# ON THE SEMIGROUP OF MONOID ENDOMORPHISMS OF THE SEMIGROUP $B_{\omega}^{\mathscr{F}}$ WITH THE TWO-ELEMENT FAMILY $\mathscr{F}$ OF INDUCTIVE NONEMPTY SUBSETS OF $\omega$ 

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#### Abstract

We study the semigroup of non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of $\omega$. We describe the structure of elements of the semigroup $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ of non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that its subsemigroup $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ of non-injective non-annihilating monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the direct product the two-element left-zero semigroup and the multiplicative semigroup of positive integers and describe Green's relations on $\boldsymbol{E n d} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$.


We shall follow the terminology of [1,2,9]. By $\omega$ we denote the set of all non-negative integers, by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{Z}$ the set of all integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of $\omega$. For any $F \in \mathscr{P}(\omega)$ and $n \in \mathbb{Z}$ we put $n F=\{n k: k \in F\}$ if $F \neq \varnothing$ and $n \varnothing=\varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called $\omega$-closed if $F_{1} \cap\left(-n+F_{2}\right) \in \mathscr{F}$ for all $n \in \omega$ and $F_{1}, F_{2} \in \mathscr{F}$. For any $a \in \omega$ we denote $[a)=\{x \in \omega: x \geqslant a\}$.

A subset $A$ of $\omega$ is said to be inductive, if $i \in A$ implies $i+1 \in A$. Obvious, that $\varnothing$ is an inductive subset of $\omega$.

Remark 1 (5]). (1) By Lemma 6 from [4] nonempty subset $F \subseteq \omega$ is inductive in $\omega$ if and only $(-1+F) \cap F=F$.
(2) Since the set $\omega$ with the usual order is well-ordered, for any nonempty inductive subset $F$ in $\omega$ there exists nonnegative integer $n_{F} \in \omega$ such that $\left[n_{F}\right]=F$.
(3) Statement (2) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in $\omega$ is a nonempty inductive subset of $\omega$.

A semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $x x^{-1} x=x$ and $x^{-1} x x^{-1}=x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the function inv : $S \rightarrow S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the inversion.

If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a band (or the band of $S$ ). Then the semigroup operation on $S$ determines the following partial order $\preccurlyeq$ on $E(S)$ : $e \preccurlyeq f$ if and only if $e f=f e=e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents.

If $S$ is an inverse semigroup then the semigroup operation on $S$ determines the following partial order $\preccurlyeq$ on $S: s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s=t e$. This order is called the natural partial order on $S$ [12].

[^0]If $S$ is a semigroup, then we shall denote the Green relations on $S$ by $\mathscr{R}, \mathscr{L}, \mathscr{J}, \mathscr{D}$ and $\mathscr{H}$ (see [1, Section 2.1]):

$$
\begin{aligned}
& a \mathscr{R} b \text { if and only if } a S^{1}=b S^{1} ; \\
& a \mathscr{L} b \text { if and only if } S^{1} a=S^{1} b ; \\
& a \mathscr{J} b \text { if and only if } S^{1} a S^{1}=S^{1} b S^{1} ; \\
& \mathscr{D}=\mathscr{L} \circ \mathscr{R}=\mathscr{R} \circ \mathscr{L} \\
& \mathscr{H}=\mathscr{L} \cap \mathscr{R}
\end{aligned}
$$

The $\mathscr{L}$-class [ $\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class, $\mathscr{J}$-class] of the semigroup $S$ containing the element $a \in S$ will be denoted by $\boldsymbol{L}_{a}\left[\boldsymbol{R}_{a}, \boldsymbol{H}_{a}, \boldsymbol{D}_{a}, \boldsymbol{J}_{a}\right]$.

The bicyclic monoid $\mathscr{C}(p, q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subjected only to the condition $p q=1$. The semigroup operation on $\mathscr{C}(p, q)$ is determined as follows:

$$
q^{k} p^{l} \cdot q^{m} p^{n}=q^{k+m-\min \{l, m\}} p^{l+n-\min \{l, m\}}
$$

It is well known that the bicyclic monoid $\mathscr{C}(p, q)$ is a bisimple (and hence simple) combinatorial $E$ unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p, q)$ is a group congruence [1].

On the set $\boldsymbol{B}_{\omega}=\omega \times \omega$ we define the semigroup operation "." in the following way

$$
\left(i_{1}, j_{1}\right) \cdot\left(i_{2}, j_{2}\right)= \begin{cases}\left(i_{1}-j_{1}+i_{2}, j_{2}\right), & \text { if } j_{1} \leqslant i_{2}  \tag{1}\\ \left(i_{1}, j_{1}-i_{2}+j_{2}\right), & \text { if } j_{1} \geqslant i_{2}\end{cases}
$$

It is well known that the bicyclic monoid $\mathscr{C}(p, q)$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}$ by the mapping $\mathfrak{h}: \mathscr{C}(p, q) \rightarrow \boldsymbol{B}_{\omega}, q^{k} p^{l} \mapsto(k, l), k, l \in \omega$ (see: [1, Section 1.12] or [11, Exercise IV.1.11(ii)]). Later we identify the bicyclic monoid $\mathscr{C}(p, q)$ with the semigroup $\boldsymbol{B}_{\omega}$ by the mapping $\mathfrak{h}$.

Next we shall describe the construction which is introduced in [4].
Let $\boldsymbol{B}_{\omega}$ be the bicyclic monoid and $\mathscr{F}$ be an $\omega$-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega} \times \mathscr{F}$ we define the semigroup operation "." in the following way

$$
\left(i_{1}, j_{1}, F_{1}\right) \cdot\left(i_{2}, j_{2}, F_{2}\right)= \begin{cases}\left(i_{1}-j_{1}+i_{2}, j_{2},\left(j_{1}-i_{2}+F_{1}\right) \cap F_{2}\right), & \text { if } j_{1} \leqslant i_{2}  \tag{2}\\ \left(i_{1}, j_{1}-i_{2}+j_{2}, F_{1} \cap\left(i_{2}-j_{1}+F_{2}\right)\right), & \text { if } j_{1} \geqslant i_{2}\end{cases}
$$

In [4] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is $\omega$-closed then $\left(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot\right)$ is a semigroup. Moreover, if an $\omega$-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I}=\{(i, j, \varnothing): i, j \in \omega\}$ is an ideal of the semigroup $\left(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot\right)$. For any $\omega$-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$
\boldsymbol{B}_{\omega}^{\mathscr{F}}= \begin{cases}\left(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot\right) / \boldsymbol{I}, & \text { if } \varnothing \in \mathscr{F} ; \\ \left(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot\right), & \text { if } \varnothing \notin \mathscr{F}\end{cases}
$$

is defined in [4]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in 4 that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is simple, 0 -simple, bisimple, 0 -bisimple, or it has the identity, are given. In particularly in [4] it is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega \times \omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if $\mathscr{F}$ consists of a non-empty inductive subset of $\omega$.

Group congruences on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its homomorphic retracts in the case when an $\omega$-closed family $\mathscr{F}$ consists of inductive non-empty subsets of $\omega$ are studied in [5]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [5], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are described. In [6] it is proved that an injective endomorphism $\varepsilon$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is the indentity transformation if and only if $\varepsilon$ has three distinct fixed points, which is equivalent to existence non-idempotent element $(i, j,[p)) \in \boldsymbol{B}_{\omega}^{\mathscr{F}}$ such that $(i, j,[p)) \varepsilon=(i, j,[p))$.

In [3, 10 the algebraic structure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is established in the case when $\omega$-closed family $\mathscr{F}$ consists of atomic subsets of $\omega$.

It is well-known that every automorphism of the bicyclic monoid $\boldsymbol{B}_{\omega}$ is the identity self-map of $\boldsymbol{B}_{\omega}$ [1], and hence the group $\operatorname{Aut}\left(\boldsymbol{B}_{\omega}\right)$ of automorphisms of $\boldsymbol{B}_{\omega}$ is trivial. In [8] it is proved that the semigroup $\operatorname{End}\left(\boldsymbol{B}_{\omega}\right)$ of all endomorphisms of the bicyclic semigroup $\boldsymbol{B}_{\omega}$ is isomorphic to the semidirect products $(\omega,+) \rtimes_{\varphi}(\omega, *)$, where + and $*$ are the usual addition and the usual multiplication on $\omega$.

In the paper [7] we study injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of $\omega$. We describe the elements of the semigroup $\boldsymbol{E n d}_{*}^{1}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that every element of the semigroup $\boldsymbol{E n d}_{*}^{1}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ has a form either $\alpha_{k, p}$ or $\beta_{k, p}$, where the endomorphism $\alpha_{k, p}$ is defined by the formulae

$$
\begin{aligned}
(i, j,[0)) \alpha_{k, p} & =(k i, k j,[0)) \\
(i, j,[1)) \alpha_{k, p} & =(p+k i, p+k j,[1))
\end{aligned}
$$

for an arbitrary positive integer $k$ and any $p \in\{0, \ldots, k-1\}$, and the endomorphism $\beta_{k, p}$ is defined by the formulae

$$
\begin{aligned}
(i, j,[0)) \beta_{k, p} & =(k i, k j,[0)) \\
(i, j,[1)) \beta_{k, p} & =(p+k i, p+k j,[0))
\end{aligned}
$$

an arbitrary positive integer $k \geqslant 2$ and any $p \in\{1, \ldots, k-1\}$. In [7] we describe the product of elements of the semigroup $\boldsymbol{E n d}_{*}^{1}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ :

$$
\begin{aligned}
\alpha_{k_{1}, p_{1}} \alpha_{k_{2}, p_{2}} & =\alpha_{k_{1} k_{2}, p_{2}+k_{2} p_{1}} ; \\
\alpha_{k_{1}, p_{1}} \beta_{k_{2}, p_{2}} & =\beta_{k_{1} k_{2}, p_{2}+k_{2} p_{1}} ; \\
\beta_{k_{1}, p_{1}} \beta_{k_{2}, p_{2}} & =\beta_{k_{1} k_{2}, k_{2} p_{1}} ; \\
\beta_{k_{1}, p_{1}} \alpha_{k_{2}, p_{2}} & =\beta_{k_{1} k_{2}, k_{2} p_{2}} .
\end{aligned}
$$

Also, here we prove that Green's relations $\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{E n d}_{*}^{1}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ coincide with the equality relation.

Later we assume that an $\omega$-closed family $\mathscr{F}$ consists of two nonempty inductive nonempty subsets of $\omega$.

This paper is a continuation of [7. We study non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. We describe the structure of elements of the semigroup $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ of all non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that its subsemigroup $\boldsymbol{E} \boldsymbol{n} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ of all non-injective non-annihilating monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the direct product the two-element left-zero semigroup and the multiplicative semigroup of positive integers and describe Green's relations on $\boldsymbol{E n d} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$.

Remark 2. By Proposition 1 of [5] for any $\omega$-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ there exists an $\omega$-closed family $\mathscr{F}^{*}$ of inductive subsets in $\mathscr{P}(\omega)$ such that $[0) \in \mathscr{F}^{*}$ and the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^{*}}$ are isomorphic. Hence without loss of generality we may assume that the family $\mathscr{F}$ contains the set [0).

If $\mathscr{F}$ is an arbitrary $\omega$-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ and $[s) \in \mathscr{F}$ for some $s \in \omega$ then

$$
\boldsymbol{B}_{\omega}^{\{[s)\}}=\{(i, j,[s)): i, j \in \omega\}
$$

is a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ [5] and by Proposition 3 of [4] the semigroup $\boldsymbol{B}_{\omega}^{\{[s)\}}$ is isomorphic to the bicyclic semigroup.
Lemma 1. Let $\mathscr{F}=\{[0),[1)\}$ and let $\mathfrak{e}$ be a monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. If $\left(i_{1}, j_{1}, F\right) \mathfrak{e}=$ $\left(i_{2}, j_{2}, F\right) \mathfrak{e}$ for distinct two elements $\left(i_{1}, j_{1}, F\right),\left(i_{2}, j_{2}, F\right)$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for some $F \in \mathscr{F}$ then $\mathfrak{e}$ is the annihilating endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

Proof. By Theorem 1 of [5] the image $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right) \mathfrak{e}$ is a subgroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. By Theorem 4(iii) of [4] every $\mathscr{H}$-class in $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a singleton, and hence $\mathfrak{e}$ is the annihilating monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.
Lemma 2. Let $\mathscr{F}=\{[0),[1)\}$. Then $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right) \mathfrak{e} \subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$ for any non-injective monoid endomorphism $\mathfrak{e}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.
Proof. Since $\mathfrak{e}$ is an monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}},(0,0,[0)) \mathfrak{e}=(0,0,[0))$. By Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_{\omega}^{\{0)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{Y}}$ is isomorphic to the bicyclic semigroup and hence by Corollary 1.32 of [1] the image $\left(\boldsymbol{B}_{\omega}^{\{[0])}\right) \mathfrak{e}$ either is isomorphic to the bicyclic semigroup or is a cyclic subgroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. If $S$ is a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ which is isomorphic to the bicyclic semigroup then by Proposition 4 of [5] there exists $F \in \mathscr{F}$ such that $S \subseteq \boldsymbol{B}_{\omega}^{\{F\}}$. Since $(0,0,[0)) \mathfrak{e}=(0,0,[0))$, Proposition 4 from [5] implies that $\left(\boldsymbol{B}_{\omega}^{\{[0)\}}\right) \mathfrak{e} \subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$ in the case when the image $\left(\boldsymbol{B}_{\omega}^{\{[0)\}}\right) \mathfrak{e}$ is isomorphic to the bicyclic semigroup. In the case when the image $\left(\boldsymbol{B}_{\omega}^{\{[0)\}}\right) \mathfrak{e}$ is isomorphic to the cyclic group we have that the equality $(0,0,[0)) \mathfrak{e}=(0,0,[0))$ implies that $\left(\boldsymbol{B}_{\omega}^{\{[0)\}}\right) \mathfrak{e} \subseteq\{(0,0,[0))\} \subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$, because by Theorem $4($ iii $)$ of [4] every $\mathscr{H}$-class in $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a singleton.

Next, by Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_{\omega}^{\{[1)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic semigroup and hence by Corollary 1.32 of [1] the image $\left(\boldsymbol{B}_{\omega}^{\{[1)\}}\right) \mathfrak{e}$ either is isomorphic to the bicyclic semigroup or is a cyclic subgroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Suppose that the image $\left(\boldsymbol{B}_{\omega}^{\{[1)\}}\right) \mathfrak{e}$ is isomorphic to the bicyclic semigroup and $\left(\boldsymbol{B}_{\omega}^{\{[1)\}}\right) \mathfrak{e} \subseteq \boldsymbol{B}_{\omega}^{\{[1)\}}$. Then monoid endomorphism $\mathfrak{e}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is injective. Indeed, injectivity of the restriction $\mathfrak{e}]_{\left.\boldsymbol{B}_{\omega}\{11)\right\}} \boldsymbol{B}_{\omega}^{\{[1)\}} \rightarrow \boldsymbol{B}_{\omega}^{\{[1)\}}$, Proposition 4 of [5], Corollary 1.32 of [1], Theorem 4(iii) of [4], and the equality $(0,0,[0)) \mathfrak{e}=(0,0,[0))$ imply that either the restriction $\mathfrak{e}]_{\boldsymbol{B}_{\omega}\{[0)\}} \boldsymbol{B}_{\omega}^{\{[0)\}} \rightarrow \boldsymbol{B}_{\omega}^{\{[0)\}}$ is an injective mapping or is an annihilating endomorphism. In the case when the restriction $\mathfrak{e}\}_{\boldsymbol{B}_{\omega}}^{\{[0)\}} \boldsymbol{B}_{\omega}^{\{[0)\}} \rightarrow \boldsymbol{B}_{\omega}^{\{[0)\}}$ is an injective mapping we get that the endomorphism $\mathfrak{e}$ is injective. If the image $\left(\boldsymbol{B}_{\omega}^{\{[0)\}}\right) \mathfrak{e}$ is a singleton then by Lemma we have that $\mathfrak{e}$ is the annihilating monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In the both cases we obtain that $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right) \mathfrak{e} \subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$.
Example 1. Let $\mathscr{F}=\{[0),[1)\}$ and $k$ be an arbitrary non-negative integer. We define a map $\gamma_{k}: \boldsymbol{B}_{\omega}^{\mathscr{F}} \rightarrow$ $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ by the formulae

$$
(i, j,[0)) \gamma_{k}=(i, j,[1)) \gamma_{k}=(k i, k j,[0))
$$

for all $i, j \in \omega$.
We claim that $\gamma_{k}: \boldsymbol{B}_{\omega}^{\mathscr{F}} \rightarrow \boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an endomorphism. Example 2 and Proposition 5 from 5 imply that the map $\gamma_{1}: \boldsymbol{B}_{\omega}^{\mathscr{F}} \rightarrow \boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a homomorphic retraction of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, and hence it is a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. By Lemma 2 of [8] every monoid endomorphism $\mathfrak{h}$ of the semigroup $\boldsymbol{B}_{\omega}$ has the following form

$$
(i, j) \mathfrak{h}=(k i, k j), \quad \text { for some } \quad k \in \omega .
$$

This implies that the map $\gamma_{k}$ is a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.
Example 2. Let $\mathscr{F}=\{[0),[1)\}$ and $k$ be an arbitrary non-negative integer. We define a map $\delta_{k}: \boldsymbol{B}_{\omega}^{\mathscr{F}} \rightarrow$ $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ by the formulae

$$
(i, j,[0)) \delta_{k}=(k i, k j,[0)) \quad \text { and } \quad(i, j,[1)) \delta_{k}=(k(i+1), k(j+1),[0))
$$

for all $i, j \in \omega$.
Proposition 1. Let $\mathscr{F}=\{[0),[1)\}$. Then for any $k \in \omega$ the map $\delta_{k}$ is an endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.
Proof. Since by Proposition 3 of [4] the subsemigroups $\boldsymbol{B}_{\omega}^{\{[0)\}}$ and $\boldsymbol{B}_{\omega}^{\{[1)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are isomorphic to the bicyclic semigroup, by Lemma 2 of [8] the restrictions $\delta_{k} 1_{\boldsymbol{B}_{\omega}}^{\{00)\}}: \boldsymbol{B}_{\omega}^{\{[0)\}} \rightarrow \boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\delta_{k} 1_{\boldsymbol{B}_{\omega}^{[1)}}: \boldsymbol{B}_{\omega}^{\{[1]\}} \rightarrow \boldsymbol{B}_{\omega}^{\mathscr{F}}$ of $\delta_{k}$ are homomorphisms. Hence it sufficient to show that the following equalities

$$
\begin{aligned}
& \left(i_{1}, j_{1},[0)\right) \delta_{k} \cdot\left(i_{2}, j_{2},[1)\right) \delta_{k}=\left(\left(i_{1}, j_{1},[0)\right) \cdot\left(i_{2}, j_{2},[1)\right)\right) \delta_{k} ; \\
& \left(i_{1}, j_{1},[1)\right) \delta_{k} \cdot\left(i_{2}, j_{2},[0)\right) \delta_{k}=\left(\left(i_{1}, j_{1},[1)\right) \cdot\left(i_{2}, j_{2},[0)\right)\right) \delta_{k},
\end{aligned}
$$

hold for any $i_{1}, j_{1}, i_{2}, j_{2} \in \omega$.
We observe that the above equalities are trivial in the case when $k=0$. Hence later we assume that $k$ is a positive integer.

Then we have that

$$
\begin{aligned}
& \left(i_{1}, j_{1},[0)\right) \delta_{k} \cdot\left(i_{2}, j_{2},[1)\right) \delta_{k}=\left(k i_{1}, k j_{1},[0)\right) \cdot\left(k\left(i_{2}+1\right), k\left(j_{2}+1\right),[0)\right)= \\
& = \begin{cases}\left(k i_{1}+k\left(i_{2}+1\right)-k j_{1}, k\left(j_{2}+1\right),\left(k j_{1}-k\left(i_{2}+1\right)+[0)\right) \cap[0)\right), & \text { if } k j_{1}<k\left(i_{2}+1\right) ; \\
\left(k i_{1}, k\left(j_{2}+1\right),[0) \cap[0)\right), & \text { if } k j_{1}=k\left(i_{2}+1\right) ; \\
\left(k i_{1}, k j_{1}+k\left(j_{2}+1\right)-k\left(i_{2}+1\right),[0) \cap\left(k\left(i_{2}+1\right)-k j_{1}+[0)\right)\right), & \text { if } k j_{1}>k\left(i_{2}+1\right)\end{cases} \\
& = \begin{cases}\left(k\left(i_{1}+i_{2}+1-j_{1}\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}<i_{2}+1 ; \\
\left(k i_{1}, k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2}+1 ; \\
\left(k i_{1}, k\left(j_{1}+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}>i_{2}+1\end{cases} \\
& = \begin{cases}\left(k\left(i_{1}+i_{2}+1-j_{1}\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}<i_{2} ; \\
\left(k\left(i_{1}+i_{2}+1-j_{1}\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k i_{1}, k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2}+1 ; \\
\left(k i_{1}, k\left(j_{1}+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}>i_{2}+1\end{cases} \\
& = \begin{cases}\left(k\left(i_{1}+i_{2}+1-j_{1}\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}<i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k i_{1}, k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2}+1 ; \\
\left(k i_{1}, k\left(j_{1}+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}>i_{2}+1,\end{cases} \\
& \left(\left(i_{1}, j_{1},[0)\right) \cdot\left(i_{2}, j_{2},[1)\right)\right) \delta_{k}= \begin{cases}\left(i_{1}+i_{2}-j_{1}, j_{2},\left(j_{1}-i_{2}+[0)\right) \cap[1)\right) \delta_{k}, & \text { if } j_{1}<i_{2} ; \\
\left(i_{1}, j_{2},[0) \cap[1)\right) \delta_{k}, & \text { if } j_{1}=i_{2} ; \\
\left(i_{1}, j_{1}+j_{2}-i_{2},[0) \cap\left(i_{2}-j_{1}+[1)\right)\right) \delta_{k}, & \text { if } j_{1}>i_{2}\end{cases} \\
& = \begin{cases}\left(i_{1}+i_{2}-j_{1}, j_{2},[1)\right) \delta_{k}, & \text { if } j_{1}<i_{2} ; \\
\left(i_{1}, j_{2},[1)\right) \delta_{k}, & \text { if } j_{1}=i_{2} ; \\
\left(i_{1}, j_{1}+j_{2}-i_{2},[0)\right) \delta_{k}, & \text { if } j_{1}>i_{2}\end{cases} \\
& = \begin{cases}\left(k\left(i_{1}+i_{2}-j_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}<i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k i_{1}, k\left(j_{1}+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}>i_{2}\end{cases} \\
& = \begin{cases}\left(k\left(i_{1}+i_{2}-j_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}<i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k i_{1}, k\left(j_{1}+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}=i_{2}+1 ; \\
\left(k i_{1}, k\left(j_{1}+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}>i_{2}+1\end{cases} \\
& = \begin{cases}\left(k\left(i_{1}+i_{2}+1-j_{1}\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}<i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k i_{1}, k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2}+1 ; \\
\left(k i_{1}, k\left(j_{1}+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}>i_{2}+1,\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(i_{1}, j_{1},[1)\right) \delta_{k} & \cdot\left(i_{2}, j_{2},[0)\right) \delta_{k}=\left(k\left(i_{1}+1\right), k\left(j_{1}+1\right),[0)\right) \cdot\left(k i_{2}, k j_{2},[0)\right)= \\
& = \begin{cases}\left(k\left(i_{1}+1\right)+k i_{2}-k\left(j_{1}+1\right), k j_{2},\left(k\left(j_{1}+1\right)-k i_{2}+[0)\right) \cap[0)\right), & \text { if } k\left(j_{1}+1\right)<k i_{2} ; \\
\left(k\left(i_{1}+1\right), k j_{2},[0) \cap[0)\right), & \text { if } k\left(j_{1}+1\right)=k i_{2} \\
\left(k\left(i_{1}+1\right), k\left(j_{1}+1\right)+k j_{2}-k i_{2},[0) \cap\left(k i_{2}-k\left(j_{1}+1\right)+[0)\right)\right), & \text { if } k\left(j_{1}+1\right)>k i_{2}\end{cases} \\
& = \begin{cases}\left(k\left(i_{1}+i_{2}-j_{1}\right), k j_{2},[0)\right), & \text { if } j_{1}+1<i_{2} ; \\
\left(k\left(i_{1}+1\right), k j_{2},[0)\right), & \text { if } j_{1}+1=i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{1}+1+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}+1>i_{2}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{cases}\left(k\left(i_{1}+i_{2}-j_{1}\right), k j_{2},[0)\right), & \text { if } j_{1}+1<i_{2} ; \\
\left(k\left(i_{1}+1\right), k j_{2},[0)\right), & \text { if } j_{1}+1=i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{1}+1+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{1}+1+j_{2}-i_{2}\right),[0)\right), & \text { if } j_{1}+1>i_{2}\end{cases} \\
&= \begin{cases}\left(k\left(i_{1}+i_{2}-j_{1}\right), k j_{2},[0)\right), & \text { if } j_{1}+1<i_{2} ; \\
\left(k\left(i_{1}+1\right), k j_{2},[0)\right), & \text { if } j_{1}+1=i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{1}+j_{2}-i_{2}+1\right),[0)\right), & \text { if } j_{1}>i_{2},\end{cases} \\
&\left(\left(i_{1}, j_{1},[1)\right) \cdot\left(i_{2}, j_{2},[0)\right)\right) \delta_{k}= \begin{cases}\left(i_{1}+i_{2}-j_{1}, j_{2},\left(j_{1}-i_{2}+[1)\right) \cap[0)\right) \delta_{k}, & \text { if } j_{1}<i_{2} ; \\
\left(i_{1}, j_{2},[1) \cap[0)\right) \delta_{k}, & \text { if } j_{1}=i_{2} ; \\
\left(i_{1}, j_{1}+j_{2}-i_{2},[1) \cap\left(i_{2}-j_{1}+[0)\right)\right) \delta_{k}, & \text { if } j_{1}>i_{2}\end{cases} \\
&= \begin{cases}\left(i_{1}+i_{2}-j_{1}, j_{2},[0)\right) \delta_{k}, & \text { if } j_{1}<i_{2} ; \\
\left(i_{1}, j_{2},[1)\right) \delta_{k}, & \text { if } j_{1}=i_{2} ; \\
\left(i_{1}, j_{1}+j_{2}-i_{2},[1)\right) \delta_{k}, & \text { if } j_{1}>i_{2}\end{cases} \\
&= \begin{cases}\left(k\left(i_{1}+i_{2}-j_{1}\right), k j_{2},[0)\right), & \text { if } j_{1}+1<i_{2} ; \\
\left(k\left(i_{1}+i_{2}-j_{1}\right), k j_{2},[0)\right), & \text { if } j_{1}=i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{2}+1\right),[0)\right), \\
\left(k\left(i_{1}+1\right), k\left(j_{1}+j_{2}-i_{2}+1\right),[0)\right), & \text { if } j_{1}>i_{2}\end{cases} \\
&= \begin{cases}\left(k\left(i_{1}+i_{2}-j_{1}\right), k j_{2},[0)\right), & \text { if } j_{1}+1<i_{2} ; \\
\left(k\left(i_{1}+1\right), k j_{2},[0)\right), & \text { if } j_{1}+1=i_{2} ; \\
\left(k\left(i_{1}+1\right), k\left(j_{2}+1\right),[0)\right), & \\
\left(k\left(i_{1}+1\right), k\left(j_{1}+j_{2}-i_{2}+1\right),[0)\right), & \text { if } j_{1}>i_{2} ;\end{cases}
\end{aligned}
$$

This completes the proof of the statement of the proposition.
Remark 3. It obvious that if $\mathfrak{e}$ is the annihilating endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ then $\mathfrak{e}=\gamma_{0}=\delta_{0}$.
By $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ we denote the semigroup of all non-injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F}=\{[0),[1)\}$.

Theorems 1 and 2 describe the algebraic structure of the semigroup $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$.
Theorem 1. If $\mathscr{F}=\{[0),[1)\}$, then for any non-injective monoid endomorphism $\mathfrak{e}$ of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ only one of the following conditions holds:
(1) $\mathfrak{e}$ is the annihilating endomorphism, i.e., $\mathfrak{e}=\gamma_{0}=\delta_{0}$;
(2) $\mathfrak{e}=\gamma_{k}$ for some positive integer $k$;
(3) $\mathfrak{e}=\delta_{k}$ for some positive integer $k$.

Proof. Fix an arbitrary non-injective monoid endomorphism $\mathfrak{e}$ of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. If $\mathfrak{e}$ is the annihilating endomorphism then statement (11) holds. Hence, later we assume that the endomorphism $\mathfrak{e}$ is not annihilating.

By Lemma 1 the restriction $\mathfrak{e}\rceil_{\left.\boldsymbol{B}_{\omega}\{0)\right\}} \boldsymbol{B}_{\omega}^{\{[0)\}} \rightarrow \boldsymbol{B}_{\omega}^{\mathscr{F}}$ of the endomorphism $\mathfrak{e}$ is an injective mapping. Since by Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_{\omega}^{\{[0)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are isomorphic to the bicyclic semigroup, the injectivity of the restriction $\mathfrak{e}\}_{\left.B_{\omega}\{0)\right\}}$ of the endomorphism $\mathfrak{e}$, Proposition 4 of [5], and Lemma 2 of [8] imply that there exists a positive integer $k$ such that

$$
\begin{equation*}
(i, j,[0)) \mathfrak{e}=(k i, k j,[0)) \tag{3}
\end{equation*}
$$

for all $i, j \in \omega$.
By Lemma 1 the restriction $\mathfrak{e}\rceil_{\left.\boldsymbol{B}_{\omega}\{11)\right\}} \boldsymbol{B}_{\omega}^{\{[1)\}} \rightarrow \boldsymbol{B}_{\omega}^{\mathscr{F}}$ of the endomorphism $\mathfrak{e}$ is an injective mapping, and by Lemma 2 we have that $\left(\boldsymbol{B}_{\omega}^{\{[1)\}}\right) \mathfrak{e} \subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$. By Proposition 1.4.21(6) of [9] a homomorphism of inverse
semigroups preserves the natural partial order, and hence the following inequalities

$$
(1,1,[0)) \preccurlyeq(0,0,[1)) \preccurlyeq(0,0,[0)),
$$

Lemma 2, and Propositions 2 of 5 imply that

$$
(k, k,[0))=(1,1,[0)) \mathfrak{e} \preccurlyeq(s, s,[0))=(0,0,[1)) \mathfrak{e} \preccurlyeq(0,0,[0))=(0,0,[0)) \mathfrak{e}
$$

for some $s \in\{0,1, \ldots, k\}$. Again by Proposition 1.4.21(6) of 9 and by Lemma 2 we get that

$$
(1,1,[1)) \mathfrak{e}=(s+p, s+p,[0))
$$

for some non-negative integer $p$. If $p=0$ then $(1,1,[1)) \mathfrak{e}=(0,0,[1)) \mathfrak{e}$. By Lemma $\square$ the endomorphism $\mathfrak{c}$ is annihilating. Hence we assume that $p$ is a positive integer.
Let $(0,1,[1) \mathfrak{e}=(x, y,[0))$ for some $x, y \in \omega$. By Proposition 1.4.21(1) of 9 and Lemma 4 of [4] we have that

$$
(1,0,[1)) \mathfrak{e}=\left((0,1,[1))^{-1}\right) \mathfrak{e}=((0,1,[1)) \mathfrak{e})^{-1}=(x, y,[0))^{-1}=(y, x,[0)) .
$$

Since

$$
(0,1,[1)) \cdot(1,0,[1))=(0,0,[1)) \quad \text { and } \quad(0,1,[1)) \cdot(1,0,[1))=(0,0,[1)) \text {, }
$$

the equalities $(0,0,[1)) \mathfrak{e}=(s, s,[0))$ and $(1,1,[1)) \mathfrak{e}=(s+p, s+p,[0))$ imply that

$$
\begin{aligned}
(s, s,[0)) & =(0,0,[1)) \mathfrak{e}=((0,1,[1)) \cdot(1,0,[1))) \mathfrak{e}=(0,1,[1)) \mathfrak{e} \cdot(1,0,[1)) \mathfrak{e}= \\
& =(x, y,[0)) \cdot(y, x,[0))=(x, x,[0))
\end{aligned}
$$

and

$$
\begin{aligned}
(s+p, s+p,[0)) & =(1,1,[1)) \mathfrak{e}=((1,0,[1)) \cdot(0,1,[1))) \mathfrak{e}=(1,0,[1)) \mathfrak{e} \cdot(0,1,[1)) \mathfrak{e}= \\
& =(y, x,[0)) \cdot(x, y,[0))=(y, y,[0)) .
\end{aligned}
$$

This and the definition of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ imply that

$$
(0,1,[1)) \mathfrak{e}=(s, s+p,[0)) \quad \text { and } \quad(1,0,[1)) \mathfrak{e}=(s+p, s,[0)) .
$$

Then for any positive integers $n_{1}$ and $n_{2}$ by usual calculations we get that
and
$\left(n_{2}, 0,[1)\right) \mathfrak{e}=(\underbrace{(1,0,[1)) \cdot \ldots \cdot(1,0,[1))}_{n_{2} \text {-times }}) \mathfrak{e}=\underbrace{(1,0,[1)) \mathfrak{e} \cdot \ldots(1,0,[1)) \mathfrak{e}}_{n_{2} \text {-times }}=(s+p, s,[0))^{n_{2}}=\left(s+n_{2} p, s,[0)\right)$, and hence

$$
\begin{equation*}
\left(n_{1}, n_{2},[1)\right) \mathfrak{e}=\left(s+n_{1} p, s+n_{2} p,[0)\right) . \tag{4}
\end{equation*}
$$

The definition of the natural partial order on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ (see Proposition 4 of [5]) imply that for any positive integer $m$ we have that

$$
(m+1, m+1,[0)) \preccurlyeq(m, m,[1)) \preccurlyeq(m, m,[0)) .
$$

Then by equalities (3), (4), and Proposition 1.4.21(6) of [9] we obtain that

$$
\begin{aligned}
(k(m+1), k(m+1),[0))=(m+1, m+1,[0)) \mathfrak{e} & \preccurlyeq(s+p m, s+p m,[0))=(m, m,[1)) \mathfrak{e} \preccurlyeq \\
& \preccurlyeq(m, m,[0)) \mathfrak{e}=(k m, k m,[0)) .
\end{aligned}
$$

The above inequalities and the definition of the natural partial order on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ (see Proposition 4 of (5) imply that $k m \leqslant s+p m \leqslant k(m+1)$ for any positive integer $m$. This implies that

$$
k \leqslant \frac{s}{m}+p \leqslant k+\frac{1}{m},
$$

and since $p$ is a positive integer we get that $p=k$. Hence by (4) we get that

$$
\begin{equation*}
\left(n_{1}, n_{2},[1)\right) \mathfrak{e}=\left(s+n_{1} k, s+n_{2} k,[0)\right) \tag{5}
\end{equation*}
$$

for all $n_{1}, n_{2} \in \omega$.
It is obvious that if $s \in\{1, \ldots, k-1\}$ then $\mathfrak{e}$ is an injective monoid endomorphism of the semigroup. Hence we have that either $s=0$ or $s=k$, Simple verifications show that

$$
\mathfrak{e}= \begin{cases}\gamma_{k}, & \text { if } s=0 \\ \delta_{k}, & \text { if } s=k\end{cases}
$$

This completes the proof of the theorem.
Theorem 2. Let $\mathscr{F}=\{[0),[1)\}$. Then for all positive integers $k_{1}$ and $k_{2}$ the following conditions hold:
(1) $\gamma_{k_{1}} \gamma_{k_{2}}=\gamma_{k_{1} k_{2}}$;
(2) $\gamma_{k_{1}} \delta_{k_{2}}=\gamma_{k_{1} k_{2}}$;
(3) $\delta_{k_{1}} \gamma_{k_{2}}=\delta_{k_{1} k_{2}}$;
(4) $\delta_{k_{1}} \delta_{k_{2}}=\delta_{k_{1} k_{2}}$.

Proof. (11) For any $i, j \in \omega$ we have that

$$
(i, j,[0)) \gamma_{k_{1}} \gamma_{k_{2}}=\left(k_{1} i, k_{1} j,[0)\right) \gamma_{k_{2}}=\left(k_{1} k_{2} i, k_{1} k_{2} j,[0)\right),
$$

and $(i, j,[1)) \gamma_{k_{1}}=(i, j,[0)) \gamma_{k_{1}}$. This implies that $\gamma_{k_{1}} \gamma_{k_{2}}=\gamma_{k_{1} k_{2}}$.
(2) Since

$$
(i, j,[0)) \gamma_{k_{1}} \delta_{k_{2}}=\left(k_{1} i, k_{1} j,[0)\right) \delta_{k_{2}}=\left(k_{1} k_{2} i, k_{1} k_{2} j,[0)\right)
$$

and $(i, j,[1)) \gamma_{k_{1}}=(i, j,[0)) \gamma_{k_{1}}$ for all $i, j \in \omega$, we get that $\gamma_{k_{1}} \delta_{k_{2}}=\gamma_{k_{1} k_{2}}$.
(3) For any $i, j \in \omega$ we have that

$$
(i, j,[0)) \delta_{k_{1}} \gamma_{k_{2}}=\left(k_{1} i, k_{1} j,[0)\right) \gamma_{k_{2}}=\left(k_{1} k_{2} i, k_{1} k_{2} j,[0)\right),
$$

and

$$
(i, j,[1)) \delta_{k_{1}} \gamma_{k_{2}}=\left(k_{1}(i+1), k_{1}(j+1),[0)\right) \gamma_{k_{2}}=\left(k_{1} k_{2}(i+1), k_{1} k_{2}(j+1),[0)\right),
$$

and hence $\delta_{k_{1}} \gamma_{k_{2}}=\delta_{k_{1} k_{2}}$.
(4) For any $i, j \in \omega$ we have that

$$
(i, j,[0)) \delta_{k_{1}} \delta_{k_{2}}=\left(k_{1} i, k_{1} j,[0)\right) \delta_{k_{2}}=\left(k_{1} k_{2} i, k_{1} k_{2} j,[0)\right)
$$

and

$$
(i, j,[1)) \delta_{k_{1}} \delta_{k_{2}}=\left(k_{1}(i+1), k_{1}(j+1),[0)\right) \delta_{k_{2}}=\left(k_{1} k_{2}(i+1), k_{1} k_{2}(j+1),[0)\right)
$$

and hence $\delta_{k_{1}} \delta_{k_{2}}=\delta_{k_{1} k_{2}}$.
By $\mathfrak{e}_{0}$ we denote the annihilating monoid endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F}=$ $\{[0),[1)\}$, i.e., $(i, j,[p)) \mathfrak{e}_{\mathbf{0}}=(0,0,[0))$ for all $i, j \in \omega$ and $p=0,1$. We put $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)=\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right) \backslash$ $\left\{\mathfrak{e}_{0}\right\}$. Theorem 2 implies that $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ is a subsemigroup of $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$.

Theorem 2 implies the following corollary.
Corollary 1. If $\mathscr{F}=\{[0),[1)\}$, then the elements $\gamma_{1}$ and $\delta_{1}$ are unique idempotents of the semigroup $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$.

Next, by $\mathfrak{L Z}_{2}$ we denote the left zero semigroup with two elements and by $\mathbb{N}_{u}$ the multiplicative semigroup of positive integers.

Proposition 2. Let $\mathscr{F}=\{[0),[1)\}$. Then the semigroup $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ is isomorphic to the direct product $\mathfrak{L Z}_{2} \times \mathbb{N}_{u}$.

Proof. Put $L Z_{2}=\{c, d\}$. We define a map $\mathfrak{I}: \boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right) \rightarrow \mathfrak{L} \mathfrak{Z}_{2} \times \mathbb{N}_{u}$ by the formula

$$
(\mathfrak{e}) \mathfrak{I}= \begin{cases}(c, k), & \text { if } \mathfrak{e}=\gamma_{k} \\ (d, k), & \text { if } \mathfrak{e}=\delta_{k}\end{cases}
$$

It is obvious that such defined map $\mathfrak{I}$ is bijective, and by Theorem 2 it is a homomorphism.
Theorem 3 describes Green's relations on the semigroup $\boldsymbol{E n d} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$. Later by $\boldsymbol{E n d} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)^{1}$ we denote the semigroup $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ with adjoined identity element.

Theorem 3. Let $\mathscr{F}=\{[0),[1)\}$. Then the following statements hold:
(1) $\gamma_{k_{1}} \mathscr{R} \gamma_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ if and only if $k_{1}=k_{2}$;
(2) $\gamma_{k_{1}} \mathscr{R} \delta_{k_{2}}$ does not hold in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ for any $\gamma_{k_{1}}, \delta_{k_{2}}$;
(3) $\delta_{k_{1}} \mathscr{R} \delta_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ if and only if $k_{1}=k_{2}$;
(4) $\gamma_{k_{1}} \mathscr{L} \gamma_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ if and only if $k_{1}=k_{2}$;
(5) $\gamma_{k_{1}} \mathscr{L} \delta_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ if and only if $k_{1}=k_{2}$;
(6) $\delta_{k_{1}} \mathscr{L} \delta_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ if and only if $k_{1}=k_{2}$;
(7) $\mathscr{H}$ is the identity relation on $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$;
(8) $\mathfrak{e}_{1} \mathscr{D} \mathfrak{e}_{2}$ in $\boldsymbol{E n d} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ if and only if there exists a positive integer $k$ such that $\mathfrak{e}_{1}, \mathfrak{e}_{2} \in\left\{\gamma_{k}, \delta_{k}\right\}$;
(9) $\mathscr{D}=\mathscr{J}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$.

Proof. (11) $(\Rightarrow)$ Suppose that $\gamma_{k_{1}} \mathscr{R} \gamma_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$. Then there exist $\mathfrak{e}_{1}, \mathfrak{e}_{2} \in \boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)^{1}$ such that
 integer $p$ such that either $\mathfrak{e}_{1}=\gamma_{p}$ or $\mathfrak{e}_{1}=\delta_{p}$. In both above cases by Theorem 2 we have that

$$
\gamma_{k_{1}}=\gamma_{k_{2}} \mathfrak{e}_{1}=\gamma_{k_{2}} \gamma_{p}=\gamma_{k_{2}} \delta_{p}=\gamma_{k_{2} p}
$$

and hence $k_{2} \mid k_{1}$. The proof of the statement that $\gamma_{k_{2}}=\gamma_{k_{1}} \mathfrak{e}_{2}$ implies that $k_{1} \mid k_{2}$ is similar. Therefore we get that $k_{1}=k_{2}$.

Implication $(\Leftarrow)$ is trivial.
Statement (22) follows from Theorem 2(2).
The proof of statement (3) is similar to (1).
(4) $(\Rightarrow)$ Suppose that $\gamma_{k_{1}} \mathscr{L} \gamma_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$. Then there exist $\mathfrak{e}_{1}, \mathfrak{e}_{2} \in \boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)^{1}$ such that $\gamma_{k_{1}}=\mathfrak{e}_{1} \gamma_{k_{2}}$ and $\gamma_{k_{2}}=\mathfrak{e}_{2} \gamma_{k_{1}}$. The equality $\gamma_{k_{1}}=\mathfrak{e}_{1} \gamma_{k_{2}}$ and Theorem 2 imply that there exists a positive integer $p$ such that $\mathfrak{e}_{1}=\gamma_{p}$. Then we have that

$$
\gamma_{k_{1}}=\mathfrak{e}_{1} \gamma_{k_{2}}=\gamma_{p} \gamma_{k_{2}}=\gamma_{p k_{2}},
$$

and hence $k_{2} \mid k_{1}$. The proof of the statement that $\gamma_{k_{2}}=\mathfrak{e}_{2} \gamma_{k_{1}}$ implies that $k_{1} \mid k_{2}$ is similar. Therefore we get that $k_{1}=k_{2}$.

Implication $(\Leftarrow)$ is trivial.
(5)) $(\Rightarrow)$ Suppose that $\gamma_{k_{1}} \mathscr{L} \delta_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$. Then there exist $\mathfrak{e}_{1}, \mathfrak{e}_{2} \in \boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)^{1}$ such that $\gamma_{k_{1}}=\mathfrak{e}_{1} \delta_{k_{2}}$ and $\delta_{k_{2}}=\mathfrak{e}_{2} \gamma_{k_{1}}$. The equality $\gamma_{k_{1}}=\mathfrak{e}_{1} \delta_{k_{2}}$ and Theorem 2 imply that there exists a positive integer $p$ such that $\mathfrak{e}_{1}=\gamma_{p}$. Then we have that

$$
\gamma_{k_{1}}=\mathfrak{e}_{1} \delta_{k_{2}}=\gamma_{p} \delta_{k_{2}}=\gamma_{p k_{2}}
$$

and hence $k_{2} \mid k_{1}$. The equality $\delta_{k_{2}}=\mathfrak{e}_{2} \gamma_{k_{1}}$ and Theorem 2 imply that there exists a positive integer $q$ such that $\mathfrak{e}_{1}=\delta_{q}$. Then we have that

$$
\delta_{k_{2}}=\mathfrak{e}_{2} \gamma_{k_{1}}=\delta_{q} \gamma_{k_{1}}=\gamma_{q k_{1}}
$$

and hence $k_{1} \mid k_{2}$. Thus we get that $k_{1}=k_{2}$.
Implication $(\Leftarrow)$ is trivial.
(6) $(\Rightarrow)$ Suppose that $\delta_{k_{1}} \mathscr{L} \delta_{k_{2}}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$. Then there exist $\mathfrak{e}_{1}, \mathfrak{e}_{2} \in \boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)^{1}$ such that $\delta_{k_{1}}=\mathfrak{e}_{1} \delta_{k_{2}}$ and $\delta_{k_{2}}=\mathfrak{e}_{2} \delta_{k_{1}}$. The equality $\delta_{k_{1}}=\mathfrak{e}_{1} \delta_{k_{2}}$ and Theorem 2 imply that there exists a positive integer $p$ such that $\mathfrak{e}_{1}=\delta_{p}$. Then we have that

$$
\delta_{k_{1}}=\mathfrak{e}_{1} \delta_{k_{2}}=\delta_{p} \delta_{k_{2}}=\delta_{p k_{2}}
$$

and hence $k_{2} \mid k_{1}$. The proof of the statement that $\delta_{k_{2}}=\mathfrak{e}_{2} \delta_{k_{1}}$ implies that $k_{1} \mid k_{2}$ is similar. Hence we get that $k_{1}=k_{2}$.

Implication $(\Leftarrow)$ is trivial.
(7) By statements (11), (2), and (3), $\mathscr{R}$ is the identity relation on the semigroup $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$. Then so is $\mathscr{H}$, because $\mathscr{H} \subseteq \mathscr{R}$.

Statement (8) follows from statements (11)-(6).
(9) Suppose to the contrary that $\mathscr{D} \neq \mathscr{J}$ in $\boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$. Since $\mathscr{D} \subseteq \mathscr{J}$, statement (8) implies that there exist $\mathfrak{e}_{1}, \mathfrak{e}_{2} \in \boldsymbol{E n d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)^{1}$ such that $\mathfrak{e}_{1} \mathscr{J} \mathfrak{e}_{2}$ and $\mathfrak{e}_{1}, \mathfrak{e}_{2} \notin\left\{\gamma_{k}, \delta_{k}\right\}$ for any positive integer $k$. Then there exist distinct positive integers $k_{1}$ and $k_{2}$ such that $\mathfrak{e}_{1} \in\left\{\gamma_{k_{1}}, \delta_{k_{1}}\right\}$ and $\mathfrak{e}_{2} \in\left\{\gamma_{k_{2}}, \delta_{k_{2}}\right\}$. Without loss of generality we may assume that $k_{1}<k_{2}$. Since $\mathfrak{e}_{1} \mathscr{J} \mathfrak{e}_{2}$ there exist $\mathfrak{e}_{1}^{\prime}, \mathfrak{e}_{2}^{\prime}, \mathfrak{e}_{1}^{\prime \prime}, \mathfrak{e}_{2}^{\prime \prime} \in \boldsymbol{E} \boldsymbol{n} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)^{1}$ such that $\mathfrak{e}_{1}=\mathfrak{e}_{1}^{\prime} \mathfrak{e}_{2} \mathfrak{e}_{1}^{\prime \prime}$ and $\mathfrak{e}_{2}=\mathfrak{e}_{2}^{\prime} \mathfrak{e}_{2} \mathfrak{e}_{2}^{\prime \prime}$. Since $\mathfrak{e}_{1} \in\left\{\gamma_{k_{1}}, \delta_{k_{1}}\right\}$ and $\mathfrak{e}_{2} \in\left\{\gamma_{k_{2}}, \delta_{k_{2}}\right\}$, the equality $\mathfrak{e}_{1}=\mathfrak{e}_{1}^{\prime} \mathfrak{e}_{2} \mathfrak{e}_{1}^{\prime \prime}$, Theorems 1 and 2 imply that $k_{2} \mid k_{1}$. This contradicts the inequality $k_{1}<k_{2}$. The obtained contradiction implies the requested statement
Remark 4. Since $\mathfrak{e}_{0}$ is zero of the semigroup $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$, the classes of equivalence of Green's relations of non-zero elements of $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ in the semigroup $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$ coincide with their corresponding classes of equivalence in $\boldsymbol{E} \boldsymbol{n d} \boldsymbol{d}^{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$, and moreover we have that $\boldsymbol{L}_{\mathfrak{c}_{0}}=\boldsymbol{R}_{\mathfrak{e}_{0}}=\boldsymbol{H}_{\mathfrak{e}_{0}}=\boldsymbol{D}_{\mathfrak{c}_{0}}=\boldsymbol{J}_{\mathfrak{e}_{0}}=\left\{\mathfrak{e}_{0}\right\}$ in the semigroup $\boldsymbol{E n d}_{*}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}}\right)$.

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