ROTATION INVARIANT WEBS FOR THREE ROW FLAMINGO SPECHT MODULES

JESSE KIM

ABSTRACT. We introduce a new rotation-invariant web basis for a family of Specht modules $S^{(d^3,1^{n-3d})}$, indexed by normal plabic graphs satisfying a degree condition and resembling A_2 webs. We show that the \mathfrak{S}_n action on our basis can be understood combinatorially via a set of skein relations. From this basis, we obtain a cyclic sieving result for a q-analog of the hook length formula for λ . Our construction extends the jellyfish invariants of Fraser, Patrias, Pechenik, and Striker and is closely related to the weblike subgraphs of Lam.

1. INTRODUCTION

This paper introduces a new web basis for a family of \mathfrak{S}_n modules indexed by partitions $(d, d, d, 1^{n-3d})$. The systematic study of web bases began with work of G. Kuperberg [10] in order to study the space of invariant tensors for simple Lie algebras and their quantum groups, though examples which are now considered web bases predate the term. What exactly constitutes a web basis differs somewhat between authors, we will use a list of properties laid out by C. Fraser, R. Patrias, O. Pechenik, and J. Striker in [6]. The properties they give are:

- (1) Each basis element is indexed by a planar diagram with n boundary vertices, embedded in a disk.
- (2) There is a topological criterion allowing identification of basis diagrams.
- (3) The action of the long cycle c = (12...n) on the basis is by rotation of diagrams.
- (4) The action of the long element $w_0 \in n(n-1) \dots 1$ on the basis is by reflection of diagrams.
- (5) There is a finite list of 'skein relations' describing the action of a simple transposition s_i on a basis diagram.

The simplest web basis is the Temperley-Lieb basis for two-row rectangle shapes (d, d), indexed by noncrossing perfect matchings of 2*d* vertices and studied by a variety of authors [23, 25, 10, 17]. One useful property of the Temperley-Lieb basis is that it makes computation of the action of \mathfrak{S}_n easy: to act by a permutation on a basis element, simply permute the matching, potentially introducing crossings, then resolve each crossing by replacing it with an uncrossing in both possible ways. This crossing resolution is called a skein relation, shown below.



Kuperberg introduced similar bases for invariant spaces for rank-two Lie algebras. In this paper we will primarily be interested in the type A_2 ; the Temperley-Lieb basis is type A_1 . In type A_2 , Kuperber's basis is indexed by bipartite trivalent planar graphs with no faces of degree less than 6, called nonelliptic SL_3 webs. For each nonelliptic SL_3 web, the corresponding element of $V^{\otimes n}$, where V is the three-dimensional defining representation of SL_3 , is defined either recursively, in terms of tensor product and contraction, or combinatorially, in terms of a weighted sum over all proper edge coloring of the web with 3 colors. These form a basis for the space of SL_3 invariants of $V^{\otimes n}$, where SL_3 acts diagonally on $V^{\otimes n}$. The symmetric group \mathfrak{S}_n acts on this invariant space by permuting tensor factors. As an \mathfrak{S}_n module, the SL_3 invariant space of $V^{\otimes n}$ is irreducible and isomorphic to the Specht module of shape (d, d, d) where n = 3d, and is 0 if n is not a multiple of 3. Various ways to generalize this construction to SL_n have been studied [4, 1], though a rotation invariant version is known only for n up to 4 [7].

Rhoades generalized the Temperley-Lieb basis in a different direction, giving a web basis for shapes $(d, d, 1^{n-2d})$ indexed by noncrossing set partitions with parts of size at least two [18]. In Rhoades' action, crossing resolution involved four skein relations based on the sizes of the crossing blocks. In previous work with Rhoades [9], we showed that this action of noncrossing set partitions could be found within the top

degree component of the fermionic diagonal coinvariant ring. One main goal of this paper is to generalize Kuperberg's SL_3 webs in an analogous way to Rhoades' generalization of the Temperley-Lieb basis.

To do so, we build upon work of R. Patrias, O. Pechenik, and J. Striker. In [12], they introduce *jellyfish invariants*, giving an alternate construction of Rhoades' basis within the homogeneous coordinate ring of a 2-step partial flag variety. Along with C. Fraser, they further develop these jellyfish invariants, reinterpreting them within the homogeneous coordinate ring of a Grassmanian [6]. They also extend their definitions to give jellyfish invariants living within a Specht modules indexed by a partition of shape $(d^r, 1^{n-rd})$. They dub these *flamingo Specht modules* as the partitions indexing them appear to stand on one leg. In the case r > 2, they do not give a basis for this module. Instead, they give a linearly independent set indexed by noncrossing set partitions and a spanning set indexed by all set partitions.

Our first result is to extend their linearly independent set to a basis of $S^{(d^r, 1^{n-r^d})}$ in the case r > 2. We do so by replacing the noncrossing condition with a weaker one based on ideas introduced by P. Pylyavskyy in [15], which we call *r-weakly noncrossing*. This basis fails to have the rotation and reflection invariance desired of a web basis, however. Our second result is to remedy the lack of rotation invariance in the case r = 3 by introducing a second basis indexed by a certain rotationally invariant set AW(n, d) of normal plabic graphs we call *augmented* SL_3 webs, as they closely resemble Kuperberg's SL_3 webs with extra edges. This resolves a conjecture made in [8]. Plabic graphs were first introduced by A. Postnikov [14] in order to study the totally nonnegative Grassmanian; we use the combinatorial machinery developed for them to show that our indexing set has the correct enumeration. To define our basis, we use a modification of proper edge colorings for SL_3 webs which we call *consistent labellings*. Consistent labellings are closely related to the weblike subgraphs introduced by T. Lam in order to define SL_3 web immanants and later used by C. Fraser, T. Lam, and I. Le to introduce a higher rank version of Postnikov's boundary measurement map [11, 5], and we make this connection explicit.

Through consideration of the combinatorics of constant labellings, we obtain skein relations for augmented SL_3 webs. These skein relations give a combinatorial description of the action of an adjacent transposition on an augmented web. The first skein relation, the crossing reduction rule shown below, shows how to expand the application of an adjacent transposition to an augmented web in the augmented web basis. The gray region represents an unknown number of edges connecting to other vertices of the web not depicted.



Note that not all terms on the right hand side of this relation are necessarily augmented webs, as black vertices of degree less than 3 or faces of degree 4 may be created. The remaining skein relations, which can be found in Section 6, show how to expand such terms when they arise.

One application of our rotationally invariant basis in the case r = 3 is that it gives us a cyclic sieving result on the indexing set. Let $X_{n,d}(q)$ denote the q-analog of the hook length formula for $\lambda = (d^3, 1^{n-3d})$,

$$X_{n,d}(q) := q^{3(d-1) + \binom{n-3(d-1)}{2}} \frac{[n]!_q}{\prod_{(i,j) \in \lambda} [h_{ij}]_q}$$

where h_{ij} denotes the hook length of a box (i, j) in the Young diagram for λ . We show that the triple $(AW(n, d), C, X_{n,d}(q))$ exhibits the cyclic sieving phenomenon when n is odd, and a signed version of cyclic sieving holds when n is even. Specializing to the case n = 3d recovers a cyclic sieving result on SL_3 webs studied by T.K. Petersen, Pylyasky, and Rhoades in [13].

The rest of the paper is organized as follows. In Section 2, we review necessary background information. In Section 3 we give a definition of *r*-weakly noncrossing set partitions and give a basis of $S^{(d^r, 1^{n-rd})}$ which extends the jellyfish invariant basis. In Section 4, we define *augmented webs* as a certain subset of normal plabic graphs and give a combinatorial bijection between them and 3-weakly noncrossing set partitions. This bijection will draw on ideas developed by J. Tymoczko and H. Russell to give a bijection between SL_3 webs and objects called *m*-diagrams, a special case of our 3-weakly noncrossing set partitions [19, 24]. In Section 5, we define an SL_3 -invariant polynomial attached to each normal plabic graph. We show that this definition extends jellyfish invariants and that the set of invariants attached to augmented webs satisfy properties (3) and (4) of a web basis. In Section 6, we show that skein relations hold for our plabic graph invariants. We use these skein relations to show that augmented web invariants are indeed a basis for $S^{(d^3, 1^{n-3d})}$. In Section 7, we show that our augmented web invariants can be interpreted in terms of weblike subgraphs. In Section 8, we discuss the cyclic sieving result for augmented webs which arises from our rotationally invariant basis. In Section 9, we discuss some possible future directions for this work.

2. Background

2.1. Specht modules. The irreducible representations of the symmetric group \mathfrak{S}_n are indexed by integer partitions of n. The irreducible indexed by a partition $\lambda \vdash n$ is denoted S^{λ} and is called the *Specht module* of shape λ . We give a construction of S^{λ} following the approach of [6]. See [20] for a more detailed treatment of Specht modules.

Let $\lambda \vdash n$, and let λ' be its transpose. Consider a matrix of $n\lambda'_1$ variables,

$$M = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{\lambda'_1,1} & x_{\lambda'_1,2} & \cdots & x_{\lambda'_1,n} \end{bmatrix}$$

The symmetric group \mathfrak{S}_n acts on this matrix, and thus on $\mathbb{C}[M]$, by permuting columns of M. Let $\pi = \{\pi_1, \pi_2, \ldots, \pi_{\lambda_1}\}$ denote a set partition of n with shape λ' , i.e. the sizes of each part of the partition are given by the rows of λ' . For each such set partition π , let p_{π} be the polynomial

$$p_{\pi} = \prod_{i=1}^{\lambda_1} M_{[\lambda_i']}^{\pi_i}$$

where $M_{[\lambda']}^{\pi_i}$ denotes the matrix minor of M whose rows are indexed by $[\lambda'_i]$ and columns are indexed by π_i .

Then the Specht module S^{λ} is the span of these polynomials as π ranges over all set partitions of shape λ .

2.2. Jellyfish invariants. Jellyfish invariants were introduced in [12] and further developed in [6] in order to study the Specht module $S^{(d^r, 1^{n-rd})}$. An (n, d, r)-jellyfish invariant is a certain element of $S^{(d^r, 1^{n-rd})}$ attached to each ordered set partition of [n] with d blocks and all blocks at least size r. We include the basic definitions from [6] below, see their paper for examples and further exposition.

An ordered set partition of n is a set partition with a total order on its blocks. Two blocks A, B of an ordered set partition cross if there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 < b_1 < a_2 < b_2$ or $b_1 < a_1 < b_2 < a_2$. An ordered set partition is noncrossing if no two of its blocks cross. Let $\mathcal{OP}(n, d, r)$ denote the set of all ordered set partitions with exactly d blocks and blocks of size at least r, and let $\mathcal{NCOP}(n, d, r)$ denote the set of all such partitions which are also noncrossing.

Definition 2.1. Let $\pi = {\pi_1, \ldots, \pi_d} \in \mathcal{OP}(n, d, r)$ be an ordered set partition. Define the set of *r*-jellyfish tableaux, $\mathcal{J}_r(\pi)$ to be the set of generalized tableau *T* with *d* columns and n - (d-1)r rows with the following constriants:

- (1) $T_{ij} \in [n]$ or T_{ij} is nonempty
- (2) If $i \in [r], T_{ij}$ is nonempty.
- (3) If i > r, there exists exactly one j such that T_{ij} is nonempty

(4) The nonempty entries in column j are exactly the elements of π_j in increasing order.

For each $T \in \mathcal{J}_r(\pi)$, define a polynomial

$$J(T) = \prod_{i=1}^d M_{R_i(T)}^{\pi_i}$$

where $R_i(T)$ is the set of rows containing an entry in π_i .

For each $\pi \in \mathcal{OP}(n, d, r)$, the *r*-jellyfish invariant, denoted $[\pi]_r$ is

$$[\pi]_r = \sum_{T \in \mathcal{J}_r(\pi)} \operatorname{sign}(T) J(T)$$

where $\operatorname{sign}(T)$ denotes the sign of the reading word of T.

Fraser, Patrias, Pechenik, and Striker prove the following about r-jellyfish invariants:

Theorem 2.2 ([6, Theorem 4.24]). For each ordered set partition $\pi \in \mathcal{OP}(n, d, r)$, the invariant $[\pi]_r$ lies in the flamingo Specht module $S^{(d^r, 1^{n-rd})}$.

Theorem 2.3 ([6, Proposition 5.11]). For any ordered set partition $\pi \in \mathcal{OP}(n, d, r)$ and any permutation $w \in \mathfrak{S}_n$, we have

$$w \cdot [\pi]_r = \operatorname{sign}(w)[w \cdot \pi]_r$$

Note that this implies the span of jellyfish invariants is closed under the action of \mathfrak{S}_n , and must therefore be equal to $S^{(d^r, 1^{n-rd})}$.

Theorem 2.4 ([6, Theorem 5.13]). For each noncrossing set partition $\gamma \in \mathcal{NC}(n, d, r)$, order the blocks in any way to create a corresponding ordered set partition π_{γ} . Then the set $\{[\pi_{\gamma}]_r : \gamma \in \mathcal{NC}(n, d, r)\}$ is linearly independent.

Fraser, Patrias, Pechenik, and Striker thus give a spanning set of $S^{(d^r, 1^{n-r^d})}$ indexed by all set partitions, and a linearly independent subset indexed by noncrossing set partitions. Thus, it is possible to choose a subset S of set partitions such that S indexes a basis and S contains all noncrossing set partitions. We will show how to do so in Section 3.

2.3. Noncrossing Tableaux. Noncrossing tableaux were introduced by P. Pylyavskyy in [15] to give a non-crossing counterpart to standard Young tableaux. Formally, noncrossing tableaux are set partitions; Pylyavskyy chose the name noncrossing tableaux to distinguish them from the more standard definition of noncrossing set partitions given in the previous subsection. As we will be using noncrossing tableaux in the context of set partitions, we will instead refer to these as *weakly* noncrossing set partitions. We will use a modification of this weaker condition to interpolate between strongly noncrossing set partitions and all set partitions.

Definition 2.5. Let $A = \{a_1 < a_2 < \cdots < a_{|A|}\}$ and $B = \{b_1 < b_2 < \cdots < b_{|B|}\}$ be two disjoint subsets of [n] with $|A| \leq |B|$. We say A and B are *weakly noncrossing* if for all $1 \leq i \leq |A| - 1$ we do not have

$$a_i < b_i < a_{i+1} < b_{i+1}$$

or

$$b_i < a_i < b_{i+1} < a_{i+1}$$

and additionally, if |A| < |B|, we do not have

$$b_{|A|} < a_{|A|} < b_{|B|}$$

A set partition π is *weakly noncrossing* if blocks of π are pairwise weakly noncrossing.

Pylyavskyy showed that weakly noncrossing set partitions of shape $\lambda \vdash n$ are in bijection with standard Young tableaux of shape λ .

2.4. **Plabic graphs.** Plabic graphs were introduced by Postnikov in order to study the totally nonnegative Grassmanian. A textbook treatment can be found in [3]. We will only need combinatorial results about plabic graphs, which we list here.

A plabic graph G is a planar graph embedded in a disk, possibly with loops and multiple edges between vertices, with interior vertices colored black and white and boundary vertices labelled clockwise 1 through n. A normal plabic graph is a plabic graph for which white vertices are degree three, boundary vertices only connect to black vertices, and same colored vertices do not share an edge. For this paper, we consider only normal plabic graphs and state results only as they apply to normal plabic graphs, rather than including the full generality.

Given a normal plabic graph G, the *trip* at i is the walk in G starting at boundary vertex i which turns right at every black vertex and left at every white vertex until it reaches the boundary at avertex we denote $\operatorname{trip}(i)$. The function defined by $i \mapsto \operatorname{trip}(i)$ is a permutation of [n] and is called the *trip permutation* of G. The *exceedances* of G are the exceedances of this permutation, i.e. those trips for which $\operatorname{trip}(i) > i$. Two normal plabic graphs are *normal move equivalent* if one can be obtained from the other via a seuquence of *normal urban renewal moves* and *normal flip moves*, which we now define. The normal urban renewal move is the move shown below, where filled in arcs represent any number of edges leading elsewhere in the graph



A normal plabic graph is *reduced* if it is not normal move equivalent to any plabic graph which contains a *forbidden configuration*, i.e. a face of degree two or a leaf vertex not adjacent to the boundary.

A bad feature of a normal plabic graph G is one of the following:

The normal flip move is

- A roundtrip: A cycle in G which turns left at every white vertex and right at every black vertex.
- An essential self-intersection: A trip in G which passes through the same edge twice.
- A bad double-crossing: Two trips in G which both pass through edge e_1 then edge e_2 in that order.

Theorem 2.6 ([3, Theorem 7.8.6]). A normal plabic graph is reduced if and only if it does not contain any bad features.

The more common use of this theorem is to test whether a plabic graph is reduced or not. The plabic graphs we are interested in, however, will be clearly reduced as their normal move equivalence class will have size 1. We will instead apply it to understand the structure of the trips of our plabic graphs.

2.5. SL_3 Webs. SL_3 webs, or A_2 webs, were introduced by Kuperberg to study SL_3 invariant tensors and the representation theory of the quantum group $U_q(\mathfrak{sl}_3)$ [10]. The following is based on both Kuperberg's work as well as a nice discussion of the topic by S. Fomin and P. Pylyavskyy in [2]. A sign string of length n is a string containing n letters, all each + or -, e.g. (+ + - - + + -). Given a sign string $s = s_1 s_2 \cdots s_n$, an SL_3 web of type s is a bipartite plabic graph with n boundary vertices in which every interior vertex has degree 3 and boundary vertex i is adjacent to a black vertex if $s_i = +$ and a white vertex if $s_i = -$. This is a slightly anachronistic version of the definition, as plabic graphs were defined after SL_3 webs, but the comparison will be useful for us later.

 SL_3 webs have representation theoretic meaning. Let V be the three-dimensional defining representation of SL_3 , with basis $\{e_1, e_2, e_3\}$, and let V^* denote its dual with dual basis $\{e_1^*, e_2^*, e_3^*\}$. An SL_3 web with sign string (+ + - - + +) e.g. represents an element of the space

$$(V \otimes V \otimes V \otimes V^* \otimes V^* \otimes V \otimes V)^{SL_3}$$

of SL_3 invariant elements of $(V \otimes V \otimes V \otimes V^* \otimes V^* \otimes V \otimes V)$ where V is the three-dimensional defining representation of SL_3 , + correspond to copies of V and – correspond to copies of V^* .

The unique SL_3 web of sign string (+++)



represents the tensor $\sum_{\sigma \in \mathfrak{S}_3} \operatorname{sign}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$ and the unique SL_3 web of sign string (--)



represents the tensor $\sum_{\sigma \in \mathfrak{S}_3} \operatorname{sign}(\sigma) e^*_{\sigma(1)} \otimes e^*_{\sigma(2)} \otimes e^*_{\sigma(3)}$. Concatenation of webs represents tensor product, and an edge between vertices represents tensor contraction.

We can also give a purely combinatorial desciption of the invariant each web represents in terms of proper edge colorings. A proper edge coloring ℓ of an SL_3 web W is a labelling of the edges by the numbers 1, 2, 3 such that no label appears more than once around each vertex. For each labelling, we get a simple basis tensor T_{ℓ} by taking the basis vector or dual basis vector e_j or e_j^* (depending on the sign string) at boundary vertex i whose incident edge has label j, and a sign sign(ℓ) given by $(-1)^{cc(\ell)}$, where $cc(\ell)$ denotes the number of interior vertices for which 1, 2, 3 appear in counterclockwise order in the labelling ℓ . The SL_3 invariant associated to W, which we denote $[W]_{SL_3}$ is

$$[W]_{SL_3} = \sum_{\text{proper labellings } \ell} \operatorname{sign}(\ell) T_{\ell}$$

A web is called *nonelliptic* if it contains no faces of degree 4 or less. The invariants for the set of all noneeliptic webs form a basis for the space of SL_3 invariant tensors.

2.6. Cyclic sieving. The cyclic sieving phenomenon was introduced by V. Reiner, D. Stanton, and D. White in order to unify a number of enumerative results in combinatorics [16]. See their paper and a survey by B. Sagan for further reference [21].

Definition 2.7. Let X be a finite set equipped with an action of the finite cyclic group $C \cong \mathbb{Z}/n\mathbb{Z}$ with generator c, let X(q) be a polynomial, and let ζ be an n^{th} root of unity. The triple (X, C, X(q) is said to exhibit the cyclic sieving phenomenon if $|X^{c^d}| = X(\zeta^d)$ for any integer d > 0, where X^{c^d} denotes the set of all elements of X fixed by c^d .

One way of obtaining cyclic sieving results is via the following, which can be found in Sagan's survey [21] and follows from a result of Springer [22].

Theorem 2.8 ([21, Theorem 8.2], [22]). Let W be a finite complex reflection group and let $C \leq W$ be cyclically generated by a regular element g. Let V be a W-module with a basis B such that gB = B. Then the triple

$$(B, C, F^V(q))$$

exhibits the cyclic sieving phenomenon, where $F^{V}(q)$ denotes the fake degree polynomial for V.

See [21] for a complete definition of the fake degree polynomial, we will only need the following.

Proposition 2.9. Let λ be a partition of n and let S^{λ} be the corresponding Specht module. The fake degree polynomial $F^{S^{\lambda}}(q)$ is given by

$$F^{S^{\lambda}}(q) = q^{b(\lambda)} \frac{[n]!_q}{\prod_{(i,j)\in\lambda} [h_{ij}]_q}$$

where $b(\lambda) = 0\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots$ and h_{ij} denotes the hook length of box (i, j) in the Young diagram of λ .

3. Weakly-noncrossing jellyfish invariants are a basis

In this section, we extend the linearly independent set given in [6] to a basis by introducing a weaker version of the noncrossing condition for ordered set partitions. The weaker version is similar to the noncrossing tableau defined by P. Pylyavskyy in [15]. Our version will differ in that it will depend on r.

Definition 3.1. Let $A = \{a_1, a_2, \dots, a_{|A|}\}$ and $B = \{b_1, b_2, \dots, b_{|B|}\}$ be two subsets of [n] each of size $\geq r$. We say that A and B are r-weakly noncrossing if the following holds:

- (1) For each $i = 1 \dots, r-2$, The arc (a_i, a_{i+1}) does not cross the arc (b_i, b_{i+1}) .
- (2) For any $j_1, j_2 \ge r$, the arc (a_{r-1}, a_{j_1}) does not cross the arc (b_{r-1}, b_{j_2}) .

An (ordered) set partition is r-weakly noncrossing if its blocks are pairwise r-weakly noncrossing.

One can think of this definition as being noncrossing in the sense of Pylyavskyy in the first r-2 entries, and noncrossing in the strong sense in the remaining entries.

Let WNC(n, d, r) denote the set of all set partitions of [n] into d blocks each of size at least r which are *r*-weakly noncrossing.

We first show that the set of r-weakly noncrossing set partitions is the correct size:

Proposition 3.2. There is a bijection between standard Young tableaux of shape $(d^r, 1^{n-rd})$ and r-weakly noncrossing set partitions in WNC(n, d, r).

Proof. We show that both sets are in bijection with a set of rectangular tableaux filled with a subset of [n]:

Definition 3.3. Let T(n, d, r) denote the set of all tableaux of shape $\lambda = (d^r)$ filled with with integers in [n] such that

- (1) Entries increase along rows and down columns.
- (2) No element of [n] appears more than once.

Example 3.4. Consider the two tableaux below.

(3) For any i which does not appear in the tableaux, the number of entries j < i appearing in row r-1strictly exceeds the number of entries j < i appearing in row r.



The tableau on the left is an element of T(16, 4, 3). The tableau on the right is not in T(16, 4, 3) because 9 does not appear as a filling and there are the same number of fillings less than 9 in the second and third rows, highlighted in gray.

The bijection between $SYT(d^r, 1^{n-rd})$ and T(n, d, r) is as follows. Let t be a standard Young tableaux of shape $(d^r, 1^{n-rd})$.

- (1) If removing the blocks in rows larger than r (which we will refer to as the *tail*) produces a tableaux in T(n, d, r), do so.
- (2) Otherwise, let i be the maximal element among the tail for which the number of elemenents j < iin row r-1 equals the number of elements j < i in row r. Remove the first block of row r and all blocks below it, shift all blocks in row r filled with j < i one space to the left, and place a block filled with i in the newly formed opening.

The maximality of i will guarantee that the third property of T(n, d, r) is satisfied for elements larger than i, and the shifting left will guarantee it is satisfied for elements smaller than i. Call the resulting tableau f(t).

To reverse this process, let t' be a tableaux in T(n, d, r). We obtain a standard Young tableaux of shape $(d^r, 1^{n-rd})$ as follows.

- (1) If the smallest element which does not appear in t' is larger than the entry in the first box of row r, simply append all integers not already appearing in the tableau in increasing order as the tail.
- (2) Otherwise, let i be the minimal filling in row r which is smaller than the filling of the box one space up and to the right, or the largest element of row r if no such element exists. Remove the box filled with i, shift all boxes to the left of it one space right, and insert the remaining entries in increasing order to form the tail.

The right shift will preserve the standard Young tableau property due to the minimality of i. Call the resulting tableau q(t').

To verify that these two maps are indeed inverses, let $t \in SYT(d^r, 1^{n-rd})$. If removing the tail of t produces a tableau in T(n, d, r), then it is clear that q(f(t)) = t. Otherwise, let i be the element inserted into row r to obtain f(t). Before this insertion, there were the same number of elements less than i in row r-1 and row r of the tableau, so the filling one space up and to the right of i in f(t) must be larger than i. Additionally, all boxes i to the left of i were shifted over, and since we started with a standard Young

JESSE KIM

tableau, the filling one space up and to the right of them is smaller than j. Therefore, i is the filling removed by g, and g(f(t)) = t.

A similar argument shows that $f \circ g$ is also the identity. Indeed, if i was the element removed from row r by g, then there must be the same number of elements j < i in row r - 1 and r of g(t'), and no other element larger than i can have this property as t' is in T(n, d, r).

Example 3.5. An example of this bijection is given below for n = 16, d = 4, r = 3. 12 is the maximal filling of the tail for which the second and third rows contain the same number of lesser fillings.



The bijection between T(n, d, r) and WNC(n, d, r) is essentially repeated applications of the standard Catalan bijection between two row rectangular standard Young tableaux and noncrossing matchings. Given a tableau $t \in T(n, d, r)$, for i = 1, ..., r, let $R_i(t)$ denote the entries in row i of t. Place the numbers 1 through n in a line, and for each i = 1, ..., r - 1, draw d arcs between elements of R_i and R_{i+1} such that

- (1) Elements of R_i are the left endpoints of arcs, and elements of R_{i+1} are the right endpoints of arcs.
- (2) There do not exist two arcs (a, b) and (c, d) such that a < c < b < d.

The standard Catalan bijection argument guarantees that this is uniquely possible. Then, for each positive integer m at most n not appearing in t, there is a unique shortest arc (a, b) created at step r - 1 such that a < m < b. The third condition of T(n, d, r) guarantees that such an arc exists, and the noncrossing condition above guarantees it is unique. Draw the arc (a, m). Finally, create a set partition π by placing all integers connected by arcs into the same block. Then $\pi \in WNC(n, d, r)$. To see that the noncrossing condition is satisfied, note that for $i = 1, \dots, r-2$ if a_i and a_{i+1} are the i^{th} and $(i+1)^{th}$ smallest elements of a block of π , then they must necessarily be connected by an arc created at step i in the above process.

The inverse of this bijection is simple, given a set partition $\pi \in T(n, d, r)$ place the smallest element of each block in increasing order in row 1, the second smallest in row 2, and so on, up to row r - 1. Finally, place the largest element of each block in row r.

Example 3.6. Consider the tableau shown below.

1	2	4	7
3	5	8	11
9	10	12	16

We get the following arc diagram. Arcs created by matching the first two rows are shown in red, arcs created by matching the second and third rows are shown in green, and arcs created by connecting missing entries are shown in blue.



The resulting set partition is $\{\{1, 11, 12\}, \{2, 3, 13, 14, 15, 16\}, \{4, 5, 6, 10\}, \{7, 8, 9\}\}$.

The second half of the proof of Proposition 3.2 also gives the following corollary, which we will need later:

Corollary 3.7. A r-weakly noncrossing set partition γ is uniquely determined by the r-1 sets

 $\{m \mid m \text{ is the } i^{th} \text{ smallest element of some block of } \gamma\}$

for $i = 1, \ldots, r - 1$, along with the set

 $\{m \mid m \text{ is the largest element of some block of } \gamma\}$

Proof. The information in these sets determines the elements of each row of the tableau in T(n, d, r) as defined in the proof of Proposition 3.2. Placing elements in increasing order within each row recovers the tableau, and thus the set partition.

Example 3.8. Suppose n = 7, d = 3, r = 2, and thus $\lambda = (2, 2, 2, 1)$. There are fourteen standard Young tableaux of shape λ , and fourteen 2-weakly noncrossing set partitions. The bijection between them is shown below, with the intermediary tableau in T(7, 3, 2) and arc diagram shown as well.

$$\begin{array}{c} 1 \\ \hline 3 \\ \hline 3 \\ \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline$$

JESSE KIM

$$\begin{array}{c} 1 & 4 \\ 2 & 6 \\ 3 & 7 \\ 5 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 6 \\ 5 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 6 \\ 5 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 \\ \end{array} \rightarrow \begin{array}{c} 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 6 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 6 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 7 \\ \end{array} \rightarrow \begin{array}{c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 5 & 7 \\ \end{array}$$

We can now define and prove our basis.

Theorem 3.9. Let $r \geq 2$. Order each weakly noncrossing set partition $\gamma \in WNC(n, d, r)$ to create an ordered set partition π_{γ} . Then the set $\{[\pi_{\gamma}]_r \mid \gamma \in WNC(n, d, r)\}$ is a basis for the flamingo Specht module $S^{(d^r, 1^{n-rd})}$.

Proof. By Proposition 3.2 and Theorem 2.2, it suffices to show that $\{[\pi_{\gamma}]_r \mid \gamma \in WNC(n, d, r)\}$ is linearly independent. To do so, we introduce a monomial order and show that under this order, each $[\pi_{\gamma}]_r$ has a unique leading term.

Recall that the r-jellyfish invariant is a polynomial in the $\nu \times n$ variables:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{\nu,1} & x_{\nu,2} & \cdots & x_{\nu,n} \end{bmatrix}$$

We order these variables in a somewhat unusual way. Define an order on these variables by $x_{i_1,j_1} < x_{i_2,j_2}$ if and only if one of the following holds:

1)
$$i_1 < i_2$$

2) $i_1 = i_2 \neq r$ and $j_1 < j_2$
3) $i_1 = i_2 = r$ and $j_1 > j_2$

In other words, we order them in reading order except we read the r^{th} row backwards. We then take the lexicogrpahic monomial order with respect to this ordering of variables. The unusual ordering is chosen to make use of Corollary 3.7. Without reversing the r^{th} row, lexicogrpahic leading terms are not unique.

Let the i^{th} block of γ be

$$\gamma_i := \{\gamma_{i,1}, \gamma_{i,2}, \dots, \gamma_{i,|\gamma_i|}\}$$

Let T be a jellyfish tableau associated to γ . Then the leading term of J(T) is straightforward to compute from the definition, we have

$$lt(J(t)) = \left(\prod_{i=1}^{d} \prod_{j=1}^{r-1} x_{j,\gamma_{i,j}}\right) \prod_{i=1}^{d} x_{r,\gamma_{i,|\gamma_i|}} \prod_{j=r+1}^{\nu} x_{j,u_j}$$

where u_j is the entry appearing in row j of J(T). In words, for $i = 1, ..., r - 1, x_{i,j}$ will appear if and only if j is the i^{th} smallest element of some block of γ , and $x_{r,j}$ will appear if and only if j is the largest element of some block of γ . The leading term of $[\pi_{\gamma}]_r$ will be the leading term of one of these J(T), and we can see that the leading term contains the information of the sets described in Corllary 3.7. Thus, the leading term of $[\pi_{\gamma}]_r$ is unique and thus $\{[\pi_{\gamma}]_r \mid \gamma \in WNC(n, d, r)\}$ is linearly independent as desired.

Remark 3.10. The property of being *r*-weakly noncrossing is not preserved under rotation, for example $\{1, 4, 5\}, \{2, 3, 6\}$ is weakly 3-noncrossing, but $\{\{1, 3, 4\}, \{2, 5, 6\}\}$ is not. So the basis given in Theorem 3.9 is not rotation invariant as desired of a web basis. The next section will give a different basis which is rotation invariant in the case r = 3.

4. Rotation invariant webs for the r = 3 case

For the rest of the paper, we specialize to the case r = 3. We will introduce a new basis for this case which is rotation and reflection invariant. To index our basis, we introduce a subset normal plabic graphs which we call *augmented webs*. We call them augmented webs to allude to the fact that they are similar to SL_3 webs, but potentially with vertices of higher degree. We will show that augmented webs are in bijection with 3-weakly noncrossing set partitions, and thus have the correct enumeration to index a basis of $S^{(d^3,1^{n-3d})}$. The benefit of working with augmented webs over 3-weakly noncrossing set paritions is that the set of augmented webs is rotation invariant. **Remark 4.2.** The term *exceedance* is chosen because the exceedance of an augmented web is also the number of exceedances in the trip permutation.



FIGURE 1. Examples of augmented in A(8,2) and A(10,3).

Remark 4.3. When an augmented web has no white vertices, it contains exactly the same information as a strongly noncrossing set partition, with the sets of all boundary vertices connected to a particular interior vertex forming the blocks.

4.1. Combinatorial properties of augmented webs. In this subsection, we develop combinatorial results for augmented webs. Our first result is that all augmented webs are reduced plabic graphs.

Proposition 4.4. Let $W \in AW(n, d)$. Then W is reduced.

Proof. As no square faces or vertices of degree two are present, normal plabic graph moves are not possible. Thus it suffices to check that W itself has no forbidden configurations and this is clear.

Next, we show that augmented webs have an inductive structure we can exploit.

Lemma 4.5. Let $W \in AW(n, d)$, and suppose W has at least one white vertex. Then for each connected component of W, there exists at least two black vertices each connected to exactly one white vertex.

Proof. Let C be a maximal cycle in W, i.e. a cycle with no edge incident to an interior face of W. Since W is reduced by Proposition 4.4, it contains no round trips or essential self-intersections. Therefore there are at least two black vertices of W exterior to C which connect to a white vertex in C. Create a graph G with two types of vertices: a vertex for every vertex of W which is on the exterior of every cycle in W, and a vertex for every maximal cycle. Add an edge between a vertex v and a maximal cycle C whenever v is adjacent to a vertex in C. Then G is a forest, and since W had at least one white vertex, G has at least one edge. So G has two leaves, and these two leaves must be black vertices connected to exactly one white vertex.

The use of Lemma 4.5 is that every augmented web with at least one white vertex can be built out of an augmented web with one fewer white vertices in the following way. Let u and v be the black vertex and its white neighbor identified by Lemma 4.5. If we remove vertices u and v, then connect the neighbors of v to the boundary by at least one edge each in a planar way, we get an augmented web W'.

We will need a notion of depth of a face, which we now define.

Definition 4.6. Let $W \in AW(n, d)$. Let f be a face of W, and let f_0 be the face connected to the section of boundary between 1 and n. The *depth* of f is the number of exceedances which separate f from f_0 .

JESSE KIM

Let e be an edge of W. We say that e is a *depth boundary edge* if the depth of the faces incident to e are not equal. Equivalently, e is a depth boundary edge if exactly one of the trips passing through e is an exceedance. We say e is a *left-to-right* depth boundary edge if, when oriented towads its black vertex endpoint, the depth of the face on the right is higher than the depth of the face on the left. Equivalently, e is a left-to-right depth boundary if only the trip passing through e from towards its black vertex is an exceedance. We define *right-to-left* deth boundary edges similarly.

Lemma 4.7. Let $W \in AW(n, d)$. Let v be an interior vertex of W. Then exactly two edges incident to v are depth boundary edges.

Proof. Consider the set of all trips t_1, \ldots, t_k passing through v, ordered cyclically. Since W is reduced, the starts of all these trips must appear in the same cyclic order around the boundary of W, since otherwise we would introduce a bad double corssing. Similarly, the ends of these trips appear in the same cyclic order around W. Therefore, the set of exceedances passing through v is a proper nonempty subset of these trips which is cyclically consecutive around v. The first and last of these trips will contribute a depth boundary edge.

Lemma 4.8. Let $W \in AW(n, d)$. Let u and v be two adjacent vertices of W. Let t_u be any trip which passes through u but does not use edge (u, v), and let t_v be any trip which passes through v but does not use edge (u, v). Then trips t_u and t_v do not share any vertices.

Proof. This follows from Euler's formula for planar graphs. Assume the contrary, that trips t_u and t_v meet at some vertex x. Let C be the cycle formed from t_u , t_v and edge (u, v), and suppose it is of length k. Consider the graph G containing all vertices and edges of W that are part of C or in its interior. Let V_{int} denote the number of vertices strictly in the interior of C, and let α be the average degree of these interior vertices. Then we have

$$|E(G)| \ge \frac{5}{4}k + \frac{\alpha}{2}V_{\text{int}} - 1$$

and thus by Euler's formula the number of faces of G not including the external face is at least $\frac{1}{4}k + \frac{\alpha-2}{2}V_{\text{int}}$. The total degree of these faces is

$$\frac{3}{2}k - 2 + \frac{\alpha}{V_{\text{int}}}$$

and thus their average degree is strictly less than 6, a contradiction.

4.2. A bijection from tableaux to augmented webs. We can now show that augmented webs are in bijection with weakly 3-noncrossing set partitions, WNC(n, d, 3). To define our bijection, we first formally define the arc diagram used in the proof of Proposition 3.2. We call these *m*-diagrams, based on the objects of the same name developed by J. Tymoczko in [24].

Definition 4.9. Let $\pi \in WNC(n, d, 3)$. To form the *m*-diagram for π , place the vertices 1 through *n* equally spaced in a line, then for each block $B = \{b_1 < b_2 < \cdots < b_k\}$, draw a semicircular arc in the lower half plane from b_1 to b_2 , and from b_2 to all other elements of *B*. We call the arc between b_1 and b_2 a first arc, and all other arcs second arcs. Note that the definition of weakly 3-noncrossing guarantees that first arcs do not cross first arcs, and second arcs do not cross second arcs. For visual clarity, we will often color first arcs in red and second arcs in black. The name *m*-diagram is due to the fact that in Tymoczko's definition, blocks were always size three and thus had a unique second arc, so the diagram appeared visually as a number of intersecting *m* shapes.

The collection of first arcs and maximal second arcs of each block divide the lower half plane into a number of regions. We define the *depth* of each region to be the number of first arcs and maximal second arcs which the region lies above.

Example 4.10. Let $\pi \in WNC(13,3,3)$ be the weakly 3-noncrossing set partition with three blocks, $\{\{1,4,6,7,8\},\{2,3,9,10\},\{5,11,12,13\}\}$. The *m*-diagram associated to π appears below.



We can now define our bijection.

Definition 4.11. The function $\varphi: WNC(n,d,3) \to AW(n,d)$ is defined as follows:

Let $\pi \in WNC(n, d, 3)$, and let M be its m diagram. For each block $B = \{b_1 < b_2 < \cdots < b_k\}$, introduce a black vertex v_B slightly above b_2 , connected to b_2 by an edge. In a small region around b_2 , modify the arcs connecting to b_2 to instead connect to v_B . Then, for every pair of blocks, if the first arc of one crosses some of the second arcs of the other, replace a small region containg all intersections as shown in figure 2.



FIGURE 2. The replacement operation used in the definition of φ . The first arc is depicted in red, and the second arcs are depicted in black.

Example 4.12. Continuing our prior example of $\pi = \{1, 4, 6, 7, 8\}, \{2, 3, 9, 10\}, \{5, 11, 12, 13\}$, the resulting web is depicted below.



Proposition 4.13. The function φ is well defined, i.e. if $\pi \in WNC(n, d, 3)$, then $\varphi(\pi)$ is indeed in AW(n, d)

Proof. We need to check that the resulting graph does not have a cycle of length 4, the other properties are clear. A 4 cycle would necessarily have two edges coming from first arcs and two edge coming from second arcs, such that no new edges are created. Orienting all edges in the *m*-diagram away from v_B for each block *B*, a 4 cycle would require that the arcs intersect with opposite orientations at each corner. But the the two first arcs would have to have different orientations, and this is not possible.

Lemma 4.14. Let $\pi \in WNC(n, d, 3)$. The first arcs and maximal second arcs of the *m* diagram for π divide the half plane into a number of regions. There is a depth preserving correspondence between faces of $\varphi(\pi)$ and these regions.

Proof. Each replacement can be made so that edges coming from first arcs and maximal second arcs stay the same except in a ϵ radius region around each intersection. Each face of $\varphi(\pi)$ is thus contained (except for an ϵ -small portion) in a unique region. The trip starting at the first arc of each m will be an exceedance of $\varphi(\pi)$, which travels left to right along first arcs, maximal second arcs, and edges introduced by intersection replacement steps. These trips either cross at each intersection, using the new edge introduced at that

JESSE KIM

intersection twice, or turn at from each other at each intersection, using the new edge introduced at that intersection 0 times. Thus, the depth boundary paths consist exactly of those edges which come from first arcs and maximal second arcs, and thus the depth of each face matches the depth of the region it is contained in. $\hfill \Box$



FIGURE 3. An example of the correspondence between *m*-diagram depth and augmented web depth. Above, an *m*-diagram with non-maximal second arcs removed and regions shaded by depth. Below, the corresponding augmented web with first and maximal second arcs mostly preserved, and faces shaded by depth.

Theorem 4.15. The function $\varphi: WNC(n, d, 3) \to AW(n, d)$ is a bijection.

Proof. To show that φ is invertible, we introduce the following definition. The idea is that we will record extra information in the process of applying φ by way of coloring the edges. This extra information will allow us to invert the φ map. We will then show that the extra information was redundant, so φ is invertible.

Definition 4.16. Given an augmented web $W \in AW(n, d)$, a valid coloring of W is a coloring of the edges of W with three colors, red, blue, and black such that the following conditions are satisfied:

- Every interior vertex is incident to exactly one red edge, exactly one blue edge, and at least one black edge. Additionally, at each interior vertex, the incident red edge shares a face with the incident blue edge.
- (2) Right-to-left depth boundaries are colored red.
- (3) Left-to-right depth boundary edges incident to a boundary vertex are colored black.
- (4) No face has three consecutive edges colored red-blue-red.

Note that we do not require this coloring to be proper, a vertex may have multiple black edges incident.

Given a weakly 3-noncrossing set partition $\pi \in WNC(n, d, 3)$, we can create a valid coloring of $\varphi(\pi)$ by initially coloring first arcs red, second arcs black, and for each block b, the connection between v_b and b_2 blue. Then, at each replacement step, color the newly introduced edge blue and preserve all other colors. To see that the coloring obtained is a valid coloring, first note that initially and at each replacement step, the created vertices satisfy property 1 of a valid coloring. Property 2 of valid colorings holds by Lemma 4.14. To see that the third property of valid colorings holds, note that initially no such face exists and at no replacement step could such a face be created.

The interior blue edges contain the information of exactly which replacement steps have been performed, so given an augmented web $\varphi(\pi)$ and the valid coloring obtained through φ , we can recover π . Therefore, it suffices to show that every augmented web w admits exactly one valid coloring.

To do so, we will give an algorithm for finding a valid coloring and show that each step is forced. Let W be an augmented web.

- (1) Consider the set of all right-to-left depth boundaries. In order to satisfy condition 2 of a valid coloring, we must color all such faces red. By Lemma 4.7, every vertex now has exactly one red edge. Similarly, color all left-to-right depth boundary edges which are incident to a boundary vertex black.
- (2) Every interior vertex is now incident to a red edge, so all remaining edges must be blue or black. Edges which do not share a face with the red edge at each of their vertices must be black by condition 1 of valid colorings, so color all such edges black.
- (3) Consider the set of yet uncolored edges adjacent to two red edges on the same face. By condition 3 of valid colorings, these edges must be black, so color all such edges black.
- (4) Consider the set of yet uncolored edges. Every interior vertex is incident to at most two of these edges, so their union consists of a set of disjoint paths and cycles. We claim that their union is in fact a disjoint union of paths with exactly one interior endpoint and odd length paths with two interior endpoints. Given this claim, there is exactly one way to satisfy condition 1 of valid colorings by coloring some of these edges blue and the rest black, that is, by coloring edges of a path alternating blue and black, starting at an interior endpoint of the path. So, if the claim holds, we are done.

To see that the claim holds, suppose towards a contradiction there is a cycle C among the uncolored edges after step 3. Let D_k be the maximal k depth boundary path sharing a vertex with C. Then D_k shares exactly one edge with C, as the uncolored edges incident to v cannot be on the same side of the depth boundary path passing through v. But this is a contradiction as that edge would have been colored black at step 3. Now suppose there is an even length path whose endpoints are both interior vertices. The path must be as shown below, though possibly longer or with the colors of vertices and trips reversed.



For the red edges to have been colored red in step 1, the trips drawn in orange must be exceedances, and the trip drawn in blue must not be. The two rightmost blue trips cannot cross more than once as W is reduced, so the face between n and 1 must lie to the right of the second rightmost blue trip. By Lemma 4.8, the leftmost orange trip and the three leftmost blue trips cannot cross eachother, and since W is reduced, the leftmost orange trip cannot cross itself. Thus, the leftmost orange trip cannot be an exceedance, but this is a contradiction. Lastly, suppose there is a boundary to boundary path among the uncolored edges. Consider the two black vertices incident to the endpoints of this path. The red edges adjacent to them must lie on the same side of the path, so one of the edges along this path incident to the boundary must be a left-to-right depth boundary edge, and this is a contradiction.

Thus, every augmented web has exactly one valid coloring, and φ is a bijection.

5. SL_3 invariants for normal plabic graphs

In this section we introduce an invariant associated to each perfectly orientable normal plabic graph in such a way that the invariants associated to augmented webs form a basis of $S^{(d^3, 1^{n-3d})}$. To do so, we first need to give an orientation to each plabic graph, which will determine the sign of our invariant.

5.1. **Perfect orientations.** A key idea in defining our augmented web invariants is that of a *perfect orientation*. Perfect orientations were introduced by A. Postnikov in [14]. Our definition will be slightly different in that our sink vertices will be interior vertices rather than boundary vertices, and we also include the information of a total order on the sinks.

Definition 5.1. Let W be a normal plabic graph. A perfect orientation \mathcal{O} of W is a choice of two things. First, an orientation of each edge of W such that each boundary edge is oriented away from the boundary, each interior white vertex has exactly one ingoing edge, each interior black vertex has *at most* one outgoing edge. There will then be a set of d black vertices with no outward edges, we refer to these as the sinks of \mathcal{O} and denote them by $S_{\mathcal{O}}$. We also require that for every vertex v in W there is a directed path from v to a sink vertex. A perfect orientation also includes the information of a total order on the sinks, i.e. a bijection $f_{\mathcal{O}}: S_{\mathcal{O}} \to \{1, \ldots, d\}$. For each perfect orientation, we call the set of edges which are oriented away from black vertices the independent set of \mathcal{O} and denote it $I(\mathcal{O})$.

If there exists at least one perfect orientation of W, we say that W is perfectly orientable.

Remark 5.2. The set of perfectly orientable plabic under our definition differs slightly from the set of perfectly orientable plabic graphs under Postnikov's definition. For example, a white vertex connected by three edges to a single black vertex which is also connected to the boundary is perfectly orientable under Postnikov's definition but not ours. However, every plabic graph with an acyclic perfect orientation per Postnikov's definition will be perfectly orientable per our definition, and Postnikov, Speyer, and Williams show that all reduced plabic graphs have an acyclic perfect orientation [?, Lemma 3.2].

We use this modified definition in order to allow for perfect orientations to be obtained from eachother via a sequence of small changes.

Definition 5.3. Let W be an augmented web with perfect orientation \mathcal{O} . Let v be a white vertex of W. A *swivel move* is a change in orientation of exactly one ingoing edge and one outgoing edge at v which connect to distinct black vertices. We have necessarily removed one sink vertex and added one sink vertex, let the new sink be in the same position in the total order as the old sink. We call this a swivel move due to the fact that the set $I(\mathcal{O})$ is being rotated around this white vertex.

Proposition 5.4. Any two perfect orientations can be transformed into each other via a sequence of swivel moves and a reordering of the sink vertices.

Proof. Let \mathcal{O}_1 and \mathcal{O}_2 be two perfect orientations. Consider the symmetric difference of independent sets of the two perfect orientations, $I(\mathcal{O}_1)\Delta I(\mathcal{O}_2)$ it is necessarily a union of disjoint cycles and paths between sinks of \mathcal{O}_1 and sinks of \mathcal{O}_2 . We induct on the number of cycles present in $I(\mathcal{O}_1)\Delta I(\mathcal{O}_2)$. If there are no cycles, performing a swivel move at each white vertex along the paths from sinks of \mathcal{O}_1 and \mathcal{O}_2 using the edges of this path will transform \mathcal{O}_1 into a perfect orientation the same as \mathcal{O}_2 up to a reordering of its sinks. If there is a cycle in $I(\mathcal{O}_1)\Delta I(\mathcal{O}_2)$, find a walk in W which starts at a sink vertex of \mathcal{O}_1 , travels to a white vertex of the cycle, travels around the cycle, then returns to the starting sink via the same path such that every other edge of this walk is in $I(\mathcal{O}_1)$. Performing a swivel move at each white vertex of this walk using the edges along this walk will result in a perfect orientation \mathcal{O}_3 for which $I(\mathcal{O}_3)\Delta I(\mathcal{O}_2)$ has one fewer cycles.

We first check that this is a sensible definition of sign.

Proposition 5.5. Let $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ be three perfect orientations of a web $W \in AW(n, d)$. Then we have

$$\operatorname{sign}(\mathcal{O}_1, \mathcal{O}_3) = \operatorname{sign}(\mathcal{O}_1, \mathcal{O}_2) \operatorname{sign}(\mathcal{O}_2, \mathcal{O}_3)$$

Proof. By Proposition 5.4, it suffices to check this holds when \mathcal{O}_2 and \mathcal{O}_3 differ by a single swivel move. Note that

$$I(\mathcal{O}_1)\Delta I(\mathcal{O}_3) = (I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))\Delta (I(\mathcal{O}_2)\Delta I(\mathcal{O}_3)),$$

and $(I(\mathcal{O}_2)\Delta I(\mathcal{O}_3))$ is size two. We split into three cases.

- Case 1: $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ and $(I(\mathcal{O}_2)\Delta I(\mathcal{O}_3))$ do not intersect. Then $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ and $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ are the same except one path is two edges longer, and $\sigma(\mathcal{O}_1, \mathcal{O}_2) = \sigma(\mathcal{O}_1, \mathcal{O}_3)$. Thus, $\operatorname{sign}(\mathcal{O}_1, \mathcal{O}_3) = -\operatorname{sign}(\mathcal{O}_1, \mathcal{O}_2)$
- Case 2: $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ and $(I(\mathcal{O}_2)\Delta I(\mathcal{O}_3))$ intersect in a single edge. Then $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ and $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ are the same except for two paths. In this case, $\sigma(\mathcal{O}_1, \mathcal{O}_2)$ and $\sigma(\mathcal{O}_1, \mathcal{O}_3)$ differ by a single transposition and sign $(\mathcal{O}_1, \mathcal{O}_3) = -\text{sign}(\mathcal{O}_1, \mathcal{O}_2)$.

• Case 3: $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ and $(I(\mathcal{O}_2)\Delta I(\mathcal{O}_3))$ interesect in two edges. Then $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ and $(I(\mathcal{O}_1)\Delta I(\mathcal{O}_2))$ are the same except one path is two edges shorter, and $\sigma(\mathcal{O}_1, \mathcal{O}_2) = \sigma(\mathcal{O}_1, \mathcal{O}_3)$. Thus, $\operatorname{sign}(\mathcal{O}_1, \mathcal{O}_3) = -\operatorname{sign}(\mathcal{O}_1, \mathcal{O}_2)$

Proposition 5.5 allows for an alternative definition of relative sign, which we find more intuitive. The relative sign between two perfect orientations is given by the parity of the number of swivel moves and the sign of the permutation of the sinks in any sequence of swivel moves and a permutation of the sinks which transforms one orientation into the other. Proposition 5.5 guarantees this is well defined. In this sense, swivel moves can be thought of as playing a similar role to adjacent transpositions in determing the sign of a permutation.

5.2. Consistent Labellings. Our invariants will be defined in terms of consistent labellings, which we now define.

Definition 5.6. Let W be a perfectly orientable normal plabic graph. A consistent labelling ℓ of W is a choice of a possibly empty subset $\ell(e)$ of $\{1, \ldots, \nu\}$ for each edge e of W, such that the following hold:

- (1) At each interior white vertex, incident edge labels are disjoint and their union is $\{1, 2, 3\}$.
- (2) At each black vertex, incident edge labels are disjoint and their union contains $\{1, 2, 3\}$.
- (3) The label at each boundary edge has size 1, and for each $i \in \{4, ..., \nu\}$, $\{i\}$ appears exactly once among boundary labels.

The edges whose labels contain 1, 2, or 3 can be thought of as determining three dimer covers of W.

The boundary word of ℓ is the word given by reading off the labels at boundary edges in order, denoted

$$\mathrm{bd}(\ell) = \mathrm{bd}(\ell)_1 \cdots \mathrm{bd}(\ell)_n$$

The boundary monomial of ℓ is the monomial

$$\mathbf{x}_{\mathrm{bd}(\ell)} = x_{\mathrm{bd}(\ell)_1,1} \cdots x_{\mathrm{bd}(\ell)_n,n}$$

The weight of a consistent labelling is

$$\operatorname{wt}(\ell) = \left(-\frac{1}{2}\right)^{\text{\#edge labels of size two}}$$

To each consistent labelling we also associate a sign, made up of a number of factors. Firstly, for each $1 \leq i \leq 3$, let E_i denote the set of edges of W whose label contains i. Consider the symmetric difference $E_i \Delta \mathcal{O}$. This will be a union of disjoint cycles and disjoint paths from sinks of \mathcal{O} to boundary vertices whose incident edge is labelled i. For each boundary vertex b with label i, let the *origin* of b be the sink vertex it connects to, denoted origin(b). An origin inversion of w is a pair of boundary vertices $b_1 < b_2$ with origin(b_1) > origin(b_2). For each i, we get a contribution to the sign of the labelling of

(-1)#origin inversions between vertices labelled *i*+#cycles of length 2 modulo 4 in $E_i \Delta O$

We also have contributions to the sign of a consistent labelling coming from the number of edges of W with labels of size 2, the number of edges of $I(\mathcal{O})$ with an even size label, and inversions in the boundary word of ℓ . The sign of ℓ with respect to orientation \mathcal{O} is given by

$$\left(\prod_{i=1}^{3} (-1)^{\text{\#cycles of length 2 modulo 4 in } E_i \Delta \mathcal{O}}\right) (-1)^{\text{\#origin inversions} + \text{inv}(\text{bd}(\ell)) + \#\{e \in I(\mathcal{O}) | |\ell(e)| \text{ is even}\}}.$$

We can also think of the sign contribution coming from origin inversions in a different way. Let the *decorated boundary word* of ℓ , $\dot{bd}(\ell)$, be the boundary word of ℓ with a subscript for the origin attached to each letter $1 \leq i \leq 3$. We can consider the decorated boundary word to be a permutation under the order

$$1_1 < 1_2 < \dots < 1_d < 2_1 < \dots < 3_d < 4 < \dots < \nu$$

Then the sign of ℓ is

$$\left(\prod_{i=1}^{3} (-1)^{\#\text{cycles of length 2 modulo 4 in } E_i \Delta \mathcal{O}}\right) \operatorname{sign}(\tilde{\operatorname{bd}}(\ell))(-1)^{\#\{e \in I(\mathcal{O}) | |\ell(e)| \text{ is even}\}}.$$

times the sign of the decorated boundary word.

Remark 5.7. The definition of weight of a labelling is a bit mysterious to us. It is chosen to make the Skein relations upcoming in Section 6 hold, and we lack any further explanation beyond that. The definition of sign of a labelling is chosen so that a change in orientation introduces a consistent change in sign among all possible consistent labellings.

Example 5.8. Consider the augmented web and consistent labelling shown below. Edges in $I(\mathcal{O})$ are highlighted in yellow, and the three sinks are labelled with their position in the total order on sinks.



The boundary word of this labelling is 4332121312, with 28 inversions. The origins of the boundary vertices for each label are:

boundary vertices labelled 1:	5	7	9
origins:	2	1	3
boundary vertices labelled 2:	4	6	10
origins:	3	1	2
boundary vertices labelled 3:	2	3	8
origins:	2	3	1

The total number of origin inversions of ℓ is 5, one from label 1 and two each from labels 2 and 3. The decorated boundary word of ℓ is $43_23_32_31_22_11_13_11_32_2$, with 33 inversions. Our orientation is acyclic, so there are no cycles to consider, and two of the highlighted edges have an even size label. The sign of this labelling is therefore

$$(-1)^{33}(-1)^2 = -1$$

There are three edges with labels of size two, so the weight of this labelling is $\left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$.

Proposition 5.9. Let W be a perfectly orientable normal plabic graph with consistent labelling ℓ . Let \mathcal{O}_1 and \mathcal{O}_2 be two distinct perfect orientations for ℓ . Then the sign of ℓ with respect to these orientations is related by

$$\operatorname{sign}(\ell, \mathcal{O}_1) = \operatorname{sign}(\mathcal{O}_1, \mathcal{O}_2)\operatorname{sign}(\ell, \mathcal{O}_2)$$

Proof. It suffices to show that this holds when \mathcal{O}_1 and \mathcal{O}_2 differ by a swivel move at a white vertex v. Let u_1 and u_2 be the sinks which vary between \mathcal{O}_1 and \mathcal{O}_2 , and let u_3 denote the third neighbor of v. The origin of each boundary vertex is the same in \mathcal{O}_1 and \mathcal{O}_2 except for the the boundary vertices whose origin path travels along edge (v, u_3) , which have swapped origins in \mathcal{O}_1 and \mathcal{O}_2 . Either we have $|\ell(v, u_3)|$ is odd,

in which case $|\ell(v, u_2)|$ and $|\ell(v, u_1)|$ are the same parity, or $|\ell(v, u_3)|$ is even, in which case $|\ell(v, u_2)|$ and $|\ell(v, u_1)|$ have opposite parity. In either case, we have

$$\operatorname{sign}(\ell, \mathcal{O}_1) = -\operatorname{sign}(\ell, \mathcal{O}_2)$$

as desired.

We can now define our invariants for normal plabic graphs.

Definition 5.10. Let W be a perfectly orientable normal plabic graph with perfect orientation \mathcal{O} . Let CL(W) denote the set of all consistent labellings of W. Define an SL_3 invariant attached to W, denoted $[W, \mathcal{O}]$ by:

$$[W, \mathcal{O}] = \sum_{\ell \in CL(W)} \operatorname{sign}(\ell, \mathcal{O}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)}$$

Example 5.11. Consider the augmented web $W \in AW(7,2)$ and perfect orientation \mathcal{O} shown below.



There are 288 consistent labellings in total, but only 2 up to graph automorphism (not necessarily boundary preserving) and permutation of $\{1, 2, 3\}$, shown below:



The left labelling has combined sign and weight of $-\frac{1}{2}$, and the right labelling has combined sign and weight 1. Let $\operatorname{Aut}(W) \subset \mathfrak{S}_n$ denote the group of automorphisms of W identified with the corresponding permutation of boundary vertices, which has size $6 \cdot 2 \cdot 2 \cdot 2 = 48$. Then we have

$$[W, \mathcal{O}] = \sum_{\sigma \in \operatorname{Aut}(W)} \sum_{\omega \in A_3} \operatorname{sign}(\sigma) (-\frac{1}{2} x_{\omega(3),\sigma(1)} x_{\omega(2),\sigma(2)} x_{\omega(1),\sigma(3)} x_{\omega(3),\sigma(4)} x_{4,\sigma(5)} x_{\omega(1),\sigma(6)} x_{\omega(2),\sigma(7)} x_{\omega(1),\sigma(6)} x_{\omega(2),\sigma(7)} x_{\omega(1),\sigma(6)} x_{\omega(2),\sigma(7)} x_{\omega(1),\sigma(6)} x_{\omega(2),\sigma(7)} x_{\omega(1),\sigma(6)} x_{\omega(2),\sigma(7)} x_{\omega(1),\sigma(7)} x_{\omega(1),\sigma($$

 $+ x_{4,\sigma(1)} x_{\omega(1),\sigma(2)} x_{\omega(3),\sigma(3)} x_{\omega(2),\sigma(4)} x_{\omega(3),\sigma(5)} x_{\omega(1),\sigma(6)} x_{\omega(2),\sigma(7)} \big)$

where A_3 is the alternating group on $\{1, 2, 3\}$.

To verify that this is a sensible definition, we first check that changing the orientation only introduces a change of sign.

Proposition 5.12. Let W be a perfectly orientable normal plabic graph with perfect orientations \mathcal{O}_1 and \mathcal{O}_2 . Then

$$[W, \mathcal{O}_1] = \operatorname{sign}(\mathcal{O}_1, \mathcal{O}_2)[W, \mathcal{O}_2]$$

Proof. By Proposition 5.5, we have

$$[W, \mathcal{O}_1] = \sum_{\ell \in CL(W)} \operatorname{sign}(\ell, \mathcal{O}_1) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)}$$

= sign($\mathcal{O}_1, \mathcal{O}_2$) $\sum_{\ell \in CL(W)} \operatorname{sign}(\ell, \mathcal{O}_2) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)}$
= sign($\mathcal{O}_1, \mathcal{O}_2$) [W, \mathcal{O}_2]

We call these invariants because the resulting polynomials will be invariant under a certain action of SL_3 . Define an action of SL_3 which acts on the matrix of 3n variables

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ x_{3,1} & x_{3,2} & \cdots & x_{3,n} \end{bmatrix}$$

via left multiplication and leaves all other variables fixed. Then we have the following:

Proposition 5.13. Let $X \in SL_3$ and $W \in AW(n, d)$ with perfect orientation \mathcal{O} . Then

 $X \cdot [W, \mathcal{O}] = [W, \mathcal{O}]$

This is clear for normal plabic graphs without interior white vertices and with only vertices of degree at least 3, i.e. jellyfish invariants. We will defer the proof for all augmented webs until Section 6, which will show that augmented web invariants live in \mathfrak{S}_n closure of jellyfish invariants. The action of \mathfrak{S}_n commutes with the action of SL_3 , so the result will follow.

We next show that this definition generalizes jellyfish invariants.

Proposition 5.14. Let $W \in AW(n, d)$ have no white vertices, let \mathcal{O} be a perfect orientation of W, and let π be the corresponding ordered set partition. Then $[W, \mathcal{O}] = [\pi]_3$.

Proof. Let v_1, \ldots, v_d denote the interior vertices of W in order. We claim that a consistent labelling of W corresponds to a choice of jellyfish tableau for π as well as a choice of permutation for the elements of each block of π . Indeed, we can create a jellyfish tableau T_ℓ for π in the following manner. For each boundary vertex $1 \leq b \leq n$, if b has label i and is connected to interior vertex v_j , fill box i, j with the entry b. Then, for each interior vertex v_j , let $R_j(\ell)$ be the set of boundary vertices connected to v_j . Let $\sigma(v_j) \in \mathfrak{S}_n$ denote the permutation which reorders the elements of R_j to have increasing labels. We claim that

(1)
$$\operatorname{sign}(\ell)\mathbf{x}_{\operatorname{bd}(\ell)} = \operatorname{sign}(J(T_{\ell}))\prod_{j=1}^{d}\operatorname{sign}(\sigma(v_{j}))\left(\prod_{i\in R_{j}}x_{\operatorname{bd}(\ell)_{i},i}\right)$$

The right side here represents one term in the monomial expansion of the product of determinants defining $J(T_{\ell})$. From the definition of $\mathbf{X}_{\mathrm{bd}(\ell)}$ we see that the variables appearing on both sides of (1) agree, so the content of this claim is that the signs match. We show this in two parts. First, we claim that

(2) #origin inversions of
$$\ell =$$
#inversions within rows of $J(T_{\ell})$

Suppose (b_1, b_2) is an origin inversion of ℓ , with label *i*. Then b_1 appears in box $(i, \operatorname{origin}(b_1))$ and b_2 appears in box $(i, \operatorname{origin}(b_2))$. So (b_1, b_2) is also an origin inversion of T_{ℓ} . Next, we claim that

(3)
$$(-1)^{\operatorname{inv}(\operatorname{bd}(\ell))} = (-1)^{\#\operatorname{inversions between rows of } J(T_{\ell})} \prod_{j=1}^{d} \operatorname{sign}(\sigma(v_j))$$

or equivalently,

(4)
$$(-1)^{\operatorname{inv}(\sigma(v_1)\sigma(v_2)\cdots\sigma(v_d)\cdot\operatorname{bd}(\ell))} = (-1)^{\#\operatorname{inversions between rows of } J(T_\ell)}.$$

If the k^{th} letter of $(\sigma(v_1)\sigma(v_2)\cdots\sigma(v_d)\cdot bd(\ell))$ is *i*, then a *k* appears in the *i*th row of J(T), so (4) holds. We therefore have

$$[W, \mathcal{O}] = [\pi]_3$$

as desired.

Augmented web invariants satisfy the rotation and reflection invariance properties laid out in [6], as well as something slightly stronger.

Proposition 5.15. Let $W \in AW(n, d)$ with perfect orientation \mathcal{O} and let $\sigma \in \mathfrak{S}_n$. Let $\sigma \cdot W$ be the graph obtained by permuting the boundary vertices of W according to σ , i.e. if b is a boundary vertex and (v, b) is an edge of W, then $(v, \sigma(b))$ will be an edge of $\sigma \cdot W$. Suppose that $\sigma \cdot W$ is planar, so it is also in AW(n, d). Then, abusively letting \mathcal{O} also be a perfect orientation of $\sigma \cdot W$, we have

$$\sigma \cdot [W, \mathcal{O}] = \operatorname{sign}(\sigma)[\sigma \cdot W, \mathcal{O}]$$

Proof. For each consistent labelling ℓ of W, we get a corresponding consistent labelling $\sigma \circ \ell$ of $\sigma \circ W$. The decorated boundary word of $\sigma \cdot \ell$ is obtained by applying σ to the decorated boundary word of ℓ , so the result follows.

Corollary 5.16. Let c be the long cycle in \mathfrak{S}_n and let w_0 be the long element. Given an augmented web $W \in AW(n, d)$ with orientation \mathcal{O}_W , let $\operatorname{rot}(W)$ and $\operatorname{rot}(\mathcal{O}_W)$ be the web obtained by rotating W and \mathcal{O}_W clockwise by $\frac{2\pi}{n}$. Let $\operatorname{refl}(W)$ and $\operatorname{refl}(\mathcal{O}_W)$ be the web and orientation obtained by reflecting W and \mathcal{O}_w across the perpendicular bisector between boundary vertices 1 and n. We have

$$c \cdot [W, \mathcal{O}] = (-1)^{n-1} [\operatorname{rot}(W), \operatorname{rot}(\mathcal{O}_W)]$$

and

$$w_0 \cdot [W, \mathcal{O}] = (-1)^{n-1} [\operatorname{refl}(W), \operatorname{refl}(\mathcal{O}_W)]$$

Remark 5.17. In the definition of $\sigma \cdot W$, we are implicitly using the fact that if a web is planar and the positions of its boundary vertices are fixed, it has a unique planar embedding in the disk up to boundary-preserving homeomorphism. This is due to a classical theorem of Whitney [26].

Our main theorem regarding augmented web invariants is that they form a basis for the flamingo Specht module. We state it now but defer its proof until the end of the next section.

Theorem 5.18. Choose a perfect orientation \mathcal{O}_W for each augmented web $W \in AW(n, d)$. Then the set $\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\}$ is a basis for the flamingo Specht module $S^{(d^3, 1^{n-3d})}$.

6. Skein relations for augmented webs

This section will introduce skein relations for normal plabic graph invariants, showing that they satisfy property (5) of web bases. Furthermore, these skein relations will demonstrate that the span of augmented web invariants is an \mathfrak{S}_n invariant module containing $S^{(d^3,n-3d)}$. Along with our combinatorial bijection from standard Young tableaux, Proposition 4.15, we will thus obtain a proof of Theorem 5.18.

We first give a diagrammatic representation of these relations. In each image below, shaded gray areas represent an unknown number of edges connecting to other vertices of the graph, and in the perfect orientation, edges are assumed to be oriented towards black vertices unless shown otherwise.



Double Edge Reduction Rule



Bivalent Vertex Reduction Rule



Boundary-Adjacent Bivalent and Leaf Reduction Rules



Proposition 6.1 (Crossing reduction rule). Let W be a perfectly orientable normal plabic graph. Suppose W has two adjacent boundary vertices i, i+1 which connect to distinct interior vertices x and y respectively. Let W_I denote the web obtained from W by removing edges (x, i) and (y, i+1), and attaching an "I" shape, i.e. adding an interior black vertex u and an interior white vertex v, then adding in edges (x, v), (y, v)(v, u), (u, i) and (u, i + 1). We have the following relation:

(5)
$$s_i \cdot [W] = [W] - [W_I] - \frac{1}{2}[W_x] - \frac{1}{2}[W_y]$$

Proof. We divide consistent labellings for our webs into classes based on a fixed choice C of labels among edges in these webs other than those between x, y, u, v, i, and i + 1, then show that within each class, equation 5 holds. Up to symmetry, there are five possible cases, we will explain the first in detail and give a table for the rest.

Case 1: Among the fixed labels of edges incident to x, 2 and 3 are present. Among the fixed labels of edges incident to y, 1, 2, and 3 are present. The missing labels around the boundary are 1 and 4. Then there is exactly one way to label the remaining edges of W, edge (x, i) must have label 1 and edge (y, i + 1) must have label 4. Call this labelling ℓ . There are two ways to label W_I , both with weight $\frac{1}{2}$: edge (x, u) must have label 1, edge (y, u) must be unlabelled, edge (u, v) must have label $\{2, 3\}$, and edges (v, i) and (v, i + 1) must have labels 1 and 4 in either order. Call these labellings $\ell_{I,1}$ and $\ell_{I,2}$. There are two ways to label W_x . There are no ways to label W_y . Note that origin inversions do not change between these labellings, and the chosen orientation of each web is compatible with the specified labelling, so the relative sign is given only by the relative change in the boundary word. Thus, there exist a fixed monomial m such that

$$s_{i} \cdot \sum_{\substack{\ell \in CL(W)\\\ell \text{ extends } C}} \operatorname{sign}(\mathcal{O}, \mathcal{O}_{\ell}) \operatorname{sign}(\ell, \mathcal{O}_{\ell}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)} = (x_{4,i}x_{1,i+1})m$$

$$\sum_{\substack{\ell \in CL(W)\\\ell \text{ extends } C}} \operatorname{sign}(\mathcal{O}, \mathcal{O}_{\ell}) \operatorname{sign}(\ell, \mathcal{O}_{\ell}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)} = (x_{1,i}x_{4,i+1})m$$

$$\sum_{\substack{\ell \in CL(W_{I})\\\ell \text{ extends } C}} \operatorname{sign}(\mathcal{O}, \mathcal{O}_{\ell}) \operatorname{sign}(\ell, \mathcal{O}_{\ell}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)} = \frac{1}{2} (x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1})m$$

$$\sum_{\substack{\ell \in CL(W_{X})\\\ell \text{ extends } C}} \operatorname{sign}(\mathcal{O}, \mathcal{O}_{\ell}) \operatorname{sign}(\ell, \mathcal{O}_{\ell}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)} = (x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1})m$$

$$\sum_{\substack{\ell \in CL(W_{X})\\\ell \text{ extends } C}} \operatorname{sign}(\mathcal{O}, \mathcal{O}_{\ell}) \operatorname{sign}(\ell, \mathcal{O}_{\ell}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)} = 0$$

Therefore, since

$$(x_{4,i}x_{1,i+1}) = (x_{1,i}x_{4,i+1}) - \frac{1}{2}(x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1}) - \frac{1}{2}(x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1}) - 0$$

among classes of labellings which fit into this case, equation 5 holds.

The remaining cases are as follows. To read the following table, first note that to condense information, we have replaced the monomial $x_{a,i}x_{b,i+1}$ with the word ab. Then let C denote a fixed way to label the edges of W, W_I , W_x and W_y other than those between x, y, u, v, i, and i + 1. The fixed labels of C at x, y, and the missing labels of C around the boundary fit into one of the cases listed in the rows of this table, up to a permutation of $\{1, 2, 3\}$ and $\{x, y\}$. Then there is a fixed monomial m such that for each column headed by a web, if the entry in that row and column headed by $s_i \cdot W$ is a, then there exists a monomial m such that

$$\sum_{\substack{\ell \in CL(W)\\\ell \text{ extends } C}} \operatorname{sign}(\mathcal{O}, \mathcal{O}_{\ell}) \operatorname{sign}(\ell, \mathcal{O}_{\ell}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)} = am$$

The first row is Case 1.

Labels at x	Labels at y	Boundary	$s_i \cdot W$	W	W_I	W_x	W_y
$\{2,3\}$	$\{1, 2, 3\}$	$\{1, 4\}$	41	14	$\frac{1}{2}(14-41)$	14 - 41	0
$\{2,3\}$	$\{2,3\}$	$\{1, 1\}$	11	11	0	0	0
$\{2,3\}$	$\{1,3\}$	$\{1, 2\}$	21	12	12 - 21	0	0
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{4, 5\}$	54 - 45	45 - 54	45 - 54	45 - 54	45 - 54
{1}	$\{1, 2, 3\}$	$\{2,3\}$	0	0	$\frac{1}{2}(21-12)$	12 - 21	0

Proposition 6.2 (Square reduction rule). Let W be a perfectly orientable normal plabic graph. Suppose W has a face of degree 4. Let v_1 , v_2 denote the white vertices of this face, and let u_1 , u_2 , u_3 , u_4 denote the neighbors of v_1, v_2 , connected by edges $(v_1, u_1), (v_1, u_2), (v_1, u_4)$ and $(v_2, u_2), (v_2, u_3), (v_3, u_4)$. Let π be a noncrossing set partition of $\{1, 2, 3, 4\}$ in which no block contains both u_2 and u_4 . Let W_{π} be the web obtained from W by first deleting u_1 or u_2 if they connect to two vertices v_i, v_j whose indices lie in the same block of π , then identifying all vertices among v_1, \ldots, v_4 whose indices are in the same block in π . We will write these set partitions without brackets and with vertical bars between blocks, e.g. $W_{1|23|4}$. Then we have the following relation:

(6)
$$\sum_{\pi} \left(-\frac{1}{2}\right)^{4-\# \text{ blocks of } \pi} [W_{\pi}] = 0$$

JESSE KIM

Proof. The proof is similar to that of the crossing rule, but there are many more cases. Again, we explain one case in detail and give a table for the rest.

Let C be a fixed way to label edges of the W_{π} other than those incident to v_1 and v_2 . Then the boundary monomial of any consistent labelling extending C is fixed, so we need to check that the coefficients satisfy equation 6. We proceed casewise, based on the fixed labels of C present at each of the four black vertices u_1, u_2, u_3 , and u_4 .

Case 1: The fixed labels at v_1 contain 1, 2, and 3. The fixed labels at v_2 contain 1 and 2 but not 3. The fixed labels at v_3 contain 3 but not 1 or 2. The fixed labels at v_4 do not contain 1, 2, or 3.

There are two consistent labellings of $W_{1|2|3|4}$ which extend C, as shown below.



Relative to eachother, the left labelling has weight and sign $-\frac{1}{2}$, and the right labelling has weight and sign $-\frac{1}{4}$.

There is one consistent labelling of each of $W_{1|23|4}$, $W_{1|2|34}$, $W_{14|2|3}$, and $W_{14|23}$ which extends C, and none for the remaining W_{π} .



Relative to our earlier labellings, these carry weights and sign -1, $-\frac{1}{2}$, $\frac{1}{2}$ and 1, respectively. We check that these satisfy equation 6:

$$0 = \left(-\frac{1}{2} - \frac{1}{4}\right) + \left(-\frac{1}{2}\right)\left(-1 + \frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{4}\right)(1)$$

The remaining cases our included in the following table, case 1 appears in row 4. Cases which can be obtained by a permutation of $\{1, 2, 3\}$ or a symmetry of the square are not included.

Labels at u_1	u_2	u_3	u_4	$W_{1 2 3 4}$	$W_{1 23 4}$	$W_{1 2 34}$	$W_{12 3 4}$	$W_{14 2 3}$	$W_{14 23}$	$W_{12 34}$	$W_{123 4}$	$W_{134 2}$
$\{1, 2, 3\}$	$\{1, 2, 3\}$	Ø	Ø	-1	-1	-1	0	1	1	0	0	1
$\{1, 2, 3\}$	$\{1, 2\}$	$\{3\}$	Ø	$-\frac{3}{4}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	0
$\{1, 2, 3\}$	$\{1, 2\}$	Ø	{3}	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	0
$\{1, 2, 3\}$	{1}	$\{2, 3\}$	Ø	$-\frac{3}{4}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	0
$\{1, 2, 3\}$	{1}	{2}	{3}	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0
$\{1, 2\}$	$\{1, 2, 3\}$	$\{3\}$	Ø	$-\frac{1}{4}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1
$\{1, 2\}$	$\{1, 2, 3\}$	Ø	{3}	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	1	0	0	1
$\{1, 2\}$	$\{1, 2\}$	{3}	{3}	$-\frac{1}{4}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	0	0
$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	Ø	0	0	-1	0	1	0	0	0	0
$\{1, 2\}$	$\{1, 3\}$	$\{2\}$	{3}	$-\frac{3}{4}$	$-\frac{1}{2}$	-1	0	$\frac{1}{2}$	1	0	0	0
$\{1, 2\}$	$\{1, 3\}$	$\{3\}$	{2}	$\frac{1}{2}$	0	1	0	0	0	0	0	0
$\{1, 2\}$	{1}	$\{2, 3\}$	{3}	$-\frac{3}{4}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	0	0	0
$\{1, 2\}$	{3}	$\{1, 2\}$	{3}	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
$\{1, 2\}$	{3}	$\{1, 2\}$	{3}	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
{1}	$\{1, 2, 3\}$	$\{1\}$	$\{2, 3\}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
{1}	$\{1, 2, 3\}$	{2}	{3}	$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1
{1}	$\{1, 2, 3\}$	Ø	$\{2, 3\}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	0	0	1
{1}	$\{1, 2\}$	{3}	$\{2, 3\}$	$\frac{1}{4}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	0	0
{1}	$\{2, 3\}$	$\{1\}$	$\{2, 3\}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
Ø	$\{1, 2, 3\}$	Ø	$\{1, 2, 3\}$	1	1	1	1	1	1	1	1	1

Proposition 6.3 (Double edge reduction rule). Let W be a perfectly orientable normal plabic graph with a white vertex v with exactly two neighbors u_1 , connected by 1 edge, and u_2 , connected by 2 edges. Let \mathcal{O} be a perfect orientation such that $I(\mathcal{O})$ contains (v, u_1) . Let W' and \mathcal{O}' be the plabic graph and orientation obtained by contracting v, u_1 , and u_2 . Then

$$[W, \mathcal{O}] = [W', \mathcal{O}']$$

Proof. Fix a labelling of edges other than those incident to v. We have 4 cases up to a permutation of $\{1, 2, 3\}$

- Case 1: Among the fixed labels at u_1 , $\{1, 2, 3\}$ appear. Then the union of the labels of the two edges between v and u_1 is $\{1, 2, 3\}$. There are two ways to label these edges with one of size 3, and these come with each with relative sign and weight -1. There are 6 ways to label the edges with one label of size 2 and the other of size 1, each with relative sign and weight *frac*12, for a total weight and sign of 1.
- Case 2: Among the fixed labels at u_2 , $\{1, 2\}$ appear. Then (v, u_1) has label 3 and the edges between v and u_2 have labels 1 and 2 split between them. There are two ways to have a label of size 2, these appear with relative sign and weight $-\frac{1}{2}$. There are two ways to have two labels of size 1, these appear with relative sign and weight 1, for a total weight and sign of 1.
- Case 3: Among the fixed labels at u_1 , only {1} appears. Then (v, u_1) has label {2,3}, and one of the edges to u_2 has label {1}. There are two ways to do this, each with relative sign and weight 1.
- Case 4: The fixed labels at u_1 are all empty. Then there is only one way to label the edges incident to v, with weight one.

In all cases W' has no choices to be made, and thus has relative sign and weight 1, so the result holds. \Box

Proposition 6.4 (Leaf vertex removal). Let W be a perfectly orientable plabic graph with a black vertex u of degree 1 connected to white vertex v. Let \mathcal{O} be a perfect orientation with the edge (u, v) oriented towards u. Let W' be the plabic graph obtained by removing vertices u and v and \mathcal{O}' be the resulting perfect orientation. Then $[W, \mathcal{O}] = [W', \mathcal{O}']$.

Proof. If we take a consistent labelling of W, and remove vertices u and v, we get a consistent labelling of W' with the same sign and weight, so the result follows.

Proposition 6.5 (Boundary adjacent leaf removal). Let W be a perfectly orientable normal plabic graph with a black vertex of degree one or two only connected to the boundary. Then [W] = 0.

Proof. There are no consistent labellings of W, so the result follows.

Proposition 6.6 (Boundary adjacent bivalent vertex removal). Let W be a perfectly orientable normal plabic graph with a degree 2 black vertex u connected to one boundary vertex and one white vertex v Let the other neighbors of v be x and y. Let \mathcal{O} be a perfect orientation of W in which (u, v) is oriented towards

v. Let W_x be the graph obtained by removing u and v and connecting x to the boundary, and let \mathcal{O}_x be the orientation obtained from \mathcal{O}_x in the same fashion. Let W_y and \mathcal{O}_y be analogous. Then we have

$$[W, \mathcal{O}] = \frac{1}{2}[W_x, \mathcal{O}_x] + \frac{1}{2}[W_y, \mathcal{O}_y]$$

Proof. A consistent labelling of W must have an edge label of size one on the boundary edge incident to u, and edge label of size 2 on the edge (u, v), and an edge label of size one on exactly one of the edges (u, x) and (u, y). Thus, consistent labellings of W are in bijection with the disjoint union of consistent labellings of W_x and W_y , and each carries relative sign and weight $\frac{1}{2}$. Thus

$$[W, \mathcal{O}] = \frac{1}{2}[W_x, \mathcal{O}_x] + \frac{1}{2}[W_y, \mathcal{O}_y]$$

as desired.

Proposition 6.7 (Bivalent vertex reduction rule). Let W be a normal plabic graph with a black vertex u of degree 2 connected to two white vertices v_1 and v_2 . Let u_1, u_2, u_3, u_4 be the other neighbors of v_1 and v_2 . Let \mathcal{O} be a perfect orientation such that (u_1, v_1) and (u, v_2) are in $I(\mathcal{O})$. For $1 \le i \le 8$, let W_i denote the plabic graphs with orientation \mathcal{O}_i as shown in the bivalent vertex reduction rule above. We have

$$[W, \mathcal{O}] = -\frac{1}{2}[W_1, \mathcal{O}_1] - \frac{1}{2}[W_2, \mathcal{O}_2] - \frac{1}{2}[W_3, \mathcal{O}_3] + \frac{1}{2}[W_4, \mathcal{O}_4] - \frac{1}{4}[W_5, \mathcal{O}_5] - \frac{1}{4}[W_6, \mathcal{O}_6] - \frac{1}{4}[W_7, \mathcal{O}_7] - \frac{1}{4}[W_8, \mathcal{O}_8] - \frac{1}{4}$$

Proof. As per the proof of the square rule, we have four cases to consider as shown in the following table

Labels at u_1	u_2	u_3	u_4	W	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	Ø	1	0	-1	-1	1	0	1	1	0
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2\}$	${3}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
$\{1, 2, 3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2,3\}$	$-\frac{1}{2}$	0	0	0	-1	0	0	0	0
$\{1, 2, 3\}$	$\{1, 2\}$	$\{1, 2, 3\}$	{3}	$-\frac{1}{4}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	-1	0

We can now give a proof of Theorem 5.18, which states the set $\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\}$ is a basis of $S^{(d^3 1^{n-3d})}$.

Proof. Let $W \in AW(n, d)$. Apply the crossing reduction rule to rewrite $s_i \cdot [W, \mathcal{O}_w]$ as a sum of invariants for normal plabic graphs G_i with coefficients c_i , i.e.

$$s_i \cdot [W, \mathcal{O}_w] = \sum_i c_i [G_i, \mathcal{O}_{G_i}]$$

If any of the G_i are not augmented webs, i.e. they have a face of degree four or a black vertex of degree less than 3, we can apply one of the other skein relations to rewrite $[G_i, \mathcal{O}_{G_i}]$. Each time we do so, we replace a plabic graph with k white vertices by a linear combination of plabic graphs with strictly fewer than k white vertices, so by iterating this process we eventually rewrite $s_i \cdot [W, \mathcal{O}_w]$ as a linear combination of augmented web invariants. Consequently,

$$\operatorname{span}(\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\})$$

is closed under the action of \mathfrak{S}_n . By Theorem 2.2 and Proposition 5.14, $\operatorname{span}(\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\})$ has a nonzero intersection with $S^{(d^3, 1^{n-3d})}$. By Theorem 4.15, the dimension of $\operatorname{span}(\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\})$ is at most the dimension of $S^{(d^3, 1^{n-3d})}$. Since $S^{(d^3, 1^{n-3d})}$ is irreducible, we have

$$\operatorname{span}(\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\}) = S^{(d^3, 1^{n-3d})}$$

and Theorem 4.15 shows that $\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\}$ is indeed a basis.

7. Augmented web invariants via weblike subgraphs

In this section we explain how to interpret our augmented web invariants in terms of the weblike subgraphs introduced by T. Lam in [11]. To do so, we first need to reinterpret out augmented web invariants as tensors rather than polynomials in $n \times \nu$ variables.

Let $V_0 \cong \mathbb{C}^{\nu}$ be a ν dimensional vector space with basis $\{e_1, \ldots, e_{\nu}\}$, let $V = \text{span}(\{e_1, e_2, e_3\})$. Consider the space W consisting of the direct sum of all tensor products of 3d copies of V and n - 3d copies of \mathbb{C} , e.g. when n = 4 and d = 1, W is

$$(V \otimes V \otimes V \otimes \mathbb{C}) \oplus (V \otimes V \otimes \mathbb{C} \otimes V) \oplus (V \otimes \mathbb{C} \otimes V \otimes V) \oplus (\mathbb{C} \otimes V \otimes V \otimes V)$$

W injects into $V_0^{\otimes n}$ by replacing the n - 3d copies of \mathbb{C} with $\bigwedge^{n-3d} \operatorname{span}(\{e_4, \ldots, e_{\nu}\})$, e.g. if $v_1, v_2, v_3 \in V$,

$$1 \otimes v_1 \otimes v_2 \otimes 1 \otimes v_3 \mapsto e_4 \otimes v_1 \otimes v_2 \otimes e_5 \otimes v_3 - e_5 \otimes v_1 \otimes v_2 \otimes e_4 \otimes v_3$$

There is also a natural injection of $V_0^{\otimes n}$ into the polynomial ring generated by the $n \times \nu$ variables

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{\nu,1} & x_{\nu,2} & \cdots & x_{\nu,n} \end{bmatrix}$$

given by

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \mapsto x_{i_1,1} x_{i_2,2} \cdots x_{i_n,r}$$

Recall that our augmented web invariants live in this polynomials ring, and furthermore, they live in the span of monomials which contain exactly one variable from each column, each with degree one. Additionally, in each augmented web invariant, the tensor factors corresponding to basis vectors e_4, \ldots, e_{ν} are alternating. Thus, augmented web invariants live in the image of the injection $\iota : W \hookrightarrow \mathbb{C}[x_{1,1}, \ldots, x_{\nu,n}]$. Denote the preimage under this injection of $[W, \mathcal{O}]$ by $[W, \mathcal{O}]$.

We can make ι into an \mathfrak{S}_n homomorphism by pulling back the action of \mathfrak{S}_n on $\mathbb{C}[x_{1,1}, \ldots, x_{\nu,n}]$ to W. Note that this pullback is *not* just simply permuting tensor factors. An adjacent transposition s_i acts on simple basis tensors by

$$s_i \cdot (v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n) = \begin{cases} -s_i \cdot (v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n) & v_1, v_{i+1} \in \mathbb{C} \\ s_i \cdot (v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n) & \text{otherwise} \end{cases},$$

i.e. it picks up a sign if both tensor factors come from \mathbb{C} .

The benefit of this viewpoint is that W is more well-studied in terms of webs. Kuperberg's work [10] gives a basis for W in terms of SL_3 webs with 0 clasps, i.e. SL_3 webs with n-3d boundary vertices without edges. We will explain how to expand $[W, \mathcal{O}]$ into this clasped web basis.

As introduced by Lam, given a normal plabic graph W a 3-weblike subgraph is an assignment of a nonnegative integer to each edge such that the sum of edges around each interior vertex is 3. A weblike subgraph can be turned into an SL_3 web with 0-clasps (i.e. unused boundary vertices) by deleting each edge assigned 0 or 3, and contracting each path of edges alternately assigned 1's and 2's to a single edge. A consistent labelling ℓ gives rise to a weblike subgraph $W'(\ell)$ via the sizes of its edge labels. We have the following

Proposition 7.1. Let W be a normal plabic graph with perfect orientation \mathcal{O} . Let W' be a 3-weblike subgraph of W with d(W') edges of multiplicity 2. Then

$$\iota^{-1}\left(\sum_{\substack{\ell \in CL(G)\\W'(\ell)=W'}} \operatorname{sign}(\ell, \mathcal{O}) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)}\right) = \pm \left(-\frac{1}{2}\right)^{d(W')} [W']_{SL_3}$$

where $[W']_{SL_3}$ denotes the usual SL_3 web invariant.

Proof. Up to a reordering of labels larger than 3, consistent labellings ℓ with $W'(\ell) = W'$ are in bijection with proper edge labellings of W'. So it suffices to check that the difference in the definition of sign for consistent labellings and proper edge labellings is the same among all such labellings. Any two proper edge labellings of W' can be transformed into eachother via swapping the labels of any path alternately labelled

 i, j, \ldots, i, j for $1 \le i \le j$. Swapping such a path will introduce a sign change both in the definition of sign for proper edge labellings and for consistent labellings.

Corollary 7.2. We thus have

$$\widetilde{[W,\mathcal{O}]} = \sum_{\text{weblike subgraphs } W' \text{ of } W} \pm (\frac{1}{2})^{d(W')} [W']_{SL_3}$$

Remark 7.3. Note that when a normal plabic graph is an SL_3 web, i.e. all vertices are degree 3, our invariants do not match the usual SL_3 invariants. Instead, we get a sum over all SL_3 webs which appear as a weblike subgraph in W.

8. Cyclic sieving

The basis $\{[W, \mathcal{O}_W] \mid W \in AW(n, d)\}$ given in Theorem 3.9 has all the necessary properties to obtain a cyclic sieving result via Springer's theorem of regular elements. The only detail left is that we need to be careful about the orientations. To address orientation, we need the following Lemma:

Lemma 8.1. Let $W \in AW(n,d)$ with perfect orientation \mathcal{O} . Suppose W is fixed by rotation by $i \ge 2$, i.e. $\operatorname{rot}^{i}(W) = W$. Then i divides n and exactly one of the following holds:

•
$$\frac{n}{i} \mid d$$

• $\frac{n}{i} \mid d-1$
• $\frac{n}{i} = 3$ and $\frac{n}{i} \mid d+1$

Let $k = \frac{di}{n}, \frac{(d-1)i}{n}, \frac{(d+1)i}{n}$ depending on which of the cases above holds. Then we have

$$\operatorname{sign}(\mathcal{O}, \operatorname{rot}^{i}(\mathcal{O})) = (-1)^{(\frac{n}{i}-1)i}$$

The relevance here is that this sign depends only on n, d, and i, not on W itself.

Proof. Note that by Lemma 5.5 it suffices to prove this for some orientation \mathcal{O} . We proceed by induction on the number of interior white vertices. If there are no interior white vertices, then a perfect orientation is simply a total order on the black vertices. Rotation by *i* induces a permutation on the black vertices with at most one cycle of size 1 and cycles of size $\frac{n}{i}$, and thus $\frac{n}{i} \mid d$ or $\frac{n}{i} \mid d - 1$. Thus,

$$\operatorname{sign}(\mathcal{O}, \operatorname{rot}^{i}(\mathcal{O})) = (-1)^{(\frac{n}{i}-1)k}$$

If there is a single white vertex v, it is necessarily fixed by rotation by i, and thus $\frac{n}{i} = 3$. Rotation by i induces a permutation of the d + 1 black vertices into cycles of size 3, and thus $3 \mid d + 1$. The orientation \mathcal{O}_W differs from $\operatorname{rot}^i(W)$ via a swivel move at v, a transposition of the two sink vertices adjacent to v, and a 3-cycle applied to each other orbit of sinks. Thus,

$$\operatorname{sign}(\mathcal{O}, \operatorname{rot}^{i}(\mathcal{O})) = 1 = (-1)^{(\frac{n}{i}-1)k}$$

If there is more than one white vertex, then by Lemma 4.5 and our rotation invariance assumption, we can find three black vertices each connected to exactly one interior white vertex such that these three white vertices are distinct. Remove these 6 vertices and connect their neighbors to the boundary in a planar and rotationally invariant way to get a web W'. From any perfect orientation \mathcal{O}' of W' we can build a perfect orientation \mathcal{O} of W by orienting the removed edges from white vertex to black vertex. We then have by inductive hypothesis

$$\operatorname{sign}(\mathcal{O}, \operatorname{rot}^{i}(\mathcal{O})) = \operatorname{sign}(\mathcal{O}', \operatorname{rot}^{i}(\mathcal{O}')) = (-1)^{(\frac{n}{i} - 1)k}.$$

So the result follows by induction.

We can now state our cyclic sieving result.

Theorem 8.2. Let $C = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with generator c acting on AW(n, d) by rotation. Let $X_{n,d}(q)$ be the fake degree polynomial for $S^{(d^3, 1^{n-3d})}$, i.e.

$$X_{n,d}(q) = q^{3(d-1) + \binom{n-3(d-1)}{2}} \frac{[n]!_q}{\prod_{(i,j) \in \lambda} [h_{ij}]_q}$$

If n is odd, then the triple

$$(AW(n,d), C, X_{n,d}(q))$$

exhibits the cyclic sieving phenomenon.

If n is even, then we have cyclic sieving up to sign, i.e.

$$|AW(n,d)^{c^i}| = |X_{n,d}(\zeta^i)|$$

where $AW(n,d)^{c^{i}}$ denotes the fixed point set of AW(n,d) under the action of c^{i} , and ζ is a primitive n^{th} root of unity.

Proof. If n is odd, then we can choose orientations \mathcal{O}_W for each web $W \in AW(n, d)$ such that

$$c \cdot [W, \mathcal{O}_W] = [\operatorname{rot}(W), \mathcal{O}_{\operatorname{rot}(W)}]$$

To do so, select a web W from each C-orbit and pick any orientation \mathcal{O}_W for it. For $1 \leq i \leq n$, let $\mathcal{O}_{\mathrm{rot}^i(W)} = \mathrm{rot}^i(\mathcal{O}_W)$. Lemma 8.1 guarantees that this is possible even if W has rotational symmetry, as $(-1)^{(\frac{n}{i}-1)k} = 1.$

Thus, $S^{(d^3,1^{n-3d})}$, $\{[W,\mathcal{O}_W] \mid W \in AW(n,d) \text{ and the rotation action of } C \text{ satisfy the hypotheses of } V \in AW(n,d) \text{ and the rotation action of } C \text{ satisfy the hypotheses of } V \in AW(n,d) \text{ and the rotation action of } C \text{ satisfy the hypotheses of } V \in AW(n,d) \text{ and the rotation action of } C \text{ satisfy the hypotheses of } V \in AW(n,d) \text{ and the rotation action of } C \text{ satisfy the hypotheses of } V \in AW(n,d) \text{ and the rotation action of } C \text{ satisfy the hypotheses of } V \in AW(n,d) \text{ satisfy the hypotheses } V \in AW(n,d) \text{ and the rotation action of } C \text{ satisfy the hypotheses } V \in AW(n,d) \text{ satisfy here } V \in AW(n,d) \text{ s$ Theorem 2.8 and the result follows.

If n is even, choose any orientation for each web $W \in AW(n,d)$. Then $\{[W, \mathcal{O}_W] \mid W \in AW(n,d) \text{ is not} \}$ necessarily fixed by the action of c, but c will act via a signed permutation matrix. Lemma 8.1 shows that the diagonal of c^i will either contain only 0's and 1's or only 0's and -1's. In either case, $|tr(c^i)| = |AW(n, d)c^i|$, and the proof of Theorem 2.8 [21] shows that

$$|AW(n,d)^{c^i}| = |X_{n,d}(\zeta^i)|$$

holds as desired.

Example 8.3. When n = 10, d = 3, then $X_{10,3}(q)$ has a rather nice form. The hook lengths are

and thus $X_{10,3}(q) = q^{12} \begin{bmatrix} 10 \\ 4 \end{bmatrix}_q$. Since $X_{10,3}(q)$ is a single q-binomial (n = 10, d = 3 is the only case for which this is true), we can verify the cyclic sieving result in this case by checking that the orbits of AW(10,3)under rotation are in size-preserving bijection with the orbits of size 4 subsets of $\{1, \ldots, 10\}$ under cyclic permutation. There are two orbits of size 5 for each set, the orbits containing sequences $\{1, 2, 6, 7\}$ and $\{1, 3, 6, 8\}$ and the orbits containing webs shown below



The remaining 20 orbits are all size 10, one web from each is shown below





9. FUTURE DIRECTIONS

We have given a rotationally invariant basis for $S^{(d^3,1^{n-3d})}$, and a natural question to ask is whether there is a way to generalize these results to r > 3. Many of the results in Section 5 as well as the crossing reduction rule in Section 6 readily generalize when r is odd (when r is even, a different treatment of signs is needed, as a change in orientation will not give a global change in sign). We can thus obtain spanning set for an \mathfrak{S}_n invariant submodule containing $S^{d^r,1^{n-rd}}$ and the question remains as to how to prune it down to a basis, as the results of Section 4 do not seem to readily generalize. There are two directions in which one might approach this problem. The first is to begin by looking for a rotationally invariant set of the right size via WNC(n,d,r):

Problem 9.1. Is there a combinatorially nice injection of WNC(n, d, r) for r > 3 into the set of normal plabic graphs, such that the image is closed under rotation?

The second approach is to determine skein relations first, and use those to reduce the set of normal plabic graphs to a basis.

Problem 9.2. Extend the definitions from Section 5 to r > 3 for r odd. What are the corresponding skein relations?

We expect these questions to likely be quite difficult, as answering both would encompass constructing a rotationally invariant basis of S^{d^r} , a question which was only recently answered in the case r = 4 by C. Gaetz, O Pechenik, S. Pfannerer, J. Striker, and J. Swanson [7] and remains open for r > 4. However, most investigation into this question has been concerned with finding a *subset* of SL_n webs which forms a basis. Towards this end, it is perhaps a feature, rather than a bug of our construction that it does not consist of genuine SL_r webs, but rather linear combinations of ones with the same underlying simple graph and its minors, as it gives a new place to search.

If we do consider the difference between our augmented web invariants and SL_3 web invariants to be something to be fixed, we can do so by constructing a poset on classical SL_3 webs with $W \leq V$ whenever W is a weblike subgraph of V, which is equivalent to the graph minor poset restricted to SL_3 webs. By Corollary ?? can thus write

$$[W] = \sum_{V \le W} h(V, W)[V]_{SL_3}$$

where h is an element of the incidence algebra for this poset which is defined up to sign in Corollary ??, and the sign is defined implicitly in the preceding exposition. Inverting h would then recover the classical SL_3 web invariants. We thus propose the following:

Problem 9.3. Extend the definition of h to the poset of perfectly orientable normal plabic graphs with order given by graph minors. Is there a simple combinatorial description of the inverse of h in the incidence algebra?

One important property of the *m*-diagram construction of SL_3 webs is that, as shown by Petersen, Pylyavskyy, and Rhoades, it intertwines promotion on rectangular tableaux and rotation of webs [13], thereby giving an algebraic proof of the cyclic sieving phenomenon for promotion on three-row rectangles. This is not the case for n > 3d, however, as promotion for tableaux of shape $(d^3, 1^{n-3d})$ for n > 3d is not so well-behaved and the order of promotion does not divide n in general. It may be interesting to investigate if there is a variant of promotion which our bijections do intertwine.

Problem 9.4. Give a combinatorial description similar to promotion of the cyclic action on standard Young tableaux of shape $(d^3, 1^{n-3d})$ given by the pullback of rotation on webs. Does the combinatorial description have a natural extension to other shapes? If so, which shapes have order dividing n?

A combinatorially defined cyclic action with order dividing n for another family of partition shapes would be good evidence for the existence of a web basis for those shapes. It is not clear that the bijection we give is necessarily the most natural, so in answering this question one may want to consider other possible bijections between standard Young tableaux and augmented webs.

10. Acknowledgements

We thank Brendon Rhoades, Oliver Pechenik, and Pasha Pylyavksyy for helpful comments and conversations.

References

- 1. Sabin Cautis, Joel Kamnitzer, and Scott Morrison, Webs and quantum skew howe duality, Mathematische Annalen 360 (2012).
- Sergey Fomin and Pavlo Pylyavskyy, Tensor diagrams and cluster algebras, Advances in Mathematics 300 (2016), 717–787, Special volume honoring Andrei Zelevinsky.
- 3. Sergey Fomin, Lauren Williams, and Andrei Zelevinsky, Intro to cluster algebras. chapter 7, (2021).
- 4. Bruce Fontaine, Generating basis webs for sl_n , Advances in Mathematics **229** (2012), no. 5, 2792–2817.
- 5. Chris Fraser, Thomas Lam, and Ian Le, From dimers to webs, Transactions of the AMS (2017).
- 6. Chris Fraser, Rebecca Patrias, Oliver Pechenik, and Jessica Striker, Web invariants for flamino specht modules, preprint (2023).
- 7. Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Joshua Swanson, Rotation invariant web bases from hourglass plabic graphs, preprint (2023).
- 8. Jesse Kim, An embedding of the skein action on set partitions into the skein action on matchings, Electronic journal of Combinatorics (2024).
- Jesse Kim and Brendon Rhoades, Set partitions, fermions, and skein relations, International Mathematics Research Notices 2023 (2022), no. 11, 9427–9480.
- 10. Greg Kuperberg, Spiders for rank 2 lie algebras, Communications in Mathematical Physics 180 (1996), 109–151.
- 11. Thomas Lam, Dimers, webs, and positroids, Journal of the London Mathematical Society 92 (2014).
- Rebecca Patrias, Oliver Pechenik, and Jessica Striker, A web basis of invariant polynomials from noncrossing partitions, Advances in Mathematics 408 (2022), 108603.
- Kyle Petersen, Pavlo Pylyavskyy, and Brendon Rhoades, Promotion and cyclic sieving via webs, Journal of Algebraic Combinatorics 30 (2008).
- 14. Postnikov, Total positivity, grassmanians, and networks, (2006).
- 15. Pavlo Pylyavskyy, Non-crossing tableaux, Annals of Combinatorics 13 (2009), 323–339.
- V. Reiner, D. Stanton, and D. White, *The cyclic sieving phenomenon*, Journal of Combinatorial Theory, Series A 108 (2004), no. 1, 17–50.
- Brendon Rhoades, Cyclic sieving, promotion, and representation theory, Journal of Combinatorial Theory, Series A 117 (2010), no. 1, 38–76.
- 18. Brendon Rhoades, A skein action of the symmetric group on noncrossing partitions, Journal of Algebraic Combinatorics **45** (2017).
- Heather Russell, An explicit bijection between semistandard tableaux and non-elliptic sl₃ webs, Journal of Algebraic Combinatorics 38 (2013), 851–862.
- 20. Bruce Sagan, The symmetric group, Springer, New York, 2001.
- 21. Bruce Sagan, The cyclic sieving phenomenon: A survey, (2010).
- 22. T.A. Springer, Regular elements of finite reflection groups, Inventiones mathematicae 25 (1974).
- 23. H. N. V. Temperley and E. H. Lieb, Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the 'percolation' problem, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **322** (1971), no. 1549, 251–280.
- Julianna Tymoczko, A simple bijection between standard 3 × n tableaux and irreducible webs for sl₃, Journal of Algebraic Combinatorics 35 (2012), 611–632.
- 25. H. Weyl, G. Rumer, and E. Teller, Eine f[']ur die valenztheorie geeignete basis der bin[']aren vektorinvarianten, Nachrichten von der Gesellschaft der Wissenschaften zu G[']ottingen, Mathematisch-Physikalische Klasse **1932** (1932), 499–504.
- 26. Hassler Whitney, 2-isomorphic graphs, American Journal of Mathematics 55 (1933), no. 1, 245–254.

DEPARTMENT OF MATHEMATICS UNIVERSITY OF CALIFORNIA, SAN DIEGO LA JOLLA, CA, 92093-0112, USA *Email address:* jvkim@ucsd.edu