Harish-Chandra Theorem for Two-parameter Quantum Groups

Naihong Hu* and Hengyi Wang

Abstract. The centre of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ is determined through the Harish-Chandra homomorphism. Based on the Rosso form and the representation theory of weight modules, we prove that when rank $\mathfrak{g}$ is even, the Harish-Chandra homomorphism is an isomorphism, and in particular, the centre of the quantum group $\breve{U}_{r,s}(\mathfrak{g})$ of the weight lattice type is a polynomial algebra $\mathbb{K}[z_{\varpi_1},\cdots,z_{\varpi_n}]$, where canonical central elements z_λ ($\lambda\in\Lambda^+$) are turned out to be uniformly expressed. For rank $\mathfrak{g}$ to be odd, we figure out a new invertible extra central generator z_*, which doesn't survive in $U_q(\mathfrak{g})$, and we get a larger centre containing $\mathbb{K}[z_{\varpi_1},\cdots,z_{\varpi_n}]\otimes_{\mathbb{K}}\mathbb{K}[z_*,z_*^{-1}]$.

1. Introduction

In mathematics and theoretical physics, by quantum groups mean a class of noncommutative and noncocommutative Hopf algebras. It was Drinfel'd [**18**] and Jimbo [**34**] who independently defined $U_q(\mathfrak{g})$ as a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$ for any semisimple Lie algebra $\mathfrak{g}$. Such q-deformed objects provide the universal solutions for the quantum Yang-Baxter equation and numerous quantum invariants for knots or links even 3-manifolds (eg. [**41, 47, 46**] etc. and references therein).

A number of works on the centre of quantum groups have been developed over the last three decades, including explicit generators and structures. Tanisaki proved that the quantum Harish-Chandra homomorphism is an isomorphism [**44, 45**]. He used the quantum Killing form of $U_q(\mathfrak{g})$ (which is also called the Rosso form [**43**]) to show the image of Harish-Chandra homomorphism, the Casimir element and a detailed proof of the existence of universal R-matrix [**45**]. Also, Joseph and Letzter proved the Harish-Chandra isomorphism theorem [**35**], they pointed out that the centre $Z(U_q)$ is not necessarily a polynomial algebra, $Gr_k(\mathcal{O}_f^q)$ and $Z(U_q)$ are not always isomorphic. A couple of years ago, Li-Xia-Zhang proved that $Z(U_q)$ is isomorphic to a polynomial algebra in types A_1, B_n, C_n, D_{2n}, E_7, E_8, F_4, G_2, while in the remaining cases it is isomorphic to a quotient of polynomial algebra [**36**]. A general description of the centres of quantum groups $\breve{U}_q(\mathfrak{g})$ of

2020 *Mathematics Subject Classification.* Primary 17B37, 81R50; Secondary 17B35.

Key words and phrases. Two-parameter quantum groups, Centre, Harish-Chandra homomorphism, Rosso form.

This work is supported by the NNSF of China (Grant No. 12171155), and in part by the Science and Technology Commission of Shanghai Municipality (Grant No. 22DZ2229014).

weight lattice types was earlier considered by Etingof in [**20**], where he concluded that $Z(\breve{U}_q) \cong \mathbb{K}[c_{\varpi_1}, \cdots, c_{\varpi_n}]$, with c_{ϖ_i} is the quantum partial trace of $\Gamma = R^{21}R$ on $L(\varpi_i)$ (coming from Reshetikhin et al [**23, 41**]). The theorem was in fact generalized to the affine cases [**20**]. Dai supplemented a detailed proof for the theorem of $Z(\breve{U}_q)$ and an explicit formula of generator c_λ [**10**] via an operator Γ from the work of R.B. Zhang et al [**22, 47**]. Recently, Luo-Wang-Ye settled the Harish-Chandra theorem for some quantum superalgebras [**39**].

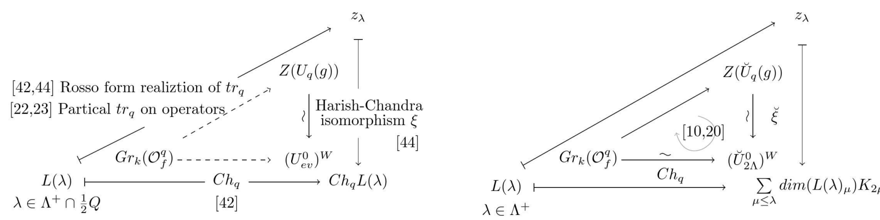

FIGURE 1. the central structure of the one-parameter quantum group

Since Benkart-Witherspoon redefined a class of two-parameter quantum groups of type A motivated by up-down algebra [**3**], and Bergeron-Gao-Hu studied the structures of two-parameter quantum groups of type B, C, D [**5**], the representation theory have progressed simultaneously [**6, 3**]. A series of work on other types have been done in [**1, 29, 31, 40, 9**], etc.

In this paper, we focus on the description of the centres of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ in the case when r, s are indeterminates. We give a uniform way to prove that the Harish-Chandra homomorphism ξ of $U_{r,s}(\mathfrak{g})$ is injective when $\mathfrak{g}$ is simple with even rank. This in particular recovers the work of Benkart-Kang-Lee for type A_{2n} [**2**], Hu-Shi for type B_{2n} [**30**] and Gan for type G_2 [**21**]. To do this, there are 3 steps to do: (i) $\xi : Z(U) \to U^0$ is injective. (ii) The image Im is in the subalgebra $\subseteq (U^0_\flat)^W$. (iii) $\xi : Z(U) \to (U^0_\flat)^W$ is surjective. With the help of the weight module theory established by [**6, 27, 40**], we find that step (i) and (ii) only rely on the non-degeneracy of matrices $R - S$ and $R + S$ derived from the the structure constant matrix. Then we construct a certain element z_λ ($\lambda \in \Lambda^+ \cap Q$) to

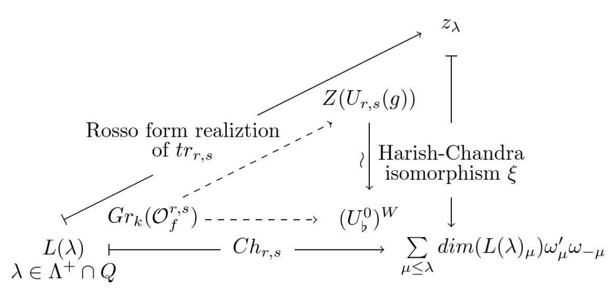

FIGURE 2. the Harish-Chandra theorem of $U_{r,s}(\mathfrak{g})$ with even rank

realize the quantum trace $\mathrm{tr}_{r,s}$ on the weight module $L(\lambda)$ by the Rosso form, that is, $\langle z_\lambda, - \rangle_U = \mathrm{tr}_{L(\lambda)}(- \circ \Theta)$. These z_λ are central elements which are used to prove (iii). Although we successfully establish the Harish-Chandra isomorphism (see the vertical map in the Fig. 2), owing to the fact that the Rosso form realization of $\mathrm{tr}_{r,s}$ and the quantum character all require $\lambda \in \Lambda^+ \cap Q$ (this is distinct from the one-parameter case: $\lambda \in \Lambda^+ \cap \frac{1}{2}Q$), they are not always well-defined on the whole $\mathrm{Gr}_k(\mathcal{O}_f^{r,s}) := \mathrm{Gr}(O_f^{r,s}) \otimes_{\mathbb{Z}} \mathbb{K}$. To overcome this difficulty, we need to extend $U_{r,s}(\mathfrak{g})$ to its weight lattice type $\breve{U}_{r,s}(\mathfrak{g})$ such that all the maps above are well-defined (see Fig. 3). It turns out that the Harish-Chandra homomorphism $\breve{\xi}$ is also an isomorphism. Furthermore, the bottom map $\mathrm{Ch}_{r,s}$ is shown to be an isomorphism of algebras. Finally, the deformation theory on weight modules [**27**] tells us that our $\mathcal{O}_f^{r,s}$ is equivalent to the category of finite-dimensional weight modules $\mathcal{O}_f^q$ of $U_q(\mathfrak{g})$ as braided tensor categories. Hence the centre $Z(\breve{U}_{r,s}) \cong (\breve{U}_\flat^0)^W \cong \mathrm{Gr}_k(O_f^{r,s}) \cong \mathrm{Gr}_k(O_f^q)$, which is also a polynomial algebra as in the one-parameter setting, that is, $Z(\breve{U}_{r,s}) \cong \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}]$.

$$
\begin{tikzcd}
L(\lambda) \arrow[ddd, mapsto, "\text{deformation theory}"'] & & & z_\lambda \arrow[ddd, mapsto] \\
& Gr_k(\mathcal{O}_f^q) \arrow[d, "\wr"] & Z(\breve{U}_{r,s}(g)) \arrow[d, "\wr"', "\breve{\xi}"] & \\
& Gr_k(\mathcal{O}_f^{r,s}) \arrow[ur] \arrow[r, "\sim"] \arrow[ur, phantom, shift right=4, "\circlearrowleft"] & (\breve{U}_\flat^0)^W & \\
\substack{L(\lambda) \\ \lambda \in \Lambda^+} \arrow[rr, mapsto, "Ch_{r,s}"'] \arrow[uuurrr, mapsto] & & \sum_{\mu \leq \lambda} dim(L(\lambda)_\mu)\omega'_\mu\omega_{-\mu} &
\end{tikzcd}
$$

FIGURE 3. the central structure of $\breve{U}_{r,s}(\mathfrak{g})$ with even rank

As for the cases of odd rank, by the fact that the matrix $R+S$ is of corank 1, it is not sufficient to prove the injectivity of the Harish-Chandra homomorphism and characterize $\mathrm{Im}(\xi)$ in the same way. However, this degeneracy provides a unique invertible central generator z_* (it will degenerate to 1 in the one-parameter case, i.e., $r = q$, $s = q^{-1}$, $\omega_i' = \omega_i^{-1}$), which is a fixed point of ξ but $z_* \notin (U_\flat^0)^W$. Thus we have $\mathrm{Im}(\xi) \supseteq (U_\flat^0)^W \otimes \mathbb{K}[z_*, z_*^{-1}]$. It recovers the case of type A_{2n+1} [**2**].

The paper is structured as follows. Section 2 recalls two-parameter quantum groups with their Rosso forms, the Harish-Chandra homomorphisms and results in the related literature. Then we introduce two matrices relevant to the structure constant matrix. Section 3 proves that the Harish-Chandra homomorphism ξ is injective when $n = \mathrm{rank}(\mathfrak{g})$ is even. Section 4 characterize the image of ξ and proves that $\mathrm{Im}(\xi)$ falls in the subalgebra $(U_\flat^0)^W$ when rank n is even. Section 5 proves that the Harish-Chandra image $\xi(Z(U)) \supseteq (U_\flat^0)^W$. Then the Harish-Chandra theorem $\xi : Z(U) \cong (U_\flat^0)^W$ holds when rank n is even. While, in the odd rank case, we construct a new extra generator z_* of the centre, which leads to the Harish-Chandra image $\xi(Z(U)) \supseteq (U_\flat^0)^W \otimes \mathbb{K}[z_*^{\pm 1}]$. We also gives an alternative description to the central elements by taking partial quantum trace. The last section introduces the extended two-parameter quantum group $\breve{U}_{r,s}(\mathfrak{g})$ of weight lattice type and proves Harish-Chandra theorem $\breve{\xi} : Z(\breve{U}_{r,s}(\mathfrak{g})) \cong (\breve{U}_\flat^0)^W$ when rank n is even. Particularly, in this case the centre $Z(\breve{U})$ is a polynomial algebra. While for the cases of odd rank n, we get a larger centre containing $\mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}] \otimes_{\mathbb{K}} \mathbb{K}[z_*, z_*^{-1}]$.

2. Preliminaries

2.1. The two-parameter quantum group $U_{r,s}(\mathfrak{g})$ and its Hopf algebra structure. A root system Φ of one complex simple Lie algebra $\mathfrak{g}$ is a finite subset of a Euclidean space E. We fix a simple root set of Φ and denote it by Π. Let W be the Weyl group of the root system Φ and $\sigma_i \in W$ be the simple reflection corresponding to the simple root α_i. For data of Φ and Π, see Carter's book [**7**, p.543]. Let $C=(c_{ij})_{n\times n}$ be the Cartan matrix, where $c_{ij}=\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$.

Denote $Q=\bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ as the root lattice of $\mathfrak{g}$. Put $\alpha_i^\vee=\frac{2\alpha_i}{(\alpha_i,\alpha_i)}$, let $\Lambda=\bigoplus_{i=1}^n \mathbb{Z}\varpi_i$ be the weight lattice and $\Lambda^+=\bigoplus_{i=1}^n \mathbb{Z}_+\varpi_i$ be the set of dominant integral weights, where $\varpi_i=\sum_{k=1}^n c^{ki}\alpha_k$ is the i-th fundamental weight, and $(c^{ij})_{n\times n}=C^{-1}$ is the inverse of the Cartan matrix.

Let r,s be two indeterminates and $\mathbb{Q}(r,s)$ be the rational functions field. Let $\mathbb{K}\supseteq\mathbb{Q}(r,s)$ be a field contains $r^{\frac{1}{m^2}}, s^{\frac{1}{m^2}}$, where $m=\min\{k\in\mathbb{Z}^+ \mid k\Lambda\subseteq Q\}=\det(C)$. Let $r_i=r^{d_i}=r^{\frac{(\alpha_i,\alpha_i)}{\ell}}$ and $s_i=s^{d_i}=s^{\frac{(\alpha_i,\alpha_i)}{\ell}}$ for $i=1,\dots,n$, where $\ell=2$ for type A_n, C_n, D_n, E_6, E_7, E_8, G_2, and $\ell=1$ for type B_n, F_4.

Let $A=(a_{ij})_{n\times n}$ be the structure constant matrix of $U_{r,s}(\mathfrak{g})$, which is given by the Euler form of $\mathfrak{g}$ as follows.

Definition 1. [**26**] *The Euler form of $\mathfrak{g}$ is the bilinear form $\langle -,-\rangle$ defined on the root lattice Q satisfying*

$$\langle i,j\rangle := \langle \alpha_i,\alpha_j\rangle = \begin{cases} d_ic_{ij} & i<j, \\ d_i & i=j, \\ 0 & i>j. \end{cases}$$

For type D, it is necessary to revise $\langle n-1,n\rangle=-1$, $\langle n,n-1\rangle=1$ ([**5, 27**]).

It can be linearly extended to the weight lattice Λ such that

$$\langle \varpi_i,\varpi_j\rangle := \sum_{k,l=1}^n c^{ki}c^{lj}\langle k,l\rangle.$$

Write $a_{ij}=r^{\langle j,i\rangle}s^{-\langle i,j\rangle}$, and $A=(a_{ij})_{n\times n}$ is the structure constant matrix of $U_{r,s}(\mathfrak{g})$. Denote $R_{ij}=\langle j,i\rangle$, $S_{ij}=-\langle i,j\rangle$. Then we have $a_{ij}=r^{R_{ij}}s^{S_{ij}}$ and $R=-S^T$.

Definition 2. [**5, 26**] *Let $U=U_{r,s}(\mathfrak{g})$ be the unital associative algebra over $\mathbb{K}$, generated by elements $e_i, f_i, \omega_i^{\pm1}, \omega_i'^{\pm1}$ $(i=1,\cdots,n)$ satisfying the following relations $(X1)$—$(X4)$:*

$$\begin{aligned} &(X1)\quad \omega_i^{\pm1}\omega_j'^{\pm1}=\omega_j'^{\pm1}\omega_i^{\pm1}, \qquad && \omega_i\omega_i^{-1}=1=\omega_j'\omega_j'^{-1},\\ &(X2)\quad \omega_ie_j\omega_i^{-1}=a_{ij}e_j, && \omega_if_j\omega_i^{-1}=a_{ij}^{-1}f_j,\\ &\qquad\quad \omega_i'e_j\omega_i'^{-1}=a_{ji}^{-1}e_j, && \omega_i'f_j\omega_i'^{-1}=a_{ji}f_j,\\ &(X3)\quad [e_i,f_j]=\delta_{ij}\frac{\omega_i-\omega_i'}{r_i-s_i}, &&\\ &(X4)\quad (\mathrm{ad}_l\, e_i)^{1-c_{ij}}(e_j)=0, && (\mathrm{ad}_r\, f_i)^{1-c_{ij}}(f_j)=0, \qquad (i\neq j) \end{aligned}$$

where the left (right)-adjoint action are as follows: for any $x,y\in U_{r,s}(\mathfrak{g})$,

$$\mathrm{ad}_l\, x(y)=\sum_{(x)} x_{(1)}yS(x_{(2)}), \qquad \mathrm{ad}_r\, x(y)=\sum_{(x)} S(x_{(1)})yx_{(2)}.$$

The comultiplication $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ *is given by Proposition 3 below.*

Proposition 3. [**5, 26**] *The algebra* $U_{r,s}(\mathfrak{g})$ *has a Hopf algebra structure* $(U_{r,s}(\mathfrak{g}),\ \Delta,\ \varepsilon,\ M,\ \iota,\ S)$ *with the comultiplication* Δ*, the counit* ϵ *and the antipode* S *defined below:*

$$\begin{aligned}
&\Delta(\omega_i^{\pm1})=\omega_i^{\pm1}\otimes\omega_i^{\pm1}, && \Delta({\omega'_i}^{\pm1})={\omega'_i}^{\pm1}\otimes{\omega'_i}^{\pm1},\\
&\Delta(e_i)=e_i\otimes 1+\omega_i\otimes e_i\ , && \Delta(f_i)=1\otimes f_i+f_i\otimes\omega'_i\ ,\\
&\varepsilon(\omega_i^{\pm1})=\varepsilon({\omega'_i}^{\pm1})=1, && \varepsilon(e_i)=\varepsilon(f_i)=0,\\
&S(\omega_i^{\pm1})=\omega_i^{\mp1}, && S({\omega'_i}^{\pm1})={\omega'_i}^{\mp1},\\
&S(e_i)=-\omega_i^{-1}e_i\ , && S(f_i)=-f_i{\omega'_i}^{-1}.
\end{aligned}$$

The two-parameter quantum group $U=U_{r,s}(\mathfrak{g})$ has the triangular decomposition $U\cong U^-\otimes U^0\otimes U^+$, where U^0 is the subalgebra generated by $\{\omega_i^{\pm},{\omega'_i}^{\pm},1\leqslant i\leqslant n\}$, U^+ is generated by $\{e_i,1\leqslant i\leqslant n\}$ and U^- is generated by $\{f_i,1\leqslant i\leqslant n\}$. Let $\mathcal{B}$ be the subalgebra of U generated by $\{e_j,\omega_j^{\pm},1\leqslant j\leqslant n\}$, and $\mathcal{B}'$ the subalgebra of U generated by $\{f_j,{\omega'_j}^{\pm},1\leqslant j\leqslant n\}$.

Proposition 4. [**5, 26**] *There exists a unique skew-dual pairing* $\langle -,-\,\rangle:\mathcal{B}'\times\mathcal{B}\to\mathbb{K}$ *of the Hopf subalgebras* $\mathcal{B}$ *and* $\mathcal{B}'$ *satisfying*

$$\begin{gathered}
\langle f_i,e_j\rangle=\delta_{ij}\frac{1}{s_i-r_i},\\
\langle\omega'_i,\omega_j\rangle=a_{ji},\\
\langle{\omega'_i}^{\pm1},\omega_j^{-1}\rangle=\langle{\omega'_i}^{\pm1},\omega_j\rangle^{-1}=\langle\omega'_i,\omega_j\rangle^{\mp1},
\end{gathered}$$

for $1\leqslant i,j\leqslant n$*, and all other pairs of generators are* 0*. Moreover,* $\langle S(a),S(b)\rangle=\langle a,b\rangle$ *for* $a\in\mathcal{B}',b\in\mathcal{B}$ *.*

For $\eta=\sum_{i=1}^n\eta_i\alpha_i\in Q$, write

$$\omega_\eta=\omega_1^{\eta_1}\cdots\omega_n^{\eta_n},\qquad\qquad \omega'_\eta={\omega'_1}^{\eta_1}\cdots{\omega'_n}^{\eta_n}.$$

Then we have

$$\begin{aligned}
\omega_\eta e_i\omega_\eta^{-1}&=\langle\omega'_i,\omega_\eta\rangle e_i, & \omega_\eta f_i\omega_\eta^{-1}&=\langle\omega'_i,\omega_\eta\rangle^{-1}f_i,\\
\omega'_\eta e_i{\omega'_\eta}^{-1}&=\langle\omega'_\eta,\omega_i\rangle^{-1}e_i, & \omega'_\eta f_i{\omega'_\eta}^{-1}&=\langle\omega'_\eta,\omega_i\rangle f_i.
\end{aligned}$$

Introduce a Q-graded structure on $U_{r,s}(\mathfrak{g})$:

$$\deg\ e_i=\alpha_i,\qquad \deg\ f_i=-\alpha_i,\qquad \deg\ \omega_i=\deg\ \omega'_i=0.$$

Namely, $U_{r,s}(\mathfrak{g})$ is a Q-graded algebra, and its subalgebras $U^{\pm}$ are $Q^{\pm}$-graded.

$$U^+=\bigoplus_{\mu\in Q^+}U^+_\mu,\qquad U^-=\bigoplus_{\mu\in Q^+}U^-_{-\mu},$$

where $U^+_\mu=U^+\cap U_\mu$, $U^-_{-\mu}=U^-\cap U_{-\mu}$ and

$$U^{\pm}_\mu=\{x\in U^{\pm}\mid \omega_\eta x\omega_\eta^{-1}=\langle\omega'_\mu,\omega_\eta\rangle x,\ \omega'_\eta x{\omega'_\eta}^{-1}=\langle\omega'_\eta,\omega_\mu\rangle^{-1}x\}.$$

And $\mathcal{B}$ (resp. $\mathcal{B}'$) is also Q^+(resp. Q^-)-graded algebras.

2.2. The matrices R and S.

Proposition 5. *The matrix $R-S$ is a symmetric Cartan matrix, i.e.*

$$R-S=DC=\frac{2}{\ell}\big((\alpha_i,\alpha_j)\big)_{n\times n}$$

where the matrix $D=\mathrm{diag}(d_1,\cdots,d_n)$, $d_i=(\alpha_i,\alpha_i)/\ell$. For type B_n and F_4 we have $\ell=1$, while $\ell=2$ for type A_n, C_n, D_n, E_6, E_7, E_8 and G_2.

Proof. By definition, we have $R-S=R+R^T=(\langle j,i\rangle+\langle i,j\rangle)_{n\times n}=DC$. $\square$

The symmetric matrix $R-S$ can be considered as a metric matrix with respect to the basis $\alpha_1,\cdots,\alpha_n$ of E, and every element of Weyl group is orthogonal, we then have

Corollary 6. *Let Σ be the matrix of $\sigma\in W$ with respect to the basis $\{\alpha_i\}$, we have $(R-S)\Sigma=\Sigma^T(R-S)$.*

Now we will discuss the non-degeneracy of another important matrix $R+S$ which is crucial in the study of the Harish-Chandra theorem of $U_{r,s}(\mathfrak{g})$.

Proposition 7. *When n is even, the matrix $R+S$ is invertible; when n is odd, we have* $\mathrm{rank}(R+S)=n-1$.

Proof. We check the conclusion case by case. For type A_n, we have

$$\det(R+S)=\begin{cases}1, & 2\mid n,\\ 0, & 2\nmid n,\end{cases}$$

where

$$A=\begin{pmatrix} rs^{-1} & s & 1 & \cdots & 1 & 1 & 1\\ r^{-1} & rs^{-1} & s & \cdots & 1 & 1 & 1\\ 1 & r^{-1} & rs^{-1} & \cdots & 1 & 1 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \cdots & rs^{-1} & s & 1\\ 1 & 1 & 1 & \cdots & r^{-1} & rs^{-1} & s\\ 1 & 1 & 1 & \cdots & 1 & r^{-1} & rs^{-1}\end{pmatrix},\quad R+S=\begin{pmatrix} & 1 & & & \\ -1 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \ddots & & 1\\ & & & -1 & & 1\\ & & & & -1 & \end{pmatrix}.$$

For type B_n, we have

$$\det(R+S)=\begin{cases}2^n, & 2\mid n,\\ 0, & 2\nmid n,\end{cases}$$

where

$$A=\begin{pmatrix} r^2s^{-2} & s^2 & \cdots & 1 & 1 & 1\\ r^{-2} & r^2s^{-2} & \ddots & 1 & 1 & 1\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 1 & 1 & \ddots & r^2s^{-2} & s^2 & 1\\ 1 & 1 & \cdots & r^{-2} & r^2s^{-2} & s^2\\ 1 & 1 & \cdots & 1 & r^{-2} & rs^{-1}\end{pmatrix},\quad R+S=\begin{pmatrix} & 2 & & & \\ -2 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \ddots & & 2\\ & & & -2 & & 2\\ & & & & -2 & \end{pmatrix}.$$

For type C_n, we have

$$\det(R+S)=\begin{cases}4, & 2\mid n,\\ 0, & 2\nmid n,\end{cases}$$

where

$$A=\begin{pmatrix} rs^{-1} & s & \cdots & 1 & 1 & 1 \\ r^{-1} & rs^{-1} & \ddots & 1 & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ddots & rs^{-1} & s & 1 \\ 1 & 1 & \cdots & r^{-1} & rs^{-1} & s^2 \\ 1 & 1 & \cdots & 1 & r^{-2} & r^2s^{-2} \end{pmatrix},\ R+S=\begin{pmatrix} & 1 & & & & \\ -1 & & \ddots & & & \\ & \ddots & & \ddots & & \\ & & \ddots & & 1 & \\ & & & -1 & & 2 \\ & & & & -2 & \end{pmatrix}.$$

For type D_n, we have

$$\det(R+S)=\begin{cases} 4, & 2 \mid n, \\ 0, & 2 \nmid n, \end{cases}$$

where

$$A=\begin{pmatrix} rs^{-1} & s & \cdots & 1 & 1 & 1 \\ r^{-1} & rs^{-1} & \ddots & 1 & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ddots & rs^{-1} & s & s \\ 1 & 1 & \cdots & r^{-1} & rs^{-1} & rs \\ 1 & 1 & \cdots & r^{-1} & r^{-1}s^{-1} & rs^{-1} \end{pmatrix},\ R+S=\begin{pmatrix} & 1 & & & & \\ -1 & & \ddots & & & \\ & \ddots & & \ddots & & \\ & & \ddots & & 1 & 1 \\ & & & -1 & & 2 \\ & & & -1 & -2 & \end{pmatrix}.$$

For the family of exceptional types E_6, E_7, E_8, it suffices to list the data of type E_8 (from which we can read off the data of type E_6, E_7).

$$A=\begin{pmatrix} rs^{-1} & 1 & s & 1 & 1 & 1 & 1 & 1 \\ 1 & rs^{-1} & 1 & s & 1 & 1 & 1 & 1 \\ r^{-1} & 1 & rs^{-1} & s & 1 & 1 & 1 & 1 \\ 1 & r^{-1} & r^{-1} & rs^{-1} & s & 1 & 1 & 1 \\ 1 & 1 & 1 & r^{-1} & rs^{-1} & s & 1 & 1 \\ 1 & 1 & 1 & 1 & r^{-1} & rs^{-1} & s & 1 \\ 1 & 1 & 1 & 1 & 1 & r^{-1} & rs^{-1} & s \\ 1 & 1 & 1 & 1 & 1 & 1 & r^{-1} & rs^{-1} \end{pmatrix},$$

$$R+S=\begin{pmatrix} & & 1 & & & & & \\ & & & 1 & & & & \\ -1 & & & 1 & & & & \\ & -1 & -1 & & 1 & & & \\ & & & -1 & & 1 & & \\ & & & & -1 & & 1 & \\ & & & & & -1 & & 1 \\ & & & & & & -1 & \end{pmatrix}.$$

Hence, for types E_6 and E_8, we have $\det(R+S)=1$; for type E_7, $\det(R+S)=0$.

For type F_4, we have $\det(R+S)=4$, where

$$A=\begin{pmatrix} r^{-2}s^{-2} & s^2 & 1 & 1 \\ r^{-2} & r^{-2}s^{-2} & s^2 & 1 \\ 1 & r^{-2} & rs^{-1} & s \\ 1 & 1 & r^{-1} & rs^{-1} \end{pmatrix},\quad R+S=\begin{pmatrix} & 2 & & \\ -2 & & 2 & \\ & -2 & & 1 \\ & & -1 & \end{pmatrix}.$$

For type G_2, we have $\det(R+S)=9$, where

$$A=\begin{pmatrix} rs^{-1} & s^3 \\ r^{-3} & r^3s^{-3} \end{pmatrix}, \qquad R+S=\begin{pmatrix} & 3 \\ -3 & \end{pmatrix}.$$

When n is odd, we have corank $(R+S)=1$. We list the unique non-zero solution (up to scalar) for each type as follow.

$A_{2k+1}:\ \mathfrak{sl}_{2k+2}$	$v^*=(1,0,1,0,\cdots,1,0,1)^T$
$B_{2k+1}:\ \mathfrak{so}_{4k+3}$	$v^*=(1,0,1,0,\cdots,1,0,1)^T$
$C_{2k+1}:\ \mathfrak{sp}_{4k+2}$	$v^*=(2,0,2,0,\cdots,2,0,1)^T$
$D_{2k+1}:\ \mathfrak{so}_{4k+2}$	$v^*=(2,0,2,0,\cdots,2,1,-1)^T$
E_7	$v^*=(0,1,0,0,1,0,1)^T$

This completes the proof. $\square$

2.3. Weight modules and the category $\mathcal{O}_f^{r,s}$. Recall the structure of weight modules studied by [**6, 40**].

Let $\varrho: U^0\to\mathbb{K}$ be an algebraic homomorphism and V^ϱ be a 1-dimensional $\mathcal{B}$-module. Denote $M(\varrho)=U\otimes_{\mathcal{B}}V^\varrho$ as the Verma module with the highest weight ϱ and $L(\varrho)$ as its unique irreducible quotient.

Let $\lambda\in\Lambda$ and rewrite it as $\lambda=\sum_{i=1}^n\lambda_i\alpha_i,\ \lambda_i\in\mathbb{Q}$. Then we can define an algebraic homomorphism $\varrho^\lambda: U^0\to\mathbb{K}$, satisfying

$$\varrho^\lambda(\omega_j):=\prod_{i=1}^n\langle\omega_i',\omega_j\rangle^{\lambda_i}=\prod_{i=1}^n a_{ji}^{\lambda_i}.$$

Clearly, it satisfies the property $\varrho^{\lambda+\mu}=\varrho^\lambda\varrho^\mu$.

When $\lambda\in Q$, one would get following relations from the Hopf paring.

$$\varrho^\lambda(\omega_j)=\langle\omega_\lambda'\ ,\omega_j\rangle,\qquad \varrho^\lambda(\omega_j')=\langle\omega_j',\omega_{-\lambda}\rangle.$$

For convenience, we denote $M(\lambda):=M(\varrho^\lambda)$ and $L(\lambda):=L(\varrho^\lambda)$ when $\lambda\in\Lambda$.

Lemma 8. [**2, 6, 40**] *For the two-parameter quantum group $U_{r,s}(\mathfrak{g})$, we have*
(1) *Let v_λ be a highest weight vector of $M(\lambda)$ for $\lambda\in\Lambda^+$. Then*

$$L(\lambda)=M(\lambda)/(\sum_{i=1}^n Uf_i^{(\lambda,\alpha_i^\vee)+1}\cdot v_\lambda).$$

is a finte dimensional irreducible $U_{r,s}(\mathfrak{g})$-module. Also, it has the decomposition of weight space $L(\lambda)=\bigoplus_{\eta\leqslant\lambda}L(\lambda)_\eta$, where

$$L(\lambda)_\eta=\{x\in L(\lambda)\mid \omega_i.x=\varrho^\eta(\omega_i)x,\ \omega_i'.x=\varrho^\eta(\omega_i')x,\ 1\leqslant i\leqslant n\}.$$

(2) *The elements e_i,f_i $(1\leqslant i\leqslant n)$ act locally nilpotently on $L(\lambda)$.*

Theorem 9. [**2, 6, 40**] *Suppose rs^{-1} is not a root of unity, when $\lambda\in\Lambda^+$, we have*

$$dim\ L(\lambda)_\eta=dim\ L(\lambda)_{\sigma(\eta)},\quad \forall\ \eta\in\Lambda,\ \sigma\in W.$$

Definition 10. [**27**] *The category $\mathcal{O}_f^{r,s}$ consists of finite-dimensional $U_{r,s}(\mathfrak{g})$-modules V (of type 1) satisfying the following conditions:*
(1) *V has a weight space decomposition $V=\bigoplus_{\lambda\in\Lambda}V_\lambda$,*

$$V_\lambda=\{v\in V\mid \omega_\eta v=r^{\langle\lambda,\eta\rangle}s^{-\langle\eta,\lambda\rangle}v,\ \omega_\eta'v=r^{-\langle\eta,\lambda\rangle}s^{\langle\lambda,\eta\rangle}v,\quad\forall\ \eta\in Q\ \},$$

where $\dim(V_\lambda)$ *is finite for all* $\lambda \in \Lambda$;

(2) *there exist a finite number of weights* $\lambda_1, \cdots, \lambda_t \in \Lambda$, *such that*

$$\mathrm{Wt}(V) \subseteq \bigcup_{i=1}^{t} D(\lambda_i),$$

where $D(\lambda) := \{\mu \in \Lambda \mid \mu < \lambda\}$. *The morphisms are* $U_{r,s}(\mathfrak{g})$*-module homomorphisms.*

2.4. Rosso form and characters.

Definition 11. [**5**] *The bilinear form* $\langle - , - \rangle_U : U \times U \to \mathbb{K}$ *defined by*

$$\left\langle f_\alpha \omega'_\mu \omega_v e_\beta, f_\theta \omega'_\sigma \omega_\delta e_\gamma \right\rangle_U = \langle \omega'_\sigma, \omega_v \rangle \left\langle \omega'_\mu, \omega_\delta \right\rangle \langle f_\theta, e_\beta \rangle \left\langle S^2(f_\alpha), e_\gamma \right\rangle$$

is called the Rosso form of the two-parameter quantum group $U = U_{r,s}(\mathfrak{g})$.

Theorem 12. [**5, 40**] *The Rosso form* $\langle -, - \rangle_U$ *on* $U \times U$ *is* ad_l*-invariant, i.e.,*

$$\langle \mathrm{ad}_l(a)b, c \rangle_U = \langle b, \mathrm{ad}_l(S(a))c \rangle_U, \quad \forall\, a, b, c \in U.$$

Theorem 13. [**6, 40**] *For any* $\beta \in Q^+$, *the restriction of the skew-pairing* $\langle -, - \rangle$ *to* $\mathcal{B}'_{-\beta} \times \mathcal{B}_\beta$ *is nondegenerate.*

Definition 14. *Define a group homomorphism* $\chi_{\eta,\phi} : Q \times Q \to \mathbb{K}^\times$ *by*

$$\chi_{\eta,\phi}(\eta', \phi') = \langle \omega'_\eta, \omega_{\phi'} \rangle \langle \omega'_{\eta'}, \omega_\phi \rangle,$$

where $(\eta, \phi), (\eta', \phi') \in Q \times Q$, $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$.

Lemma 15. *If* $\chi_{\eta,\phi} = \chi_{\eta',\phi'}$, *then* $(\eta, \phi) = (\eta', \phi')$.

Proof. Let $\eta = \sum_{i=1}^n \eta_i \alpha_i$, $\eta' = \sum_{i=1}^n \eta'_i \alpha_i$, and $\phi, \phi' \in Q$, for $j = 1, \cdots, n$ we have

$$\chi_{\eta,\phi}(0, \alpha_j) = \langle \omega'_\eta, \omega_j \rangle = \prod_{i=1}^{n} \langle \omega'_i, \omega_j \rangle^{\eta_i}, \qquad \chi_{\eta',\phi'}(0, \alpha_j) = \prod_{i=1}^{n} \langle \omega'_i, \omega_j \rangle^{\eta'_i}.$$

Since $\chi_{\eta,\phi} = \chi_{\eta',\phi'}$, we have

$$\begin{aligned} 1 = \frac{\chi_{\eta,\phi}(0, \alpha_j)}{\chi_{\eta',\phi'}(0, \alpha_j)} &= \prod_{i=1}^{n} \langle \omega'_i, \omega_j \rangle^{\eta_i - \eta'_i} = \prod_{i=1}^{n} a_{ji}^{\eta_i - \eta'_i} \\ &= \prod_{i=1}^{n} r^{R_{ji}(\eta_i - \eta'_i)} s^{S_{ji}(\eta_i - \eta'_i)}, \end{aligned}$$

which in turn gives $(R - S)(\eta - \eta') = 0$. Finally, Proposition 5 yields $\eta = \eta'$. A similar argument leads to the conclusion $\phi = \phi'$. □

Theorem 16. *The Rosso form* $\langle -, - \rangle_U$ *of* $U_{r,s}(\mathfrak{g})$ *is nondegenerate.*

Proof. Since the skew-pairing $\langle -, - \rangle$ has orthogonality for the grading, it suffices to check the case when $u \in U^-_{-\nu} U^0 U^+_\mu$, if $\langle u, v \rangle_U = 0$ holds for all $v \in U^-_{-\mu} U^0 U^+_\nu$, then $u = 0$.

Denote $d_\mu = \dim\, U^+_\mu$. Let $\{u^\mu_1, \cdots, u^\mu_{d_\mu}\}$ be a basis of U^+_μ and $\{v^\mu_1, \cdots, v^\mu_{d_\mu}\}$ be its dual basis in $U^-_{-\mu}$ with respect to the Rosso form by Theorem 13, that is,

$\langle v_i^\mu, u_j^\mu\rangle_U = \delta_{ij}$. Hence $U^-_{-\nu}U^0U^+_\mu = \mathrm{span}_{\mathbb{K}}\{v_i^\nu\omega'_\eta\omega_\phi u_j^\mu \mid 1 \leqslant i \leqslant d_\nu,\ 1 \leqslant j \leqslant d_\mu\}$. Notice that

$$\begin{aligned}\langle v_i^\nu\omega'_\eta\omega_\phi u_j^\mu, v_k^\mu\omega'_{\eta'}\omega_{\phi'}u_l^\nu\rangle_U &= \langle\omega'_\eta,\omega_{\phi'}\rangle\langle\omega'_{\eta'},\omega_\phi\rangle\langle v_k^\mu,u_j^\mu\rangle\langle S^2(v_i^\nu),u_l^\nu\rangle\\ &= \delta_{kj}\delta_{il}(rs^{-1})^{\frac{2}{\ell}(\rho,\nu)}\langle\omega'_\eta,\omega_{\phi'}\rangle\langle\omega'_{\eta'},\omega_\phi\rangle,\end{aligned}$$

where ρ is the half sum of positive roots. Let $u = \sum_{i,j,\eta,\phi}k_{i,j,\eta,\phi}v_i^\nu\omega'_\eta\omega_\phi u_j^\mu$, $v = v_k^\mu\omega'_{\eta'}\omega_{\phi'}u_l^\nu$, where $1 \leqslant k \leqslant d_\mu,\ 1 \leqslant l \leqslant d_\nu,\ \eta',\phi' \in Q$. Suppose $\langle u,v\rangle_U = 0$, we have

$$\begin{aligned}0 &= \sum_{\eta,\phi}k_{k,l,\eta,\phi}(rs^{-1})^{\frac{2}{\ell}(\rho,\nu)}\langle\omega'_\eta,\omega_{\phi'}\rangle\langle\omega'_{\eta'},\omega_\phi\rangle\\ &= \sum_{\eta,\phi}k_{k,l,\eta,\phi}(rs^{-1})^{\frac{2}{\ell}(\rho,\nu)}\chi_{\eta,\phi}(\eta',\phi'),\end{aligned}$$

By Dedekind theorem, we have $k_{l,k,\eta,\phi} = 0$. so $u = 0$. □

2.5. Harish-Chandra homomorphism. Let $Z(U)$ be the centre of $U_{r,s}(\mathfrak{g})$. It follows that $Z(U) \subseteq U_0$. Now we define an algebra homomorphism $\gamma^{-\rho}: U^0 \to U^0$ as

$$\gamma^{-\rho}(\omega'_\eta\omega_\phi) = \varrho^{-\rho}(\omega'_\eta\omega_\phi)\omega'_\eta\omega_\phi.$$

Particularly, we have

$$\gamma^{-\rho}(\omega'_i\omega_i^{-1}) = (r_is_i^{-1})^{(\rho,\alpha_i^\vee)}\omega'_i\omega_i^{-1}.$$

Definition 17. *Denote $\xi : Z(U) \to U^0$ as the restricted map $\gamma^{-\rho}\circ\pi|_{Z(U)}$,*

$$\gamma^{-\rho}\pi : U_0 \to U^0 \to U^0,$$

where $\pi : U_0 \to U^0$ is the canonical projection. We call ξ the Harish-Chandra homomorphism of U.

Define a subalgebra $U^0_\flat = \bigoplus_{\eta\in Q}\mathbb{K}\omega'_\eta\omega_{-\eta}$ and let the Weyl group W act on it

$$\sigma(\omega'_\eta\omega_{-\eta}) = \omega'_{\sigma(\eta)}\omega_{-\sigma(\eta)},\ \forall\ \sigma \in W,\ \eta \in Q.$$

Theorem 18. [**2, 5, 21**] *For types A_n, B_n and G_2, when rank n is even, the Harish-Chandra homomorphism $\xi : Z(U) \to (U^0_\flat)^W$ is an algebra isomorphism.*

3. Harish-Chandra homomorphism ξ is injective with even rank

Lemma 19. *For all $\lambda \in \Lambda^+$, we have*
(i) *If $\varrho^\lambda(\omega'_\eta\omega_\phi) = 1$, then $\eta = \phi$.*
(ii) *When rank n is even, then $\varrho^\lambda(\omega'_\eta\omega_\phi) = 1$ if and only if $(\eta,\phi) = (0,0)$.*

Proof. (i) Fix $\eta,\phi \in Q,\ \lambda \in \Lambda^+$, and write them as $\eta = \sum\limits_{i=1}^n\eta_i\alpha_i,\ \phi = \sum\limits_{i=1}^n\phi_i\alpha_i,\ \lambda = \sum\limits_{i=1}^n\lambda_i\alpha_i,\quad \eta_i,\phi_i \in \mathbb{Z},\ \lambda_i \in \mathbb{Q}$, and denote $\eta_\alpha,\phi_\alpha,\lambda_\alpha$ as their column vectors with respect to the basis $\{\alpha_i\}$, respectively. Then

$$\varrho^\lambda(\omega_\phi) = \prod_{j=1}^{n}(\varrho^\lambda(\omega_j))^{\phi_i} = \prod_{j,i=1}^{n} a_{ji}^{\lambda_i\phi_j} \tag{1}$$

$$= \prod_{j,i=1}^{n} r^{R_{ji}\lambda_i\phi_j} s^{S_{ji}\lambda_i\phi_j} = r^{\phi_\alpha^T R\lambda_\alpha} \cdot s^{\phi_\alpha^T S\lambda_\alpha}.$$

Similarly, we have $\varrho^\lambda(\omega'_\eta) = s^{\eta_\alpha^T R\lambda_\alpha} r^{\eta_\alpha^T S\lambda_\alpha}$, then

$$1 = \varrho^\lambda(\omega'_\eta\omega_\phi) = r^{\lambda_\alpha^T(R^T\phi_\alpha + S^T\eta_\alpha)} \cdot s^{\lambda_\alpha^T(S^T\phi_\alpha + R^T\eta_\alpha)}.$$

It follows that

$$R^T\phi_\alpha + S^T\eta_\alpha = 0, \tag{2}$$

$$S^T\phi_\alpha + R^T\eta_\alpha = 0. \tag{3}$$

$$(R-S)^T(\phi_\alpha - \eta_\alpha) = 0. \tag{4}$$

Then by Proposition 5 we know that $R-S$ is invertible, hence $\phi = \eta$.

(ii) Let $\phi = \eta$, then (2) yields

$$(R+S)^T\eta_\alpha = 0.$$

When n is even, $\eta = 0 = \phi$ holds since $R+S$ is invertible by Proposition 7. □

Theorem 20. *The Harish-Chandra homomorphism $\xi : Z(U) \to U^0$ is injective when $n = \mathrm{rank}(\mathfrak{g})$ is even.*

Proof. Consider the triangular decomposition of $U = U^- U^0 U^+$ and set $K = \bigoplus_{v>0} U^-_{-v} U^0 U^+_v$. which is a two-sided ideal of $U_0 = U^0 \oplus K$ and $K = ker(\pi)$, the following argument shows that when $z \in Z(U)$ and $\xi(z) = 0$, we will get $z = 0$.

Let $z = \sum_{v \in Q^+} z_v$, where $z_v \in U^-_{-v} U^0 U^+_v$. Fix a minimal root $v \in Q^+$ that $z_v \neq 0$, and choose bases $\{x_l\}$ and $\{y_k\}$ for spaces U^+_{+v} and U^-_{-v} respectively. Now we write $z_v = \sum_{k,l} y_k t_{k,l} x_l,\ t_{k,l} \in U^0$. Then

$$\begin{aligned} 0 &= e_i z - z e_i \\ &= \sum_{\gamma\neq\nu}(e_i z_\gamma - z_\gamma e_i) + (e_i z_\nu - z_\nu e_i) \\ &= \sum_{\gamma\neq\nu}(e_i z_\gamma - z_\gamma e_i) + \sum_{k,l}(e_i y_k - y_k e_i)\, t_{k,l} x_l + \sum_{k,l} y_k\,(e_i t_{k,l} x_l - t_{k,l} x_l e_i)\,. \end{aligned}$$

Since $e_i y_k - y_k e_i \in U^-_{-(v-\alpha_i)} U^0$, only the second term in the equation falls in $U^-_{-(v-\alpha_i)} U^0 U^+_v$, so it is forced to be 0, that is

$$\sum_{k,l}(e_i y_k - y_k e_i)\, t_{k,l} x_l = 0.$$

Based on the triangular decomposition of U and the fact that $\{x_l\}$ is a set of basis of U^+_v, the equation $\sum_k e_i y_k t_{k,l} = \sum_k y_k e_i t_{k,l}$ holds for each pair (l,i). Fix a l with a $\lambda \in \Lambda^+$ and take an element $m = (\sum_k y_k t_{k,l}) \cdot v_\lambda \notin L(\lambda)_\lambda$ in the irreducible module $L(\lambda)$, for each i we have

$$e_i m = \sum_k e_i y_k t_{k,l} \cdot v_\lambda = \sum_k y_k e_i t_{k,l} \cdot v_\lambda = 0.$$

This shows that m generates a proper submodule of $L(\lambda)$ which contradicts the irreducibility of $L(\lambda)$, hence

$$0=m=\sum_k y_k t_{k,l}\cdot v_\lambda=\sum_k y_k.\varrho^\lambda(t_{k,l})v_\lambda.$$

Let $v=k_1\alpha_1+\cdots+k_n\alpha_n\in Q^+\setminus 0$. Taking $N=max\{k_i\mid 1\leqslant i\leqslant n\}$ and $\lambda=N\rho$, we have $(\lambda,\alpha_i^\vee)\geq k_i$. By the fact that the map $\sum_k y_k\varrho^\lambda(t_{k,l})\mapsto(\sum_k y_k\varrho^\lambda(t_{k,l}))\cdot v_\lambda$ is injective in [**6**, Theorem 2.12] and [**40**, Corollary 38], we have $\sum_k y_k\varrho^\lambda(t_{k,l})=0$. Then $\varrho^\lambda(t_{k,l})=0$ for each k since $\{y_k\}$ is a basis of U^-_{-v}.

Now let $t_{k,l}=\sum_{\eta,\phi}k_{\eta,\phi}\omega'_\eta\omega_\phi,\ k_{\eta,\phi}\in\mathbb{K}$. Since $\varrho^\lambda(t_{k,l})=0$ and there are only finitely many non-zero terms in $t_{k,l}=\sum_{\eta,\phi}k_{\eta,\phi}\omega'_\eta\omega_\phi$. Count the number of non-zero terms by p. Taking $m=1,2,\cdots,p$, we obtain

$$0=\varrho^{m\lambda}(t_{k,l})=\sum_{\eta,\phi}\varrho^\lambda(\omega'_\eta\omega_\phi)^m k_{\eta,\phi},$$

whose coefficients $\varrho^\lambda(\omega'_\eta\omega_\phi)$ form a Vandermonde matrix. It follows from Lemma 19 that when n is even, for each $\lambda\in\Lambda^+$ we have $\varrho^\lambda(\omega'_\eta\omega_\phi)=\varrho^\lambda(\omega'_\zeta\omega_\psi)$, $\Longleftrightarrow$ $(\eta,\phi)=(\zeta,\psi)$. Hence all $k_{\eta,\phi}=0$, that is, $t_{k,l}=0$. $\square$

Remark 21. *The assumption on rank n to be even is a sufficient condition for ξ to be injective. Note that in the case when rank n is odd, the proof of Propostion 5.2* (*see p. 458, -line 3* [**2**]) *for type A_n really contains a gap, as was pointed out in Remark 3.5* [**30**].

4. The image of the Harish-Chandra homomorphism ξ with even rank

Define an algebra homomorphisms $\varrho^{\lambda,\mu}$ from U^0 to $\mathbb{K}$ as $\varrho^{\lambda,\mu}=\varrho^{0,\lambda}\varrho^{\mu,0},\lambda,\mu\in\Lambda$, where

$$\begin{aligned}\varrho^{0,\mu}&:\omega'_\eta\omega_\phi\mapsto(rs^{-1})^{(\eta+\phi,\mu)},\\ \varrho^{\lambda,0}&:\omega'_\eta\omega_\phi\mapsto\varrho^\lambda(\omega'_\eta\omega_\phi).\end{aligned}$$

Lemma 22. *Suppose n is even, then*
(1) *let $u=\omega'_\eta\omega_\phi,\ \eta,\phi\in Q$. If $\varrho^{\lambda,\mu}(u)=1,\ \forall\ \lambda,\mu\in\Lambda$, then $u=1$;*
(2) *if $u\in U^0$ satisfying $\varrho^{\lambda,\mu}(u)=0$ for all $\lambda,\mu\in\Lambda$, then $u=0$.*

Proof. (1) Again, denote $\eta_\alpha,\ \phi_\alpha$ and λ_α as coordinates of $\eta,\ \phi\in Q$ and λ with respect to the basis $\{\alpha_i\}$, respectively. By $1=\varrho^{\lambda,\mu}(u)=\varrho^\lambda(u)\varrho^{0,\mu}(u)$ and the formula (1) in the proof of Lemma 19, we have

$$\begin{aligned}\varrho^\lambda(\omega'_\eta\omega_\phi)&=r^{\lambda_\alpha^T(R^T\phi_\alpha+S^T\eta_\alpha)}s^{\lambda_\alpha^T(S^T\phi_\alpha+R^T\eta_\alpha)},\\ \varrho^{0,\mu}(\omega'_\eta\omega_\phi)&=(rs^{-1})^{(\eta+\phi,\mu)}.\end{aligned}$$

It follows that

$$\begin{cases}\ \ (\eta+\phi,\mu)+\lambda_\alpha^T(R^T\phi_\alpha+S^T\eta_\alpha)=0,\\ -(\eta+\phi,\mu)+\lambda_\alpha^T(S^T\phi_\alpha+R^T\eta_\alpha)=0.\end{cases}$$

Set $\lambda=0$, we have $\eta+\phi=0$. Similarly, it leads to $\eta=\phi=0$, that is, $u=\omega'_\eta\omega_\phi=1$.

(2) Fixing a pair $(\eta,\phi)\in Q\times Q$, one can define a character $\kappa_{\eta,\phi}$ on the group $\Lambda\times\Lambda$ to be $\kappa_{\eta,\phi}:(\lambda,\mu)\mapsto\varrho^{\lambda,\mu}(\omega'_\eta\omega_\phi)$.

Let $u = \sum_{(\eta,\phi)} k_{\eta,\phi}\omega'_\eta\omega_\phi,\ k_{\eta,\phi} \in \mathbb{K}$. Then

$$0 = \varrho^{\lambda,\mu}(u) = \sum_{(\eta,\phi)} k_{\eta,\phi}\varrho^{\lambda,\mu}(\omega'_\eta\omega_\phi) = \sum_{(\eta,\phi)} k_{\eta,\phi}\kappa_{\eta,\phi}(\lambda,\mu).$$

Since characters $\{\kappa_{\eta,\phi}\}$ are different from each other, we have $k_{\eta,\phi} = 0$, that is, $u = 0$. □

PROPOSITION 23. *$\varrho^{\sigma(\lambda),\mu}(u) = \varrho^{\lambda,\mu}(\sigma^{-1}(u))$, for $u \in U^0_\flat,\ \sigma \in W,\ \lambda,\ \mu \in \Lambda$.*

PROOF. It is sufficient to check the case that $u = \omega'_\eta\omega_{-\eta}$ and $\sigma = \sigma_i$ a simple reflection. As we did earlier, use subscript α to denote ones column vectors with respect to the basis $\{\alpha_i\}$. By Corollary 6, we have

$$\begin{aligned}
\varrho^{\lambda,0}(\sigma_i^{-1}(u)) &= \varrho^{\lambda}(\omega'_{\sigma_i(\eta)}\omega_{-\sigma_i(\eta)})\\
&= r^{\lambda_\alpha^T(R^T(\sigma_i(\eta))_\alpha + S^T(\sigma_i(-\eta))_\alpha)} s^{\lambda_\alpha^T(S^T(\sigma_i(\eta))_\alpha + R^T(\sigma_i(-\eta))_\alpha)}\\
&= r^{\lambda_\alpha^T(R-S)\Sigma_i\cdot\eta_\alpha} s^{\lambda_\alpha^T(S-R)\Sigma_i\cdot\eta_\alpha}\\
&= r^{\lambda_\alpha^T\Sigma_i^T\cdot(R-S)\eta_\alpha} s^{\lambda_\alpha^T\Sigma_i^T\cdot(S-R)\eta_\alpha}\\
&= r^{(\sigma_i(\lambda))_\alpha^T(R^T\eta_\alpha + S^T(-\eta)_\alpha)} s^{(\sigma_i(\lambda))_\alpha^T(S^T\eta_\alpha + R^T(-\eta)_\alpha)}\\
&= \varrho^{\sigma_i(\lambda)}(\omega'_\eta\omega_{-\eta}) = \varrho^{\sigma_i(\lambda),0}(u),\\
\varrho^{0,\mu}(\sigma^{-1}(u)) &= \varrho^{0,\lambda}(\sigma^{-1}(\omega'_\eta\omega_{-\eta}))\\
&= (rs^{-1})^{(\sigma^{-1}(\eta)+\sigma^{-1}(-\eta),\mu)}\\
&= (rs^{-1})^{(0,\mu)} = \varrho^{0,\mu}(\omega'_\eta\omega_{-\eta}).
\end{aligned}$$

So $\varrho^{\lambda,\mu}(\sigma^{-1}(u)) = \varrho^{\sigma(\lambda),\mu}(u)$. □

Define a subalgebra $(U^0_\flat)^W = \{u \in U^0_\flat \mid \sigma(u) = u, \forall\sigma \in W\}$ and characters $\kappa_{\eta,\phi} : (\lambda,\mu) \mapsto \varrho^{\lambda,\mu}(\omega'_\eta\omega_\phi)$, on $\Lambda\times\Lambda$ for each $(\eta,\phi) \in Q\times Q$. Further we define $\kappa^i_{\eta,\phi} : (\lambda,\mu) \mapsto \varrho^{\sigma_i(\lambda),\mu}(\omega'_\eta\omega_\phi)$.

LEMMA 24. *Suppose that rank n is even, $\sigma \in W,\ \lambda,\mu \in \Lambda$. If $u \in U^0$ satisfies that $\varrho^{\sigma(\lambda),\mu}(u) = \varrho^{\lambda,\mu}(u)$, then $u \in (U^0_\flat)^W$.*

PROOF. Let $u = \sum_{(\eta,\phi)} k_{\eta,\phi}\omega'_\eta\omega_\phi \in U^0$, then

$$\sum_{(\eta,\phi)} k_{\eta,\phi}\varrho^{\lambda,\mu}(\omega'_\eta\omega_\phi) = \sum_{(\zeta,\psi)} k_{\zeta,\psi}\varrho^{\sigma(\lambda),\mu}(\omega'_\zeta\omega_\psi),$$

hence we have an equation for characters:

$$\sum_{(\eta,\phi)} k_{\eta,\phi}\kappa_{\eta,\phi} = \sum_{(\zeta,\psi)} k_{\zeta,\psi}\kappa^i_{\zeta,\psi}.$$

Comparing the two sides of the equation, for each $k_{\eta,\phi} \neq 0$, there exists one $(\zeta,\psi) \in Q\times Q$, such that $\kappa_{\eta,\phi} = \kappa^i_{\zeta,\psi}$ and $k_{\zeta,\psi} = k_{\eta,\phi}$. Then

$$\begin{aligned}
\kappa_{\eta,\phi}(0,\varpi_j) &= \varrho^{0,\varpi_j}(\omega'_\eta\omega_\phi) = (rs^{-1})^{(\eta+\phi,\varpi_j)}\\
&= \kappa^i_{\zeta,\phi}(0,\varpi_j)\\
&= \varrho^{0,\varpi_j}(\omega'_\zeta\omega_\psi) = (rs^{-1})^{(\zeta+\phi,\varpi_j)}
\end{aligned}$$

yields $\eta+\phi=\zeta+\psi$, and

$$\varrho^{\varpi_i}(\omega'_\eta\omega_\phi)=\varrho^{\sigma_i(\varpi_i)}(\omega'_\zeta\omega_\phi)=\varrho^{\varpi_i-\alpha_i}(\omega'_\zeta\omega_\phi).$$

Since $\phi=\zeta+\psi-\eta$, it follows that

$$\varrho^{\varpi_i}(\omega'_{\eta-\zeta}\omega_{-(\eta-\zeta)})\varrho^{\alpha_i}(\omega'_\zeta\omega_{-\zeta})=\varrho^{-\alpha_i}(\omega_{\zeta+\phi}).$$

Rewrite the equation in the form of $r^ks^l=1$, for $i=1,\cdots,n$, we have

$$\begin{aligned}(\varpi_i)^T_\alpha(R-S)^T(\eta-\zeta)_\alpha+\varepsilon_i^T(R-S)^T\zeta_\alpha+\varepsilon_i^TS^T(\zeta+\psi)_\alpha=0,\\ (\varpi_i)^T_\alpha(S-R)^T(\eta-\zeta)_\alpha+\varepsilon_i^T(S-R)^T\zeta_\alpha+\varepsilon_i^TR^T(\zeta+\psi)_\alpha=0,\end{aligned}$$

where ε_i is the i-th unit vector. Adding two equations, one has $\varepsilon_i^T(R+S)^T(\zeta+\psi)_\alpha=0$, $i=1,\cdots,n$. That is, $(R+S)^T(\zeta+\psi)_\alpha=0$. Since n is even, $\det(R+S)\neq 0$, we have $\zeta+\psi=0=\phi+\eta$. Then

$$u=\sum_{(\eta,-\eta)}k_{\eta,-\eta}\omega'_\eta\omega_{-\eta}\in U^0_\flat.$$

Finally, by Lemma 23,

$$\varrho^{\lambda,\mu}(\sigma^{-1}(u))=\varrho^{\sigma(\lambda),\mu}(u)=\varrho^{\lambda,\mu}(u),\quad\forall\,\lambda,\mu\in\Lambda,\ \sigma\in W,$$

which yields $u=\sigma^{-1}(u)$ for all $\sigma\in W$. So $u\in(U^0_\flat)^W$. □

THEOREM 25. *Suppose rank n is even, then $\varrho^{\lambda+\rho,\mu}(\xi(z))=\varrho^{\sigma(\lambda+\rho),\mu}(\xi(z))$, $\forall z\in Z(U)$, $\sigma\in W$, $\lambda,\mu\in\Lambda$. As a result, we have $\mathrm{Im}(\xi)\subseteq(U^0_\flat)^W$.*

PROOF. Let $z\in Z(U)$ and $\mu\in\Lambda$, take a $\lambda\in\Lambda$ such that $(\lambda,\alpha_i^\vee)\geq 0$ for some fixed i. Let $v_{\lambda,\mu}$ be the highest weight vector of the Verma module $M(\varrho^{\lambda,\mu})$. Then

$$z\cdot v_{\lambda,\mu}=\pi(z)\cdot v_{\lambda,\mu}=\varrho^{\lambda,\mu}(\pi(z))v_{\lambda,\mu}=\varrho^{\lambda+\rho,\mu}(\xi(z))v_{\lambda,\mu}.$$

That is, z acts on $M(\varrho^{\lambda,\mu})$ by scalar $\varrho^{\lambda+\rho,\mu}(\xi(z))$. On the other hand, by [**5**, Corollary 2.6] and [**40**, Property 37], let $[m]_i=\frac{r_i^m-s_i^m}{r_i-s_i}$, then we have

$$e_if_i^{(\lambda,\alpha_i^\vee)+1}\cdot v_{\lambda,\mu}=[(\lambda,\alpha_i^\vee)+1]_i\ f_i^{(\lambda,\alpha_i^\vee)}\frac{r_i^{-(\lambda,\alpha_i^\vee)}\omega_i-s_i^{-(\lambda,\alpha_i^\vee)}\omega'_i}{r_i-s_i}\cdot v_{\lambda,\mu}.$$

Notice that

$$(r_i^{-(\lambda,\alpha_i^\vee)}\omega_i-s_i^{-(\lambda,\alpha_i^\vee)}\omega'_i)\cdot v_{\lambda,\mu}=\Big(r_i^{-(\lambda,\alpha_i^\vee)}\varrho^{\lambda,\mu}(\omega_i)-s_i^{-(\lambda,\alpha_i^\vee)}\varrho^{\lambda,\mu}(\omega'_i)\Big)\cdot v_{\lambda,\mu}.$$

Since $r_i^{-(\lambda,\alpha_i^\vee)}\varrho^{\lambda,0}(\omega_i)=s_i^{-(\lambda,\alpha_i^\vee)}\varrho^{\lambda,0}(\omega'_i)$, it follows that

$$\begin{aligned}e_jf_i^{(\lambda,\alpha_i^\vee)+1}\cdot v_{\lambda,\mu}&=0, && j=1,\cdots,n,\\ zf_i^{(\lambda,\alpha_i^\vee)+1}\cdot v_{\lambda,\mu}&=\pi(z)f_i^{(\lambda,\alpha_i^\vee)+1}\cdot v_{\lambda,\mu}\\ &=\varrho^{\sigma_i(\lambda+\rho)-\rho,\mu}(\pi(z))f_i^{(\lambda,\alpha_i^\vee)+1}\cdot v_{\lambda,\mu}\\ &=\varrho^{\sigma_i(\lambda+\rho),\mu}(\xi(z))f_i^{(\lambda,\alpha_i^\vee)+1}\cdot v_{\lambda,\mu}, && \forall\ z\in Z(U).\end{aligned}$$

Hence, z acts on $M(\varrho^{\lambda,\mu})$ by scalar $\varrho^{\sigma_i(\lambda+\rho),\mu}(\xi(z))$. we have

$$\varrho^{\lambda+\rho,\mu}(\xi(z))=\varrho^{\sigma_i(\lambda+\rho),\mu}(\xi(z)).\tag{5}$$

In fact, equation (5) holds for any $\lambda \in \Lambda$. This is because if $(\lambda, \alpha_i^\vee) = -1$, then $\lambda + \rho = \sigma_i(\lambda' + \rho)$ such that (5) holds. If $(\lambda, \alpha_i^\vee) < -1$, let $\lambda' = \sigma_i(\lambda + \rho) - \rho$, then $(\lambda', \alpha_i^\vee) \geq 0$ such that (5) holds for λ'. Relacing λ' with $\sigma_i(\lambda + \rho) - \rho$ into the result that (5) holds for λ in this case. Finally, since (5) holds for each σ_i, so it holds for all $\sigma \in W$, which implies that Im $(\xi) \subseteq (U_\flat^0)^W$, by Lemma 24. $\square$

5. Central elements and the Harish-Chandra theorem

In this section, we will deal with arbitary rank.

5.1. The Harish-Chandra theorem. In what follows, we aim to prove that the subspace $(U_\flat^0)^W$ is in $\mathrm{Im}(\xi)$.

LEMMA 26. *Let $z \in U$, then $z \in Z(U)$ if and only if $\mathrm{ad}_l(x)z = \varepsilon(x)z$, $\forall\, x \in U$.*

PROOF. Suppose $z \in Z(U)$, then for all $x \in U$, we have

$$\mathrm{ad}_l(x)z = \sum_{(x)} x_{(1)} z S(x_{(2)}) = z \sum_{(x)} x_{(1)} S(x_{(2)}) = \varepsilon(x)z.$$

Conversely, if $\mathrm{ad}_l(x)z = \varepsilon(x)z$ holds for all $x \in U$, then

$$\omega_i z \omega_i^{-1} = \mathrm{ad}_l(w_i)z = \varepsilon(\omega_i)z = z.$$

Since for each generator e_i and f_i of $U_{r,s}(\mathfrak{g})$, we have

$$\begin{aligned} 0 &= \varepsilon(e_i)z = \mathrm{ad}_l(e_i)z \\ &= e_i z + \omega_i z S(e_i) = e_i z - (\omega_i z \omega_i^{-1})e_i = e_i z - z e_i, \\ 0 &= \varepsilon(f_i)z = \mathrm{ad}_l(f_i)z \\ &= zS(f_i) + f_i z S(\omega_i') = (-z f_i + f_i z)\omega_i'^{-1}. \end{aligned}$$

So $z \in Z(U)$. $\square$

LEMMA 27. *Given a bilinear form $\Psi:\ U^-_{-\mu} \times U^+_v \to \mathbb{K}$ and a pair $(\eta, \phi) \in Q \times Q$, then there exists an element $u \in U^-_{-\mu} U^0 U^+_v$ such that for any $x \in U^+_v$, $y \in U^-_{-\mu}$ and $(\zeta, \psi) \in Q \times Q$,*

$$\langle u, y\omega_\zeta' \omega_\psi x\rangle_U = \langle \omega_\zeta', \omega_\phi\rangle \langle \omega_\eta', \omega_\psi\rangle \Psi(y, x).$$

PROOF. Let $\mu \in Q^+$ and $\{u_1^\mu, \cdots, u_{d_\mu}^\mu\}$ be a basis of U^+_μ, then take a dual basis $\{v_1^\mu, \cdots, v_{d_\mu}^\mu\}$ in $U^-_{-\mu}$ with respect to $\langle -, -\rangle$. Take that

$$u = \sum_{i,j} (rs^{-1})^{-\frac{2}{\ell}(\rho,\nu)}\ \Psi(v_j^\mu, u_i^\nu)\ v_i^\nu \omega_\eta' \omega_\phi u_j^\mu,$$

then u satisfies the identity in the lemma. $\square$

DEFINITION 28. (1) *Define a U-module structure on U^* by*

$$(x \cdot f)(v) = f(\mathrm{ad}_l(S(x))v), \quad \forall\, x \in U,\ f \in U^*.$$

Then we define a morphsim $\beta:\ U \to U^*$, $u \mapsto \langle u, -\rangle_U$. It follows that β is an injective morphism of U-module since Rosso form is nondegenerate and ad-invariant.

(2) *Let M be a finite-dimensional U-module. For each $m \in M$, $f \in M^*$, define the matrix coefficient by $C_{f,m} \in U^*$, $C_{f,m}(v) = f(v \cdot m)$, $\forall\, v \in U$.*

THEOREM 29. *Let M be a finite-dimensional U-module. Decompose $M = \bigoplus_{\lambda\in\Pi(M)} M_\lambda$, where*

$$M_\lambda = \{m \in M \mid \omega_i \cdot m = \varrho^\lambda(\omega_i)m,\ \omega_i' \cdot m = \varrho^\lambda(\omega_i')m\}.$$

If the weight set of M satisfies $\mathrm{Wt}(M) \subseteq Q$, then for $f \in M^*$, $m \in M$, there exists a unique $u \in U$ such that

$$C_{f,m}(v) = \langle u, v\rangle_U, \quad \forall\, v \in U.$$

PROOF. We start by proving the existence of u. Since $C_{f,m}$ is linear with respect to $m \in M$, it is sufficient to check the case of $m \in M_\lambda$ for each λ. Suppose v is a monomial $v = y\omega_\eta'\omega_\phi x$, where $x \in U_\nu^+$, $y \in U_{-\mu}^-$, then for each $f \in M^*$, we have

$$\begin{aligned} C_{f,m}(v) &= C_{f,m}(y\omega_\eta'\omega_\phi x) = f(y\omega_\eta'\omega_\phi x.m)\\ &= \varrho^{\lambda+\nu}(\omega_\eta'\omega_\phi)f(y.x.m)\\ &= \langle\omega_\eta', \omega_{-(\lambda+\nu)}\rangle\langle\omega_{\lambda+\nu}', \omega_\phi\rangle f(y.x.m), \end{aligned}$$

Since $\Psi : (y, x) \mapsto f(y.x.m)$ is a bilinear form, by Lemma 27, there exists a unique $u_{\nu\mu}$ such that for all $v \in U_{-\mu}^- U^0 U_\nu^+$, $C_{f,m}(v) = \langle u_{\nu\mu}, v\rangle_U$ holds.

More generally, let $v \in U$ with $v = \sum_{(\mu,\nu)} v_{\mu\nu}$, where $v_{\mu\nu} \in U_{-\mu}^- U^0 U_\nu^+$. Since M is finite-dimensional, there exists a finite set Ω of pairs $(\mu, \nu) \in Q \times Q$, such that

$$C_{f,m}(v) = C_{f,m}\Big(\sum_{(\mu,\nu)} v_{\mu\nu}\Big), \quad \forall\, \nu \in U.$$

Let $u = \sum_{(\mu,\nu)\in\Omega} u_{\nu\mu}$, then

$$\begin{aligned} \langle u, v\rangle_U &= \sum_{(\mu,\nu),(\mu',\nu')\in\Omega} \langle u_{\nu'\mu'}, v_{\mu\nu}\rangle_U\\ &= \sum_{(\mu,\nu)\in\Omega} \langle u_{\nu\mu}, v_{\mu\nu}\rangle_U\\ &= \sum_{(\mu,\nu)\in\Omega} \langle u_{\mu\nu}, v\rangle_U = C_{f,m}(v). \end{aligned}$$

Here the second row of the equations holds by Lemma 13. □

LEMMA 30. *Let (M, ζ) be a weight module, where $\zeta : U \to \mathrm{End}(M)$ and define a linear map $\Theta :\ M \to M$ by*

$$m \mapsto (rs^{-1})^{-\frac{2}{\ell}(\rho,\lambda)}m, \quad \forall\, m \in M_\lambda,\ \lambda \in \Lambda.$$

Then for all $u \in U$, $\Theta \circ \zeta(u) = \zeta(S^2(u)) \circ \Theta$. That is,

$$\Theta(u.m) = S^2(u).\Theta(m), \quad \forall\, m \in M.$$

PROOF. It is sufficient to check it for the generators e_i, f_i. Notice that

$$\begin{aligned} \langle\omega_i', \omega_i\rangle &= a_{ii} = (rs^{-1})^{d_i} = (rs^{-1})^{\frac{(\alpha_i,\alpha_i)}{\ell}}\\ &= (rs^{-1})^{\frac{2}{\ell}(\varpi_i,\alpha_i)} = (rs^{-1})^{\frac{2}{\ell}(\rho,\alpha_i)}\ , \end{aligned}$$

which yields for any $m \in M_\lambda$,

$$\begin{aligned}
S^2(e_i).\Theta(m) &= (rs^{-1})^{-\frac{2}{\ell}(\rho,\lambda)}(S^2(e_i).m)\\
&= (rs^{-1})^{-\frac{2}{\ell}(\rho,\lambda)}\langle\omega_i',\omega_i\rangle^{-1}e_i.m\\
&= (rs^{-1})^{-\frac{2}{\ell}(\rho,\lambda+\alpha_i)}e_i.m = \Theta(e_i.m),\\
S^2(f_i).\Theta(m) &= S^2(f_i)(rs^{-1})^{-\frac{2}{\ell}(\rho,\lambda)}.m\\
&= \langle\omega_i',\omega_i\rangle(rs^{-1})^{-\frac{2}{\ell}(\rho,\lambda)}f_i.m\\
&= (rs^{-1})^{-\frac{2}{\ell}(\rho,\lambda-\alpha_i)}f_i.m = \Theta(f_i.m).
\end{aligned}$$

This completes the proof. $\square$

Proposition 31. *For $\lambda \in \Lambda^+$, define a quantum trace $t_\lambda \in U^*$ as $t_\lambda(u) := \mathrm{tr}_{L(\lambda)}(u\Theta)$. If $\lambda \in \Lambda^+ \cap Q$, then $t_\lambda \in \mathrm{Im}(\beta)$, and $z_\lambda := \beta^{-1}(t_\lambda) \in Z(U)$.*

Proof. (1) Since $\beta(u) = \langle u, -\rangle_U$ is injective and $t_\lambda = \mathrm{tr}_{L(\lambda)}(-\circ\Theta) \in U^*$, denote the dimension of $L(\lambda)$ by d, take a basis $\{m_i\}_{i=1}^d$ in $L(\lambda)$ and its dual basis $\{f_i\}_{i=1}^d$ in $L(\lambda)^*$, we have

$$v.\Theta(m_i) = \sum_{j=1}^d f_j(v.\Theta(m_i))m_j = \sum_{j=1}^d C_{f_j,\Theta(m_i)}(v)m_j,$$
$$t_\lambda(v) = \mathrm{tr}_{L(\lambda)}(v\circ\Theta) = \sum_{i=1}^d C_{f_i,\Theta(m_i)}(v).$$

By Theorem 27, for each $i = 1, \cdots, d$, there exists $u_i \in U$ that realizes the matrix coefficient by $\langle u_i, v\rangle_U = C_{f_i,\Theta(m_i)}(v)$. Write $u = \sum_{i=1}^d u_i$, then

$$\begin{aligned}
\beta(u)(v) = \langle u, v\rangle_U &= \sum_{i=1}^d \langle u_i, v\rangle_U\\
&= \sum_{i=1}^d C_{f_i,\Theta(m_i)}(v) = t_\lambda(v).
\end{aligned}$$

It follows that $t_\lambda \in \mathrm{Im}(\beta)$.

(2) Since U^* has a U-module structure $(x\cdot f)(v) = f(\mathrm{ad}_l(S(x))v)$, $\forall\, x \in U,\ f \in U^*,$ then for any $x, u \in U$, we have

$$\begin{aligned}
(S^{-1}(x).t_\lambda)(u) &= t_\lambda(\mathrm{ad}_l(x)u) = \mathrm{tr}_{L(\lambda)}\big(\sum_{(x)} x_{(1)}uS(x_{(2)})\Theta\big)\\
&= \mathrm{tr}_{L(\lambda)}\big(u\sum_{(x)} S(x_{(2)})\Theta x_{(1)}\big) = \mathrm{tr}_{L(\lambda)}\big(u\sum_{(x)} S(x_{(2)})S^2(x_{(1)})\Theta\big)\\
&= \mathrm{tr}_{L(\lambda)}\big(uS\big(\sum_{(x)} S(x_{(1)})x_{(2)}\big)\Theta\big) = (\iota\circ\varepsilon)(x)\,\mathrm{tr}_{L(\lambda)}(u\Theta)\\
&= (\iota\circ\varepsilon)(x)t_\lambda(u).
\end{aligned}$$

Substituting $S(x)$ for x above, we get $x.t_\lambda = (\iota \circ \varepsilon)(x)t_\lambda$, $\forall\, x \in U$. Notice the fact $t_\lambda \in \operatorname{Im}\beta$, we define $z_\lambda := \beta^{-1}(t_\lambda)$, then

$$x.t_\lambda = x.(\beta(z_\lambda))) = (x.\beta)(z_\lambda) = \beta(\mathrm{ad}_l(x)(z_\lambda)),$$
$$(\iota \circ \varepsilon)(x)t_\lambda = (\iota \circ \varepsilon)(x)\beta(z_\lambda) = \beta((\iota \circ \varepsilon)(x)z_\lambda).$$

Since β is injective, $\mathrm{ad}_l(x)(z_\lambda) = (\iota \circ \varepsilon)(x)z_\lambda$, $\forall\, x \in U$. Therefore, by Lemma 26, it follows that $z_\lambda \in Z(U)$. $\square$

As described in Theorem 29, for each $\lambda \in \Lambda$, there is the unique simple weight module $L(\lambda)$ with the weight space decomposition $\bigoplus_{\tau \le \lambda} L(\lambda)_\tau$. Define $d_\tau = \dim\, L(\lambda)_\tau$ and set a basis $\{m_i^\tau\}_{i=1}^{d_\tau}$ for this weight space. Then $L(\lambda)$ has a basis $\{m_i^\tau\}_{\tau,i}$ and $L(\lambda)^*$ has the canonical dual basis $\{f_\tau^i\}_{\tau,i}$. Since $L(\lambda)$ has finite dimension, there exists a finite set $\Omega \subseteq Q \times Q$ such that for any $v \in U = U^-U^0U^+, m \in L(\lambda)$, we have $v \cdot m = \sum_{(\mu,\nu)\in\Omega} v_{\mu,\nu} \cdot m$, where $v_{\mu,\nu} \in U^-_{-\mu}U^0U^+_\nu$.

Theorem 32. *For each $\lambda \in \Lambda^+ \bigcap Q$, the central element z_λ is*

$$z_\lambda = \sum_{\tau \le \lambda} \sum_{\mu \in Q^+} \sum_{i,j} (rs^{-1})^{-\frac{2}{\ell}(\rho,\tau+\mu)} tr(v_j^\mu u_i^\mu \circ P_\tau)\; v_i^\mu \omega'_{\tau+\mu}\omega^{-1}_{\tau+\mu} u_j^\mu,$$

where $\{u_j^\mu\}_{j=1}^{d_\mu}$ is a basis of U^+_μ and $\{v_i^\mu\}_{i=1}^{d_\mu}$ is the dual basis of $U^-_{-\mu}$ with respect to the restriction of $\langle -, - \rangle$ to $U^-_{-\mu} \times U^+_\mu$, and P_τ is the projector from $L(\lambda)$ to $L(\lambda)_\tau$.

Proof. We have shown that for any $v \in U$,

$$\mathrm{tr}_{L(\lambda)}(v \circ \Theta) = \sum_{\tau,l} f_\tau^l(v \cdot \Theta(m_l^\tau)) = \sum_{\tau,l} C_{f_\tau^l,\Theta(m_l^\tau)}(v).$$

Firstly, we restrict v to any graded space $U^-_{-\mu}U^0U^+_\nu$ and take a monomial $v = y\omega'_\eta\omega_\phi x$, $x \in U^+_\nu, y \in U^-_{-\mu}$, then by Theorem 29, we have

$$\begin{aligned} C_{f_\tau^l,\Theta(m_l^\tau)}(v) &= f_\tau^l(y\omega'_\eta\omega_\phi x \cdot \Theta(m_l^\tau)) \\ &= \varrho^{\tau+\nu}(\omega'_\eta\omega_\phi) f_\tau^l(yx \cdot \Theta(m_i^\tau)) \\ &= \langle \omega'_\eta, \omega_{-(\tau+\nu)}\rangle \langle \omega'_{\tau+\nu}, \omega_\phi\rangle f_\tau^l(yx \cdot \Theta(m_l^\tau)). \end{aligned}$$

Then by Lemma 27, put $\Psi(y,x) = f_\tau^l(yx \cdot \Theta(m_i^\tau))$, then we get an element

$$\begin{aligned} z_{\nu,\mu}^{(\tau,l)} &= \sum_{i,j} (rs^{-1})^{-\frac{2}{\ell}(\rho,\nu)} \Psi(v_j^\mu, u_i^\nu)\; v_i^\nu \omega'_{\tau+\nu}\omega_{-(\tau+\nu)} u_j^\mu \\ &= \sum_{i,j} (rs^{-1})^{-\frac{2}{\ell}(\rho,\tau+\nu)}\; f_\tau^l(v_j^\mu u_i^\nu . m_l^\tau)\; v_i^\nu \omega'_{\tau+\nu}\omega^{-1}_{\tau+\nu} u_j^\mu, \end{aligned}$$

so that $\langle z_{\nu,\mu}^{(\tau,l)}, v\rangle_U = C_{f_\tau^i,\Theta(m_i^\tau)}(v)$, for any $v \in U^-_{-\mu}U^0U^+_\nu$.

Further, we add up all the elements labeled by the finite set Ω and write $z^{(\tau,l)} = \sum_{(\nu,\mu)\in\Omega} z_{\nu,\mu}^{(\tau,l)}$, then

$$\langle z^{(\tau,l)}, v\rangle_U = C_{f_\tau^l,\Theta(m_l^\tau)}(v), \quad \forall\, v \in U.$$

Finally, since $z_\lambda = \sum_{\tau,l} z^{(\tau,l)}$, we get the expression

$$z_\lambda = \sum_{\tau \le \lambda} \sum_{l=1}^{d^\tau} \sum_{(\nu,\mu)\in\Omega} \sum_{(i,j)=(1,1)}^{(d_\nu,d_\mu)} (rs^{-1})^{-\frac{2}{\ell}(\rho,\tau+\nu)}\; f_\tau^l(v_j^\mu u_i^\nu \cdot m_l^\tau)\; v_i^\nu \omega'_{\tau+\nu}\omega^{-1}_{\tau+\nu} u_j^\mu.$$

Since $\{m_i^\tau\}_{i=1}^{d_\tau}$ and $\{f_\tau^i\}_{\tau,i}$ are dual to each other, when $\nu \neq \mu$ the component in $z_{\nu,\mu}^{(\tau,l)} = 0$. Also, we can simplify the expression of z_λ by projectors P_τ, then

$$z_\lambda = \sum_{\tau\leq\lambda}\sum_{\mu\in Q^+}\sum_{i,j}(rs^{-1})^{-\frac{2}{\ell}(\rho,\tau+\mu)}\ tr(v_j^\mu u_i^\mu \circ P_\tau)\ v_i^\mu \omega'_{\tau+\mu}\omega_{\tau+\mu}^{-1}u_j^\mu.$$

This completes the proof. $\square$

Theorem 33. *Suppose rank n is odd, there is one extra invertible central generator $z_* := \prod_{i=1}^n(\omega_i\omega_i')^{v_i^*}$ in $U_{r,s}(\mathfrak{g})$ (which doesn't survive in $U_q(\mathfrak{g})$), where the vector v^* is given in Proposition 7.*

Proof. It suffices to show z_* commutes with generators e_i and f_i.

$$\begin{aligned}
z_*e_i = \big(\prod_{j=1}^n(\omega_j\omega_j')^{v_j^*}\big)e_i &= e_i\big(\prod_{j=1}^n(\omega_j\omega_j')^{v_j^*}\big)\prod_{j=1}^n(a_{ji}a_{ij}^{-1})^{v_j^*}\\
&= e_iz_*(rs)^{\sum_{j=1}^n(\langle i,j\rangle-\langle j,i\rangle)v_j^*}\\
&= e_iz_*(rs)^{(R+S)v^*} = e_iz_*,
\end{aligned}$$

$$\begin{aligned}
z_*f_i &= f_i\big(\prod_{j=1}^n(\omega_j\omega_j')^{v_j^*}\big)\prod_{j=1}^n(a_{ji}a_{ij}^{-1})^{-v_j^*}\\
&= fz_*(rs)^{-(R+S)v^*} = f_iz_*.
\end{aligned}$$

This completes the proof. $\square$

Proposition 34. *When rank n is odd, the central element z_* is a fixed point of the Harish-Chandra homomorphism ξ, and $\xi(z_*) = z_* \notin (U_\flat^0)^W$.*

Proof. The first statement is proved directly as follow.

$$\begin{aligned}
\xi(z_*) &= (\prod_{j=1}^n\varrho^{-\rho}((\omega_j\omega_j')^{v_j^*}))z_* = (\prod_{i,j=1}^n(a_{ji}^{-1}a_{ij})^{v_j^*/2})z_*\\
&= (rs)^{\frac{1}{2}\sum_{i,j=1}^n(R+S)_{ij}v_j^*}z_* = z_*.
\end{aligned}$$

Since $U_{r,s}$ is also a Q-bigraded Hopf algebra [**27**] where $e_i \in (U_{r,s})_{(\alpha_i,0)}$, $f_i \in (U_{r,s})_{(0,-\alpha_i)}$, $\omega_i,\omega_i' \in (U_{r,s})_{(\alpha_i,-\alpha_i)}$. Thus all generators $\omega'_\eta\omega_{-\eta}$ of $U_\flat^0$ have the same bigrade $(0,0)$, they can not generate an element graded by $(\eta,-\eta),\eta\in Q\setminus 0$, which leads to the second statement. $\square$

Theorem 35. *If rank n is even, then the Harish-Chandra homomorphism $\xi : Z(U_{r,s}(\mathfrak{g})) \to (U_\flat^0)^W$ is an algebra isomorphism. If rank n is odd, we have $\mathrm{Im}(\xi) \supseteq (U_\flat^0)^W \otimes \mathbb{K}[z_*,z_*^{-1}]$.*

Proof. Firstly by Theorem 32, for each z_λ, $\lambda\in\Lambda^+\cap Q$, we have

$$z_\lambda^0 = \sum_{\mu\leqslant\lambda}(rs^{-1})^{-\frac{2}{\ell}(\rho,\mu)}\dim(L(\lambda)_\mu)\omega'_\mu\omega_{-\mu}.$$

Therefore, by definition in section 2.5, we have

$$\begin{aligned}\xi(z_\lambda)=\gamma^{-\rho}(z_\lambda^0)&=\sum_{\mu\leqslant\lambda}(rs^{-1})^{-\frac{2}{\ell}(\rho,\mu)}\dim(L(\lambda)_\mu)\ \varrho^{-\rho}(\omega'_\mu\omega_{-\mu})\omega'_\mu\omega_{-\mu}\\&=\sum_{\mu\leqslant\lambda}\dim(L(\lambda)_\mu)\omega'_\mu\omega_{-\mu}.\end{aligned}$$

Notice that $\{\xi(z_\lambda)\mid\lambda\in\Lambda^+\cap Q\}\subseteq\operatorname{Im}\xi\subseteq(U_\flat^0)^W$ in Theorem 24. It is sufficient to show $(U_\flat^0)^W\subseteq\operatorname{Im}(\xi)$. Since for each $\eta\in Q$, there exists a unique $\sigma\in W$ such that $\sigma(\eta)\in\Lambda^+\cap Q$, it is clear that the elements

$$\operatorname{av}(\lambda)=\frac{1}{|w|}\sum_{\sigma\in W}\sigma(\omega'_\lambda\omega_{-\lambda})=\frac{1}{|w|}\sum_{\sigma\in W}\omega'_{\sigma(\lambda)}\omega_{-\sigma(\lambda)},\ \lambda\in\Lambda^+\cap Q$$

form a basis of $(U_\flat^0)^W$.

It only remains to prove that $\operatorname{av}(\lambda)\in\operatorname{Im}(\xi)$. By induction on the height of λ, if $\lambda=0$, then $\operatorname{av}(\lambda)=1\in\operatorname{Im}(\xi)$. Assume that $\lambda>0$, by the fact that $\dim(L(\lambda)_\mu)=\dim(L(\lambda)_{\sigma(\mu)})$, $\forall\,\sigma\in W$ and $\dim(L(\lambda)_\lambda)=1$, we have

$$\begin{aligned}\xi(z_\lambda)&=\sum_{\mu\leqslant\lambda}\dim(L(\lambda)_\mu)\omega'_\mu\omega_{-\mu}\\&=|W|\operatorname{av}(\lambda)+|W|\sum_{\mu<\lambda,\ \mu\in\Lambda^+\cap Q}\dim(L(\lambda)_\mu)\operatorname{av}(\mu).\end{aligned}$$

By the induction hypothesis, we get

$$\operatorname{av}(\lambda)=\frac{1}{|W|}\xi(z_\lambda)-\sum_{\mu<\lambda,\ \mu\in\Lambda^+\cap Q}\dim(L(\lambda)_\mu)\operatorname{av}(\mu)\in\operatorname{Im}\xi.$$

Therefore,

$$(U_\flat^0)^W=\operatorname{span}_{\mathbb{K}}\{\operatorname{av}(\lambda)\mid\lambda\in\Lambda^+\cap Q\}\subseteq\operatorname{Im}(\xi).$$

When rank n is even, combining Theorem 25, we have that ξ is isomorphic to its image. When rank n is odd, we have $(U_\flat^0)^W\otimes\mathbb{K}[z_*,z_*^{-1}]\subset\operatorname{Im}(\xi)$ by Theorem 33 and Proposition 34. $\square$

5.2. Alternative approach to central elements. Similar to [**47**], we can also construct central elements by taking quantum partial trace.

PROPOSITION 36. *Let $\lambda\in\Lambda^+$ and $\zeta:U_{r,s}(\mathfrak{g})\to\operatorname{End}(L(\lambda))$ be the weight representation. If there is an operator $\Gamma\in U_{r,s}(\mathfrak{g})\otimes\operatorname{End}(L(\lambda))$ such that*

$$\Gamma\circ(id\otimes\zeta)\Delta(x)=(id\otimes\zeta)\Delta(x)\circ\Gamma,\quad\forall\,x\in U_{r,s}(\mathfrak{g}),$$

then the element $c=\operatorname{tr}_2(\Gamma(1\otimes\Theta))\in Z(U_{r,s}(\mathfrak{g}))$.

PROOF. It is enough to check this element commutes with all generators of U. In convenience we view the operator $(id\otimes\zeta)\Delta(x)$ as $\Delta(x)$, and write $\Gamma=$

$\sum_\Gamma \Gamma_{(1)}\otimes\Gamma_{(2)}$. Since $[\Gamma,\Delta(\omega_i^{\pm1})]=0$, and $S^2(u)\Theta=\Theta u$, $\forall\, u\in U$, we have

$$\begin{aligned}
0&=\mathrm{tr}_2([\Gamma,\Delta(\omega_i^{\pm1})]\cdot(1\otimes\Theta\omega_i^{\mp1}))\\
&=\sum_\Gamma \Gamma_{(1)}\omega_i^{\pm1}\otimes\mathrm{tr}(\Gamma_{(2)}\omega_i^{\pm1}\Theta\omega_i^{\mp1})-\sum_\Gamma\omega_i^{\pm1}\Gamma_{(1)}\otimes\mathrm{tr}(\omega_i^{\pm1}\Gamma_{(2)}\Theta\omega_i^{\mp1})\\
&=\sum_\Gamma \Gamma_{(1)}\omega_i^{\pm1}\otimes\mathrm{tr}(\Gamma_{(2)}\Theta)-\sum_\Gamma\omega_i^{\pm1}\Gamma_{(1)}\otimes\mathrm{tr}(\Gamma_{(2)}\Theta)\\
&=\sum_\Gamma[\Gamma_{(1)},\omega_i^{\pm1}]\otimes\mathrm{tr}(\Gamma_{(2)}\Theta)\\
&=[\mathrm{tr}_2(\Gamma(1\otimes\Theta)),\omega_i^{\pm1}]=[c,\omega_i^{\pm1}].
\end{aligned}$$

Similarly, we have $[c,{\omega_i'}^{\pm1}]=0$.

To prove $[c,e_i]=0$, we have to check $[c,e_i\omega_i^{-1}]=0$,

$$\begin{aligned}
0&=\mathrm{tr}_2([\Gamma,\Delta(e_i\omega_i^{-1})]\cdot(1\otimes\Theta\omega_i))\\
&=\mathrm{tr}_2([\Gamma,e_i\omega_i^{-1}\otimes\omega_i^{-1}+1\otimes e_i\omega_i^{-1}](1\otimes\Theta\omega_i))\\
&=\mathrm{tr}_2(\Gamma(e_i\omega_i^{-1})\otimes\Theta+\Gamma(1\otimes e_i\Theta)\\
&\qquad-(e_i\omega_i^{-1}\otimes\omega_i^{-1})\Gamma(1\otimes\Theta\omega_i)-(1\otimes e_i\omega_i^{-1})\Gamma(1\otimes\Theta\omega_i))\\
&=\mathrm{tr}_2(\Gamma(1\otimes\Theta)(e_i\omega_i^{-1}\otimes1)+\Gamma(1\otimes e_i\Theta)\\
&\qquad-(e_i\omega_i^{-1}\otimes1)\Gamma(1\otimes\Theta)-\Gamma(1\otimes\Theta\omega_ie_iw_i^{-1}))\\
&=[\mathrm{tr}_2(\Gamma(1\otimes\Theta)),e_i\omega_i^{-1}]=[c,e_i\omega_i^{-1}].
\end{aligned}$$

The fifth equation holds since $\Theta\omega_ie_iw_i^{-1}=\omega_iS^2(e_i)w_i^{-1}\Theta=e_i\Theta$.

Similarly, we have $[c,f_i]=0$. □

6. Centre of $\breve{U}_{r,s}(\mathfrak{g})$ of weight lattice type

In this section, we will add some group-like elements to $U_{r,s}(\mathfrak{g})$ to get the two-parameter quantum group $\breve{U}_{r,s}(\mathfrak{g})$ of weight lattice type.

Definition 37. *The algebra $\breve{U}_{r,s}(\mathfrak{g})$ of the so-called weight lattice form of $U_{r,s}(\mathfrak{g})$, is the unital associative algebra generated by elements e_i, f_i, $\omega_{\varpi_i}^{\pm1}$, ${\omega'_{\varpi_i}}^{\pm1}$ $(i=1,\cdots,n)$ over $\mathbb{K}$. Set $\omega_i:=\prod_{l=1}^n{\omega_{\varpi_l}}^{c_{li}}$, $\omega'_j:=\prod_{k=1}^n{\omega'_{\varpi_k}}^{c_{kj}}$, the generators satisfy the following relations:*

$$\begin{array}{lll}
(X1,\Lambda) & \omega_{\varpi_i}^{\pm1}{\omega'_{\varpi_j}}^{\pm1}={\omega'_{\varpi_j}}^{\pm1}\omega_{\varpi_i}^{\pm1}, & \omega_{\varpi_i}\omega_{\varpi_i}^{-1}=1=\omega'_{\varpi_j}{\omega'_{\varpi_j}}^{-1},\\
(X2,\Lambda) & \omega_{\varpi_i}e_j\omega_{\varpi_i}^{-1}=r^{\langle j,\varpi_i\rangle}s^{-\langle\varpi_i,j\rangle}e_j, & \omega_{\varpi_i}f_j\omega_{\varpi_i}^{-1}=r^{-\langle j,\varpi_i\rangle}s^{\langle\varpi_i,j\rangle}f_j,\\
 & \omega'_{\varpi_i}e_j{\omega'}^{-1}_{\varpi_i}=r^{-\langle\varpi_i,j\rangle}s^{\langle j,\varpi_i\rangle}e_j, & \omega'_{\varpi_i}f_j{\omega'}^{-1}_{\varpi_i}=r^{\langle\varpi_i,j\rangle}s^{-\langle j,\varpi_i\rangle}f_j,\\
(X3) & [e_i,f_j]=\delta_{ij}\frac{\omega_i-\omega'_i}{r_i-s_i}, & \\
(X4) & (\mathrm{ad}_l\,e_i)^{1-c_{ij}}(e_j)=0,\qquad(\mathrm{ad}_r\,f_i)^{1-c_{ij}}(f_j)=0, & (i\neq j).
\end{array}$$

Here $\breve{U}_{r,s}(\mathfrak{g})$ naturally extends the Hopf structure of $U_{r,s}(\mathfrak{g})$ (elements $\omega_{\varpi_i},\omega'_{\varpi_i}$ are group-likes).

Proposition 38. *There exists a unique skew-dual pairing* $\langle\, -, - \,\rangle : \breve{\mathcal{B}}' \times \breve{\mathcal{B}} \to \mathbb{K}$ *of the Hopf subalgebra* $\breve{\mathcal{B}}$ *and* $\breve{\mathcal{B}}'$ *satisfying*

$$\begin{aligned}
&\langle f_i, e_j\rangle = \delta_{ij}\frac{1}{s_i - r_i}, && 1 \leqslant i,j \leqslant n,\\
&\langle \omega'_{\varpi_i}, \omega_{\varpi_j}\rangle = r^{\langle \varpi_i, \varpi_j\rangle} s^{-\langle \varpi_j, \varpi_i\rangle}, && 1 \leqslant i,j \leqslant n,\\
&\langle {\omega'_i}^{\pm 1}, \omega_j^{-1}\rangle = \langle {\omega'_i}^{\pm 1}, \omega_j\rangle^{-1} = \langle \omega'_i, \omega_j\rangle^{\mp 1}, && 1 \leqslant i,j \leqslant n,
\end{aligned}$$

and all other pairs of generators are 0.

Corollary 39. *For any* $\lambda_1, \lambda_2 \in \Lambda$, *we have* $\langle \omega'_{\lambda_1}, \omega_{\lambda_2}\rangle = r^{\langle \lambda_1, \lambda_2\rangle} s^{-\langle \lambda_2, \lambda_1\rangle}$.

It is obvious that $U_{r,s}$ is a Hopf subalgebra of $\breve{U}_{r,s}$, and these two Hopf algebras have the same category of the weight modules, namely $\mathcal{O}_f^{r,s}$. Notice that our approach to the Harish-Chandra isomorphism of $\breve{U}$ is exactly the same as in the previous sections, so it is sufficient to show that the lemmas and properties affected by the extension still hold.

Lemma 40. *The Rosso form* $\langle -, - \rangle_{\breve{U}}$ *defined as in Definition 11 is non-degenerate on* $\breve{U}$.

Proof. The only difference with Theorem 14 is the character. Define a group homomorphism $\chi_{\eta,\phi} : \Lambda \times \Lambda \to \breve{\mathbb{K}}^\times$ by

$$\chi_{\eta,\phi}(\nu, \mu) = \langle \omega'_\eta, \omega_\mu\rangle \langle \omega'_\nu, \omega_\phi\rangle.$$

Parallel to Lemma 15, we only need to prove that if $\chi_{\eta,\phi} = \chi_{\eta',\phi'}$, then $(\eta, \phi) = (\eta', \phi')$.

Let $\eta = \sum_{i=1}^n \eta_i\alpha_i,\ \eta' = \sum_{i=1}^n \eta'_i\alpha_i$, for $j = 1, \cdots, n$, we have

$$\begin{aligned}
1 = \frac{\chi_{\eta,\phi}(0, \varpi_j)}{\chi_{\eta',\phi'}(0, \varpi_j)} &= \prod_{i=1}^n \langle \omega'_i, \omega_{\varpi_j}\rangle^{\eta_i - \eta'_i} = \prod_{i=1}^n \prod_{k=1}^n \langle \omega'_i, \omega_k\rangle^{c^{kj}(\eta_i - \eta'_i)}\\
&= r^{\sum_{i,k=1}^n c^{kj} R_{ki}(\eta_i - \eta'_i)} s^{\sum_{i,k=1}^n c^{kj} S_{ki}(\eta_i - \eta'_i)},
\end{aligned}$$

which leads to $C^{-T}(R-S)(\eta - \eta') = 0$ and yields $\eta = \eta'$. A similar argument leads to $\phi = \phi'$. $\square$

Lemma 41. *When rank* n *is even, the Harish-Chandra homomorphism* $\breve{\xi} : Z(\breve{U}) \to \breve{U}^0$ *is injective .*

Proof. It is sufficient to show Lemma 19 holds in $\breve{U}$, that is for all $\lambda \in \Lambda^+$ and $\eta, \phi \in \Lambda$, one has

(1) If $\varrho^\lambda(\omega'_\eta\omega_\phi) = 1$, then $\eta = \phi$.

(2) When n is even, then $\varrho^\lambda(\omega'_\eta\omega_\phi) = 1$ if and only if $(\eta, \phi) = (0, 0)$.

Notice that here $\eta = \sum\limits_{i=1}^n \eta_i\alpha_i,\ \phi = \sum\limits_{i=1}^n \phi_i\alpha_i,\ \lambda = \sum\limits_{i=1}^n \lambda_i\alpha_i,\quad \eta_i, \phi_i, \lambda_i \in \mathbb{Q}$, and the proof of Lemma 19 is independent of where the coefficients are taken from, it follows that these two propositions also hold true. $\square$

The same reason can be used to show that Lemmas 22-24, 27 still hold in $\breve{U}$, which leads to Theorem 25, that is $\mathrm{Im}(\breve{\xi}) \subseteq (\breve{U}^0_\flat)^W$, where $\breve{U}^0_\flat := \bigoplus_{\eta \in \Lambda} \mathbb{K}\omega'_\eta\omega_{-\eta}$. Parallel to Proposition 31 and Theorem 32, there are enough group-like elements to define all $z_\lambda \in \breve{U}, \forall\ \lambda \in \Lambda^+$. Finally by the fact that $(\breve{U}^0_\flat)^W$ has a basis $\{\mathrm{av}(\lambda) \mid \lambda \in \Lambda^+\}$, repeating the proof of Theorem 35, one gets:

THEOREM 42. *The Harish-Chandra homomorphism* $\breve{\xi} : Z(\breve{U}_{r,s}(\mathfrak{g})) \to (\breve{U}^0_\flat)^W$ *is an isomorphism of algebras when rank* n *is even. In particular, for each* $\lambda \in \Lambda^+$,

$$\breve{\xi}(z_\lambda) = \sum_{\mu\leq\lambda} \dim(L(\lambda)_\mu)\, \omega'_\mu \omega_{-\mu}.$$

When rank n *is odd,* $\mathrm{Im}(\xi) \supseteq (\breve{U}^0_\flat)^W \otimes \mathbb{K}[z_*, z_*^{-1}]$.

Now we are able to construct the character map to study the centre $Z(\breve{U}_{r,s}(\mathfrak{g}))$.

PROPOSITION 43. *Let* $K(U_{r,s}) := Gr(\mathcal{O}^{r,s}_f)\otimes_{\mathbb{Z}}\mathbb{K}$, *then the character map* $\mathrm{Ch}_{r,s} : K(U_{r,s}) \to (\breve{U}^0_\flat)^W$ *is an isomorphism of algebras with*

$$\mathrm{Ch}_{r,s}([V]) = \sum_{\mu\leq\lambda} \dim(V_\mu)\, \omega'_\mu \omega_{-\mu}, \quad \forall\, V \in \mathcal{O}^{r,s}_f.$$

PROOF. Since $(V\otimes W)_\mu = \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes W_{\mu-\lambda}$ still holds in $\mathcal{O}^{r,s}_f$, it is clear that $\mathrm{Ch}_{r,s}$ is a homomorphism of algebras. Next, if $\mathrm{Ch}_{r,s}([V]) = \mathrm{Ch}_{r,s}([W])$ for some $V, W \in \mathcal{O}^{r,s}_f$, then

$$\sum_{\mu\in\Pi(V)} \dim(V_\mu)\, \omega'_\mu\omega_{-\mu} = \sum_{\nu\in\Pi(W)} \dim(W_\nu)\, \omega'_\nu\omega_{-\nu},$$

where $\Pi(V)$ is the weight set of V. Since $\omega'_\mu\omega_{-\mu}$ with $\mu\in\Lambda$ are linear independent, we have $\dim(V_\mu) = \dim(W_\mu)$ for any μ, i.e., $[V] = [W]$. Finally, for any $\lambda \in \Lambda^+$, we have $\mathrm{Ch}_{r,s}([L(\lambda)]) = \breve{\xi}(z_\lambda)$. Parallel to the proof of Theorem 35, $\mathrm{Ch}_{r,s}$ is also surjective. □

In [**27**], one of us and Pei have studied the deformation theory of the representations of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$, where $\mathcal{O}^{r,s}_f$ is defined as the category of finite-dimensional weight modules (of type 1) of $U_{r,s}(\mathfrak{g})$, and proved that

THEOREM 44. *Assume that* $rs^{-1} = q^2$, *there is an equivalence as braided tensor categories that*

$$\mathcal{O}^{r,s}_f \simeq \mathcal{O}^{q,q^{-1}}_f \simeq \mathcal{O}^q_f,$$

where $\mathcal{O}^q_f$ *is the category of the finite-dimensional weight modules* (*of type* 1) *of the quantum group* $U_q(\mathfrak{g})$, *and the equivalence takes* $L(\lambda) \in \mathcal{O}^q_f$ *to a deformed weight module in* $\mathcal{O}^{r,s}_f$ *which is just* $L(\lambda)$ *defined in Lemma 8.*

Combining the results in [**20, 10**]:

THEOREM 45. *The* $K(\breve{U}_q) := Gr(\mathcal{O}^q_f) \otimes_{\mathbb{Z}} \mathbb{K}$, *the Grothendieck ring* (*over* $\mathbb{K}$) *of the category* $\mathcal{O}^q_f$ *of the quantum group* $\breve{U}_q(\mathfrak{g})$, *is a polynomial algebra. More precisely, let* $\{\varpi_i\}_{i=1}^n$ *be the set of fundamental weights of* $\mathfrak{g}$, *then*

$$Z(\breve{U}_q) \cong K(\breve{U}_q) = \mathbb{K}[\,[L(\varpi_1)], \cdots, [L(\varpi_n)]\,].$$

with Theorem 44, we arrive at the following

THEOREM 46. *The algebra* $K(\breve{U}_{r,s}) := Gr(\mathcal{O}^{r,s}_f) \otimes_{\mathbb{Z}} \mathbb{K}$ *is a polynomial algebra. When rank* n *is even, the centre of the extended two-parameter quantum group* $\breve{U}_{r,s}(\mathfrak{g})$ *is a polynomial algebra* $Z(\breve{U}_{r,s}) = \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}]$. *When rank* n *is odd, the centre* $Z(\breve{U}_{r,s}) \supseteq \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}] \otimes \mathbb{K}[z_*, z_*^{-1}]$.

Remark 47. *When $n = rank(\mathfrak{g})$ is odd, we still cannot claim if the centre of $U_{r,s}(\mathfrak{g})$ or $\breve{U}_{r,s}(\mathfrak{g})$ is equal to $\mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}] \otimes \mathbb{K}[z_*, z_*^{-1}]$ or not. It remains an open question.*

References

[1] X. Bai and N. Hu, *Two-parameter quantum groups of exceptional type E-series and convex PBW-type basis*, Algebra Colloq. **15** (4) (2008), 619-636.

[2] G. Benkart, S.J. Kang and K.H. Lee, *On the centre of two-parameter quantum groups*, Proc. Roy. Soc. Edinb. Sec. A **136** (3) (2006), 445-472.

[3] G. Benkart and S. Witherspoon, *A Hopf structure for down-up algebras*, Math. Zeit. **238** (3) (2001), 523-553.

[4] —, *Two-parameter quantum groups and Drinfel'd doubles*, Algebras and Rep. Theory **7** (3) (2004), 261-286.

[5] N. Bergeron, Y. Gao and N. Hu, *Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups*, J. Algebra, **301** (1) (2006), 378-405.

[6] —, *Representations of two-parameter quantum orthogonal groups and symplectic groups*, in: AMS/IP Stud. Adv. Math., 39, Amer. Math. Soc., Providence, RI, 2007, 1-21.

[7] R. W. Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics, 96. Cambridge University Press, Cambridge, 2005.

[8] Z. Chang and Y. Wang, *Central extensions of generalized orthosymplectic Lie seralgebras*, Sci. China Math. **60** (2), 223-260.

[9] X. Chen, N. Hu, and X. Wang, *Convex PBW-type Lyndon bases and restricted two-parameter quantum group of type F_4*, Acta. Math. Sin.-Eng. Ser. **39** (2023), 1053-1084.

[10] Y. Dai, *Explicit generators of the centre of the quantum group*, Commun. Math. Stat. **11** (2023), 541-562.

[11] Y. Dai and Y. Zhang, *Explicit generators and relations for the centre of the quantum group*, ArXiv: 2102.07407.

[12] C. De Concini and V. G. Kac, *Representations of quantum groups at roots of* 1, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), 471-506, Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990.

[13] C. De Concini, V. G. Kac and C. Procesi, *Quantum coadjoint action*, J. Amer. Math. Soc. **5** (1) (1992), 151-189.

[14] —, *Some remarkable degenerations of quantum groups*, Comm. Math. Phys. **157** (2) (1993), 405-427.

[15] —, *Some quantum analogues of solvable Lie groups*, Geometry and analysis (Bombay, 1992), 41-65, Tata Inst. Fund. Res., Bombay, 1995.

[16] C. De Concini and C. Procesi, *Quantum groups. in D-modules, representation theory, and quantum groups*, Lecture Notes in Math. **1565**, Springer, Berlin, 1993.

[17] J. Ding and P. Etingof, *Center of a quantum affine algebra at the critical level*, Math. Res. Lett. (1) **4** (1994), 469-480.

[18] V. G. Drinfeld, *Quantum groups*, In: Proceedings of International Congress of Mathematicans, Providence: Amer Math Soc , 1986, 798-820.

[19] J. Duan, *On the center of two-parameter quantum group $U_{r.s}(so_5)$*, Master's thesis, East China Normal University, 2009.

[20] P. I. Etingof, *Central elements for quantum affine algebras and affine Macdonald's operators*, Math. Research Lett. (2) (1995), 611–628.

[21] S. Gan, *On the center of two-parameter quantum group $U_{r.s}(G_2)$*, Master's thesis, East China Normal University, 2010.

[22] M. D. Gould, R. B. Zhang and A. J. Bracken, *Generalized Gel'fand invariants and characteristic identities for quantum groups*, J. Math. Phys. **32** (9) (1991), 2298-2303.

[23] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, In: Algebraic Analysis, vol. 1, 129-139, Academic Press, Boston, MA, 1988.

[24] J. Hu and Y. Zhang, *Quantum double of $U_q((sl_2)^{\leqslant 0})$* , J. Algebra **317** (1) (2007), 87-110.

[25] N. Hu, *Quantum divided power algebra, q-derivatives, and some new quantum groups*, J. Algebra **232** (2) (2000), 507-540.

[26] N. Hu and Y. Pei, *Notes on two-parameter groups (I)*, Sci. in China, Ser. A. **51** (6) (2008), 1101-1110.

[27] —, *Notes on two-parameter groups (II)*, Comm. in Algebra **40** (9) (2012), 3202-3220.
[28] N. Hu, M. Rosso and H. Zhang, *Two-parameter quantum affine algebra $U_{r,s}(\tilde{\mathfrak{sl}}_n)$, Drinfel'd realization and quantum affine Lyndon basis*, Comm. Math. Phys. **278** (2) (2008), 453-486.
[29] N. Hu and Q. Shi, *Two-parameter Quantum group of exceptional type G_2 and Lusztig's symmetries*, Pacific J. Math. **230** (2007), 327-345.
[30] N. Hu and Y. Shi, *On the centre of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ for type B_n with n even*, J. Geom. Phys. **86** (2014), 422-433.
[31] N. Hu and X. Wang, *Convex PBW-type Lyndon basis and restricted two-parameter quantum groups of type G_2*, Pacific J. of Math. **241** (2) (2009), 243-273.
[32] —, *Convex PBW-type Lyndon bases and restricted two-parameter quantum groups of type B*, J. Geom. Phys. **60** (3) (2010), 430-453.
[33] J. C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Math., 6, Amer. Math. Soc, Providence, RI, 1996.
[34] M. Jimbo, *A q-difference anologue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1986), 63-69.
[35] A. Joseph and G. Letzter, *Local finiteness of the adjoint action for quantized enveloping algebras*, Journal of Algebra **153** (1992), 289-318.
[36] L. Li, L. Xia and Y. Zhang, *On the center of the quantized enveloping algebra of a simple Lie algebra*, ArXiv: 1607.00802.
[37] —, *On The Centers of quantum groups of A_n type*, Sci. China Math. **61** (2) (2018), 287-294.
[38] F. Liu, N. Hu and N. Jing, *Quantum supergroup $U_{r,s}(osp(1,2))$, scasimir operators, and Dickson polynomials*, J. Algebra Appl. **23** (1) (2024), No. 2450003, 18 pages.
[39] Y. Luo, Y. Wang and Y. Ye, *On the Harish-Chandra homomorphism for quantum superalgebras*, Comm. Math. Phys. **393** (3) (2022), 1483–1527.
[40] Y. Pei, N. Hu and M. Rosso, *Multi-parameter quantum groups and quantum shuffles (I)*, Contemp. Math., **506**, A. M. S., Providence, RI, 2010, 145-171.
[41] N. Yu. Reshetikhin, *Quasitriangular Hopf algebras and invariants of links*, Algebra i Analiz, vol. 1, issue 2 (1989), 169–188.
[42] N. Yu. Reshetikhin and M. A. Semenov-Tian-Shansky, *Central extensions of quantum current groups*, Lett. Math. Phys. **19** (1990), 133-142.
[43] M. Rosso, *Analogues de la forme de Killing et du théorème d'Harish-Chandra pour les groupes quantiques*, Annales scientifiques de l'É.N.S. 4e série, **23** (3) (1990), 445-467.
[44] T. Tanisaki, *Harish-Chandra isomorphisms for quantum algebras*, Commun. Math. Phys. **127** (1990), 555-571.
[45] T. Tanisaki, *Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras*, Infinite analysis, Part A, B (Kyoto, 1991), 941-961, Adv. Ser. Math. Phys., **16**, World Sci. Publ., River Edge, NJ, 1992.
[46] V.G. Turaev, Quantum invariants of knots and 3-manifolds, De Gruyter Studies in Mathematics, **18**, Walter de Gruyter & Co., Berlin, x+588 pp., 1994.
[47] R. B. Zhang, M. D. Gould and A. J. Bracken, *Quantum group invariants and link polynomials*, Commun. Math. Phys. **137** (1991), 13-27.

School of Mathematical Sciences, MOE Key Laboratory of Mathematics and Engineering Applications & Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China

Email address: `nhhu@math.ecnu.edu.cn`

School of Mathematical Sciences, MOE Key Laboratory of Mathematics and Engineering Applications & Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China

Email address: `52265500001@stu.ecnu.edu.cn`