - S'$. Consequently, it follows that $|S'| \leq p$. Let $S = S' \cap V(G)$ and D be an arbitrary connected component of $G - S$. Since $|S| \leq |S'| \leq p$, it suffices to prove that $w(D) \leq \ell$. Let D' be the connected component of $G' - S'$ such that $D \subseteq D'$. Since S' does not contain any vertices of degree 1, it holds that for each vertex of D , D' contains the corresponding leaf attached to it, and thus, $w(D') = w(D) + p \cdot w(D) = (p + 1)w(D)$. Since $w(D') \leq k = \ell p + \ell + p < (p + 1)(\ell + 1)$, it follows that $w(D) < \ell + 1$.

Finally, to conclude the proof, notice that $\text{fes}(G') = \text{fes}(G)$ as well as $\Delta(G') \leq \Delta(G) + 1$, since the only vertices added are one leaf per vertex and vertices of degree 0. $\blacktriangleleft$

► **Theorem 5.** SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY is $W[1]$ -hard parameterized by $\text{fes} + \Delta$.

Proof. Let (A, k, f) be an instance of RESTRICTED UNARY BIN PACKING, where $A = \{a_1, \dots, a_n\}$ denotes the set of items given in unary, k is the number of bins, and $f : A \rightarrow \binom{[k]}{2}$ dictates the bins an item may be placed into. In the following, consider $B = \Sigma(A)/k$, as well as $M = kB + 1$ and $L = 8k^2BM$. Notice that $M > kB$, while $L/(4k) \in \mathbb{N}$ and $L/(4k) > (k - 1) \cdot 2BM + B$. We will reduce (A, k, f) to an equivalent instance (G, w, ℓ, p) of SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY, where $\ell = L + (k - 1) \cdot 2BM + B$ and $p = 3\binom{k}{2}$ denote the maximum component weight and size of the deletion set respectively.

For every $i \in [k]$, we introduce a clique on vertex set $\hat{C}_i$, which is comprised of $4k$ vertices, each of weight $L/(4k)$. Fix i and j such that $1 \leq i < j \leq k$, and let $H_{i,j} = \{a \in A : f(a) = \{i, j\}\}$ denote the subset of items which can be placed either on bin i or bin j , where $\Sigma(H_{i,j}) \leq 2B$. Let $\mathcal{S}(H_{i,j}) = \{\Sigma(H) : H \subseteq H_{i,j}\}$ denote the set of all subset sums of $H_{i,j}$, and notice that since every element of $H_{i,j}$ is in unary, $\mathcal{S}(H_{i,j})$ can be computed in polynomial time using e.g. Bellman's classical DP algorithm [2].

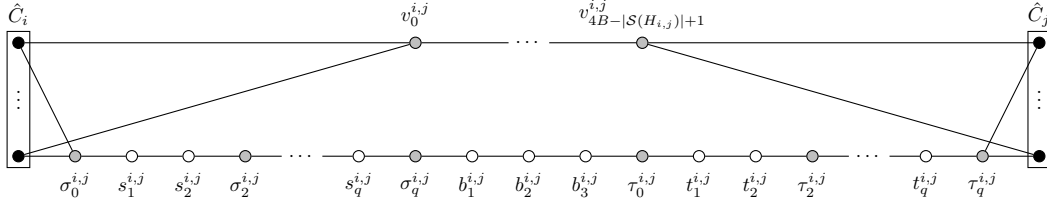
Next, we will construct two paths connecting the vertices of $\hat{C}_i$ and $\hat{C}_j$. First, introduce vertex set $\hat{U}_{i,j} = \{v_q^{i,j} : q \in [0, 4B - |\mathcal{S}(H_{i,j})| + 1]\}$, where $w(v) = M$ for all $v \in \hat{U}_{i,j}$. Add edges $(v_q^{i,j}, v_{q+1}^{i,j})$ for all $q \in [0, 4B - |\mathcal{S}(H_{i,j})|]$, as well as $(v_1, v_0^{i,j})$ and $(v_{4B - |\mathcal{S}(H_{i,j})| + 1}^{i,j}, v_2)$, for all $v_1 \in \hat{C}_i$ and $v_2 \in \hat{C}_j$. Next, introduce vertices

- $\{s_q^{i,j}, t_q^{i,j} : q \in [\Sigma(H_{i,j})]\}$, each of weight 1,
- $\{\sigma_q^{i,j}, \tau_q^{i,j} : q \in \mathcal{S}(H_{i,j})\}$, each of weight M ,
- $\hat{D}_{i,j}^3 = \{b_1^{i,j}, b_2^{i,j}, b_3^{i,j}\}$, where $w(b_1^{i,j}) = w(b_3^{i,j}) = L/2$, and $w(b_2^{i,j}) = (k - 1) \cdot 2BM + B - \Sigma(H_{i,j}) - (|\mathcal{S}(H_{i,j})| - 1) \cdot M$,

and set $\hat{D}_{i,j} = \hat{D}_{i,j}^1 \cup \hat{D}_{i,j}^2 \cup \hat{D}_{i,j}^3$, where $\hat{D}_{i,j}^1 = \{s_q^{i,j} : q \in [\Sigma(H_{i,j})]\} \cup \{\sigma_q^{i,j} : q \in \mathcal{S}(H_{i,j})\}$ and $\hat{D}_{i,j}^2 = \{t_q^{i,j} : q \in [\Sigma(H_{i,j})]\} \cup \{\tau_q^{i,j} : q \in \mathcal{S}(H_{i,j})\}$. Then, add the following edges:

- $(v_1, \sigma_0^{i,j})$ and $(\tau_{\Sigma(H_{i,j})}^{i,j}, v_2)$, for all $v_1 \in \hat{C}_i$ and $v_2 \in \hat{C}_j$,
- $(\sigma_{\Sigma(H_{i,j})}^{i,j}, b_1^{i,j})$, $(b_1^{i,j}, b_2^{i,j})$, $(b_2^{i,j}, b_3^{i,j})$ and $(b_3^{i,j}, \tau_0^{i,j})$,
- for $q \in \mathcal{S}(H_{i,j})$, add edges $(s_q^{i,j}, \sigma_q^{i,j})$ and $(t_q^{i,j}, \tau_q^{i,j})$ if $q \neq 0$, and edges $(\sigma_q^{i,j}, s_{q+1}^{i,j})$ and $(\tau_q^{i,j}, t_{q+1}^{i,j})$ if $q \neq \Sigma(H_{i,j})$,
- for $q \in [\Sigma(H_{i,j})] \setminus \mathcal{S}(H_{i,j})$, add edges $(s_q^{i,j}, s_{q+1}^{i,j})$ and $(t_q^{i,j}, t_{q+1}^{i,j})$.

This concludes the construction of G . See Figure 4 for an illustration.



■ **Figure 4** Rectangles denote cliques of size $4k$. Here we assume that $1 \leq i < j \leq k$, $2 \in \mathcal{S}(H_{i,j})$, while $q = \Sigma(H_{i,j})$. It holds that $w(b_1^{i,j}) = w(b_3^{i,j}) = L/2$, while $w(b_2^{i,j}) = (k-1) \cdot 2BM + B - \Sigma(H_{i,j}) - (|\mathcal{S}(H_{i,j})| - 1) \cdot M$. For the rest of the vertices, the white, gray, or black color indicates weight of 1, M , or $L/(4k)$ respectively.

► **Lemma 17.** *If (A, k, f) is a Yes-instance of RESTRICTED UNARY BIN PACKING, then (G, w, ℓ, p) is a Yes-instance of SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY.*

Proof. Let $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a partition of A such that for all $i \in [k]$ it holds that (i) $\Sigma(\mathcal{A}_i) = B$, and (ii) $\forall a \in \mathcal{A}_i, i \in f(a)$. Fix $1 \leq i < j \leq k$, and let $\mathcal{H}_{i,j} \subseteq H_{i,j}$ such that $\mathcal{A}_i \cap H_{i,j} = \mathcal{H}_{i,j}$, while $\mathcal{A}_j \cap H_{i,j} = H_{i,j} \setminus \mathcal{H}_{i,j}$. Note that for all $i \in [k]$, it holds that

$$\sum_{j \in [i-1]} (\Sigma(H_{j,i}) - \Sigma(\mathcal{H}_{j,i})) + \sum_{j \in [i+1, k]} \Sigma(\mathcal{H}_{i,j}) = B. \quad (1)$$

Additionally, let $r_{i,j} = |\mathcal{S}(H_{i,j}) \cap [0, \Sigma(\mathcal{H}_{i,j}) - 1]|$ denote the number of subset sums of $H_{i,j}$ of sum at most $\Sigma(\mathcal{H}_{i,j}) - 1$. Set $S = \{\sigma_{\Sigma(\mathcal{H}_{i,j})}^{i,j}, \tau_{\Sigma(\mathcal{H}_{i,j})}^{i,j}, v_{2B-r_{i,j}}^{i,j} : 1 \leq i < j \leq k\}$. It holds that $|S| = 3 \binom{k}{2} = p$. In the following, we will prove that every connected component of $G - S$ has weight at most ℓ .

Notice that the vertices of $\hat{C}_i$ and $\hat{C}_j$ are in different connected components of $G - S$, and let $\hat{C}_i$ denote the connected component of $G - S$ that contains the vertices of $\hat{C}_i$, for all $i \in [k]$. Additionally, for every $1 \leq i < j \leq k$, let $\hat{P}_{i,j}$ denote the connected component of $G - S$ that contains the vertices of $\hat{D}_{i,j}^3$.

Fix $1 \leq i < j \leq k$. Notice that $S \cap \hat{D}_{i,j} = \{\sigma_{\Sigma(\mathcal{H}_{i,j})}^{i,j}, \tau_{\Sigma(\mathcal{H}_{i,j})}^{i,j}\}$. Consequently, it holds that $\hat{P}_{i,j}$ contains $\Sigma(H_{i,j})$ vertices of weight 1, $|\mathcal{S}(H_{i,j})| - 1$ vertices of weight M , as well as vertices $b_1^{i,j}$, $b_2^{i,j}$, and $b_3^{i,j}$. Therefore, it follows that $w(\hat{P}_{i,j}) = \Sigma(H_{i,j}) + (|\mathcal{S}(H_{i,j})| - 1) \cdot M + L/2 + L/2 + (k-1) \cdot 2BM + B - \Sigma(H_{i,j}) - (|\mathcal{S}(H_{i,j})| - 1) \cdot M$, thus $w(\hat{P}_{i,j}) = L + (k-1) \cdot 2BM + B = \ell$.

Additionally, it holds that $\hat{C}_i$ contains $\Sigma(\mathcal{H}_{i,j})$ vertices of weight 1 belonging to $\hat{D}_{i,j}$, as well as $r_{i,j} + 2B - r_{i,j} = 2B$ vertices of weight M from $\hat{D}_{i,j} \cup \hat{U}_{i,j}$. As for $\hat{C}_j$, it contains $\Sigma(H_{i,j}) - \Sigma(\mathcal{H}_{i,j})$ vertices of weight 1 belonging to $\hat{D}_{i,j}$, as well as $|\mathcal{S}(H_{i,j})| - 1 - r_{i,j} + 4B - |\mathcal{S}(H_{i,j})| + 1 - (2B - r_{i,j}) = 2B$ vertices of weight M from $\hat{D}_{i,j} \cup \hat{U}_{i,j}$. For any fixed $i \in [k]$, it follows that

$$\begin{aligned} w(\hat{C}_i) &= L + \sum_{j \in [i-1]} (\Sigma(H_{i,j}) - \Sigma(\mathcal{H}_{i,j}) + 2BM) + \sum_{j \in [i+1, k]} (\Sigma(\mathcal{H}_{i,j}) + 2BM) \\ &= L + (k-1) \cdot 2BM + B \\ &= \ell, \end{aligned}$$

where the second equality is due to Equation (1). This concludes the proof. ◀

► **Lemma 18.** *If (G, w, ℓ, p) is a Yes-instance of SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY, then (A, k, f) is a Yes-instance of RESTRICTED UNARY BIN PACKING.*

Proof. Let $S_0 \subseteq V(G)$ such that $|S_0| \leq p = 3\binom{k}{2}$, and for every connected component of $G - S_0$ it holds that the sum of the weights of its vertices is at most $\ell = L + (k-1) \cdot 2BM + B$. The following claim shows that there exists $S \subseteq V(G)$ such that $S \cap \hat{C}_i = \emptyset$, for all $i \in [k]$; in fact S contains either 1 or 2 vertices per path between cliques.

▷ **Claim 19.** There exists a set $S \subseteq V(G)$ such that $|S| \leq p$, and for every connected component of $G - S$ it holds that the sum of the weights of its vertices is at most ℓ . Additionally, for all $1 \leq i < j \leq k$, it holds that $|S \cap \hat{U}_{i,j}| = |S \cap \hat{D}_{i,j}^1| = |S \cap \hat{D}_{i,j}^2| = 1$.

Proof. To prove the statement we will first introduce the following reduction rule.

Rule (†). Let $Z \subseteq V(G)$ be a deletion set such that $|Z \cap \hat{C}_i| \geq 2(k-1)$ for some $i \in [k]$, while every connected component of $G - Z$ has weight at most ℓ . Then, replace Z with $Z' = (Z \setminus \hat{C}_i) \cup N(\hat{C}_i)$, where $N(\hat{C}_i) = \bigcup_{v \in \hat{C}_i} N(v) \setminus \hat{C}_i$.

It is easy to see that $|Z'| \leq |Z|$, since $|N(\hat{C}_i)| = 2(k-1)$. Moreover, it holds that every connected component of $G - Z'$ has weight at most ℓ . To see this, it suffices to consider the connected component that contains the vertices of $\hat{C}_i \setminus Z$. Let $\hat{C}_i$ denote the set of vertices of the connected component of $G - Z$ where $\hat{C}_i \setminus Z \subseteq \hat{C}_i$, and $\hat{C}_i' = \hat{C}_i \setminus (\hat{C}_i \cup N(\hat{C}_i))$. It holds that $w(\hat{C}_i') \leq w(\hat{C}_i) \leq \ell$, as well as $w(\hat{C}_i) = L \leq \ell$, while in $G - Z'$ the connected component $\hat{C}_i$ is split into the connected component $\hat{C}_i'$ as well as a partition of $\hat{C}_i'$.

Starting from S_0 , let $S_1 \subseteq V(G)$ be the set obtained after applying Rule (†) exhaustively. Note that this process will finish in at most k steps. Notice that it holds that $|S_1 \cap \hat{D}_{i,j}| \geq 1$, for all $1 \leq i < j \leq k$, since $w(\hat{D}_{i,j}) > \ell$. Moreover, observe that since $L/(4k) > (k-1) \cdot 2BM + B$, it follows that $\ell < L/(4k) \cdot (4k+1)$, thus at most $4k$ vertices of weight $L/(4k)$ may be in the same connected component of $G - S_1$. In an analogous way, at most $2k$ vertices of weight $L/(4k)$ may be in the same connected component of $G - S_1$ with a vertex of weight $L/2$.

Assume there exist $1 \leq i < j \leq k$ such that $S_1 \cap \hat{U}_{i,j} = \emptyset$. In that case, the vertices of $(\hat{C}_i \cup \hat{C}_j) \setminus S_1$ are in the same connected component of $G - S_1$, and since every such vertex is of weight $L/(4k)$, it follows that $|S_1 \cap (\hat{C}_i \cup \hat{C}_j)| \geq 4k$. Assume without loss of generality that $0 \leq |S_1 \cap \hat{C}_i| \leq |S_1 \cap \hat{C}_j| \leq 4k$, thus $|S_1 \cap \hat{C}_j| \geq 2k$ follows, which is a contradiction.

Next, assume there exists $\hat{D}_{i,j}$ such that $|S_1 \cap \hat{D}_{i,j}| = 1$. In that case, it holds that either $b_1^{i,j} \notin S_1$ or $b_3^{i,j} \notin S_1$, and is in the same connected component of $G - S_1$ as the vertices of either $\hat{C}_i \setminus S_1$ or $\hat{C}_j \setminus S_1$. Assume without loss of generality that $b_1^{i,j}$ is in the same connected component of $G - S_1$ as the vertices of $\hat{C}_i \setminus S_1$. Then, since $w(b_1^{i,j}) = L/2$ and every vertex of $\hat{C}_i$ is of weight $L/(4k)$, it follows that $|S_1 \cap \hat{C}_i| \geq 2k$, which is a contradiction.

Consequently, for all $1 \leq i < j \leq k$ it holds that $|S_1 \cap \hat{D}_{i,j}| \geq 2$, as well as $S_1 \cap \hat{U}_{i,j} \neq \emptyset$. Since $|S_1| \leq 3\binom{k}{2}$, it follows that $|S_1 \cap \hat{D}_{i,j}| = 2$, and $|S_1 \cap \hat{U}_{i,j}| = 1$.

Notice that if $b_2^{i,j} \in S_1$ or $S_1 \cap \hat{D}_{i,j} \subseteq \hat{D}_{i,j}^1 \cup \{b_1^{i,j}\}$ or $S_1 \cap \hat{D}_{i,j} \subseteq \hat{D}_{i,j}^2 \cup \{b_3^{i,j}\}$, then it follows once again that either $b_1^{i,j} \notin S_1$ or $b_3^{i,j} \notin S_1$, and is in the same connected component of $G - S_1$ as the vertices of either $\hat{C}_i \setminus S_1$ or $\hat{C}_j \setminus S_1$, leading to contradiction.

Let $\hat{P}_{i,j} \subseteq \hat{D}_{i,j}$ denote the connected component (path) of $G - S_1$ such that $b_2^{i,j} \in \hat{P}_{i,j}$. Assume $b_1^{i,j} \in S_1$ and consider $S_2 = (S_1 \setminus \{b_1^{i,j}\}) \cup \{\sigma_{\Sigma(H_{i,j})}^{i,j}\}$. Then, it follows that $\hat{P}_{i,j} \subseteq (\{b_2^{i,j}, b_3^{i,j}\} \cup \hat{D}_{i,j}^2) \setminus \{\tau_{\Sigma(H_{i,j})}^{i,j}\}$, therefore $w(\hat{P}_{i,j}) \leq \ell - L/2$. Consequently, every connected component of $G - S_2$ has weight at most ℓ . Using analogous arguments, one can substitute $b_3^{i,j}$ with $\tau_0^{i,j}$ in case $b_3^{i,j}$ belongs to the deletion set. Let S be the set obtained by those substitutions. ◁

Let $\hat{C}_i$ denote the connected component of $G - S$ containing the vertices of $\hat{C}_i$. Additionally, let $\hat{P}_{i,j} \subseteq \hat{D}_{i,j}$ denote the connected component (path) of $G - S$ such that $\hat{D}_{i,j}^3 \subseteq \hat{P}_{i,j}$.

▷ **Claim 20.** For all $1 \leq i < j \leq k$, it holds that $w(\hat{\mathcal{P}}_{i,j}) = \ell$, while both vertices of $S \cap \hat{D}_{i,j}$ are of weight M . Moreover, it holds that $w(\hat{\mathcal{C}}_i) = \ell$, for all $i \in [k]$.

Proof. Let $\hat{Q}_{i,j} = \hat{\mathcal{P}}_{i,j} \cup (S \cap \hat{D}_{i,j})$. Notice that since $\hat{D}_{i,j}^3 \subseteq \hat{\mathcal{P}}_{i,j}$, it holds that the vertices of $S \cap \hat{D}_{i,j}$ are of weight at most M , thus $w(\hat{Q}_{i,j}) \leq \ell + 2M$. For the first statement, it suffices to prove that it holds $w(\hat{Q}_{i,j}) = \ell + 2M$.

In the following we prove that $w(\hat{Q}_{i,j}) \geq \ell + 2M$. Notice that there are exactly k additional connected components $\hat{\mathcal{C}}_1, \dots, \hat{\mathcal{C}}_k$ in $G - S$, apart from all components $\hat{\mathcal{P}}_{i,j}$. Moreover, their weight sums up to

$$\sum_{i \in [k]} w(\hat{\mathcal{C}}_i) = w(G) - \sum_{1 \leq i < j \leq k} w(\hat{Q}_{i,j}) - \binom{k}{2} \cdot M, \quad (2)$$

where the last term is due to the vertices in $S \cap \hat{U}_{i,j}$ which are of weight M , therefore

$$w(G) - \sum_{1 \leq i < j \leq k} w(\hat{Q}_{i,j}) - \binom{k}{2} \cdot M \leq k \cdot \ell. \quad (3)$$

In order to compute $w(G)$, notice that

- $w(\hat{\mathcal{C}}_i) = L$, for all $i \in [k]$,
- $w(\hat{U}_{i,j}) = (4B - |\mathcal{S}(H_{i,j})| + 2) \cdot M$, for all $1 \leq i < j \leq k$,
- $w(\hat{D}_{i,j}) = 2\Sigma(H_{i,j}) + 2|\mathcal{S}(H_{i,j})| \cdot M + L + (k-1) \cdot 2BM + B - \Sigma(H_{i,j}) - (|\mathcal{S}(H_{i,j})| - 1) \cdot M$, for all $1 \leq i < j \leq k$,

and adding up the last two items gives

$$w(\hat{U}_{i,j} \cup \hat{D}_{i,j}) = \Sigma(H_{i,j}) + L + B + (k-1) \cdot 2BM + (4B+3) \cdot M.$$

Consequently, it follows that

$$\begin{aligned} w(G) &= kL + kB + \binom{k}{2} (L + B + (k-1) \cdot 2BM + (4B+3) \cdot M) \\ &= k(L + B + (k-1) \cdot 2BM) + \binom{k}{2} (L + B + (k-1) \cdot 2BM + 3M) \\ &= k\ell + \binom{k}{2} (\ell + 3M), \end{aligned}$$

which due to Equation (3) gives

$$k\ell + \binom{k}{2} (\ell + 3M) - \sum_{1 \leq i < j \leq k} w(\hat{Q}_{i,j}) - \binom{k}{2} \cdot M \leq k \cdot \ell,$$

thus

$$\binom{k}{2} (\ell + 2M) \leq \sum_{1 \leq i < j \leq k} w(\hat{Q}_{i,j}),$$

and since $w(\hat{Q}_{i,j}) \leq \ell + 2M$, it follows that $w(\hat{Q}_{i,j}) = \ell + 2M$, for all $1 \leq i < j \leq k$.

As for the weight of the components $\hat{\mathcal{C}}_i$, due to $w(\hat{\mathcal{C}}_i) \leq \ell$ and Equation (2), it follows that $w(\hat{\mathcal{C}}_i) = \ell$ for all $i \in [k]$. $\triangleleft$

Due to Claim 20, it follows that $S \cap \hat{D}_{i,j} = \{\sigma_q^{i,j}, \tau_q^{i,j}\}$, for some $q \in \mathcal{S}(H_{i,j})$. Since $w(\hat{C}_i) = \ell$, while $kB < M$, it follows that $\hat{C}_i$ contains exactly $(k-1) \cdot 2B$ vertices of weight M , as well as exactly B vertices of weight 1. Let $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a partition of A defined in the following way: for all $1 \leq i < j \leq k$, if $\sigma_q^{i,j} \in S$, then $\mathcal{A}_i \cap H_{i,j} = \mathcal{H}_{i,j}$ and $\mathcal{A}_j \cap H_{i,j} = H_{i,j} \setminus \mathcal{H}_{i,j}$, where $\mathcal{H}_{i,j} \subseteq H_{i,j}$ such that $\Sigma(\mathcal{H}_{i,j}) = q$. Notice that $\Sigma(\mathcal{A}_i)$ is equal to the number of vertices of weight 1 in $\hat{C}_i$, therefore $\Sigma(\mathcal{A}_i) = B$ follows. $\blacktriangleleft$

► **Lemma 21.** *It holds that $\text{fes}(G) = \mathcal{O}(k^3)$ and $\Delta(G) = \mathcal{O}(k)$.*

Proof. Let $F \subseteq E(G)$ contain all edges between vertices of $\hat{C}_i$, for all $i \in [k]$, as well as all edges which are adjacent to the endpoints of the paths $\hat{U}_{i,j}$ and $\hat{D}_{i,j}$. Notice that

$$|F| = k \cdot \binom{4k}{2} + 4 \binom{k}{2} \cdot 4k = \mathcal{O}(k^3),$$

while the graph remaining after the deletion of the edges in F is a forest.

For the maximum degree, notice that $|N(v)| = 4k - 1 + 2(k-1) = \mathcal{O}(k)$, for all $v \in \bigcup_{i \in [k]} \hat{C}_i$. As for the vertices of the paths, they are all of degree 2, apart from the endpoints which are neighbors with all the vertices of a single clique, therefore of degree $\mathcal{O}(k)$. $\blacktriangleleft$

Due to Lemmas 17, 18, and 21, the statement follows. $\blacktriangleleft$

B Proofs for Section 5 (Max-Leaf Number)

► **Lemma 8.** *In any graph G , the set X of vertices of degree at least 3 has size at most $|X| \leq 12\text{ml}(G) + 32$.*

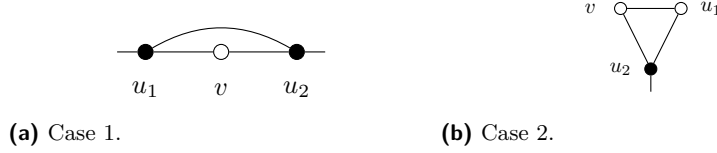
Proof. The statement is trivially true if the maximum degree of the graph is less than 3, so in the following assume that this is not the case. Kleitman and West actually proved that in a graph G with n vertices and minimum degree 3 or more, $\text{ml}(G) \geq n/4$. Given a graph G , let X_G and Y_G denote the set of vertices of degree at least 3 and at most 2 respectively, i.e. $V(G) = X_G \cup Y_G$ where $X_G = \{v \in V(G) : \deg_G(v) \geq 3\}$ and $Y_G = \{v \in V(G) : \deg_G(v) \leq 2\}$. Start from an arbitrary (connected) graph G , and greedily contract any edge in G , as long as the number of vertices of degree at least 3 is not reduced. Call the resulting graph G' , where $|X_{G'}| \geq |X_G|$. Notice that the contraction of an edge does not increase the max-leaf number, thus $\text{ml}(G') \leq \text{ml}(G)$. In case $Y_{G'} = \emptyset$ the statement immediately follows by the result of Kleitman and West, so in the following assume that $Y_{G'} \neq \emptyset$. Moreover, for all $v \in Y_{G'}$, it holds that $N_{G'}(v) \cap X_{G'} \neq \emptyset$; if that were not the case, then contracting an edge incident to v does not decrease the degree of any vertex in $X_{G'}$, which is a contradiction since in G' any such edge has already been contracted.

▷ **Claim 22.** *It holds that $|Y_{G'}| \leq \text{ml}(G')$.*

Proof. It suffices to prove that there exists a spanning tree of G' where every vertex of $Y_{G'}$ is a leaf. Let $v \in Y_{G'}$, and notice that if $\deg_{G'}(v) = 1$, then v is a leaf in any spanning tree of G' . In the following, assume that $N_{G'}(v) = \{u_1, u_2\}$, where $N_{G'}(v) \cap X_{G'} \neq \emptyset$. Moreover, we claim that $\{u_1, u_2\} \in E(G')$. Assume otherwise, and notice that in that case, the contraction of any edge incident on v does not change the degree of any other vertex, which is a contradiction.

First consider the case where $u_1, u_2 \in X_{G'}$. Given a spanning tree of G' that contains both edges $\{v, u_1\}$ and $\{v, u_2\}$, one can obtain another spanning tree that contains edges $\{v, u_1\}$ and $\{u_1, u_2\}$ instead.

For the remaining case, assume without loss of generality that $u_1 \in Y_{G'}$ and $u_2 \in X_{G'}$. Then, given a spanning tree of G' that contains the edges $\{v, u_1\}$ and either $\{v, u_2\}$ or $\{u_1, u_2\}$, one can obtain another spanning tree that contains edges $\{v, u_2\}$ and $\{u_1, u_2\}$ instead.



■ **Figure 5** Black vertices belong to $X_{G'}$.

Consequently, there exists a spanning tree of G' where every vertex of $Y_{G'}$ is a leaf, thus $|Y_{G'}| \leq \text{ml}(G')$ follows. $\triangleleft$

Lastly, we apply the result of Kleitman and West. If $|Y_{G'}| \geq 3$, add at most $|Y_{G'}| \leq \text{ml}(G')$ edges connecting vertices of $Y_{G'}$, so that the resulting graph G'' has minimum degree at least 3. Each such addition increases the max-leaf number by at most 2, therefore it holds that $\text{ml}(G'') \leq \text{ml}(G') + 2\text{ml}(G')$, while $|X_{G''}| = |X_{G'} \cup Y_{G'}| > |X_{G'}|$, and since $|X_{G''}| \leq 4\text{ml}(G'')$ it follows that $|X_G| \leq 12\text{ml}(G)$. It remains to consider the case where $1 \leq |Y_{G'}| \leq 2$. Then, it holds that $|X_{G'}| \geq 3$, since otherwise it follows that either $X_{G'} = \emptyset$ or $Y_{G'} = \emptyset$. Consequently, add at most 2 edges per vertex of $Y_{G'}$, so that the resulting graph G'' has minimum degree 3, therefore $\text{ml}(G'') \leq \text{ml}(G') + 8$, while $|X_{G''}| = |X_{G'} \cup Y_{G'}| > |X_{G'}|$, and since $|X_{G''}| \leq 4\text{ml}(G'')$ it follows that $|X_G| \leq 4\text{ml}(G) + 32$. $\blacktriangleleft$

► **Lemma 9.** *Let $G = (V, E)$ be a graph and X be a set of vertices such that all vertices of $V \setminus X$ have degree at most 2 in G . For all positive integers ℓ, p , if there exists a separator S of size at most p such that all components of $G - S$ have size at most ℓ , then there exists such a separator S that also satisfies the following property: for every cycle C of G with $C \cap X \neq \emptyset$ we either have $C \cap S = \emptyset$ or $C \cap X \cap S \neq \emptyset$.*

Proof. We will show that if we have a separator S that does not satisfy the property, then we can obtain an equally good separator that contains strictly more elements of X . Applying this argument exhaustively will yield the lemma.

Fix some separator S of size at most p such that all components of $G - S$ have size at most ℓ . Suppose that we have a cycle C in G of length t , say $C = v_0, v_1, \dots, v_{t-1}, v_0$ such that C contains some vertices of X and some vertices of S , but no vertex that belongs to both X and S . Suppose that $C \cap S = \{v_{i_0}, v_{i_1}, \dots, v_{i_{r-1}}\}$, where $r = |C \cap S|$ and $i_0 < i_1 < \dots < i_{r-1}$. We claim that the set $S' = (S \setminus C) \cup \{v_{i_0+1}, v_{i_1+1}, \dots, v_{i_{r-1}+1}\}$ (where additions are modulo t) is a separator of the same size as S which also leaves components of size at most ℓ in $G - S'$. Informally, what we are doing is replacing every vertex of $C \cap S$ with the next vertex in the cycle C . It is clear that $|S| = |S'|$. Furthermore, because $C \cap S \cap X = \emptyset$, all vertices of $C \cap S$ have degree 2 in G and one of their neighbors is in S' . Let D_j be the component of $G - S$ that is adjacent to v_{i_j} and $v_{i_{j-1}}$ (where subtraction is done modulo r). We observe that $G - S'$ has the component $D_j \cup \{v_{i_j}\} \setminus \{v_{i_{j-1}+1}\}$, which has the same size as D_j . Here we are using the fact that v_{i_j} has degree 2 and its neighbor $v_{i_{j+1}}$ is in S' . Hence, this procedure does not increase the size of any component (in fact, if $C \cap S' \cap X \neq \emptyset$, the

size of some components, which previously contained vertices of $C \cap S' \cap X$, is reduced, since such components are divided in $G - S'$). If the new separator S' has $C \cap S' \cap X \neq \emptyset$, we have increased the intersection of the separator with X . If not, we repeat this process. Since at each step the intersection between the separator and X cannot decrease, we eventually obtain a separator that satisfies the condition of the lemma. $\blacktriangleleft$

► **Theorem 10.** UNWEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(\text{ml})} n^{\mathcal{O}(1)}$.

Proof. We will solve COMPONENT ORDER CONNECTIVITY for a given maximum component size ℓ . As explained, we can reduce UNWEIGHTED VERTEX INTEGRITY to this problem by trying out all values of ℓ . We are given a graph G and let X be the set of vertices of degree at least 3. By Lemma 8, $|X| \leq 12\text{ml}(G) + 32$. Let S be a separator of minimum size such that $G - S$ has only components of size at most ℓ . Among all such separators, select an S such that $S \cap X$ is maximized. Fix this separator S for the analysis of the rest of the algorithm.

Our algorithm as a first step guesses $S \cap X$, removes these vertices from the graph and sets $p := p - |S \cap X|$. In the remainder, we focus on the case where this guess was correct. To simplify the rest of the presentation we assume that some vertices of the graph may be marked as undeleteable, meaning that we are only looking for separators that do not contain such vertices. Initially, the vertices of $X \setminus S$ are undeleteable.

We now perform a series of polynomial-time simplification steps. First, if the current graph contains a connected component of size at most ℓ , we remove this component. This is clearly correct. Second, if the graph contains a component where all vertices have degree at most 2, we compute in polynomial time an optimal deletion set for this component, update p appropriately, and remove this component.

We are now in a situation where every cycle C in our current graph must contain a vertex of X . By Lemma 9 any such cycle C must have $C \cap S = \emptyset$. We now contract all the vertices of C into a single vertex and attach to this vertex $|C| - 1$ leaves. We mark this new vertex as undeleteable, that is, we will only look for separators S that do not contain it. We also place this vertex in X . It is not hard to see that this transformation is correct, since if the optimal solution does not place any vertex of C into S , we can replace C with a single undeleteable vertex which contributes $|C|$ to the size of its component.

Performing the above exhaustively results in a graph that contains no cycles. The problem can now easily be solved optimally via dynamic programming in polynomial time. $\blacktriangleleft$

C Proofs for Section 6 (Modular Width)

► **Lemma 11.** Let $G = (V, E)$ be an instance of WEIGHTED VERTEX INTEGRITY. There exists an optimal solution using a separator S such that for all connected components D of $G - S$ and modules M of G we have one of the following: (i) $M \cap D = \emptyset$, (ii) $M \subseteq D$, or (iii) $D \subseteq M$.

Proof. Let S be a separator giving an optimal solution, that is a solution minimizing the sum of the weight of S and the weight of the heaviest component of $G - S$. Our strategy is to show that if S does not satisfy the condition of the lemma, then S contains some vertex z which in the terminology of [22] is redundant, that is, z has neighbors in at most one connected component of $G - S$. As observed in [22], this implies that $S \setminus \{z\}$ is a separator that gives a solution that is at least as good as that given by S , so we can remove z from S . Repeating this exhaustively will produce a separator S that satisfies the conditions of the lemma.

Suppose then that S is a separator such that for some component D of $G - S$ and some module M we have $M \cap D \neq \emptyset$, $D \setminus M \neq \emptyset$, and $M \setminus D \neq \emptyset$. Let $x \in D \cap M$ and $y \in D \setminus M$ such that x, y are adjacent (such x, y must exist, since D is connected). Since M is a module and $y \notin M$, we have that y is adjacent to all of M . Hence, $M \setminus S \subseteq D$ because all vertices of $M \setminus S$ are adjacent to $y \in D$. It follows that each vertex $z \in M \setminus D$ must belong to the separator S . We claim that z is redundant, that is, z only has neighbors in D and in no other connected component of $G - S$. This is not hard to see, since each neighbor of z is either in M (so is adjacent to $y \in D$) or is adjacent to all of M (hence also to x). ◀

► **Theorem 13.** *There exists an algorithm that solves WEIGHTED VERTEX INTEGRITY in time $2^{\mathcal{O}(\text{mw})} n^{\mathcal{O}(1)}$, where mw is the modular width of the input graph, n is the size of the input, and weights are allowed to be written in binary.*

Proof. Fix some optimal separator S satisfying the conditions of Lemma 11. Our strategy is to use these conditions to guess the weight of the heaviest component of $G - S$. If this weight is ℓ , then we can simply call the algorithm of Theorem 12. More precisely, we will show that by considering $\mathcal{O}(2^{\text{mw}} n)$ distinct values, we are guaranteed to find ℓ . Hence, by executing the algorithm of Theorem 12 $\mathcal{O}(2^{\text{mw}} n)$ times and picking the best solution we find the optimal solution.

Consider the modular decomposition tree associated with G , which can be computed in polynomial time [30] and is recursively constructed as follows: if $G = (V, E)$ has at most k vertices, then G has a root labeled V with k children, each corresponding to a vertex of V ; while if $|V| > k$, then V can be partitioned into $k' \leq k$ modules $V_1, \dots, V_{k'}$, so we construct a root labeled V and give it as children the roots of the trees constructed for $G[V_i]$, for $i \in [k']$. Each node of the constructed tree is labeled with a module of G .

Let D be the heaviest component of $G - S$. Find the node of the modular decomposition that is as far from the root as possible and satisfies that the corresponding module M has $D \subseteq M$. (Such a node exists, since $D \subseteq V$.) The module M can be decomposed into $k' \leq k$ modules $M_1, \dots, M_{k'}$. For each such module we have either $M_i \cap D = \emptyset$ or $M_i \subseteq D$, by Lemma 11 and because if we had $D \subseteq M_i$, this would contradict our choice of M . It must therefore be the case that for some $I \subseteq [k']$ we have $D = \bigcup_{i \in I} M_i$.

We now observe that we have $\mathcal{O}(n)$ choices of M (since the modular decomposition has $\mathcal{O}(n)$ nodes), and 2^{mw} choices for I . Hence, there are $\mathcal{O}(2^{\text{mw}} n)$ possible values of the weight of the heaviest component ℓ . ◀

D Proofs for Section 7 (Vertex Cover Number)

► **Theorem 14.** UNWEIGHTED VERTEX INTEGRITY can be solved in time $5^{\text{vc}} n^{\mathcal{O}(1)}$.

Proof. Let C be a vertex cover of G of size $|C| = \text{vc}$. Without loss of generality, we assume that $k < \text{vc} + 1$, since otherwise C is a separator that realizes the vertex integrity at most k . For the analysis, fix an optimal separator S , which by Proposition 1 can be assumed to be irredundant. We first guess $S \cap C$, delete its vertices, and decrease k by $|S \cap C|$. In the remainder, assume that $S \cap C = \emptyset$. Let $I = V \setminus C$ be the independent set of the graph. We have to decide for each vertex of I whether to place it in S or not.

We keep track of the vertices of I which have been marked as belonging to S , denoted by I_S , those that have been marked as not belonging to S , denoted by $I_{\bar{S}}$, and those which are undecided, denoted by I_U (initially $I = I_U$). We also keep track of the connected components of the graph induced by $C \cup I_{\bar{S}}$. Now, as long as there exists an undecided vertex $v \in I_U$ such that all its neighbors in C are in the same connected component, we mark that v is not

in S and move it to $I_{\bar{S}}$ (because v would be redundant). Consider then an undecided $v \in I_U$ that has neighbors in C which are in distinct components of the graph induced by $C \cup I_{\bar{S}}$. We consider two cases: $v \in S$ and $v \notin S$. In the first case, we remove v from the graph and decrease k by 1. In the latter, we observe that the number of connected components made up of vertices of $C \cup I_{\bar{S}}$ has decreased. We continue this branching procedure until $k < 0$ (in which case we reject); or all vertices have been decided (in which case we evaluate the solution). The algorithm is correct, assuming that our guess of $S \cap C$ was correct, because for all vertices of I we either know that we have made the correct choice (because S is irredundant) or we consider both possible choices.

For the running time, we define a potential function as the sum of k plus the number of connected components induced by $C \cup I_{\bar{S}}$. Then for both branches, the potential function decreases by at least 1. Moreover, the value of the potential function is bounded by $k - |C \cap S| + |C \setminus S| \leq 2vc - 2|C \cap S|$. Note that the branching algorithm is applied to the graph after deleting $C \cap S$. Therefore, the running time is bounded by

$$\begin{aligned}
 \sum_{S \cap C \subseteq C} 2^{2vc - 2|C \cap S|} n^{\mathcal{O}(1)} &= \sum_{S \subseteq C} 4^{vc - |C \cap S|} n^{\mathcal{O}(1)} \\
 &= \sum_{S \subseteq C} 4^{vc - |C \cap S|} 1^{|C \cap S|} n^{\mathcal{O}(1)} \\
 &= \sum_{i=0}^{vc} \binom{vc}{i} 4^{vc-i} 1^i n^{\mathcal{O}(1)} \\
 &= 5^{vc} n^{\mathcal{O}(1)},
 \end{aligned}$$

and the statement follows. $\blacktriangleleft$

► **Theorem 15.** ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER *can be solved in time $2^{\mathcal{O}(|C|)} n^{\mathcal{O}(1)}$.*

Proof. Let $\mathcal{I} = (G, C, w_{\max})$ be an instance of ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER, where $\text{OPT}(\mathcal{I})$ denotes the weight of the optimal solution S , provided such a set S exists. Additionally, let $k = |C|$ and $I = V \setminus C$.

Assume that $N(v) = \emptyset$ for some $v \in I$. We claim that if $w(v) > w_{\max}$, then there is no irredundant wvi-set $S \subseteq V \setminus C$ such that $w(D) \leq w_{\max}$ for all $D \in \text{cc}(G - S)$ and we report NO. Assume there existed such a set S . Then, since $w(v) > w_{\max}$, it follows that $v \in S$. However, since $N(v) = \emptyset$, it follows that no component of $G - S$ contains neighbors of v , thus v is redundant, which is a contradiction. If on the other hand it holds that $w(v) \leq w_{\max}$, then reduce $\mathcal{I}$ to $\mathcal{I}' = (G - v, C, w_{\max})$, where $\text{OPT}(\mathcal{I}) = \text{OPT}(\mathcal{I}')$. The correctness of the reduction rule is easy to see. In the following, assume $N(v) \neq \emptyset$, for all $v \in I$. We perform branching on whether there exists a component D in $G - S$ such that $|D \cap C| \geq k/10$ or not.

In the first case, we guess $D \cap C$. Since S is irredundant, it holds that $D \setminus C = \{v \in I : N(v) \subseteq D \cap C\}$. Moreover, let $I_R \subseteq I$ be the set of vertices with neighbors in both $D \cap C$ and $C \setminus D$. If D is not connected or $w(D) > w_{\max}$, we immediately reject. Otherwise, notice that all vertices $v \in I_R$ must belong to S , thus we obtain a smaller instance $\mathcal{I}' = (G - (D \cup I_R), C \setminus D, w_{\max})$, where $|C \setminus D| \leq 9k/10$ and $\text{OPT}(\mathcal{I}) = w(I_R) + \text{OPT}(\mathcal{I}')$.

In the second case, i.e. when for all components $D \in \text{cc}(G - S)$ it holds that $|D \cap C| < k/10$, we first show the following claim.

► **Claim 23.** There exists a set $A \subseteq C$ of size $k/2 \leq |A| \leq 3k/5$ such that every component D of $G - S$ satisfies either $D \cap C \subseteq A$ or $D \cap C \subseteq C \setminus A$.

Proof. Let $\mathcal{D} = \{D_1, \dots, D_p\}$ be the set of connected components of $G - S$. Note that $\bigcup_{D \in \mathcal{D}} (D \cap C) = C$ since $S \cap C = \emptyset$. By the assumption, $|D \cap C| < k/10$ holds for all $D \in \mathcal{D}$. Set $A = \bigcup_{i=1}^j (D_i \cap C)$ where j is the maximum index such that $|A \setminus D_j| < k/2$. By the definition of A , $|A| \geq k/2$ holds. Moreover, since $|D_j| < k/10$, $|A| \leq k/2 + k/10 = 3k/5$. Thus, the claim holds. $\triangleleft$

Next, we guess $A \subseteq C$ by considering all possible subsets of C whose sizes are between $k/2$ and $3k/5$. Since every component D of $G - S$ satisfies either $D \cap C \subseteq A$ or $D \cap C \subseteq C \setminus A$, all vertices of $I_R \subseteq I$ must belong to S , where $v \in I_R$ if it has neighbors in both A and $C \setminus A$. In that case, we obtain two separate instances $\mathcal{I}_1 = (G[A \cup I_A], A, w_{\max})$ and $\mathcal{I}_2 = (G[(C \setminus A) \cup I_{C \setminus A}], C \setminus A, w_{\max})$, where $I_Z = \{v \in I : N(v) \subseteq Z\}$ for $Z \in \{A, C \setminus A\}$, and $\text{OPT}(\mathcal{I}) = w(I_R) + \text{OPT}(\mathcal{I}_1) + \text{OPT}(\mathcal{I}_2)$.

Consequently, for both cases we obtain smaller instances and hence, we only have to recursively compute the annotated problems. Finally, we analyze the running time of our algorithm. Let $T(k)$ denote the running time when $|C| = k$. Then it holds that $T(k) \leq 2^k n^{\mathcal{O}(1)} \cdot T(9k/10) + 2^k n^{\mathcal{O}(1)} \cdot 2T(3k/5) + n^{\mathcal{O}(1)}$, thus $T(k) = 2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ follows. $\blacktriangleleft$

► **Theorem 16.** WEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(\text{vc})} n^{\mathcal{O}(1)}$.

Proof. Let C be a vertex cover of G of size $|C| = \text{vc}$ and $I = V \setminus C$. For the analysis, fix an optimal separator S , which by Proposition 1 can be assumed to be irredundant. We first guess $S \cap C$, and set $C' = C \setminus S$. In the following, let for $v \in I$, $N(v)$ denote the neighborhood of v in G , while $N'(v) = N(v) \cap C'$. Let $D_{\max}$ denote a connected component of $G - S$ of maximum weight $w(D_{\max}) = w_{\max}$, and set $D_{\max} \cap C' = A \subseteq C'$. We argue that given $S \cap C$, there are at most $2^{|C'|} + n \leq 2^{\text{vc}} + n$ cases regarding $D_{\max}$, where $n = |V|$. If $A = \emptyset$, then $D_{\max} = \{v\}$ for some vertex $v \in I$ with $N(v) \subseteq S \cap C$, for a total of at most $|I| \leq n$ choices. Otherwise, it holds that $A \neq \emptyset$, and due to the irredundancy of S it follows that $D_{\max} \setminus A = \{v \in I : \emptyset \neq N'(v) \subseteq A\}$, while any vertex of $I_R = \{v \in I : N'(v) \cap A, N'(v) \cap (C' \setminus A) \neq \emptyset\}$ must belong to S .

Consequently, it suffices to guess $S \cap C$ and $D_{\max}$, assure that $D_{\max}$ is indeed connected as intended, and then compute the expression

$$\text{OPT}(\mathcal{I}) + w(S \cap C) + w(I_R) + w_{\max},$$

where $\mathcal{I} = (G - (D_{\max} \cup I_R \cup (S \cap C)), C', w_{\max})$ is an instance of ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER and $\text{OPT}(\mathcal{I})$ the weight of its optimal solution. Due to Theorem 15, $\text{OPT}(\mathcal{I})$ can be computed in time $2^{\mathcal{O}(\text{vc})} n^{\mathcal{O}(1)}$, therefore the total running time is $2^{\text{vc}} \cdot (2^{\text{vc}} + n) \cdot n^{\mathcal{O}(1)} \cdot 2^{\mathcal{O}(\text{vc})} n^{\mathcal{O}(1)} = 2^{\mathcal{O}(\text{vc})} n^{\mathcal{O}(1)}$. $\blacktriangleleft$