ON THE SEMIGROUP OF MONOID ENDOMORPHISMS OF THE SEMIGROUP $B_{\omega}^{\mathscr{F}}$ WITH THE TWO-ELEMENT FAMILY \mathscr{F} OF INDUCTIVE NONEMPTY SUBSETS OF ω

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ABSTRACT. We study the semigroup of non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family \mathscr{F} of inductive nonempty subsets of ω . We describe the structure of elements of the semigroup $\boldsymbol{End}_*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that its subsemigroup $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of non-injective non-annihilating monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the direct product the two-element left-zero semigroup and the multiplicative semigroup of positive integers and describe Green's relations on $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

We shall follow the terminology of [1,2,9]. By ω we denote the set of all non-negative integers, by \mathbb{N} the set of all positive integers, and by \mathbb{Z} the set of all integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω . For any $F \in \mathscr{P}(\omega)$ and $n \in \mathbb{Z}$ we put $nF = \{nk : k \in F\}$ if $F \neq \emptyset$ and $n\emptyset = \emptyset$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called ω -closed if $F_1 \cap (-n + F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a] = \{x \in \omega : x \geqslant a\}$.

A subset A of ω is said to be *inductive*, if $i \in A$ implies $i + 1 \in A$. Obvious, that \emptyset is an inductive subset of ω .

Remark 1 ([5]). (1) By Lemma 6 from [4] nonempty subset $F \subseteq \omega$ is inductive in ω if and only $(-1+F) \cap F = F$.

- (2) Since the set ω with the usual order is well-ordered, for any nonempty inductive subset F in ω there exists nonnegative integer $n_F \in \omega$ such that $[n_F) = F$.
- (3) Statement (2) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in ω is a nonempty inductive subset of ω .

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function inv: $S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by E(S). If S is an inverse semigroup, then E(S) is closed under multiplication and we shall refer to E(S) as a band (or the band of S). Then the semigroup operation on S determines the following partial order \leq on E(S): $e \leq f$ if and only if ef = fe = e. This order is called the natural partial order on E(S). A semilattice is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order \leq on S: $s \leq t$ if and only if there exists $e \in E(S)$ such that s = te. This order is called the *natural* partial order on S [12].

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If S is a semigroup, then we shall denote the Green relations on S by \mathcal{R} , \mathcal{L} , \mathcal{J} , \mathcal{D} and \mathcal{H} (see [1, Section 2.1]):

$$a\mathcal{R}b$$
 if and only if $aS^1 = bS^1$;
 $a\mathcal{L}b$ if and only if $S^1a = S^1b$;
 $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$;
 $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$;
 $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

The \mathcal{L} -class [\mathcal{R} -class, \mathcal{H} -class, \mathcal{D} -class, \mathcal{I} -class] of the semigroup S containing the element $a \in S$ will be denoted by L_a [R_a , H_a , D_a , J_a].

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition pq = 1. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\mathbf{B}_{\omega} = \omega \times \omega$ we define the semigroup operation "." in the following way

(1)
$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2. \end{cases}$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is isomorphic to the semigroup \mathbf{B}_{ω} by the mapping $\mathfrak{h}: \mathscr{C}(p,q) \to \mathbf{B}_{\omega}, q^k p^l \mapsto (k,l), k,l \in \omega$ (see: [1, Section 1.12] or [11, Exercise IV.1.11(ii)]). Later we identify the bicyclic monoid $\mathscr{C}(p,q)$ with the semigroup \mathbf{B}_{ω} by the mapping \mathfrak{h} .

Next we shall describe the construction which is introduced in [4].

Let \mathbf{B}_{ω} be the bicyclic monoid and \mathscr{F} be an ω -closed subfamily of $\mathscr{P}(\omega)$. On the set $\mathbf{B}_{\omega} \times \mathscr{F}$ we define the semigroup operation "·" in the following way

(2)
$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases}$$

In [4] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω -closed then $(\mathbf{B}_{\omega} \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω -closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set \varnothing then the set $\mathbf{I} = \{(i, j, \varnothing) : i, j \in \omega\}$ is an ideal of the semigroup $(\mathbf{B}_{\omega} \times \mathscr{F}, \cdot)$. For any ω -closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$m{B}_{\omega}^{\mathscr{F}} = \left\{ egin{array}{ll} (m{B}_{\omega} imes \mathscr{F}, \cdot) / m{I}, & ext{if } \varnothing \in \mathscr{F}; \\ (m{B}_{\omega} imes \mathscr{F}, \cdot), & ext{if } \varnothing \notin \mathscr{F} \end{array}
ight.$$

is defined in [4]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [4] that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is simple, 0-bisimple, 0-bisimple, or it has the identity, are given. In particularly in [4] it is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigroup of $\omega \times \omega$ -matrix units if and only if \mathscr{F} consists of a singleton set and the empty set, and $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if \mathscr{F} consists of a non-empty inductive subset of ω .

Group congruences on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω -closed family \mathscr{F} consists of inductive non-empty subsets of ω are studied in [5]. It is proven that a congruence \mathfrak{C} on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [5], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are described. In [6] it is proved that an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is the indentity transformation if and only if ε has three distinct fixed points, which is equivalent to existence non-idempotent element $(i,j,[p)) \in \boldsymbol{B}_{\omega}^{\mathscr{F}}$ such that $(i,j,[p))\varepsilon = (i,j,[p))$.

In [3,10] the algebraic structure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is established in the case when ω -closed family \mathscr{F} consists of atomic subsets of ω .

It is well-known that every automorphism of the bicyclic monoid \mathbf{B}_{ω} is the identity self-map of \mathbf{B}_{ω} [1], and hence the group $\mathbf{Aut}(\mathbf{B}_{\omega})$ of automorphisms of \mathbf{B}_{ω} is trivial. In [8] it is proved that the semigroup $\mathbf{End}(\mathbf{B}_{\omega})$ of all endomorphisms of the bicyclic semigroup \mathbf{B}_{ω} is isomorphic to the semidirect products $(\omega, +) \rtimes_{\varphi} (\omega, *)$, where + and * are the usual addition and the usual multiplication on ω .

In the paper [7] we study injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family \mathscr{F} of inductive nonempty subsets of ω . We describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that every element of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ has a form either $\alpha_{k,p}$ or $\beta_{k,p}$, where the endomorphism $\alpha_{k,p}$ is defined by the formulae

$$(i, j, [0))\alpha_{k,p} = (ki, kj, [0)),$$

 $(i, j, [1))\alpha_{k,p} = (p + ki, p + kj, [1)),$

for an arbitrary positive integer k and any $p \in \{0, ..., k-1\}$, and the endomorphism $\beta_{k,p}$ is defined by the formulae

$$(i, j, [0))\beta_{k,p} = (ki, kj, [0)),$$

 $(i, j, [1))\beta_{k,p} = (p + ki, p + kj, [0)),$

an arbitrary positive integer $k \ge 2$ and any $p \in \{1, \dots, k-1\}$. In [7] we describe the product of elements of the semigroup $\operatorname{End}^1_*(B^{\mathscr{F}}_{\omega})$:

$$\begin{split} &\alpha_{k_1,p_1}\alpha_{k_2,p_2} = \alpha_{k_1k_2,p_2+k_2p_1};\\ &\alpha_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,p_2+k_2p_1};\\ &\beta_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,k_2p_1};\\ &\beta_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1k_2,k_2p_1}. \end{split}$$

Also, here we prove that Green's relations \mathscr{R} , \mathscr{L} , \mathscr{H} , \mathscr{D} , and \mathscr{J} on $\operatorname{\boldsymbol{End}}^1_*(\boldsymbol{B}^{\mathscr{F}}_{\omega})$ coincide with the equality relation.

Later we assume that an ω -closed family ${\mathscr F}$ consists of two nonempty inductive nonempty subsets of ω .

This paper is a continuation of [7]. We study non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. We describe the structure of elements of the semigroup $\boldsymbol{End}_*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that its subsemigroup $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all non-injective non-annihilating monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the direct product the two-element left-zero semigroup and the multiplicative semigroup of positive integers and describe Green's relations on $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Remark 2. By Proposition 1 of [5] for any ω -closed family \mathscr{F} of inductive subsets in $\mathscr{P}(\omega)$ there exists an ω -closed family \mathscr{F}^* of inductive subsets in $\mathscr{P}(\omega)$ such that $[0) \in \mathscr{F}^*$ and the semigroups $B_{\omega}^{\mathscr{F}}$ and $B_{\omega}^{\mathscr{F}^*}$ are isomorphic. Hence without loss of generality we may assume that the family \mathscr{F} contains the set [0).

If \mathscr{F} is an arbitrary ω -closed family \mathscr{F} of inductive subsets in $\mathscr{P}(\omega)$ and $[s) \in \mathscr{F}$ for some $s \in \omega$ then

$$\mathbf{B}_{\omega}^{\{[s)\}} = \{(i, j, [s)) : i, j \in \omega\}$$

is a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ [5] and by Proposition 3 of [4] the semigroup $\boldsymbol{B}_{\omega}^{\{[s)\}}$ is isomorphic to the bicyclic semigroup.

Lemma 1. Let $\mathscr{F} = \{[0), [1)\}$ and let \mathfrak{e} be a monoid endomorphism of the semigroup $\mathbf{B}_{\omega}^{\mathscr{F}}$. If $(i_1, j_1, F)\mathfrak{e} = (i_2, j_2, F)\mathfrak{e}$ for distinct two elements $(i_1, j_1, F), (i_2, j_2, F)$ of $\mathbf{B}_{\omega}^{\mathscr{F}}$ for some $F \in \mathscr{F}$ then \mathfrak{e} is the annihilating endomorphism of $\mathbf{B}_{\omega}^{\mathscr{F}}$.

Proof. By Theorem 1 of [5] the image $(\boldsymbol{B}_{\omega}^{\mathscr{F}})\mathfrak{e}$ is a subgroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. By Theorem 4(iii) of [4] every \mathscr{H} -class in $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a singleton, and hence \mathfrak{e} is the annihilating monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

Lemma 2. Let $\mathscr{F} = \{[0), [1)\}$. Then $(\mathbf{B}_{\omega}^{\mathscr{F}})\mathfrak{e} \subseteq \mathbf{B}_{\omega}^{\{[0)\}}$ for any non-injective monoid endomorphism \mathfrak{e} of $\mathbf{B}_{\omega}^{\mathscr{F}}$.

Proof. Since \mathfrak{e} is an monoid endomorphism of $\mathbf{B}_{\omega}^{\mathscr{F}}$, $(0,0,[0))\mathfrak{e}=(0,0,[0))$. By Proposition 3 of [4] the subsemigroup $\mathbf{B}_{\omega}^{\{[0)\}}$ of $\mathbf{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic semigroup and hence by Corollary 1.32 of [1] the image $(\mathbf{B}_{\omega}^{\{[0)\}})\mathfrak{e}$ either is isomorphic to the bicyclic semigroup or is a cyclic subgroup of $\mathbf{B}_{\omega}^{\mathscr{F}}$. If S is a subsemigroup of $\mathbf{B}_{\omega}^{\mathscr{F}}$ which is isomorphic to the bicyclic semigroup then by Proposition 4 of [5] there exists $F \in \mathscr{F}$ such that $S \subseteq \mathbf{B}_{\omega}^{\{F\}}$. Since $(0,0,[0))\mathfrak{e}=(0,0,[0))$, Proposition 4 from [5] implies that $(\mathbf{B}_{\omega}^{\{[0)\}})\mathfrak{e} \subseteq \mathbf{B}_{\omega}^{\{[0)\}}$ in the case when the image $(\mathbf{B}_{\omega}^{\{[0)\}})\mathfrak{e}$ is isomorphic to the bicyclic semigroup. In the case when the image $(\mathbf{B}_{\omega}^{\{[0)\}})\mathfrak{e}$ is isomorphic to the cyclic group we have that the equality $(0,0,[0))\mathfrak{e}=(0,0,[0))$ implies that $(\mathbf{B}_{\omega}^{\{[0)\}})\mathfrak{e} \subseteq \{(0,0,[0))\}\subseteq \mathbf{B}_{\omega}^{\{[0)\}}$, because by Theorem 4(iii) of [4] every \mathscr{H} -class in $\mathbf{B}_{\omega}^{\mathscr{F}}$ is a singleton.

Next, by Proposition 3 of [4] the subsemigroup $B_{\omega}^{\{[1)\}}$ of $B_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic semigroup and hence by Corollary 1.32 of [1] the image $(B_{\omega}^{\{[1)\}})\mathfrak{e}$ either is isomorphic to the bicyclic semigroup or is a cyclic subgroup of $B_{\omega}^{\mathscr{F}}$. Suppose that the image $(B_{\omega}^{\{[1)\}})\mathfrak{e}$ is isomorphic to the bicyclic semigroup and $(B_{\omega}^{\{[1)\}})\mathfrak{e} \subseteq B_{\omega}^{\{[1)\}}$. Then monoid endomorphism \mathfrak{e} of $B_{\omega}^{\mathscr{F}}$ is injective. Indeed, injectivity of the restriction $\mathfrak{e}|_{B_{\omega}^{\{[1)\}}}B_{\omega}^{\{[1)\}}\to B_{\omega}^{\{[1)\}}$, Proposition 4 of [5], Corollary 1.32 of [1], Theorem 4(iii) of [4], and the equality $(0,0,[0))\mathfrak{e}=(0,0,[0))$ imply that either the restriction $\mathfrak{e}|_{B_{\omega}^{\{[0)\}}}B_{\omega}^{\{[0)\}}\to B_{\omega}^{\{[0)\}}$ is an injective mapping or is an annihilating endomorphism. In the case when the restriction $\mathfrak{e}|_{B_{\omega}^{\{[0)\}}}B_{\omega}^{\{[0)\}}\to B_{\omega}^{\{[0)\}}$ is an injective mapping we get that the endomorphism \mathfrak{e} is injective. If the image $(B_{\omega}^{\{[0)\}})\mathfrak{e}$ is a singleton then by Lemma 1 we have that \mathfrak{e} is the annihilating monoid endomorphism of $B_{\omega}^{\mathscr{F}}$. In the both cases we obtain that $(B_{\omega}^{\mathscr{F}})\mathfrak{e}\subseteq B_{\omega}^{\{[0)\}}$.

Example 1. Let $\mathscr{F} = \{[0), [1)\}$ and k be an arbitrary non-negative integer. We define a map $\gamma_k \colon \mathbf{B}_{\omega}^{\mathscr{F}} \to \mathbf{B}_{\omega}^{\mathscr{F}}$ by the formulae

$$(i, j, [0))\gamma_k = (i, j, [1))\gamma_k = (ki, kj, [0))$$

for all $i, j \in \omega$.

We claim that $\gamma_k \colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an endomorphism. Example 2 and Proposition 5 from [5] imply that the map $\gamma_1 \colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a homomorphic retraction of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, and hence it is a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. By Lemma 2 of [8] every monoid endomorphism \mathfrak{h} of the semigroup \boldsymbol{B}_{ω} has the following form

$$(i,j)\mathfrak{h} = (ki,kj),$$
 for some $k \in \omega$.

This implies that the map γ_k is a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

Example 2. Let $\mathscr{F} = \{[0), [1)\}$ and k be an arbitrary non-negative integer. We define a map $\delta_k \colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ by the formulae

$$(i, j, [0))\delta_k = (ki, kj, [0))$$
 and $(i, j, [1))\delta_k = (k(i+1), k(j+1), [0))$

for all $i, j \in \omega$.

Proposition 1. Let $\mathscr{F} = \{[0), [1)\}$. Then for any $k \in \omega$ the map δ_k is an endomorphism of the monoid $\mathbf{B}_{\omega}^{\mathscr{F}}$.

Proof. Since by Proposition 3 of [4] the subsemigroups $\boldsymbol{B}_{\omega}^{\{[0)\}}$ and $\boldsymbol{B}_{\omega}^{\{[1)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are isomorphic to the bicyclic semigroup, by Lemma 2 of [8] the restrictions $\delta_k \upharpoonright_{\boldsymbol{B}_{\omega}^{\{[0)\}}} : \boldsymbol{B}_{\omega}^{\{[0)\}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\delta_k \upharpoonright_{\boldsymbol{B}_{\omega}^{[1)}} : \boldsymbol{B}_{\omega}^{\{[1)\}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ of δ_k are homomorphisms. Hence it sufficient to show that the following equalities

$$(i_1, j_1, [0))\delta_k \cdot (i_2, j_2, [1))\delta_k = ((i_1, j_1, [0)) \cdot (i_2, j_2, [1)))\delta_k;$$

$$(i_1, j_1, [1))\delta_k \cdot (i_2, j_2, [0))\delta_k = ((i_1, j_1, [1)) \cdot (i_2, j_2, [0)))\delta_k,$$

hold for any $i_1, j_1, i_2, j_2 \in \omega$.

We observe that the above equalities are trivial in the case when k = 0. Hence later we assume that k is a positive integer.

Then we have that

$$\begin{aligned} &(i_1,j_1,[0])\delta_k \cdot (i_2,j_2,[1])\delta_k = (ki_1,kj_1,[0]) \cdot (k(i_2+1),k(j_2+1),[0]) = \\ &= \begin{cases} &(ki_1+k(i_2+1)-kj_1,k(j_2+1),(kj_1-k(i_2+1)+[0)) \cap [0)), & \text{if } kj_1 < k(i_2+1);\\ &(ki_1,k(j_2+1),[0]) \cap [0)), & \text{if } kj_1 < k(i_2+1);\\ &(ki_1,k(j_2+1),[0]) \cap (k(i_2+1)-kj_1+[0))), & \text{if } kj_1 > k(i_2+1);\\ &(ki_1,k(j_2+1),k(j_2+1),[0)), & \text{if } j_1 < i_2 + 1;\\ &(ki_1,k(j_2+1),k(j_2+1),[0)), & \text{if } j_1 < i_2;\\ &(ki_1+i_2+1-j_1),k(j_2+1),[0)), & \text{if } j_1 < i_2;\\ &(k(i_1+i_2+1-j_1),k(j_2+1),[0)), & \text{if } j_1 < i_2;\\ &(k(i_1+i_2+1-j_1),k(j_2+1),[0)), & \text{if } j_1 = i_2 + 1;\\ &(ki_1,k(j_2+1),k(j_2+1),k(j_2+1),[0]), & \text{if } j_1 < i_2;\\ &(k(i_1+j_2+1-j_1),k(j_2+1),k(j_2+1),[0]), & \text{if } j_1 < i_2;\\ &(k(i_1+j_2+1-j_1),k(j_2+1),k(j_$$

and

$$\begin{aligned} &(i_{1},j_{1},[1))\delta_{k}\cdot(i_{2},j_{2},[0))\delta_{k} = (k(i_{1}+1),k(j_{1}+1),[0))\cdot(ki_{2},kj_{2},[0)) = \\ &= \begin{cases} &(k(i_{1}+1)+ki_{2}-k(j_{1}+1),kj_{2},(k(j_{1}+1)-ki_{2}+[0))\cap[0)), & \text{if } k(j_{1}+1) < ki_{2}; \\ &(k(i_{1}+1),kj_{2},[0)\cap[0)), & \text{if } k(j_{1}+1) = ki_{2}; \\ &(k(i_{1}+1),k(j_{1}+1)+kj_{2}-ki_{2},[0)\cap(ki_{2}-k(j_{1}+1)+[0))), & \text{if } k(j_{1}+1) > ki_{2} \end{cases} \\ &= \begin{cases} &(k(i_{1}+i_{2}-j_{1}),kj_{2},[0)), & \text{if } j_{1}+1 < i_{2}; \\ &(k(i_{1}+1),kj_{2},[0)), & \text{if } j_{1}+1 = i_{2}; \\ &(k(i_{1}+1),k(j_{1}+1+j_{2}-i_{2}),[0)), & \text{if } j_{1}+1 > i_{2} \end{aligned}$$

$$= \begin{cases} (k(i_1+i_2-j_1),kj_2,[0)), & \text{if } j_1+1 < i_2; \\ (k(i_1+1),kj_2,[0)), & \text{if } j_1+1 = i_2; \\ (k(i_1+1),k(j_1+1+j_2-i_2),[0)), & \text{if } j_1 = i_2; \\ (k(i_1+1),k(j_1+1+j_2-i_2),[0)), & \text{if } j_1+1 > i_2 \end{cases}$$

$$= \begin{cases} (k(i_1+i_2-j_1),kj_2,[0)), & \text{if } j_1+1 < i_2; \\ (k(i_1+1),kj_2,[0)), & \text{if } j_1+1 = i_2; \\ (k(i_1+1),k(j_2+1),[0)), & \text{if } j_1 = i_2; \\ (k(i_1+1),k(j_1+j_2-i_2+1),[0)), & \text{if } j_1 > i_2, \end{cases}$$

$$((i_1, j_1, [1)) \cdot (i_2, j_2, [0))) \delta_k = \begin{cases} (i_1 + i_2 - j_1, j_2, (j_1 - i_2 + [1)) \cap [0)) \delta_k, & \text{if } j_1 < i_2; \\ (i_1, j_2, [1) \cap [0)) \delta_k, & \text{if } j_1 = i_2; \\ (i_1, j_1 + j_2 - i_2, [1) \cap (i_2 - j_1 + [0))) \delta_k, & \text{if } j_1 > i_2 \end{cases}$$

$$= \begin{cases} (i_1 + i_2 - j_1, j_2, [0)) \delta_k, & \text{if } j_1 < i_2; \\ (i_1, j_2, [1)) \delta_k, & \text{if } j_1 = i_2; \\ (i_1, j_1 + j_2 - i_2, [1)) \delta_k, & \text{if } j_1 > i_2 \end{cases}$$

$$= \begin{cases} (k(i_1 + i_2 - j_1), kj_2, [0), & \text{if } j_1 + 1 < i_2; \\ (k(i_1 + i_2 - j_1), kj_2, [0)), & \text{if } j_1 = i_2; \\ (k(i_1 + 1), k(j_2 + 1), [0)), & \text{if } j_1 = i_2; \\ (k(i_1 + 1), k(j_1 + j_2 - i_2 + 1), [0)), & \text{if } j_1 + 1 < i_2; \\ (k(i_1 + 1), k(j_2 + 1), [0)), & \text{if } j_1 + 1 < i_2; \\ (k(i_1 + 1), k(j_2 + 1), [0)), & \text{if } j_1 = i_2; \\ (k(i_1 + 1), k(j_1 + j_2 - i_2 + 1), [0)), & \text{if } j_1 = i_2; \end{cases}$$

This completes the proof of the statement of the proposition.

Remark 3. It obvious that if \mathfrak{e} is the annihilating endomorphism of the monoid $\mathbf{B}_{\omega}^{\mathscr{F}}$ then $\mathfrak{e} = \gamma_0 = \delta_0$.

By $End_*(B_{\omega}^{\mathscr{F}})$ we denote the semigroup of all non-injective monoid endomorphisms of the monoid $B_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0), [1)\}.$

Theorems 1 and 2 describe the algebraic structure of the semigroup $\boldsymbol{End}_*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Theorem 1. If $\mathscr{F} = \{[0), [1)\}$, then for any non-injective monoid endomorphism \mathfrak{e} of the monoid $\mathbf{B}_{\omega}^{\mathscr{F}}$ only one of the following conditions holds:

- (1) \mathfrak{e} is the annihilating endomorphism, i.e., $\mathfrak{e} = \gamma_0 = \delta_0$;
- (2) $\mathfrak{e} = \gamma_k$ for some positive integer k;
- (3) $\mathfrak{e} = \delta_k$ for some positive integer k.

Proof. Fix an arbitrary non-injective monoid endomorphism \mathfrak{e} of the monoid $\mathbf{B}_{\omega}^{\mathscr{F}}$. If \mathfrak{e} is the annihilating endomorphism then statement (1) holds. Hence, later we assume that the endomorphism \mathfrak{e} is not annihilating.

By Lemma 1 the restriction $\mathfrak{e}_{B_{\omega}^{\{[0)\}}} B_{\omega}^{\{[0)\}} \to B_{\omega}^{\mathscr{F}}$ of the endomorphism \mathfrak{e} is an injective mapping. Since by Proposition 3 of [4] the subsemigroup $B_{\omega}^{\{[0)\}}$ of $B_{\omega}^{\mathscr{F}}$ are isomorphic to the bicyclic semigroup, the injectivity of the restriction $\mathfrak{e}_{B_{\omega}^{\{[0)\}}}$ of the endomorphism \mathfrak{e} , Proposition 4 of [5], and Lemma 2 of [8] imply that there exists a positive integer k such that

(3)
$$(i, j, [0)) \mathfrak{e} = (ki, kj, [0)),$$

for all $i, j \in \omega$.

By Lemma 1 the restriction $\mathfrak{e}_{B_{\omega}^{\{[1)\}}} = B_{\omega}^{\{[1)\}} \to B_{\omega}^{\mathscr{F}}$ of the endomorphism \mathfrak{e} is an injective mapping, and by Lemma 2 we have that $(B_{\omega}^{\{[1)\}})\mathfrak{e} \subseteq B_{\omega}^{\{[0)\}}$. By Proposition 1.4.21(6) of [9] a homomorphism of inverse

semigroups preserves the natural partial order, and hence the following inequalities

$$(1, 1, [0)) \preceq (0, 0, [1)) \preceq (0, 0, [0)),$$

Lemma 2, and Propositions 2 of [5] imply that

$$(k, k, [0)) = (1, 1, [0))\mathfrak{e} \preceq (s, s, [0)) = (0, 0, [1))\mathfrak{e} \preceq (0, 0, [0)) = (0, 0, [0))\mathfrak{e}$$

for some $s \in \{0, 1, ..., k\}$. Again by Proposition 1.4.21(6) of [9] and by Lemma 2 we get that

$$(1,1,[1))\mathfrak{e} = (s+p,s+p,[0))$$

for some non-negative integer p. If p = 0 then $(1, 1, [1))\mathfrak{e} = (0, 0, [1))\mathfrak{e}$. By Lemma 1 the endomorphism \mathfrak{e} is annihilating. Hence we assume that p is a positive integer.

Let $(0,1,[1))\mathfrak{e}=(x,y,[0))$ for some $x,y\in\omega$. By Proposition 1.4.21(1) of [9] and Lemma 4 of [4] we have that

$$(1,0,[1))\mathfrak{e} = ((0,1,[1))^{-1})\mathfrak{e} = ((0,1,[1))\mathfrak{e})^{-1} = (x,y,[0))^{-1} = (y,x,[0)).$$

Since

$$(0,1,[1)) \cdot (1,0,[1)) = (0,0,[1))$$
 and $(0,1,[1)) \cdot (1,0,[1)) = (0,0,[1)),$

the equalities (0,0,[1))e = (s,s,[0)) and (1,1,[1))e = (s+p,s+p,[0)) imply that

$$(s, s, [0)) = (0, 0, [1))\mathfrak{e} = ((0, 1, [1)) \cdot (1, 0, [1)))\mathfrak{e} = (0, 1, [1))\mathfrak{e} \cdot (1, 0, [1))\mathfrak{e} = (x, y, [0)) \cdot (y, x, [0)) = (x, x, [0))$$

and

$$(s+p,s+p,[0)) = (1,1,[1))\mathfrak{e} = ((1,0,[1)) \cdot (0,1,[1)))\mathfrak{e} = (1,0,[1))\mathfrak{e} \cdot (0,1,[1))\mathfrak{e} = (y,x,[0)) \cdot (x,y,[0)) = (y,y,[0)).$$

This and the definition of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ imply that

$$(0,1,[1))\mathfrak{e} = (s,s+p,[0))$$
 and $(1,0,[1))\mathfrak{e} = (s+p,s,[0)).$

Then for any positive integers n_1 and n_2 by usual calculations we get that

$$(0, n_1, [1))\mathfrak{e} = (\underbrace{(0, 1, [1)) \cdot \ldots \cdot (0, 1, [1))}_{n_1 \text{-times}})\mathfrak{e} = \underbrace{(0, 1, [1))\mathfrak{e} \cdot \ldots \cdot (0, 1, [1))\mathfrak{e}}_{n_1 \text{-times}} = (s, s + p, [0))^{n_1} = (s, s + n_1 p, [0))$$

and

$$(n_2, 0, [1))\mathfrak{e} = (\underbrace{(1, 0, [1)) \cdot \ldots \cdot (1, 0, [1))}_{n_2 \text{-times}})\mathfrak{e} = \underbrace{(1, 0, [1))\mathfrak{e} \cdot \ldots \cdot (1, 0, [1))\mathfrak{e}}_{n_2 \text{-times}} = (s + p, s, [0))^{n_2} = (s + n_2 p, s, [0)),$$

and hence

(4)
$$(n_1, n_2, [1))\mathfrak{e} = (s + n_1 p, s + n_2 p, [0)).$$

The definition of the natural partial order on the semigroup $\mathbf{B}_{\omega}^{\mathscr{F}}$ (see Proposition 4 of [5]) imply that for any positive integer m we have that

$$(m+1, m+1, [0)) \preceq (m, m, [1)) \preceq (m, m, [0)).$$

Then by equalities (3), (4), and Proposition 1.4.21(6) of [9] we obtain that

$$(k(m+1),k(m+1),[0)) = (m+1,m+1,[0))\mathfrak{e} \preccurlyeq (s+pm,s+pm,[0)) = (m,m,[1))\mathfrak{e} \preccurlyeq (m,m,[0))\mathfrak{e} = (km,km,[0)).$$

The above inequalities and the definition of the natural partial order on the semigroup $\mathbf{B}_{\omega}^{\mathscr{F}}$ (see Proposition 4 of [5]) imply that $km \leq s + pm \leq k(m+1)$ for any positive integer m. This implies that

$$k \leqslant \frac{s}{m} + p \leqslant k + \frac{1}{m},$$

and since p is a positive integer we get that p = k. Hence by (4) we get that

(5)
$$(n_1, n_2, [1))\mathfrak{e} = (s + n_1 k, s + n_2 k, [0)),$$

for all $n_1, n_2 \in \omega$.

It is obvious that if $s \in \{1, ..., k-1\}$ then \mathfrak{e} is an injective monoid endomorphism of the semigroup. Hence we have that either s = 0 or s = k, Simple verifications show that

$$\mathfrak{e} = \left\{ \begin{array}{ll} \gamma_k, & \text{if } s = 0; \\ \delta_k, & \text{if } s = k. \end{array} \right.$$

This completes the proof of the theorem.

Theorem 2. Let $\mathscr{F} = \{[0], [1]\}$. Then for all positive integers k_1 and k_2 the following conditions hold:

- $(1) \gamma_{k_1} \gamma_{k_2} = \gamma_{k_1 k_2};$
- (2) $\gamma_{k_1}\delta_{k_2} = \gamma_{k_1k_2};$
- (3) $\delta_{k_1} \gamma_{k_2} = \delta_{k_1 k_2};$
- $(4) \ \delta_{k_1} \delta_{k_2} = \delta_{k_1 k_2}.$

Proof. (1) For any $i, j \in \omega$ we have that

$$(i, j, [0))\gamma_{k_1}\gamma_{k_2} = (k_1i, k_1j, [0))\gamma_{k_2} = (k_1k_2i, k_1k_2j, [0)),$$

and $(i, j, [1))\gamma_{k_1} = (i, j, [0))\gamma_{k_1}$. This implies that $\gamma_{k_1}\gamma_{k_2} = \gamma_{k_1k_2}$.

(2) Since

$$(i, j, [0))\gamma_{k_1}\delta_{k_2} = (k_1i, k_1j, [0))\delta_{k_2} = (k_1k_2i, k_1k_2j, [0)),$$

and $(i, j, [1))\gamma_{k_1} = (i, j, [0))\gamma_{k_1}$ for all $i, j \in \omega$, we get that $\gamma_{k_1}\delta_{k_2} = \gamma_{k_1k_2}$.

(3) For any $i, j \in \omega$ we have that

$$(i, j, [0))\delta_{k_1}\gamma_{k_2} = (k_1i, k_1j, [0))\gamma_{k_2} = (k_1k_2i, k_1k_2j, [0)),$$

and

$$(i, j, [1))\delta_{k_1}\gamma_{k_2} = (k_1(i+1), k_1(j+1), [0))\gamma_{k_2} = (k_1k_2(i+1), k_1k_2(j+1), [0)),$$

and hence $\delta_{k_1}\gamma_{k_2}=\delta_{k_1k_2}$.

(4) For any $i, j \in \omega$ we have that

$$(i, j, [0))\delta_{k_1}\delta_{k_2} = (k_1i, k_1j, [0))\delta_{k_2} = (k_1k_2i, k_1k_2j, [0)),$$

and

$$(i, j, [1))\delta_{k_1}\delta_{k_2} = (k_1(i+1), k_1(j+1), [0))\delta_{k_2} = (k_1k_2(i+1), k_1k_2(j+1), [0)),$$

and hence $\delta_{k_1}\delta_{k_2} = \delta_{k_1k_2}$.

By \mathfrak{e}_0 we denote the annihilating monoid endomorphism of the monoid $\mathbf{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0), [1)\}$, i.e., $(i, j, [p))\mathfrak{e}_0 = (0, 0, [0))$ for all $i, j \in \omega$ and p = 0, 1. We put $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}}) = \mathbf{End}_*(\mathbf{B}_{\omega}^{\mathscr{F}}) \setminus \{\mathfrak{e}_0\}$. Theorem 2 implies that $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ is a subsemigroup of $\mathbf{End}_*(\mathbf{B}_{\omega}^{\mathscr{F}})$.

Theorem 2 implies the following corollary.

Corollary 1. If $\mathscr{F} = \{[0), [1)\}$, then the elements γ_1 and δ_1 are unique idempotents of the semigroup $End^*(B_{\omega}^{\mathscr{F}})$.

Next, by $\mathfrak{L}\mathfrak{Z}_2$ we denote the left zero semigroup with two elements and by \mathbb{N}_u the multiplicative semigroup of positive integers.

Proposition 2. Let $\mathscr{F} = \{[0), [1)\}$. Then the semigroup $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ is isomorphic to the direct product $\mathfrak{L}\mathfrak{Z}_2 \times \mathbb{N}_u$.

Proof. Put $LZ_2 = \{c, d\}$. We define a map $\mathfrak{I}: \mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}}) \to \mathfrak{L}\mathfrak{Z}_2 \times \mathbb{N}_u$ by the formula

$$(\mathfrak{e})\mathfrak{I} = \left\{ \begin{array}{ll} (c,k), & \text{if } \mathfrak{e} = \gamma_k; \\ (d,k), & \text{if } \mathfrak{e} = \delta_k. \end{array} \right.$$

It is obvious that such defined map \mathfrak{I} is bijective, and by Theorem 2 it is a homomorphism.

Theorem 3 describes Green's relations on the semigroup $End^*(B_{\omega}^{\mathscr{F}})$. Later by $End^*(B_{\omega}^{\mathscr{F}})^1$ we denote the semigroup $End^*(B_{\omega}^{\mathscr{F}})$ with adjoined identity element.

Theorem 3. Let $\mathscr{F} = \{[0), [1)\}$. Then the following statements hold:

- (1) $\gamma_{k_1} \mathscr{R} \gamma_{k_2}$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ if and only if $k_1 = k_2$;
- (2) $\gamma_{k_1} \mathscr{R} \delta_{k_2}$ does not hold in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ for any $\gamma_{k_1}, \delta_{k_2}$;
- (3) $\delta_{k_1} \mathscr{R} \delta_{k_2}$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ if and only if $k_1 = k_2$; (4) $\gamma_{k_1} \mathscr{L} \gamma_{k_2}$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ if and only if $k_1 = k_2$;
- (5) $\gamma_{k_1} \mathscr{L} \delta_{k_2}$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ if and only if $k_1 = k_2$;
- (6) $\delta_{k_1} \mathscr{L} \delta_{k_2}$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ if and only if $k_1 = k_2$;
- (7) \mathscr{H} is the identity relation on $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$;
- (8) $\mathfrak{e}_1 \mathscr{D} \mathfrak{e}_2$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$ if and only if there exists a positive integer k such that $\mathfrak{e}_1, \mathfrak{e}_2 \in \{\gamma_k, \delta_k\}$;
- (9) $\mathscr{D} = \mathscr{J} \text{ in } \mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}}).$

Proof. (1) (\Rightarrow) Suppose that $\gamma_{k_1} \mathscr{R} \gamma_{k_2}$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})^1$ such that $\gamma_{k_1} = \gamma_{k_2} \mathfrak{e}_1$ and $\gamma_{k_2} = \gamma_{k_1} \mathfrak{e}_2$. The equality $\gamma_{k_1} = \gamma_{k_2} \mathfrak{e}_1$ and Theorem 2 imply that there exists a positive integer p such that either $\mathfrak{e}_1 = \gamma_p$ or $\mathfrak{e}_1 = \delta_p$. In both above cases by Theorem 2 we have that

$$\gamma_{k_1} = \gamma_{k_2} \mathfrak{e}_1 = \gamma_{k_2} \gamma_p = \gamma_{k_2} \delta_p = \gamma_{k_2 p},$$

and hence $k_2|k_1$. The proof of the statement that $\gamma_{k_2} = \gamma_{k_1} \mathfrak{e}_2$ implies that $k_1|k_2$ is similar. Therefore we get that $k_1 = k_2$.

Implication (\Leftarrow) is trivial.

Statement (2) follows from Theorem 2(2).

The proof of statement (3) is similar to (1).

(4) (\Rightarrow) Suppose that $\gamma_{k_1} \mathcal{L} \gamma_{k_2}$ in $\mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathbf{End}^*(\mathbf{B}_{\omega}^{\mathscr{F}})^1$ such that $\gamma_{k_1} = \mathfrak{e}_1 \gamma_{k_2}$ and $\gamma_{k_2} = \mathfrak{e}_2 \gamma_{k_1}$. The equality $\gamma_{k_1} = \mathfrak{e}_1 \gamma_{k_2}$ and Theorem 2 imply that there exists a positive integer p such that $\mathfrak{e}_1 = \gamma_p$. Then we have that

$$\gamma_{k_1} = \mathfrak{e}_1 \gamma_{k_2} = \gamma_p \gamma_{k_2} = \gamma_{pk_2},$$

and hence $k_2|k_1$. The proof of the statement that $\gamma_{k_2} = \mathfrak{e}_2 \gamma_{k_1}$ implies that $k_1|k_2$ is similar. Therefore we get that $k_1 = k_2$.

Implication (\Leftarrow) is trivial.

(5) (\Rightarrow) Suppose that $\gamma_{k_1} \mathscr{L} \delta_{k_2}$ in $\operatorname{End}^*(B_{\omega}^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \operatorname{End}^*(B_{\omega}^{\mathscr{F}})^1$ such that $\gamma_{k_1} = \mathfrak{e}_1 \delta_{k_2}$ and $\delta_{k_2} = \mathfrak{e}_2 \gamma_{k_1}$. The equality $\gamma_{k_1} = \mathfrak{e}_1 \delta_{k_2}$ and Theorem 2 imply that there exists a positive integer p such that $\mathfrak{e}_1 = \gamma_p$. Then we have that

$$\gamma_{k_1} = \mathfrak{e}_1 \delta_{k_2} = \gamma_p \delta_{k_2} = \gamma_{pk_2},$$

and hence $k_2|k_1$. The equality $\delta_{k_2} = \mathfrak{e}_2 \gamma_{k_1}$ and Theorem 2 imply that there exists a positive integer q such that $\mathfrak{e}_1 = \delta_q$. Then we have that

$$\delta_{k_2} = \mathfrak{e}_2 \gamma_{k_1} = \delta_q \gamma_{k_1} = \gamma_{qk_1},$$

and hence $k_1|k_2$. Thus we get that $k_1 = k_2$.

Implication (\Leftarrow) is trivial.

(6) (\Rightarrow) Suppose that $\delta_{k_1} \mathscr{L} \delta_{k_2}$ in $\operatorname{End}^*(B_{\omega}^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \operatorname{End}^*(B_{\omega}^{\mathscr{F}})^1$ such that $\delta_{k_1} = \mathfrak{e}_1 \delta_{k_2}$ and $\delta_{k_2} = \mathfrak{e}_2 \delta_{k_1}$. The equality $\delta_{k_1} = \mathfrak{e}_1 \delta_{k_2}$ and Theorem 2 imply that there exists a positive integer p such that $\mathfrak{e}_1 = \delta_p$. Then we have that

$$\delta_{k_1} = \mathfrak{e}_1 \delta_{k_2} = \delta_p \delta_{k_2} = \delta_{pk_2},$$

and hence $k_2|k_1$. The proof of the statement that $\delta_{k_2} = \mathfrak{e}_2 \delta_{k_1}$ implies that $k_1|k_2$ is similar. Hence we get that $k_1 = k_2$.

Implication (\Leftarrow) is trivial.

(7) By statements (1), (2), and (3), \mathscr{R} is the identity relation on the semigroup $\operatorname{\boldsymbol{End}}^*(B_{\omega}^{\mathscr{F}})$. Then so is \mathscr{H} , because $\mathscr{H} \subseteq \mathscr{R}$.

Statement (8) follows from statements (1)–(6).

(9) Suppose to the contrary that $\mathscr{D} \neq \mathscr{J}$ in $\operatorname{\boldsymbol{End}}^*(B_\omega^{\mathscr{F}})$. Since $\mathscr{D} \subseteq \mathscr{J}$, statement (8) implies that there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \operatorname{\boldsymbol{End}}^*(B_\omega^{\mathscr{F}})^1$ such that $\mathfrak{e}_1 \mathscr{J} \mathfrak{e}_2$ and $\mathfrak{e}_1, \mathfrak{e}_2 \notin \{\gamma_k, \delta_k\}$ for any positive integer k. Then there exist distinct positive integers k_1 and k_2 such that $\mathfrak{e}_1 \in \{\gamma_{k_1}, \delta_{k_1}\}$ and $\mathfrak{e}_2 \in \{\gamma_{k_2}, \delta_{k_2}\}$. Without loss of generality we may assume that $k_1 < k_2$. Since $\mathfrak{e}_1 \mathscr{J} \mathfrak{e}_2$ there exist $\mathfrak{e}'_1, \mathfrak{e}'_2, \mathfrak{e}''_1, \mathfrak{e}''_2 \in \operatorname{\boldsymbol{End}}^*(B_\omega^{\mathscr{F}})^1$ such that $\mathfrak{e}_1 = \mathfrak{e}'_1\mathfrak{e}_2\mathfrak{e}''_1$ and $\mathfrak{e}_2 = \mathfrak{e}'_2\mathfrak{e}_2\mathfrak{e}''_2$. Since $\mathfrak{e}_1 \in \{\gamma_{k_1}, \delta_{k_1}\}$ and $\mathfrak{e}_2 \in \{\gamma_{k_2}, \delta_{k_2}\}$, the equality $\mathfrak{e}_1 = \mathfrak{e}'_1\mathfrak{e}_2\mathfrak{e}''_1$, Theorems 1 and 2 imply that $k_2|k_1$. This contradicts the inequality $k_1 < k_2$. The obtained contradiction implies the requested statement

Remark 4. Since \mathfrak{e}_0 is zero of the semigroup $End_*(B_\omega^{\mathscr{F}})$, the classes of equivalence of Green's relations of non-zero elements of $End_*(B_\omega^{\mathscr{F}})$ in the semigroup $End_*(B_\omega^{\mathscr{F}})$ coincide with their corresponding classes of equivalence in $End^*(B_\omega^{\mathscr{F}})$, and moreover we have that $L_{\mathfrak{e}_0} = R_{\mathfrak{e}_0} = H_{\mathfrak{e}_0} = D_{\mathfrak{e}_0} = J_{\mathfrak{e}_0} = \{\mathfrak{e}_0\}$ in the semigroup $End_*(B_\omega^{\mathscr{F}})$.

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