Explicit formula for the Benjamin–Ono equation with square integrable and real valued initial data and applications to the zero dispersion limit

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Abstract

In this paper, we extend the Gérard's formula for the solution of the Benjamin–Ono equation on the line to square integrable and real valued initial data. Combined with this formula, we also extend the Gérard's formula for the zero dispersion limit of the Benjamin–Ono equation on the line to more singular initial data. In the derivation of the extension of the formula for the zero dispersion limit, we also find an interesting integral equality, which might be useful in other contexts.

Keywords Benjamin–Ono equation, Explicit formula, Zero dispersion limit.

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Contents

1 Introduction **2**
- 1.1 The Benjamin–Ono equation . . . 2
- 1.2 The Lax pair . . . 2
- 1.3 The explicit formula . . . 4
- 1.4 Zero dispersion limit . . . 4
- 1.5 Maximally dissipative operators and the Kato-Rellich theorem . . . 7
- 1.6 Structure of the paper . . . 8

2 Proof of the extension of the explicit formula **8**

3 Proof of the extension of the formula for the zero dispersion limit **13**

4 Final comments and open problems **22**

1 Introduction

1.1 The Benjamin–Ono equation

The Benjamin–Ono equation is a nonlinear partial integro-differential equation which describes one-dimensional internal waves in deep water. It was introduced by Benjamin in [1](see also Davis–Acrivos [3], Ono [17]). On the line, it reads

$$\begin{aligned} &\partial_t u = \partial_x \left(|D|u - u^2\right), \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \\ &u(0,x) = u_0. \end{aligned} \tag{1.1}$$

Here $u = u(t,x)$ denotes a real valued function. We refer to the book by Klein and Saut [12] for a recent survey of this equation. In this paper, we denote by H^s_r (or L^p_r with $p = 2, \infty$) the Sobolev (or Lebesgue) space of real valued functions.

The global well-posedness of (1.1) in $H^s_r(\mathbb{R})$ with $s \geq 0$ was proved in [10][15] by a synthesis of Tao's gauge transformation [19] and $X^{s,b}$ techniques. In [9], M. Ifrim and D. Tataru have provided a much simpler proof of the local well-posedness of (1.1) in $L^2_r(\mathbb{R})$. Recently, R. Killip, T. Laurens and M. Vişan have proved the global well-posedness of (1.1) in $H^s_r(\mathbb{R})$ with $-\frac{1}{2} < s < 0$ [11]. The unconditional uniqueness in $H^s(\mathbb{R})$ with $s > 3 - \sqrt{33/4}$ has been recently proved in [16].

Theorem 1.1 ([10], [15], [9] [16], [11])**.** *For every $u_0 \in H^s_r(\mathbb{R})$ with $s > 3 - \sqrt{33/4}$, there exists a unique solution $u \in C(\mathbb{R}, H^s_r(\mathbb{R}))$ of (1.1) with $u(0) = u_0$. Also, for every $T > 0$, the flow map $u_0 \in H^s(\mathbb{R}) \mapsto u \in C([-T,T], H^s(\mathbb{R}))$ is continuous. Moreover, this flow map $u_0 \in H^s(\mathbb{R}) \mapsto u \in C([-T,T], H^s(\mathbb{R}))$ can be continuously extended to $H^s(\mathbb{R})$ for any $s > -\frac{1}{2}$.*

Our aim in this paper is to give an explicit formula of the solution $u(t)$ in terms of any initial data u_0 in $L^2_r(\mathbb{R})$. Before presenting our main results, we need to introduce the Lax pair structure for (1.1).

1.2 The Lax pair

In this paper, we denote by $L^2_+(\mathbb{R})$ the Hardy space corresponding to $L^2(\mathbb{R})$ functions having a Fourier transform supported in the domain $\xi \geq 0$. Recall that the space $L^2_+(\mathbb{R})$

identifies to holomorphic functions on the upper-half plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ such that

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx < +\infty.$$

The Riesz-Szegő projector Π is the orthogonal projector from $L^2(\mathbb{R})$ onto $L^2_+(\mathbb{R})$. It is given by

$$\forall f \in L^2(\mathbb{R}), \quad \forall z \in \mathbb{C}_+, \quad \Pi f(z) = \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{f(y)}{y-z} dy. \tag{1.2}$$

The Toeplitz operator on $L^2_+(\mathbb{R})$ associated to a function $b \in L^\infty(\mathbb{R})$ is defined by

$$T_b f := \Pi(bf), \quad f \in L^2_+(\mathbb{R}).$$

We notice that for $b \in L^\infty_r(\mathbb{R})$, T_b is a self-adjoint operator on $L^2_+(\mathbb{R})$. As shown in [2, Proposition 2.2], we also remark that

$$T_b \in \mathscr{L}\left(L^2_+(\mathbb{R})\right) \text{ if and only if } b \in L^\infty(\mathbb{R}). \tag{1.3}$$

For $u \in L^2_r(\mathbb{R})$, the operator L_u is defined by

$$\forall f \in \operatorname{Dom}(L_u) = H^1_+ := H^1(\mathbb{R}) \cap L^2_+(\mathbb{R}), \quad L_u f := Df - T_u f \text{ with } D := \frac{1}{i}\frac{d}{dx}.$$

We notice that L_u is a semi–bounded selfadjoint operator on $L^2_+(\mathbb{R})$.

Also, we recall the definition of G in [7],

$$\forall f \in \operatorname{Dom}(G) := \left\{ f \in L^2_+(\mathbb{R}) : \hat{f} \in H^1(0,\infty) \right\}, \quad \widehat{Gf}(\xi) := i\frac{d}{d\xi}[\hat{f}(\xi)]\mathbf{1}_{\xi>0}.$$

Here G is the adjoint of the operator of multiplication by x on $L^2_+(\mathbb{R})$, and we notice that $(\operatorname{Dom}(G), -iG)$ is maximally dissipative. We also notice that

$$\forall f \in \operatorname{Dom}(G), \left|\hat{f}\left(0^+\right)\right|^2 = -4\pi \operatorname{Im}\langle Gf \mid f\rangle \le 4\pi \|Gf\|_{L^2}\|f\|_{L^2}.$$

Therefore, we can define

$$\forall f \in \operatorname{Dom}(G), \quad I_+(f) := \hat{f}\left(0^+\right).$$

In fact, as observed in [11, Lemma 3.4], the resolvent of G is given by

$$\forall z \in \mathbb{C}_+, \quad \forall f \in L^2_+(\mathbb{R}), \quad (G - z\,\mathrm{Id})^{-1} f(x) = \frac{f(x) - f(z)}{x-z}, \tag{1.4}$$

and we have

$$\forall z \in \mathbb{C}_+, \quad \forall f \in L^2_+(\mathbb{R}), \quad f(z) = \frac{1}{2i\pi} I_+\left((G - z\,\mathrm{Id})^{-1} f\right). \tag{1.5}$$

1.3 The explicit formula

The explicit formula for the solution of (1.1) with the initial data $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ has been introduced by P. Gérard in [7]. In this paper, we extend the explicit formula of the solution to (1.1) to the initial data $u_0 \in L^2_r(\mathbb{R})$.

Theorem 1.2. *For $u_0 \in L^2_r(\mathbb{R})$, let $u \in C\left(\mathbb{R}, L^2_r(\mathbb{R})\right)$ be the corresponding solution of (1.1) in the sense of the continuous extension of the flow map as shown in Theorem 1.1. Then $u(t) = \Pi u(t) + \overline{\Pi u}(t)$, with*

$$\Pi u(t,z) = \frac{1}{2i\pi} I_+ \left(\left(G - 2tL_{u_0} - z\mathrm{Id}\right)^{-1} \Pi u_0 \right), \quad \forall z \in \mathbb{C}_+, \tag{1.6}$$

where

$$\left(G - 2tL_{u_0} - z\mathrm{Id}\right)^{-1} : L^2_+(\mathbb{R}) \to \mathrm{Dom}(G) \text{ is well-defined for every } z \in \mathbb{C}_+.$$

Remark 1.3. *In [11], R. Killip, T. Laurens and M. Vişan have obtained another explicit formula for a Hamiltonian system corresponding to (1.1)(see [11, Theorem 6.1] for details). Also, from this formula, they have recovered the formula (1.6) with the initial data $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$, which has been firstly obtained in [7].*

1.4 Zero dispersion limit

In [8], P. Gérard considered the Benjamin–Ono equation on the line with a small dispersion $\varepsilon > 0$,

$$\begin{aligned} &\partial_t u = \partial_x \left(\varepsilon |D| u - u^2\right), \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ &u^\varepsilon(0,x) = u_0(x). \end{aligned} \tag{1.7}$$

Observe that the L^2 norm of $u^\varepsilon(t)$ is independent of t, equal to the L^2 norm of u_0, so there exists a subsequence ε_j tending to 0 such that $u_{\varepsilon_j}(t)$ has a weak limit in $L^2(\mathbb{R})$, and we want to show that all these weak limits coincide under certain initial data conditions. If all these weak limits coincide, we call this weak limit the zero dispersion limit. Combined with the explicit formula (1.6) for $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$, P. Gérard has obtained the explicit formula for the zero dispersion limit in [8]. In this paper, since we have obtained the explicit formula (1.6) with $u_0 \in L^2_r(\mathbb{R})$, we can extend the explicit formula for the zero dispersion limit to more singular initial data.

Theorem 1.4. *Let $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ with $\lim_{x\to\infty} \frac{|u_0(x)|}{|x|} = 0$. Then for every $t \in \mathbb{R}$, the corresponding solution $u^\varepsilon(t)$ to (1.7) converges weakly in $L^2(\mathbb{R})$ to $ZD\left[u_0\right](t)$, characterized by*

$$\forall x \in \mathbb{R}, \quad ZD\left[u_0\right](t,x) = \Pi ZD\left[u_0\right](t,x) + \overline{\Pi ZD\left[u_0\right](t,x)}$$

and

$$\forall z \in \mathbb{C}_+, \quad \Pi ZD\left[u_0\right](t,z) = \frac{1}{2i\pi} I_+\left(\left(G + 2tT_{u_0} - z\mathrm{Id}\right)^{-1}\Pi u_0\right) \\ = \frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right) dy, \tag{1.8}$$

where Log *denotes the principal value of the logarithm, and*

$$\left(G + 2tT_{u_0} - z\mathrm{Id}\right)^{-1} : L^2_+(\mathbb{R}) \to \mathrm{Dom}(G) \text{ is well-defined for every } z \in \mathbb{C}_+.$$

Remark 1.5. *We observe that $u_0 \in L^2_r(\mathbb{R})$ with $|u_0(x)| \le C\langle x\rangle^k (k < 1)$ satisfies the initial data condition in Theorem 1.4, so we can give the formula of the zero dispersion limit for every $t \in \mathbb{R}$ with such an initial datum.*

Remark 1.6. *In [8], P. Gérard has also obtained the following description of the zero dispersion limit: Assume that the initial data $u_0 \in L^2_r(\mathbb{R}) \cap C^1(\mathbb{R})$ with $|u_0(x)| + |u_0'(x)| \to 0$, then for every $t \in \mathbb{R}$, the set $K_t(u_0)$ of critical values of the function*

$$y \in \mathbb{R} \mapsto y + 2tu_0(y)$$

is a compact subset of measure 0. For every connected component Ω of $K_t\left(u_0\right)^c$, there exists a nonnegative integer ℓ such that, for every $x \in \Omega$, the equation

$$y + 2tu_0(y) = x$$

has $2\ell + 1$ simple real solutions

$$y_0(t,x) < y_1(t,x) < \cdots < y_{2\ell}(t,x),$$

and the zero dispersion limit is given by

$$ZD\left[u_0\right](t,x) = \sum_{k=0}^{2\ell} (-1)^k u_0\left(y_k(t,x)\right). \tag{1.9}$$

Formula (1.9) was proved by Miller-Wetzel [13](see also Miller-Xu [14]) in the special case of a rational Klaus–Shaw initial potential, and by L. Gassot [4][5] in the special case of a general bell shaped initial potential with periodic boundary conditions.

Remark 1.7. *In [8], P. Gérard has obtained (1.8) with the initial data $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In the derivation of the second equality in (1.8), P. Gérard first considered the rational initial data to deduce this equality, and then extend this equality to $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$. However, this proof is not a direct derivation. In this paper, we provide a direct proof of the second equality of (1.8), and this direct approach allows us to extend this equality to $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ with $\lim_{x\to\infty} \frac{|u_0(x)|}{|x|} = 0$.*

In the direct derivation of the second equality of (1.8), we also find an interesting integral equality (1.10), which might be useful in other contexts. We summarize this interesting equality in the following lemma.

Lemma 1.8. *For $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $n \in \mathbb{N}_{\geq 1}$, we have*

$$\begin{aligned}&\int_{\mathbb{R}^n} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n\\ =&(n+1)\int_{\{\forall 1\leq j\leq n, y_j>0\}} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n.\end{aligned} \tag{1.10}$$

With a slight modification of the proof of Theorem 1.4, we can obtain the following zero dispersion limit result for $u_0 \in L^2_r(\mathbb{R})$ with $|u_0(x)| \leq C\langle x\rangle$ in a short time.

Corollary 1.9. *Let $u_0 \in L^2_r(\mathbb{R})$ with $|u_0(x)| \leq C\langle x\rangle$. Then for every $|t| < \frac{1}{2C}$, the corresponding solution $u^\varepsilon(t)$ to (1.7) converges weakly in $L^2(\mathbb{R})$ to $ZD[u_0](t)$, characterized by*

$$\forall x \in \mathbb{R}, \quad ZD[u_0](t,x) = \Pi ZD[u_0](t,x) + \overline{\Pi ZD[u_0](t,x)}$$

and

$$\begin{aligned}\forall z \in \mathbb{C}_+, \quad \Pi ZD[u_0](t,z) &= \frac{1}{2i\pi} I_+\left((G + 2tT_{u_0} - z\mathrm{Id})^{-1}\Pi u_0\right)\\ &= \frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right)dy,\end{aligned} \tag{1.11}$$

where Log *denotes the principal value of the logarithm, and*

$$(G + 2tT_{u_0} - z\mathrm{Id})^{-1} : L^2_+(\mathbb{R}) \to \mathrm{Dom}(G) \text{ is well-defined for every } z \in \mathbb{C}_+.$$

Remark 1.10. *Even for $u_0 \in L^2_r(\mathbb{R})$, we know that $\frac{2tu_0}{y-z} \notin \mathbb{R}$ for all $z \in \mathbb{C}_+$ and for all $t \in \mathbb{R}$, so $\mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right)$ is well defined in $\mathbb{C}_+$. We also notice that*

$$\frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right)dy = \frac{1}{2i\pi}\int_{\mathbb{R}}\int_0^1 \frac{u_0(y)}{y - z + 2stu_0(y)}dsdy,$$

since

$$\frac{1}{y - z + 2stu_0(y)} \in L^\infty_s(0,1)L^2_y(\mathbb{R}) \cap L^\infty_s(0,1)L^\infty_y(\mathbb{R}),$$

we can deduce that $\frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right)dy$ is well defined and holomorphic in $\mathbb{C}_+$. This tells us the formula for the zero dispersion limit

$$\Pi ZD[u_0](t,z) = \frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right)dy$$

might be extended to $u_0 \in L^2_r(\mathbb{R})$ for every $t \in \mathbb{R}$, but the difficulty lies in solving the problem of switching the order of a double limit, see also Section 4 for details.

1.5 Maximally dissipative operators and the Kato-Rellich theorem

In this paper, we mainly apply the Kato-Rellich theorem to show that operators remain maximally dissipative after some perturbations. To present the Kato-Rellich theorem for maximally dissipative operators, we first need to introduce the following definition of the dissipative and maximally dissipative operators in Hilbert spaces.

Definition 1.11. *Let $(D(A), A)$ be an operator in a Hilbert space $\mathscr{H}$.*

1. We say that A is dissipative if for all $g \in D(A)$ and all $\lambda > 0$,

$$\|(\lambda I - A)g\| \geq \lambda \|g\|.$$

2. We say that A is maximal dissipative if it is dissipative and for all $h \in \mathscr{H}$ and for all $\lambda > 0$, there exists $g \in D(A)$ such that $(\lambda I - A)g = h$.

Remark 1.12. *In fact, an operator $(D(A), A)$ in a Hilbert space $\mathscr{H}$ is dissipative if and only if for all $g \in D(A)$, $\Re\langle Ag, g\rangle \leq 0$.*

Remark 1.13. *Let $(D(A), A)$ be a maximally dissipative operator in a Hilbert space $\mathscr{H}$. From the definition of maximally dissipative operators, we can deduce that, for all $\lambda > 0$, we have*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(\mathscr{H})} \leq \frac{1}{\lambda}, \qquad \|A(\lambda I - A)^{-1}\|_{\mathcal{L}(\mathscr{H})} \leq 1. \tag{1.12}$$

Since the Kato-Rellich theorem involves the related notion of the perturbation of operators, we give the following definition of the relative bound of an operator with respect to another operator (see also the definition in [18]).

Definition 1.14. *Let $(D(A), A)$ and $(D(B), B)$ be densely defined linear operators on a Hilbert space $\mathscr{H}$. Suppose that:*

(i) $D(A) \subset D(B)$;
(ii) For some a and b in $\mathbb{R}$ and all $\varphi \in D(A)$,

$$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\|.$$

Then B is said to be A-bounded. The infimum of such a is called the relative bound of B with respect to A. If the relative bound is 0, we say that B is infinitesimally small with respect to A.

Then we state the Kato-Rellich theorem for maximally dissipative operators.

Theorem 1.15 (Kato-Rellich theorem)**.** *Let $(D(A), A)$ be a maximally dissipative operator which is densely defined on a Hilbert space $\mathscr{H}$ and assume $(D(B), B)$ to be dissipative and A-bounded with the relative bound smaller than 1. Then $(D(A), A + B)$ is also a maximally dissipative operator.*

We refer to [18, Theorem X.12] for the proof of the Kato-Rellich theorem for self-adjoint operators. The readers can also see the proof of Corollary 2.2.

1.6 Structure of the paper

In Section 2, for $u_0 \in L^2_r(\mathbb{R})$, we apply the Kato-Rellich theorem 1.15 to show that $(G - 2tL_{u_0} - z\mathrm{Id})^{-1}$ is well-defined on $L^2_+(\mathbb{R})$ for every $z \in \mathbb{C}_+$, then we can extend the explicit formula (1.6) to $u_0 \in L^2_r(\mathbb{R})$ and prove Theorem 1.2.

In Section 3, for $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ with $\lim_{x\to\infty} \frac{|u_0(x)|}{|x|} = 0$, we can still apply the Kato-Rellich theorem 1.15 to show that $(G + 2tT_{u_0} - z\mathrm{Id})^{-1}$ is well-defined on $L^2_+(\mathbb{R})$ for every $z \in \mathbb{C}_+$. Also, we prove Lemma 1.8 and then adapt the equality (1.10) to prove the second equality of (1.8). Finally, we show that the zero dispersion limit exists and complete the proof of Theorem 1.4.

In Section 4, we discuss the difficulties in further extensions of the explicit formula (1.6) and of the formula (1.8) for the zero dispersion limit. We also introduce briefly the results and the open problem on the zero dispersion limit for the Benjamin–Ono equation on the torus.

2 Proof of the extension of the explicit formula

In this section, we will show why the formula (1.6) can be extended to the initial data in $L^2_r(\mathbb{R})$. In fact, P. Gérard proved directly the formula (1.6) for $u_0 \in H^2_r(\mathbb{R})$ in [7], and then he extended this formula to $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let us firstly recall the sketch of proof of the generalized formula for $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We consider the following operator

$$A_t := -iG + 2itD, \text{ with } \mathrm{Dom}\,(A_t) := \left\{ f \in L^2_+(\mathbb{R}) : \mathrm{e}^{it\xi^2} \hat{f} \in H^1(0, \infty) \right\}.$$

In fact, we observe that

$$A_t = -iG + 2itD = \mathrm{e}^{-itD^2}(-iG)\mathrm{e}^{itD^2},$$

so we can easily deduce that $(\mathrm{Dom}\,(A_t), A_t)$ is maximally disspative. Then, for $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we know that $\left(L^2_+(\mathbb{R}), T_{u_0}\right)$ is a bounded and self-adjoint operator, so by a classical perturbation theory, we can deduce that $A_t - 2itT_{u_0} = -iG + 2itL_{u_0}$ is also maximally dissipative, and then by approximation, we conclude that (1.6) holds for $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

However, for $u_0 \in L^2_r(\mathbb{R})$, we cannot expect that T_{u_0} to remain bounded and dissipative on $L^2_+(\mathbb{R})$, so we cannot adapt directly the argument in [7] in this case.

Fortunately, we can adapt another approach to verify the formula (1.6) for $u_0 \in L^2_r(\mathbb{R})$. In fact, we observe that for $f \in \mathrm{Dom}\,(A_t)$, we have

$$-iGf + 2itL_{u_0}f = A_t f - 2itT_{u_0}f = \mathrm{e}^{-itD^2}\left(-iG - 2it\mathrm{e}^{itD^2}T_{u_0}\mathrm{e}^{-itD^2}\right)\mathrm{e}^{itD^2}f.$$

Then we consider the operator

$$\mathcal{G}_t := -iG - 2it\mathrm{e}^{itD^2}T_{u_0}\mathrm{e}^{-itD^2}$$

with

$$\mathrm{Dom}\,(\mathcal{G}_t) = \mathrm{Dom}\,(G) := \left\{f \in L^2_+(\mathbb{R}) : \hat{f} \in H^1(0,\infty)\right\}.$$

We recall that $(\mathrm{Dom}\,(G)\,, -iG)$ is maximally dissipative. Now we are going to prove that $B^t_{u_0} := -2it\mathrm{e}^{itD^2}T_{u_0}\mathrm{e}^{-itD^2}$ with the domain $\mathrm{Dom}\,(G)$ is dissipative and is infinitesimally small with respect to G, and then we can apply Theorem 1.15 to show that $(\mathrm{Dom}\,(G)\,, \mathcal{G}_t)$ is maximally dissipative, and so is $(\mathrm{Dom}\,(A_t)\,, -iG + 2itL_{u_0})$.

Lemma 2.1. *Given $u_0 \in L^2_r(\mathbb{R})$, for any $t \in \mathbb{R}$, the operator $B^t_{u_0} := -2it\mathrm{e}^{itD^2}T_{u_0}\mathrm{e}^{-itD^2}$ with the domain $\mathrm{Dom}\,(G)$ is dissipative and is infinitesimally small with respect to G.*

Proof. Firstly, we show that $\left(\mathrm{Dom}\,(G)\,, B^t_{u_0}\right)$ is dissipative. In fact, if we can show that

$$\forall f \in \mathrm{Dom}\,(G) \ \text{ and } \forall 0 < t < \infty, \quad t^{\frac{1}{2}}\mathrm{e}^{-itD^2}f \in L^\infty(\mathbb{R}). \tag{2.1}$$

Then for all $f \in \mathrm{Dom}\,(G)$ and for all $0 < t < \infty$, we have

$$\begin{aligned}\Re\left\langle -2it\mathrm{e}^{itD^2}T_{u_0}\mathrm{e}^{-itD^2}f, g\right\rangle &= 2\Im\left\langle T_{u_0}t^{\frac{1}{2}}\mathrm{e}^{-itD^2}f, t^{\frac{1}{2}}\mathrm{e}^{-itD^2}f\right\rangle\\ &= 2\Im\left\langle t^{\frac{1}{2}}\mathrm{e}^{-itD^2}f, T_{u_0}t^{\frac{1}{2}}\mathrm{e}^{-itD^2}f\right\rangle\\ &= -\Re\left\langle \mathrm{e}^{-itD^2}f, -2itT_{u_0}\mathrm{e}^{-itD^2}f\right\rangle,\end{aligned}$$

which implies that

$$\forall f \in \mathrm{Dom}\,(G) \ \text{ and } \forall 0 < t < \infty, \quad \Re\left\langle B^t_{u_0}f, f\right\rangle = 0.$$

From Remark 1.12, we can deduce that $\left(\mathrm{Dom}\,(G)\,,B_{u_0}^t\right)$ is dissipative. So the point is to prove (2.1).

Before proving (2.1), we define a function $g \in L^2_+(\mathbb{R})$ by

$$\widehat{g}(\xi) := \mathbf{1}_{\xi \geq 0}\, \mathrm{e}^{-\xi}$$

with

$$I_+(g) = 1.$$

We recall that

$$|I_+(f)|^2 = -4\pi \,\mathrm{Im}\, \langle Gf \mid f \rangle \leq 4\pi \|Gf\|_{L^2} \|f\|_{L^2}.$$

Then we have

$$\begin{aligned}\left\| t^{\frac{1}{2}} \mathrm{e}^{-itD^2} f \right\|_{L^\infty} &\leq |t|^{\frac{1}{2}} \left\| \mathrm{e}^{-itD^2} \left(f - I_+(f) g\right) \right\|_{L^\infty} + |t|^{\frac{1}{2}} \left\| \mathrm{e}^{-itD^2} \left(I_+(f) g\right) \right\|_{L^\infty} \\ &:= I_1 + I_2.\end{aligned}$$

By Young's convolution inequality, we have the following estimate for I_2,

$$I_2 \leq |t|^{\frac{1}{2}} |I_+(f)| \|\widehat{g}\|_{L^1_\xi} \leq C |t|^{\frac{1}{2}} \|Gf\|_{L^2}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}}.$$

For I_1, from the dispersive estimate, we have

$$\begin{aligned} I_1 &\leq |t|^{\frac{1}{2}} \left\| \mathrm{e}^{-itD^2} \left(f - I_+(f) g\right) \right\|_{L^\infty} \\ &\leq C \left\| f - I_+(f) g \right\|_{L^1} \\ &\leq C \left\| f - I_+(f) g \right\|_{L^2}^{\frac{1}{2}} \left\| x \left(f - I_+(f) g\right) \right\|_{L^2}^{\frac{1}{2}} \\ &\leq C \left\| f - I_+(f) g \right\|_{L^2}^{\frac{1}{2}} \left\| G \left(f - I_+(f) g\right) \right\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

Here, $x\left(f - I_+(f)g\right) = G\left(f - I_+(f)g\right)$ since we have $\mathbf{1}_{\xi \geq 0} \left(\widehat{f}(\xi) - I_+(f)\widehat{g}(\xi)\right)$ is continuous at $\xi = 0$.

Then we have

$$\begin{aligned} \left\| f - I_+(f) g \right\|_{L^2} &\leq \|f\|_{L^2} + |I_+(f)| \|g\|_{L^2} \\ &\leq \|f\|_{L^2} + C \|Gf\|_{L^2}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}} \end{aligned}$$

and

$$
\begin{aligned}
\|G(f-I_+(f)g)\|_{L^2} &\leq \|Gf\|_{L^2}+|I_+(f)|\|Gg\|_{L^2} \\
&\leq \|Gf\|_{L^2}+C\|Gf\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

From the above estimates for I_1 and I_2, we can deduce that

$$
\begin{aligned}
\left\|t^{\frac{1}{2}}\mathrm{e}^{-itD^2}f\right\|_{L^\infty} \leq\ & C|t|^{\frac{1}{2}}\|Gf\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}+C\|Gf\|_{L^2}^{\frac{1}{4}}\|f\|_{L^2}^{\frac{3}{4}} \\
&+C\|Gf\|_{L^2}^{\frac{3}{4}}\|f\|_{L^2}^{\frac{1}{4}},
\end{aligned}
\tag{2.2}
$$

which verifies (2.1).

Then we prove that $B_{u_0}^t$ is infinitesimally small with respect to G. It is equivalent to show the following argument: Let $0<t<\infty$ (fixed) and $u_0\in L_r^2(\mathbb{R})$, for any $\varepsilon>0$, we have

$$
\|B_{u_0}^t f\|_{L^2}\leq \varepsilon\|Gf\|_{L^2}+C_\varepsilon\|f\|_{L^2},\quad \forall f\in \mathrm{Dom}\,(G)\,. \tag{2.3}
$$

In fact, we can combine (2.2) with the Young's inequality for products and then we can deduce (2.3). The proof of Lemma 2.1 is complete. □

With Lemma 2.1, we can now adapt directly Theorem 1.15 to show that $(\mathrm{Dom}\,(G)\,,\mathcal{G}_t)$ is maximally dissipative, and so is $(\mathrm{Dom}\,(A_t)\,,-iG+2itL_{u_0})$. For the readers' convenience, we reproduce the proof of Theorem 1.15 in the proof of Corollary 2.2.

Corollary 2.2. *Let $u_0\in L_r^2(\mathbb{R})$, for any $t\in\mathbb{R}$, $(\mathrm{Dom}\,(G)\,,\mathcal{G}_t)$ and $(\mathrm{Dom}\,(A_t)\,,-iG+2itL_{u_0})$ are maximally dissipative.*

Proof. We recall that

$$
-iG+2itL_{u_0}=A_tf-2itT_{u_0}=\mathrm{e}^{-itD^2}\mathcal{G}_t\mathrm{e}^{itD^2},
$$

so we only need to show that $(\mathrm{Dom}\,(G)\,,\mathcal{G}_t)$ is maximally dissipative.

Since $(\mathrm{Dom}\,(G)\,,-iG)$ and $\left(\mathrm{Dom}\,(G)\,,B_{u_0}^t\right)$ are dissipative, we know that $(\mathrm{Dom}\,(G)\,,\mathcal{G}_t)$ is dissipative.

Then we only need to show that $\mathcal{G}_t+iz\,\mathrm{Id}:\mathrm{Dom}\,(G)\to L_+^2(\mathbb{R})$ is bijective for some $z\in\mathbb{C}_+$. We write

$$
\mathcal{G}_t+iz\,\mathrm{Id}=-iG+B_{u_0}^t+iz\,\mathrm{Id}=\left(\mathrm{Id}+B_{u_0}^t(-iG+iz\,\mathrm{Id})^{-1}\right)(-iG+iz\,\mathrm{Id})\,.
$$

Since $B_{u_0}^t$ is infinitesimally small with respect to G, for $z \in i\mathbb{R}_{>0}$ and for any $\varepsilon > 0$, we have

$$\begin{aligned}\forall f \in L^2_+(\mathbb{R}), \quad \left\|B_{u_0}^t(-iG + iz\,\mathrm{Id})^{-1} f\right\|_{L^2} &\leq \varepsilon \left\|G(-iG + iz\,\mathrm{Id})^{-1} f\right\|_{L^2} + C_\varepsilon \left\|(-iG + iz\,\mathrm{Id})^{-1} f\right\|_{L^2} \\ &\leq \left(\varepsilon + \frac{C_\varepsilon}{\Im(z)}\right) \|f\|_{L^2}.\end{aligned}$$

The last inequality above comes from (1.12).

Then we choose $\varepsilon = \frac{1}{4}$ and $z \in i\mathbb{R}_{>0}$ such that $C_{\frac{1}{4}}/\Im(z) < \frac{1}{4}$, and we have

$$\left\|B_{u_0}^t(G + iz\,\mathrm{Id})^{-1} f\right\|_{L^2} < \frac{1}{2}\|f\|_{L^2}.$$

Since G is maximally dissipative, we can deduce that $\mathcal{G}_t + iz\,\mathrm{Id} : \mathrm{Dom}\,(G) \to L^2_+(\mathbb{R})$ is bijective for some $z \in i\mathbb{R}_{>0}$, which provides that $(\mathrm{Dom}\,(G)\,, \mathcal{G}_t)$ is maximally dissipative, so is $(\mathrm{Dom}\,(A_t)\,, -iG + 2itL_{u_0})$. $\square$

By Corollary 2.2, we know that $(\mathrm{Dom}\,(A_t)\,, -iG + 2itL_{u_0})$ is maximally dissipative, thus for every $z \in \mathbb{C}_+$, the operator $(G - 2tL_{u_0} - zId)^{-1}$ is well-defined on $L^2_+(\mathbb{R})$.

Now we are able to present the following proof of Theorem 1.2.

Proof of Theorem 1.2. For $u_0 \in L^2_r(\mathbb{R})$, we can take $u_0^n \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which tends to u_0 in $L^2(\mathbb{R})$, then we can easily deduce that Πu_0^n tends to Πu_0 in $L^2_+(\mathbb{R})$. We denote the solutions of (1.1) by $u^n(t)$ and $u(t)$ corresponding to u_0^n and u_0. By the continuity of the flow map, we can deduce that $u^n(t)$ tends to $u(t)$ in $L^2(\mathbb{R})$. Then for $z \in \mathbb{C}_+$, we have

$$|\Pi u^n(t, z) - \Pi u(t, z)| \leq \int_0^\infty \mathrm{e}^{-\xi \Im(z)} |\widehat{u^n}(t, \xi) - \widehat{u}(t, \xi)| d\xi \leq C \left\|u^n(t) - u(t)\right\|_{L^2(\mathbb{R})} \to 0,$$

which implies the pointwise convergence of $\Pi u^n(t, z)$ to $\Pi u(t, z)$ for all $z \in \mathbb{C}_+$. Moreover, by Lemma 2.1 and Corollary 2.2, we can easily deduce that

$$B_{u_0^n}^t \left(G - zId\right)^{-1} \to B_{u_0}^t \left(G - zId\right)^{-1} \text{ in } \mathscr{L}\left(L^2_+(\mathbb{R})\right), \quad \forall z \in \mathbb{C}_+,$$

which implies that

$$\left(G - 2tL_{u_0^n} - z\mathrm{Id}\right)^{-1} \to \left(G - 2tL_{u_0} - z\mathrm{Id}\right)^{-1} \text{ in } \mathscr{L}\left(L^2_+(\mathbb{R})\right), \quad \forall z \in \mathbb{C}_+.$$

Then we recall the following explicit formula of $\Pi(u^n(t, z))$,

$$\Pi(u^n(t, z)) = \frac{1}{2i\pi} I_+ \left(\left(G - 2tL_{u_0^n} - z\mathrm{Id}\right)^{-1} \Pi u_0^n\right), \quad \forall z \in \mathbb{C}_+. \tag{2.4}$$

From the previous arguments, we can conclude that the formula (2.4) converges pointwisely in $\mathbb{C}_+$ to

$$\Pi(u(t,z)) = \frac{1}{2i\pi} I_+ \left((G - 2tL_{u_0} - z\mathrm{Id})^{-1} \Pi u_0 \right), \quad \forall z \in \mathbb{C}_+.$$

The proof is complete.

3 Proof of the extension of the formula for the zero dispersion limit

In this section, we will show why the formula (1.8) can be extended to the initial data $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ with $\lim_{x\to\infty} \frac{|u_0(x)|}{|x|} = 0$. Before proving Theorem 1.4, let us first give an important observation.

We consider the equation (1.7) with $u_0 \in L^2_r(\mathbb{R})$. By an elementary scaling argument, the solution u^ε of (1.7) is given by

$$u^\varepsilon(t,x) = \varepsilon v^\varepsilon(\varepsilon t, x),$$

where v^ε is the solution of the Benjamin–Ono equation (1.1) with the initial data

$$v^\varepsilon(0,x) = \frac{1}{\varepsilon} u_0(x).$$

By applying the explicit formula (1.6) to v^ε, we infer, for every $z \in \mathbb{C}_+$,

$$\Pi u^\varepsilon(t,z) = \frac{1}{2i\pi} I_+ \left(\left(G + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0} \mathrm{e}^{i\varepsilon t\partial_x^2} - z\mathrm{Id} \right)^{-1} \mathrm{e}^{-i\varepsilon t\partial_x^2} \Pi u_0 \right). \tag{3.1}$$

In fact, we have

$$\left(G + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0} \mathrm{e}^{i\varepsilon t\partial_x^2} - z\mathrm{Id} \right) = \left(Id + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0} \mathrm{e}^{i\varepsilon t\partial_x^2} (G - zId)^{-1} \right)(G - zId).$$

We observe that, for any $z \in \mathbb{C}_+$,

$$\text{as } \varepsilon \to 0, \quad 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0} \mathrm{e}^{i\varepsilon t\partial_x^2} (G - zId)^{-1} \text{ has a limit in } \mathscr{L}\left(L^2_+(\mathbb{R})\right)$$

if and only if

$$\forall f \in L^2_+(\mathbb{R}), \quad T_{u_0}(G - zId)^{-1} f \in L^2_+(\mathbb{R}). \tag{3.2}$$

We recall the formula (1.4),

$$\forall f \in L^2_+(\mathbb{R}), \quad (G - zId)^{-1} f(x) = \frac{f(x) - f(z)}{x - z},$$

In fact, for any $z \in \mathbb{C}_+$, $\frac{f(z)}{x-z} \in L^\infty_x(\mathbb{R})$, so we already have $T_{u_0}\frac{f(z)}{\cdot - z} \in L^2_+(\mathbb{R})$. Then we can deduce that (3.2) is equivalent to

$$\forall z \in \mathbb{C}_+, \quad \forall f \in L^2_+(\mathbb{R}), \quad T_{u_0}\frac{f(\cdot)}{\cdot - z} \in L^2_+(\mathbb{R}). \tag{3.3}$$

Since (3.3) holds for all $f \in L^2_+(\mathbb{R})$, from (1.3), we know that (3.3) is equivalent to

$$\forall z \in \mathbb{C}_+, \quad \frac{u_0(x)}{x-z} \in L^\infty_x(\mathbb{R}). \tag{3.4}$$

We can also observe that (3.4) is equivalent to

$$|u_0(x)| \le C\langle x\rangle \quad \text{with} \quad \langle x\rangle := (1+x^2)^{\frac{1}{2}}. \tag{3.5}$$

From the previous arguments, we can deduce that (3.5) is a necessary condition for $\left(G + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2}T_{u_0}\mathrm{e}^{i\varepsilon t\partial_x^2} - z\mathrm{Id}\right)$ to have a limit in $\mathscr{L}\left(\mathrm{Dom}\left(G\right), L^2_+(\mathbb{R})\right)$ as $\varepsilon \to 0$. So we may only expect (1.8) to hold for initial data in $L^2_r(\mathbb{R})$ which satisfies at least the condition (3.5). So far, for $u_0 \in L^2_r(\mathbb{R})$ with $|u_0(x)| \le C\langle x\rangle$, we cannot show that the zero dispersion limit exists and obtain the formula (1.8) for every $t \in \mathbb{R}$, but we can still show that this argument holds for $|t| < \frac{1}{2C}$. Moreover, with $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ satisfying $\lim_{x\to\infty}\frac{|u_0(x)|}{|x|} = 0$, which is a slightly stronger condition than (3.5), we can deduce that the zero dispersion limit exists and obtain the formula (1.8) for every $t \in \mathbb{R}$.

Now we deal with the proof of Theorem 1.4. To prove Theorem 1.4, first we show that $(\mathrm{Dom}(G), -iG - 2itT_{u_0})$ is maximally dissipative.

Lemma 3.1. *For $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ with $\lim_{x\to\infty}\frac{|u_0(x)|}{|x|} = 0$, $(\mathrm{Dom}(G), -iG - 2itT_{u_0})$ is maximally dissipative.*

Proof. Since $(\mathrm{Dom}(G), -iG)$ is maximally dissipative, it suffices to prove that, for $0 < t < \infty$ fixed, $-2itT_{u_0}$ is infinitesimally small with respect to G. It is equivalent to show that, for any $\varepsilon > 0$, we have

$$\|T_{u_0}f\|_{L^2} \le \varepsilon\|Gf\|_{L^2} + C_\varepsilon\|f\|_{L^2}, \quad \forall f \in \mathrm{Dom}(G). \tag{3.6}$$

We follow an approach which we used in the proof of Lemma 2.1. We recall the definition of g,

$$\widehat{g}(\xi) := \mathbf{1}_{\xi\ge 0}\,\mathrm{e}^{-\xi}.$$

Since u_0 satisfies $\lim_{x\to\infty}\frac{|u_0(x)|}{|x|} = 0$, then for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\frac{|u_0(x)|}{|x|} < \varepsilon \quad \text{for all} \quad |x| \ge R_\varepsilon.$$

Also, since $u_0 \in L^\infty_{loc}(\mathbb{R})$, there exists $M_\varepsilon > 0$ such that

$$\|u_0\|_{L^\infty(|x|<R_\varepsilon)} \leq M_\varepsilon.$$

Then for $f \in \mathrm{Dom}(G)$, we have

$$\begin{aligned}
&\|T_{u_0} f\|_{L^2(\mathbb{R})} \\
\leq &\|T_{u_0}\left(f - I_+(f)g\right)\|_{L^2(\mathbb{R})} + \|T_{u_0}\left(I_+(f)g\right)\|_{L^2(\mathbb{R})} \\
\leq &\|T_{u_0}\left(f - I_+(f)g\right)\|_{L^2(|x|<R_\varepsilon)} + \|T_{u_0}\left(f - I_+(f)g\right)\|_{L^2(|x|\geq R_\varepsilon)} + \|T_{u_0}\left(I_+(f)g\right)\|_{L^2(\mathbb{R})} \\
\leq &M_\varepsilon \|f - I_+(f)g\|_{L^2(\mathbb{R})} + \varepsilon \|x\left(f - I_+(f)g\right)\|_{L^2(\mathbb{R})} + \|T_{u_0}\left(I_+(f)g\right)\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Since $\mathbf{1}_{\xi \geq 0}\left(\widehat{f}(\xi) - I_+(f)\widehat{g}(\xi)\right)$ is continuous at $\xi = 0$, we have

$$\|x(f - I_+(f)g)\|_{L^2(\mathbb{R})} = \|G(f - I_+(f)g)\|_{L^2(\mathbb{R})}.$$

Then we have

$$\|f - I_+(f)g\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})} + |I_+(f)|\|g\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})} + C\|Gf\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}},$$

$$\begin{aligned}
\|G(f - I_+(f)g)\|_{L^2(\mathbb{R})} &\leq \|Gf\|_{L^2(\mathbb{R})} + |I_+(f)|\|Gg\|_{L^2(\mathbb{R})} \\
&\leq \|Gf\|_{L^2(\mathbb{R})} + C\|Gf\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}
\end{aligned}$$

and

$$\|T_{u_0}\left(I_+(f)g\right)\|_{L^2(\mathbb{R})} \leq |I_+(f)|\|u_0\|_{L^2(\mathbb{R})}\|\hat{g}\|_{L^1_\xi(\mathbb{R})} \leq C\|Gf\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

Combined with the Young's inequality, we can verify (3.6), which implies that $-2itT_{u_0}$ is infinitesimally small with respect to G. Then from the Kato-Rellich theorem 1.15, we can deduce that $(\mathrm{Dom}(G), -iG - 2itT_{u_0})$ is maximally dissipative, the proof is complete. $\square$

Remark 3.2. *Since $(\mathrm{Dom}(G), -iG - 2itT_{u_0})$ is a maximally dissipative operator, we know that $(G + 2tT_{u_0} - zId)^{-1}$ is well-defined for every $z \in \mathbb{C}_+$. By applying (1.5), we can deduce that*

$$\begin{aligned}
\frac{1}{2i\pi} I_+\left((G + 2tT_{u_0} - z\mathrm{Id})^{-1}\,\Pi u_0\right) &= \frac{1}{2i\pi} I_+\left((G - zId)^{-1}\left(Id + 2tT_{u_0}(G - zId)^{-1}\right)^{-1}\Pi u_0\right) \\
&= \left[\left(Id + 2tT_{u_0}(G - zId)^{-1}\right)^{-1}\Pi u_0\right](z)
\end{aligned}$$

is well-defined and holomorphic in $\mathbb{C}_+$.

In Theorem 1.4, the point is to prove the existence of the zero dispersion limit and show the formula (1.8) of this zero dispersion limit. In the following derivation, we first prove the the second equality of (1.8), and then show the existence of the zero dispersion limit.

To show the second equality of (1.8), we need the following integral equality, which has also been introduced in Lemma 1.8.

Lemma 3.3. *For $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $n \in \mathbb{N}_{\geq 1}$, we have*

$$\begin{aligned}&\int_{\mathbb{R}^n} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n\\ =&(n+1)\int_{\{\forall 1\leq j\leq n, y_j>0\}} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n.\end{aligned} \tag{3.7}$$

Proof. For $j \in \mathbb{N}_{\geq 0}$ and $0 \leq j \leq n$, we define

$$A_j := \{(y_1, y_2, ..., y_n) \in \mathbb{R}^n | \text{ there are } j \text{ negative elements in } (y_1, y_2, ..., y_n)\}.$$

We claim that, for $1 \leq j \leq n$, we have

$$\begin{aligned}&\int_{A_0} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n\\ =&\int_{A_j} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n.\end{aligned} \tag{3.8}$$

We notice that, if we obtain (3.8), since the integral on the null set is always equal to 0, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n\\ =&\sum_{j=0}^{n}\int_{A_j} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n\\ =&(n+1)\int_{A_0} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n,\end{aligned}$$

which implies (3.7). So the point is to prove (3.8).

Now we prove (3.8). For $1 \leq i, j \leq n$ and $0 \leq k \leq n$, we define

$$B_{k,i,j} := \{(y_1, y_2, ..., y_n) \in A_k | y_i \text{ is the } j\text{-th smallest element}\}.$$

For $(y_1, y_2, ..., y_n) \in B_{0,i,j}$, we make the following change of variables

$$\begin{cases} z_\ell = y_{\ell+i} - y_i \quad 1 \le \ell \le n-i, \\ z_{n+1-i} = -y_i, \\ z_\ell = y_{\ell+i-n-1} - y_i, \quad n+2-i \le \ell \le n. \end{cases}$$

We notice that $(z_1, z_2, ..., z_n) \in B_{j,n+1-i,1}$, so this linear transformation is from $B_{0,i,j}$ to $B_{j,n+1-i,1}$, and the absolute value of the determinant of this linear transformation is 1. We also observe that the inverse of this transformation

$$\begin{cases} y_k = z_{k+1+n-i} - z_{n+1-i} \quad 1 \le k \le i-1, \\ y_i = -z_{n+1-i}, \\ y_k = z_{k-i} - z_{n+1-i}, \quad i+1 \le k \le n. \end{cases}$$

is from $B_{j,n+1-i,1}$ to $B_{0,i,j}$, so this transformation is bijective from $B_{0,i,j}$ to $B_{j,n+1-i,1}$. Then we have

$$\begin{aligned} &\int_{B_{0,i,j}} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n \\ =&\int_{B_{j,n+1-i,1}} f(z_1)f(z_2-z_1)...f(z_n-z_{n-1})f(-z_n)dz_1dz_2...dz_n. \end{aligned}$$

Combining the above equality, we can deduce that

$$\begin{aligned} &\int_{A_0} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n \\ =&\sum_{i=1}^{n}\int_{B_{0,i,j}} f(y_1)f(y_2-y_1)...f(y_n-y_{n-1})f(-y_n)dy_1dy_2...dy_n \\ =&\sum_{i=1}^{n}\int_{B_{j,n+1-i,1}} f(z_1)f(z_2-z_1)...f(z_n-z_{n-1})f(-z_n)dz_1dz_2...dz_n \\ =&\int_{A_j} f(z_1)f(z_2-z_1)...f(z_n-z_{n-1})f(-z_n)dz_1dz_2...dz_n, \end{aligned}$$

which implies (3.8). The proof of (3.7) is complete. ∎

Remark 3.4. *We notice that the left hand side of (3.7) represents the value of the convolution of* $(n+1)$*-functions* $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *at the point 0, and the right hand side of (3.7) represents the value of the convolution of these* $(n+1)$*-*f *restricted in the support of positive half-line at the point 0. As observed in the proof of Lemma 3.3, (3.7) is derived from (3.8), and (3.8) is also interesting since it gives the equality between two convolutions at the point 0 with different supports of these* f*.*

Now we are able to prove the second equality of (1.8).

Lemma 3.5. *For $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ with $\lim_{x\to\infty} \frac{|u_0(x)|}{|x|} = 0$, we have*

$$\forall z \in \mathbb{C}_+, \quad \frac{1}{2i\pi} I_+ \left((G + 2tT_{u_0} - z\mathrm{Id})^{-1} \Pi u_0 \right) = \frac{1}{4i\pi t} \int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right) dy, \tag{3.9}$$

where Log *denotes the principal value of the logarithm.*

Proof. By applying (1.5), for any $z \in \mathbb{C}_+$, we have

$$\begin{aligned}\frac{1}{2i\pi} I_+ \left((G + 2tT_{u_0} - z\mathrm{Id})^{-1} \Pi u_0 \right) &= \frac{1}{2i\pi} I_+ \left((G - zId)^{-1} \left(Id + 2tT_{u_0} (G - zId)^{-1} \right)^{-1} \Pi u_0 \right) \\ &= \left[\left(Id + 2tT_{u_0} (G - zId)^{-1} \right)^{-1} \Pi u_0 \right] (z).\end{aligned} \tag{3.10}$$

Then we only need to show that

$$\forall z \in \mathbb{C}_+, \quad \left[\left(Id + 2tT_{u_0} (G - zId)^{-1} \right)^{-1} \Pi u_0 \right] (z) = \frac{1}{4i\pi t} \int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right) dy. \tag{3.11}$$

Since $-2itT_{u_0}$ is infinitesimally small with respect to G, we have

$$\begin{aligned}\left\| 2tT_{u_0}(G - zId)^{-1} \right\|_{\mathscr{L}(L^2_+)} &\le \varepsilon \left\| G(G - zId)^{-1} \right\|_{\mathscr{L}(L^2_+)} + C_\varepsilon \left\| (G - zId)^{-1} \right\|_{\mathscr{L}(L^2_+)} \\ &\le \varepsilon + \frac{C_\varepsilon}{\Im(z)}.\end{aligned}$$

The last inequality above comes from (1.12).

Then we choose $\varepsilon = \frac{1}{4}$ and $z \in i\mathbb{R}_{>0}$ such that $C_{\frac{1}{4}}/\Im(z) < \frac{1}{4}$, and we have

$$\left\| 2tT_{u_0}(G - zId)^{-1} \right\|_{\mathscr{L}(L^2_+)} < \frac{1}{2}.$$

Thus, we can develop $\left(Id + 2tT_{u_0}(G - zId)^{-1}\right)^{-1}$ into the series with such these z. We have

$$\left[\left(Id + 2tT_{u_0}(G - zId)^{-1} \right)^{-1} \Pi u_0 \right] (z) = \sum_{n=1}^{\infty} (-2t)^{n-1} \left[\left(T_{u_0} (G - zId)^{-1} \right)^{n-1} \Pi u_0 \right] (z). \tag{3.12}$$

We recall the formula (1.2) for $\Pi u_0(z)$,

$$\Pi u_0(z) = \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{u_0(y)}{y-z} dy, \tag{3.13}$$

which is the case of $n=1$.

When $n \geq 2$, we are going to prove

$$\left[\left(T_{u_0}\left(G-zId\right)^{-1}\right)^{n-1}\Pi u_0\right](z)=\frac{1}{2i\pi}\int_{\mathbb{R}} f_z(y)T_{f_z}^{n-2}\Pi f_z(y)dy \tag{3.14}$$

with

$$f_z(y):=\frac{u_0(y)}{y-z}.$$

We now adapt the mathematical induction to deduce (3.14). When $n=2$, by applying (1.2) and (1.4), we have

$$\begin{aligned}
\left[(G-zId)^{-1}\Pi u_0\right](x) &= \frac{\Pi u_0(x)-\Pi u_0(z)}{x-z}\\
&= \frac{1}{2i\pi(x-z)}\left(\lim_{\delta>0,\delta\to 0}\int_{\mathbb{R}}\frac{u_0(y)}{y-x-i\delta}dy-\int_{\mathbb{R}}\frac{u_0(y)}{y-z}dy\right)\\
&= \frac{1}{2i\pi}\lim_{\delta>0,\delta\to 0}\int_{\mathbb{R}}\frac{u_0(y)}{(y-x-i\delta)(y-z)}dy\\
&= \Pi f_z(x).
\end{aligned}$$

Thus we have

$$\left[T_{u_0}\left(G-zId\right)^{-1}\Pi u_0\right](z)=\left[T_{u_0}\Pi f_z\right](z)=\frac{1}{2i\pi}\int_{\mathbb{R}} f_z(y)\Pi f_z(y)dy,$$

which yields (3.14) with $n=2$.

Then we suppose that (3.14) holds for $n=k(k\geq 2)$. For $n=k+1$, we have

$$\left[\left(T_{u_0}\left(G-zId\right)^{-1}\right)^{k}\Pi u_0\right](z)=\left[T_{u_0}\left(G-zId\right)^{-1}\left(T_{u_0}\left(G-zId\right)^{-1}\right)^{k-1}\Pi u_0\right](z).$$

We note

$$g_k(z):=\left[\left(T_{u_0}\left(G-zId\right)^{-1}\right)^{k-1}\Pi u_0\right](z),$$

by the assumption, we have

$$g_k(z)=\frac{1}{2i\pi}\int_{\mathbb{R}} f_z(y)T_{f_z}^{k-2}\Pi f_z(y)dy.$$

Then we have

$$
\begin{aligned}
&(G-zId)^{-1}g_k(x)\\
=&\frac{g_k(x)-g_k(z)}{x-z}\\
=&\frac{1}{2i\pi(x-z)}\left(\lim_{\delta>0,\delta\to 0}\int_{\mathbb{R}}\frac{u_0(y)}{y-x-i\delta}T_{f_z}^{k-2}\Pi f_z(y)dy-\frac{1}{2i\pi}\int_{\mathbb{R}}\frac{u_0(y)}{y-z}T_{f_z}^{k-2}\Pi f_z(y)dy\right)\\
=&\frac{1}{2i\pi}\lim_{\delta>0,\delta\to 0}\int_{\mathbb{R}}\frac{u_0(y)}{(y-x-i\delta)(y-z)}T_{f_z}^{k-2}\Pi f_z(y)dy\\
=&T_{f_z}^{k-1}\Pi f_z(x).
\end{aligned}
$$

Thus we have

$$\left[\left(T_{u_0}(G-zId)^{-1}\right)^k\Pi u_0\right](z)=\left[T_{u_0}T_{f_z}^{k-1}\Pi f_z\right](z)=\frac{1}{2i\pi}\int_{\mathbb{R}}f_z(y)T_{f_z}^{k-1}\Pi f_z(y)dy,$$

which yields (3.14) with $n=k+1$. By the induction, we complete the proof of (3.14).

In fact, we can easily observe that $f_z\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, so $\widehat{f_z}\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. Then for $n\geq 2$, by Lemma 3.3, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}f_z(y)T_{f_z}^{n-2}\Pi f_z(y)dy\\
=&\mathcal{F}_{y\to\eta}\left(f_zT_{f_z}^{n-2}\Pi f_z\right)(0)\\
=&\int_{\{\forall 1\leq j\leq n-1,\eta_j>0\}}\widehat{f_z}(\eta_1)\widehat{f_z}(\eta_2-\eta_1)...\widehat{f_z}(\eta_{n-1}-\eta_{n-2})\widehat{f_z}(-\eta_{n-1})d\eta_1d\eta_2...d\eta_{n-1}\\
=&\frac{1}{n}\int_{\mathbb{R}^{n-1}}\widehat{f_z}(\eta_1)\widehat{f_z}(\eta_2-\eta_1)...\widehat{f_z}(\eta_{n-1}-\eta_{n-2})\widehat{f_z}(-\eta_{n-1})d\eta_1d\eta_2...d\eta_{n-1}\\
=&\frac{1}{n}\mathcal{F}_{y\to\eta}\left(f_z^n\right)(0)\\
=&\frac{1}{n}\int_{\mathbb{R}}f_z^n(y)dy.
\end{aligned}
\tag{3.15}
$$

For $t\in\mathbb{R}$ fixed, since u_0 satisfies $\lim_{x\to\infty}\frac{|u_0(x)|}{|x|}=0$, then for any $\varepsilon>0$, there exists $R_\varepsilon>0$ such that

$$\frac{2|t||u_0(x)|}{|x|}<\varepsilon\quad\text{ for all }\quad |x|\geq R_\varepsilon.$$

Also, since $u_0\in L^\infty_{loc}(\mathbb{R})$, there exists $M_\varepsilon>0$ such that

$$2|t|\|u_0\|_{L^\infty(|x|<R_\varepsilon)}\leq M_\varepsilon.$$

We fix $\varepsilon = \frac{1}{4}$, and take $z \in i\mathbb{R}_{>0}$ such that $\Im(z) > 4M_{\frac{1}{4}}$, and then we have

$$2|t| \, \|f_z\|_{L^\infty} < \frac{1}{2}. \tag{3.16}$$

Thus, for $z \in i\mathbb{R}_{>0}$ with $\Im(z)$ large enough, combining (3.12), (3.13), (3.14), (3.15) and (3.16), we can deduce that

$$\begin{aligned}
&\frac{1}{2i\pi} I_+ \left((G + 2tT_{u_0} - z\mathrm{Id})^{-1} \, \Pi u_0 \right) \\
&= \sum_{n=1}^{\infty} (-2t)^{n-1} \left[\left(T_{u_0} \, (G - zId)^{-1} \right)^{n-1} \Pi u_0 \right] (z) \\
&= \frac{1}{4i\pi t} \int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (2tf_z(y))^n dy \\
&= \frac{1}{4i\pi t} \int_{\mathbb{R}} \mathrm{Log}(1 + 2tf_z(y)) dy \\
&= \frac{1}{4i\pi t} \int_{\mathbb{R}} \mathrm{Log}(1 + \frac{2tu_0}{y - z}) dy,
\end{aligned}$$

which implies (3.9) for $z \in i\mathbb{R}_{>0}$ with $\Im(z)$ large enough. By Remark 1.10 and Remark 3.2, we know that the functions (with respect to z) on both sides of (3.9) are holomorphic in $\mathbb{C}_+$, then from the isolated zeros theorem, we can deduce (3.9) on the whole upper half-plane $\mathbb{C}_+$. The proof is complete. $\square$

Remark 3.6. *In fact, (3.15) implies that, for every $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and for every $n \geq 2$,*

$$\int_{\mathbb{R}} f(y) T_f^{n-2} \Pi f(y) dy = \frac{1}{n} \int_{\mathbb{R}} f^n(y) dy, \tag{3.17}$$

so we have obtained an integral equality (3.17) related to the Toeplitz operator T_f, which is derived from (3.7).

Combining Lemma 3.1 and Lemma 3.5, we give the following proof of Theorem 1.4.

Proof of Theorem 1.4. We consider the equation (1.7) with $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$ satisfying $\lim_{x \to \infty} \frac{|u_0(x)|}{|x|} = 0$. By the L^2 conservation law for (1.7), we know that

$$\forall t \in \mathbb{R}, \quad \|u^\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2} .$$

Consequently, the family $u^\varepsilon(t)$ has weak limits in $L^2(\mathbb{R})$ as $\varepsilon \to 0$. Our task therefore consists in proving that there is only one such weak limit ω_t . Since u^ε is real valued, so is ω_t, hence $w_t = \Pi w_t + \overline{\Pi w_t}$ on the real line. In Lemma 3.5, we have already shown the second equality in (1.8), we are therefore reduced to proving the identity

$$\forall z \in \mathbb{C}_+, \quad \Pi w_t(z) = \frac{1}{2i\pi} I_+ \left((G + 2tT_{u_0} - z\mathrm{Id})^{-1} \Pi u_0 \right), \tag{3.18}$$

since this identity clearly characterises ω_t.

We then recall the formula (3.1) for $\Pi u^\varepsilon(t,z)$,

$$\Pi u^\varepsilon(t,z) = \frac{1}{2i\pi} I_+ \left(\left(G + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0} \mathrm{e}^{i\varepsilon t\partial_x^2} - z\mathrm{Id} \right)^{-1} \mathrm{e}^{-i\varepsilon t\partial_x^2} \Pi u_0 \right). \tag{3.19}$$

Since $\mathrm{e}^{\pm i\varepsilon t\partial_x^2}$ is convergent to Id in $\mathscr{L}\left(L^2_+(\mathbb{R})\right)$ and since $(\mathrm{Dom}(G), -iG - 2itT_{u_0})$ is maximally dissipative, then for $z \in \mathbb{C}_+$, the function

$$g_z^\varepsilon := \left(G + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0} \mathrm{e}^{i\varepsilon t\partial_x^2} - z\mathrm{Id} \right)^{-1} \mathrm{e}^{-i\varepsilon t\partial_x^2} \Pi u_0$$

is strongly convergent to

$$g_z^0 := (G + 2tT_{u_0} - z\mathrm{Id})^{-1} \Pi u_0$$

in $L^2_+(\mathbb{R})$, and therefore $I_+(g_z^\varepsilon)$ converges to $I_+\left(g_z^0\right)$ pointwisely in $\mathbb{C}_+$.

Also, from the weak convergence of $u^\varepsilon(t)$ to ω_t in $L^2(\mathbb{R})$, we have

$$\forall z \in \mathbb{C}_+, \quad \Pi u^\varepsilon(t,z) - \Pi\omega_t(z) = \int_0^\infty \mathrm{e}^{iz\xi} \left(\widehat{u^\varepsilon}(t,\xi) - \widehat{\omega}_t(\xi) \right) d\xi \to 0,$$

and thus (3.18) follows. The proof is complete.

Remark 3.7. *In fact, for $u_0 \in L^2_r(\mathbb{R})$ with $|u_0(x)| \le C\langle x \rangle$ and for $|t| < \frac{1}{2C}$, by applying the method in the proof of Lemma 3.1, we can deduce that $-2itT_{u_0}$ is G-bounded with the relative bound smaller than 1. Then by the Kato-Rellich theorem 1.15, we can conclude that $(\mathrm{Dom}(G), -iG - 2itT_{u_0})$ is maximally dissipative. With a slight modification of the proof of Lemma 3.5, we can also show (3.9) for $u_0 \in L^2_r(\mathbb{R})$ with $|u_0(x)| \le C\langle x \rangle$ in a short time range $|t| < \frac{1}{2C}$. Finally, by following the same approach used in the proof of Theorem 1.4, we can deduce Corollary 1.9.*

4 Final comments and open problems

Let us briefly give some comments related to the previous sections.

1. Recently, R. Killip, T. Laurens and M. Vişan have extended continuously the flow map of (1.1) to $u_0 \in H_r^s(\mathbb{R})$ with $-\frac{1}{2} < s < 0$ [11]. However, so far we have not been able to extend the explicit formula (1.6) to $u_0 \in H_r^s(\mathbb{R})$ with $-\frac{1}{2} < s < 0$. In fact, we cannot apply directly the perturbation argument used in Section 2 to this case, and we do not know if $(G - 2tL_{u_0} - z\mathrm{Id})^{-1}$ exists on $H_+^s(\mathbb{R})$ with $-\frac{1}{2} < s < 0$. So far no other suitable approach has been found to give an explicit formula for the solution of (1.1) in this case. We remark that we have also the global well-posedness of the Benjamin–Ono equation on the torus in $H_r^s(\mathbb{T})$ with $-\frac{1}{2} < s < 0$ [6][11], and the explicit formula for the Benjamin–Ono equation on the torus has been successfully extended to $u_0 \in H_r^s(\mathbb{T})$ with $-\frac{1}{2} < s < 0$ [7].

2. As explained in Remark 1.10, we know that the expression

$$\frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right) dy$$

makes sense if $u_0 \in L_r^2(\mathbb{R})$. Nevertheless, this does not imply that the zero dispersion limit exists in this case. For $u_0 \in L_r^2(\mathbb{R})$, let u^ε be the corresponding solution to (1.7) with the initial data u_0, and we take a sequence $u_0^n \in L_r^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which converges to u_0 in $L^2(\mathbb{R})$. In fact, from (1.8) and Remark 1.10, we know that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} \Pi u_n^\varepsilon(t,z) = \lim_{n\to\infty} \frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0^n(y)}{y-z}\right) dy = \frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right) dy,$$

where u_n^ε denotes the corresponding solution to (1.7) with the initial data u_0^n.

To show the existence of the zero dispersion limit with the initial data $u_0 \in L_r^2(\mathbb{R})$, we only need to show that $\lim_{\varepsilon\to 0}\lim_{n\to\infty} \Pi u_n^\varepsilon(t,z)$ exists. A natural idea is to show that these two limits can be exchanged in order, which would then imply that

$$\lim_{\varepsilon\to 0} \Pi u^\varepsilon(t,z) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \Pi u_n^\varepsilon(t,z) = \lim_{n\to\infty}\lim_{\varepsilon\to 0} \Pi u_n^\varepsilon(t,z) = \frac{1}{4i\pi t}\int_{\mathbb{R}} \mathrm{Log}\left(1 + \frac{2tu_0(y)}{y-z}\right) dy.$$

However, we lack certain uniform conditions for this double limit to prove the order exchangeability, so the existence for the zero dispersion limit with the initial data $u_0 \in L_r^2(\mathbb{R})$ is still unknown even in a short time.

Also, as observed in (3.5), the condition

$$u_0 \in L_r^2(\mathbb{R}) \text{ with } |u_0(x)| \le C\langle x\rangle$$

is a necessary condition for $\left(G + 2t\mathrm{e}^{i\varepsilon t\partial_x^2} T_{u_0} \mathrm{e}^{-i\varepsilon t\partial_x^2} - z\mathrm{Id}\right)$ to have a limit in $\mathscr{L}\left(\mathrm{Dom}(G), L_+^2(\mathbb{R})\right)$ as $\varepsilon \to 0$. With this condition we can only deduce the existence of the zero dispersion limit in a short time, and the existence of the zero dispersion limit in a long time is still unknown for the same reason explained in the previous paragraph. A natural idea to solve this difficulty is to apply the Kato-Rellich theorem to show that

$$(G + 2tT_{u_0} - z\mathrm{Id})^{-1}$$

exists on $L^2_+(\mathbb{R})$ for every $z \in \mathbb{C}_+$, but the perturbation argument fails in a long time range since we cannot deduce that the relative bound of $-2itT_{u_0}$ with respect to G is smaller than 1 for every $t \in \mathbb{R}$.

3. The zero dispersion limit for the Benjamin–Ono equation on the torus was studied by L. Gassot in [4][5]. In [5], the explicit formula for the Benjamin–Ono equation on the torus established in [7] was used to prove the existence of the zero dispersion limit for every initial datum in $L^\infty(\mathbb{T})$. The existence of the zero–dispersion limit for more singular initial data is still an open problem. As introduced in Remark 1.6, L. Gassot has also obtained the formula (1.9) in the special case of a general bell shaped initial datum in [4][5].

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