FREE ENERGY EXPANSIONS OF A CONDITIONAL GinUE AND LARGE DEVIATIONS OF THE SMALLEST EIGENVALUE OF THE LUE

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Abstract. We consider a planar Coulomb gas ensemble of size N with the inverse temperature $\beta = 2$ and external potential $Q(z) = |z|^2 - 2c \log |z - a|$, where $c > 0$ and $a \in \mathbb{C}$. Equivalently, this model can be realised as N eigenvalues of the complex Ginibre matrix of size $(c+1)N \times (c+1)N$ conditioned to have deterministic eigenvalue a with multiplicity cN. Depending on the values of c and a, the droplet reveals a phase transition: it is doubly connected in the post-critical regime and simply connected in the pre-critical regime. In both regimes, we derive precise large-N expansions of the free energy up to the $O(1)$ term, providing a non-radially symmetric example that confirms the Zabrodin-Wiegmann conjecture made for general planar Coulomb gas ensembles. As a consequence, our results provide asymptotic behaviours of moments of the characteristic polynomial of the complex Ginibre matrix, where the powers are of order $O(N)$. Furthermore, by combining with a duality formula, we obtain precise large deviation probabilities of the smallest eigenvalue of the Laguerre unitary ensemble. A key ingredient for the proof lies in the fine asymptotic behaviour of a planar orthogonal polynomial, extending a result of Betola et al [16]. This result holds its own interest and is based on a refined Riemann-Hilbert analysis using the partial Schlesinger transform.

1. Introduction

1.1. **Models and numerology.** In this work, we obtain precise asymptotic behaviours up to the $O(1)$ term in the context of the following interrelated topics.

- Zabrodin-Wiegmann prediction on the partition functions of planar Coulomb gas ensembles: a case study for a conditional complex Ginibre ensemble breaking the rotational symmetry.
- Asymptotic behaviours of moments of the characteristic polynomial of the complex Ginibre ensemble.
- Large deviation probabilities of the smallest eigenvalue of the Laguerre unitary ensemble.

Due to a certain duality relation (Proposition 1.1) these topics are indeed equivalent, and readers may find a particular viewpoint most interesting based on their individual interests. In the aforementioned topics, a specific phase transition occurs, yielding distinct geometric/probabilistic implications in each context, see Subsection 1.3 for details, cf. Figures 1 and 2.

Let us be more precise in introducing our models and formulations.

1.1.1. *GinUE and its characteristic polynomials.* We begin with the complex Ginibre ensemble (GinUE) $\mathbf{G}_N$, an $N \times N$ matrix whose entries are given by independent centered complex Gaussian random variables with variance $1/N$, see [32, 33] for recent reviews. It is well known that the eigenvalues $\{z_j\}_{j=1}^N$ of $\mathbf{G}_N$ follow the joint probability distribution

$$(1.1) \qquad \frac{1}{Z_N^{\mathrm{Gin}}} \prod_{j>k=1}^{N} |z_j - z_k|^2 \prod_{j=1}^{N} e^{-N|z_j|^2} \, dA(z_j), \qquad Z_N^{\mathrm{Gin}} = N! \frac{G(N+1)}{N^{N(N+1)/2}},$$

where $dA(z) = d^2z/\pi$ is the area measure and G is the Barnes G-function [101, Section 5.17]. Here, Z_N^{Gin} is the normalisation constant, known as the partition function that makes (1.1) a probability measure. As $N \to \infty$, the eigenvalues $\{z_j\}_{j=1}^N$ tend to be uniformly distributed on the unit disk, known as the circular law, see e.g. [75] for a recent progress. Note here that we have a simple weight function $e^{-N|z|^2}$ in (1.1), which enables explicit computations of the GinUE statistics. For instance, the evaluation of Z_N^{Gin} follows from Andréief's formula together with the norm of the associated orthogonal polynomial (which is in this case monomial).

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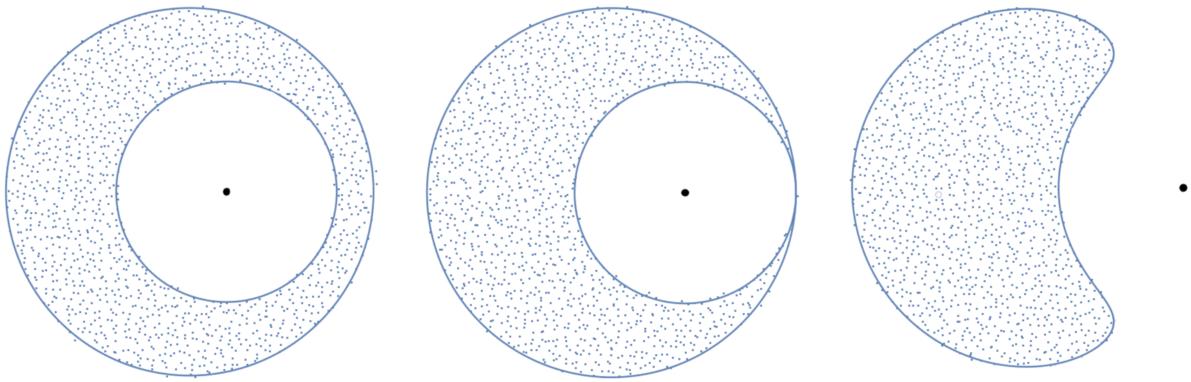


(A) Post-critical; $a = 1/4$ (B) Critical; $a = 1/2$ (C) Pre-critical; $a = 1$

FIGURE 1. Illustration of the droplet, where $c = 9/16$. The black dot indicates the point a.

For the GinUE model, we shall investigate the moment of its characteristic polynomial

$$\mathbb{E}\Big|\det(\mathbf{G}_N - z)\Big|^{2\gamma}, \qquad \gamma \geq 0. \tag{1.2}$$

For a fixed value of $\gamma \geq 0$, the asymptotic behaviours of (1.2) were recently obtained by Webb and Wong [121] for the bulk regime $|z| < 1$, and by Deaño and Simm [55] for the edge regime $|z| = 1 + O(1/\sqrt{N})$. These asymptotic behaviours can be applied to construct a Gaussian multiplicative chaos measure [105], see also [86]. We mention that the moment of the characteristic polynomial in Hermitian random matrix theory has been extensively studied, see e.g. [22,38,42,43,83,87,120]. Along the similar spirit of [55,121], we study the asymptotic behaviours of (1.2), but instead of a fixed γ, we consider the exponentially varying regime $\gamma = O(N)$. Namely, we examine the regime where γ is scaled as $\gamma = cN$ for a fixed parameter $c > 0$. In this case, a phase transition occurs at the critical value $|z| = \sqrt{c+1} - \sqrt{c}$, and we will investigate all the regimes that arise.

1.1.2. *Partition function of a determinantal Coulomb gas.* As a second formulation, we consider a conditional point process. For this purpose, we fix a parameter $c > 0$ and consider the GinUE of size $(c+1)N \times (c+1)N$ conditioned to have deterministic eigenvalue $a \geq 0$ with multiplicity cN. Then the remaining N random eigenvalues $\{z_j\}_{j=1}^N$ follow the distribution

$$\frac{1}{Z_N(a,c)} \prod_{j>k=1}^N |z_j - z_k|^2 \prod_{j=1}^N |z_j - a|^{2cN} e^{-N|z_j|^2} \, dA(z_j). \tag{1.3}$$

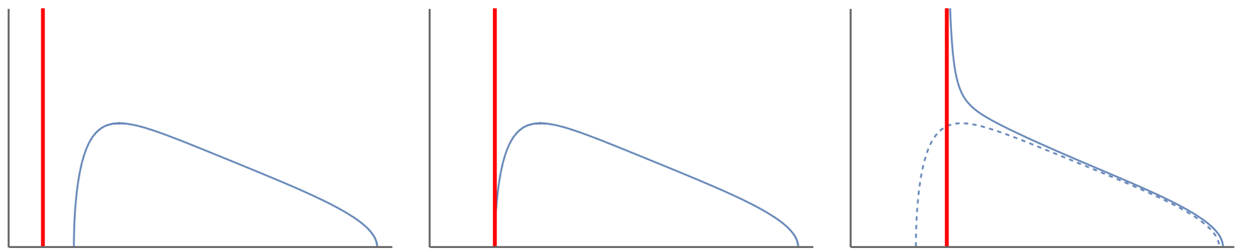
This model is called the induced GinUE [14, 59] and its partition function $Z_N(a,c)$ is also called a massive partition function in the quantum chromodynamics related literature [6]. Note that if $c > 0$, the ensemble is rotationally symmetric only for the case $a = 0$, and in this case, again explicit computations lead to

$$Z_N(0,c) = N! \frac{G(N + cN + 1)}{G(cN+1)} N^{-(c+\frac{1}{2})N^2 - \frac{1}{2}N}. \tag{1.4}$$

From the statistical physics viewpoint, the model (1.3) can be realised as a planar Coulomb gas with inverse temperature $\beta = 2$ (also known as the random normal matrix model [12, 44]) and the external potential

$$Q(z) = |z|^2 - 2c \log|z - a|. \tag{1.5}$$

We mention that potentials of this type are sometimes called Hele-Shaw potentials, see [15] for a recent work. As $N \to \infty$, the ensemble (1.3) tends to be uniformly distributed on a certain droplet $S \equiv S_Q$, see Figure 1. The precise shape of the droplet is characterised in [16], and it reveals a topological phase transition, see Subsection 1.3. In the theory of Coulomb gases, the large-N expansion of the free energy $\log Z_N(a,c)$ is a fundamental topic [90, 126], as the coefficients of this expansion provide essential potential theoretic/geometric properties of the model, see Subsection 1.2 for more details.



(A) Pulled; $t = \lambda_- - 1$ (B) Critical; $t = \lambda_-$ (C) Pushed; $t = \lambda_- + 1$

FIGURE 2. Illustration of the Marchenko-Pastur law (1.7) and constrained spectral density (2.45), where $\alpha = 5$. Here, the vertical (full red) line indicates the hard wall $x = t$ of the LUE.

1.1.3. *Laguerre unitary ensemble and its smallest eigenvalue.* The third formulation is in the context of a classical Hermitian random matrix model [61]. We consider an $N \times N$ Wishart matrix, also known as the Laguerre unitary ensemble (LUE) $\mathbf{W}_N = \mathbf{R}_N \mathbf{R}_N^*$, where $\mathbf{R}_N$ of size $N \times (\alpha+1)N$ is a rectangular complex Ginibre matrix. Here $\alpha \geq 0$ is the rectangular parameter. Then the joint probability distribution of eigenvalues $\{\lambda_j\}_{j=1}^N$ of $\mathbf{W}_N$ is proportional to

$$\prod_{j>k=1}^N |\lambda_j - \lambda_k|^2 \prod_{j=1}^N \lambda_j^{\alpha N} e^{-N\sum_{j=1}^N \lambda_j}, \qquad (\lambda_N > \cdots > \lambda_1 > 0). \tag{1.6}$$

It is well known that the as $N \to \infty$, the empirical measure of the LUE follows the Marchenko-Pastur distribution

$$\frac{1}{2\pi} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{x} \cdot \mathbb{1}_{[\lambda_-,\lambda_+]}(x), \qquad \lambda_\pm := (\sqrt{\alpha+1} \pm 1)^2. \tag{1.7}$$

In this context, we focus on the statistics of the smallest eigenvalue λ_1. Such a statistic of the LUE, or a more general sample covariance matrix where the Gaussian entries are replaced by an i.i.d. random variable, finds several applications in random matrix theory, see e.g. [18,69,124] and references therein.

1.1.4. *Duality relation.* We now discuss the equivalence of the above three formulations. First, notice that by definition, the characteristic polynomial and partition functions are related as

$$\mathbb{E}\Big|\det(G_N - z)\Big|^{2cN} = \frac{1}{Z_N^{\rm Gin}} \int_{\mathbb{C}^N} \prod_{j>k=1}^N |z_j - z_k|^2 \prod_{j=1}^N |z - z_j|^{2cN} e^{-N|z_j|^2} \, dA(z_j) = \frac{Z_N(|z|,c)}{Z_N^{\rm Gin}}. \tag{1.8}$$

The following equivalence was introduced in [64,100]. This is a restatement of [55, Proposition 3.1] after several transformations, and for the reader's convenience, we provide the details in Section 6.

Proposition 1.1 (Duality relation). *Let λ_1 be the smallest eigenvalue of the LUE in* (1.6)*. For a fixed $c > 0$, we put $\alpha = 1/c$. Then for any $x \in \mathbb{R}$, we have*

$$\mathbb{P}\Big[\lambda_1 > \frac{x^2}{c}\Big] = e^{-cN^2x^2} \frac{Z_N(x,c)}{Z_N(0,c)}\Big|_{N \to N/c}. \tag{1.9}$$

Due to this duality relation and the well-known asymptotic expansion (3.16) of the Barnes G-function, it becomes evident that the asymptotic behaviours of the three aforementioned formulations are equivalent. We mention that the duality relation originates from the supersymmetry method, cf. [66,68,100]. Remarkably, this relation expresses the integral (1.2) over an $N \times N$ non-Hermitian random matrix in terms of an integral over a $\gamma \times \gamma$ Hermitian random matrix. In our present case, where $\gamma = cN$, we further make the change of variables $cN \mapsto N$, resulting in the formula (1.9) with the parameter $\alpha = 1/c$ of the LUE. It is noteworthy that such a duality relation finds application in various contexts of random matrix theory, see e.g. [95,110,111].

1.2. **Free energy expansion; Zabrodin-Wiegmann prediction.** Among the above three formulations, let us discuss the free energy expansion from a viewpoint of a more general Coulomb gas theory. In general, the partition function of the random normal matrix model is given by

$$Z_N^V := \int_{\mathbb{C}^N} \prod_{j>k=1}^N |z_j - z_k|^2 \prod_{j=1}^N e^{-N\,V(z_j)}\,dA(z_j), \tag{1.10}$$

where $V : \mathbb{C} \to \mathbb{R}$ is a given external potential.

To describe the asymptotic behaviour of Z_N^V, we recall some potential theoretic notions [106]. Given a compactly supported probability measure μ on $\mathbb{C}$, the weighted logarithmic energy $I_V[\mu]$ associated with the potential V is given by

$$I_V[\mu] := \int_{\mathbb{C}^2} \log \frac{1}{|z-w|}\,d\mu(z)\,d\mu(w) + \int_{\mathbb{C}} V\,d\mu. \tag{1.11}$$

For a general V, there exists a unique minimizer σ_V called the equilibrium measure. Furthermore, due to Frostman's theorem, it is of the form

$$d\sigma_V(z) = \Delta V(z)\,\mathbb{1}_{S_V}(z)\,dA, \qquad \Delta = \partial\bar{\partial}, \tag{1.12}$$

where the compact support S_V is called the droplet. It is conjectured that if the droplet is *connected*, the partition function Z_N^V has the asymptotic expansion of the form

$$\begin{aligned} \log Z_N^V = -I_V[\sigma_V]N^2 + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1 - \frac{1}{2}\int_{\mathbb{C}} \log(\Delta V)\,d\sigma_V\Big)N \\ + \frac{6-\chi}{12}\log N + \frac{\log(2\pi)}{2} + \chi\,\zeta'(-1) + \mathcal{F}_V + o(1). \end{aligned} \tag{1.13}$$

Here χ is the Euler characteristic of the droplet and ζ is the Riemann zeta function. We refer the reader to [34, Section 1.1] and [32, Section 5.4] for the development of the expansion (1.13).

Among the variety of literature on the expansion (1.13), we mention that Leblé and Serfaty [90] proved the expansion (1.13) up to the order $O(N)$ in the context of a more general β ensemble, see also [20, 113] for quantitative error bounds. The topology-dependence of the $O(\log N)$ term was introduced in the work of Jancovici et al. [76, 116] through exactly solvable examples such as (induced) Ginibre and spherical ensembles. The $O(1)$ term reflects a conformal geometric property of the droplet. More precisely, in [126], Zabrodin and Wiegmann made use of the Ward's identities in conformal field theory (see e.g. [78, Appendix 6]) and proposed a remarkable prediction, suggesting that the term $\mathcal{F}_V$ can be expressed in terms of the zeta-regularized determinant of the exterior droplet. In particular, for a quasi-harmonic potential V (i.e. ΔV is a constant), this conjecture reads

$$\mathcal{F}_V = -\frac{1}{2}\log\det{}_\zeta(\Delta_{\mathbb{C}\setminus S_V}), \tag{1.14}$$

cf. Remark 2.3. For a general potential, one needs an additional term related to the first correction of the global density.

The expansion of the form (1.13) was obtained in a recent work [34] for a radially symmetric potential V with $\Delta V > 0$ in $\mathbb{C}$. This strictly sub-harmonic assumption is crucial in [34], as it leads to a droplet that is either a disc or an annulus. The asymptotic behaviour of the partition function Z_N^V has been further investigated in [9] to include the case where the ensemble exhibits a spectral gap (i.e. V can be such that $\Delta V < 0$). In particular, it was shown in [9] that if the droplet has multiple components, a non-trivial oscillatory term (a new "displacement term") emerges in the $O(1)$ expansion, indicating that the expansion of the form (1.13) does not hold. Furthermore, in [9], the case with the harmonic measure perturbation of the order $O(1/N)$ in the potential and Fisher-Hartwig singularities [82] has been investigated, which are closely related to the fluctuation theorem [8].

Remark 1.2 (Potential with a hard edge)*.* In the above discussions, we have focused on the case that V is supported on the whole complex plane, leading to a droplet with a soft edge. In contrast, if V is confined to a subset of the droplet, the expansion of the free energy takes on a notably different form. This hard edge regime finds applications in different contexts including the counting statistics [2, 10, 11, 31, 39] and hole probabilities [1, 5, 40, 41, 60, 67], cf. Remark 2.4. Moreover, when V is supported on a Jordan curve, the asymptotic behaviour of the free energy has also been investigated, see e.g. [46, 77, 122] and references therein.

In this work, we obtain a precise expansion in the form of (1.13) for the potential given by (1.5). To our knowledge, this provides the first non-radially symmetric example (other than exactly solvable models such as the elliptic GinUE) that confirms this conjecture. Indeed, the elliptic GinUE [33, Section 2.3], indexed by the non-Hermiticity parameter $\tau \in [0, 1)$, is the only known non-radially symmetric example for which the free energy expansion in the form of (1.13) is known. However, in this case, the free energy is simply the same as that of the GinUE up to an additive constant of $\frac{N}{2}\log(1-\tau^2)$.

Due to the lack of the rotational symmetry, it requires a different approach compared to [9, 34], and we implement the partial Schlesinger transform [25] to refine the Riemann-Hilbert analysis in [16, 84, 94]. Furthermore, as explained in the previous subsection, it follows from Proposition 1.1 that this result implies the asymptotic behaviours of moments of the characteristic polynomials of the GinUE, as well as the large-deviation probabilities of the smallest eigenvalue of the LUE.

1.3. **Phase transition.** As previously mentioned, the model we consider undergoes an interesting phase transition. We now precisely describe such a transition. Let $S \equiv S_Q$ be the droplet in (1.12) associated with the potential Q given in (1.5). The droplet reveals a topological phase transition at the critical value

$$c_{\rm cri} := \frac{(1-a^2)^2}{4a^2}, \qquad a_{\rm cri} := \sqrt{c+1} - \sqrt{c}. \tag{1.15}$$

In the post- and pre-critical regimes, the parameters a and c are assumed to be fixed (i.e. independent of N). The droplet is then given as follows [16, Section 2].

- **Post-critical regime**: $c < c_{\rm cri}$, i.e. $a < a_{\rm cri}$. In this case, the droplet is given by
$$S = \overline{\mathbb{D}(0, \sqrt{1+c})} \setminus \mathbb{D}(a, \sqrt{c}), \tag{1.16}$$
where $\mathbb{D}(z, r)$ is a disk with center z and radius $r > 0$.
- **Pre-critcal regime**[1]: $c > c_{\rm cri}$, i.e. $a > a_{\rm cri}$. In this case, the droplet is a simply connected domain whose boundary is given by the image of the unit circle under the conformal map
$$f(z) = R\,z - \frac{\kappa}{z-q} - \frac{\kappa}{q}, \qquad R = \frac{1+a^2q^2}{2aq}, \qquad \kappa = \frac{(1-q^2)(1-a^2q^2)}{2aq}. \tag{1.17}$$
Note that $R > 0$ is the conformal radius of the droplet. Here, $q \equiv q(a)$ satisfies $f(1/q) = a$ and it is given by a unique solution of the algebraic equation
$$q^6 - \Big(\frac{a^2+4c+2}{2a^2}\Big)q^4 + \frac{1}{2a^4} = 0 \tag{1.18}$$
such that $0 < q < 1$ and $\kappa > 0$.

See Figure 1 for the shape of the droplet. From the above description of the droplet, one can see that the Euler characteristic χ of the droplet S_Q is given by

$$\chi = \begin{cases} 0 & \text{for the post-critical case,} \\ 1 & \text{for the pre-critical case.} \end{cases} \tag{1.19}$$

This will play an important role in the free energy expansion.

Let us mention that there are some more examples of the droplets revealing a topological phase transition, see e.g. [3, 17, 30, 47]. A notable feature of the phase transition is the emergence of a singular boundary point. In our present case, it is a merging (double) point that falls into the class of Sakai's regularity theory [91, 107]. Such singular boundaries are of particular interest from the viewpoint of the non-standard universality classes, and have been studied for several different models, see e.g. [13, 24, 27, 35, 99] and references therein.

Beyond the post- and pre-critical regimes, it is also natural to study the behaviour of the ensemble (1.3) in the critical regime, see [84] for a recent work on the local statistics. Let us first define the critical regime.

[1]Compared to [16], we have replaced the notations: $\alpha \to q$ and $\rho \to R$.

Definition 1.3 (Critical scaling regime). The critical case corresponds to the scaling regime

$$1 - a(a + 2\sqrt{c}) = O(N^{-2/3}).$$

We consider a parameter $s \in \mathbb{R}$ describing the critical regime, see [16, Eq.(1.36)]. Then the parameter a satisfies

$$a = a_{\rm cri} - \frac{(\sqrt{c+1} - \sqrt{c})^{1/3}}{2c^{1/6}(c+1)^{1/6}} \frac{s}{N^{2/3}} + O\Big(\frac{1}{N^{4/3}}\Big). \tag{1.20}$$

As $s \to +\infty$, the limit coincides with the post-critical regime, while the opposite limit as $s \to -\infty$ coincides with the pre-critical regime.

Turning to the LUE statistics, note that in the setup in Proposition 1.1 with $x = a \geq 0$, it follows from (1.7) that

$$\frac{x^2}{c} < \lambda_- = \Big(\sqrt{\frac{c+1}{c}} - 1\Big)^2, \quad \text{if and only if} \quad a < \sqrt{c+1} - \sqrt{c} = a_{\rm cri}. \tag{1.21}$$

Hence, the topological phase transition of the droplet can be naturally interpreted from the viewpoint of lower and upper large deviation probabilities. This transition is also called the push-to-pulled transition, see Figure 2. Furthermore, while the order $O(N^{-2/3})$ in the critical scaling regime might be less intuitive in the two-dimensional model (1.3), it becomes clear if we interpret this in the context of the LUE given the standard square root decay of the Marchenko-Pastur law (1.7) at the soft edge. In this context, a universal pulled-to-pushed transition of the third order has been observed, see e.g. [48, 96]. See Remark 2.6 for more details.

2. Main results

In this section, we present our main results. Recall that the partition function $Z_N(a,c)$ is given by

$$Z_N(a,c) \equiv Z_N^Q := \int_{\mathbb{C}^N} \prod_{j>k=1}^N |z_j - z_k|^2 \prod_{j=1}^N e^{-NQ(z_j)} \, dA(z_j), \tag{2.1}$$

where Q is given by (1.5). As discussed in Subsection 1.2, the logarithmic energy (1.11) should appear in the leading term of the free energy expansion. We first evaluate the energy associated with the potential Q explicitly.

Proposition 2.1 (**Evaluation of the energy**). *Let Q be given by* (1.5). *Then we have the following.*

- *For the post-critical case, we have*

$$I_Q[\sigma_Q] = \mathcal{I}^{\rm post}(a,c) := \frac{3}{4} + \frac{3c}{2} + \frac{c^2}{2}\log c - \frac{(c+1)^2}{2}\log(c+1) - ca^2. \tag{2.2}$$

- *For the pre-critical case, we have*

$$\begin{aligned} I_Q[\sigma_Q] = \mathcal{I}^{\rm pre}(a,c) &:= \frac{3}{8} + \frac{a^2}{8} + \frac{3}{8a^2q^4} - \frac{5}{8q^2} + \Big(\frac{3}{4} + \frac{a^2}{8}\Big)a^2q^2 - \frac{3a^4q^4}{8} \\ &\quad + \log(2aq) + 2c\log(2aq^2) + \log\frac{(1 + a^2q^2 - 2a^2q^4)^{c^2}}{(1+a^2q^2)^{(c+1)^2}}, \end{aligned} \tag{2.3}$$

where $q = q(a)$ is given by (1.18).

Furthermore, for a fixed $c > 0$, suppose that $a > a_{\rm cri}$. Then we have

$$\mathcal{I}^{\rm post}(a,c) \leq \mathcal{I}^{\rm pre}(a,c), \tag{2.4}$$

where the equality holds when $a = a_{\rm cri}$.

See Figure 3 for the graphs of the energy. We mention that the inequality (2.4) plays an important role in Theorem 2.9 below. The evaluation of energy is closely related to the equilibrium measure problem. Using potential theory and conformal mapping methods, Proposition 2.1 will be established in Section 4.

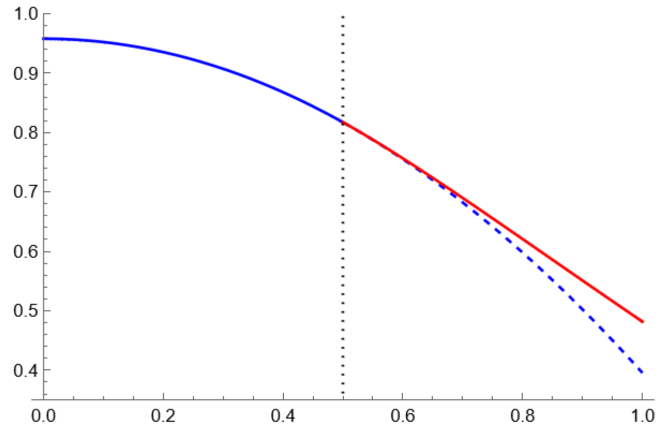


FIGURE 3. The plot shows the graph of the energy $a \mapsto I_Q[\sigma_Q]$, where $c = 9/16$. Here, the dotted vertical line represents $a = a_{\rm cri} = 1/2$. The graph (full line) for $a < a_{\rm cri}$ follows (2.2), while for $a > a_{\rm cri}$ it follows (2.3). The dotted line for $a > a_{\rm cri}$ is the continuation of the graph (2.2).

Recall that the Bernoulli number B_k is given in terms of the generating function as

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{2.5}$$

We are now ready to state our main result.

Theorem 2.2 (Free energy expansion for the post- and pre-critical cases). *Let Q be given by* (1.5). *Then as $N \to \infty$, we have*

$$\begin{aligned} \log Z_N(a,c) = &-I_Q[\sigma_Q]N^2 + \frac{1}{2} N \log N + \Big(\frac{\log(2\pi)}{2} - 1\Big) N \\ &+ \frac{6-\chi}{12} \log N + \frac{\log(2\pi)}{2} + \chi\, \zeta'(-1) + \mathcal{F}(a,c) + \mathcal{E}_N, \end{aligned} \tag{2.6}$$

where $I_Q[\mu_Q]$ is the energy given in Proposition 2.1, χ is the Euler characteristic of the droplet S_Q given in (1.19) *and ζ is the Riemann zeta function. Here $\mathcal{F}(a,c)$ and the error term $\mathcal{E}_N$ are given as follows.*

- *For the post-critical case, we have*

$$\mathcal{F}(a,c) = \mathcal{F}^{\rm post}(a,c) := \frac{1}{12} \log \Big(\frac{c}{1+c}\Big) \tag{2.7}$$

and

$$\mathcal{E}_N = \sum_{k=1}^{M} \Big(\frac{B_{2k}}{2k(2k-1)} \frac{1}{N^{2k-1}} + \frac{B_{2k+2}}{4k(k+1)} \Big(\frac{1}{(c+1)^{2k}} - \frac{1}{c^{2k}} \Big) \frac{1}{N^{2k}} \Big) + O(\frac{1}{N^{2M+1}}) \tag{2.8}$$

for any $M > 0$, where B_k is the Bernoulli number.

- *For the pre-critical case, we have*

$$\mathcal{F}(a,c) = \mathcal{F}^{\rm pre}(a,c) := \frac{1}{24} \log \Big(\frac{(1+a^2q^2-2a^2q^4)^4}{(1+a^2q^2)^4(1-q^2)^3(1-a^4q^6)} \Big) \tag{2.9}$$

and $\mathcal{E}_N = O(N^{-1})$.

As explained above, up to the $O(N)$ term, Theorem 2.2 is a special case of [90, Corollary 1.1]. It is obvious that the entropy in the $O(N)$ term of (1.13) vanishes since $\Delta Q = 1$. The constant term $\mathcal{F}(a,c)$ coincides with the prediction (1.13), as we discuss now.

Remark 2.3 (Regularized determinant of Laplacian). Let $0 < \lambda_{1,D} \le \lambda_{2,D} \le \dots$ be the eigenvalues of the Dirichlet Laplacian Δ on a domain $D \subset \mathbb{C}$. Then the spectral zeta function is defined by

$$\zeta_\Delta(s;D) := \sum \lambda_{j,D}^{-s}. \tag{2.10}$$

This is a building block to define the zeta-regularized determinant of Δ:

$$\det{}_\zeta(\Delta_D) := \exp(-\zeta_\Delta'(0;D)), \tag{2.11}$$

see e.g. [70] for more details. The spectral determinant can also be expressed in terms of several different domain functionals in conformal geometry, such as Brownian loop measure [88, 89], Loewner energy [118, 119], and the Grunsky operator [77, 115]. It is also used to describe large deviation principles (see e.g. [102, 118]), aligning with the same spirit as our result, especially from the viewpoint of the LUE statistics.

If the derivative of the potential ∂V is rational, $\det_\zeta(\Delta_{\mathbb{C}\setminus S_V})$ can be made more explicit as discussed in [126, Section 6.1]. As a consequence, for the post-critical case, it can be observed that

$$\log \det{}_\zeta(\Delta_{\mathbb{C}\setminus S}) = -\frac{1}{6}\log\Big(\frac{c}{1+c}\Big) \tag{2.12}$$

since the boundary of the droplet is a union of two circles. For the pre-critical case[2], let

$$z_\pm = q \pm i\sqrt{\frac{\kappa}{R}} \tag{2.13}$$

be the critical points of the conformal map f given in (1.17). Then we have

$$\log \det{}_\zeta(\Delta_{\mathbb{C}\setminus S}) = \frac{1}{12}\log\Big(\frac{R^4 f'(1/z_+)f'(1/z_-)}{f'(1/q)^2}\Big) = -\frac{1}{12}\log\Big(\frac{(1+a^2q^2-2a^2q^4)^4}{(1-q^2)^3(1-a^4q^6)(1+a^2q^2)^4}\Big). \tag{2.14}$$

Therefore, one can observe that $\mathcal{F}(a,c) = -\frac{1}{2}\log\det_\zeta(\Delta_{\mathbb{C}\setminus S})$, confirming the prediction of Zabrodin and Wiegmann.

We mention that in our present case, since we consider the quasi-harmonic potential $\Delta Q \equiv 1$, the equilibrium measure (1.12) has a flat metric. On the other hand, if ΔQ is not a constant, the determinant of Laplacian with respect to a non-trivial conformal metric or its conformal transformation law can be obtained via the Polyakov-Alvarez conformal anomaly formula [7, 104].

Remark 2.4 (Absence of the $O(\sqrt{N})$ term for $\beta = 2$). As a side remark, it is worth mentioning the belief for a general β ensemble that there exists an $O(\sqrt{N})$ term, with a coefficient proportional to $\log(\beta/2)$ called the surface tension. This conjecture was made in an unpublished note of Lutsyshin and first appeared in [37], see also [81, 108, 113]. However, since the coefficient is expected to be proportional to $\log(\beta/2)$, the absence of the $O(\sqrt{N})$ term in the determinantal case $\beta = 2$ is, in this context, a statement of prediction. The absence of the $O(\sqrt{N})$ term has been verified for the rotational symmetry case [9, 34]. However, one might question whether this absence truly results from $\beta = 2$ or from rotational symmetry. Nonetheless, as per Theorem 2.2, one can observe that even without the rotational symmetry, there is no $O(\sqrt{N})$ term for $\beta = 2$. (It is worth noting however that the $O(\sqrt{N})$ term does arise for $\beta = 2$ when considering the hard wall constraints of the potential [10, 31, 39, 40]).

We now discuss the critical regime. It can be expected from the duality relation (Proposition 1.1) that the expansion of the free energy in the critical regime is closely related to the Tracy-Widom distribution:

$$F_{\mathrm{TW}}(t) := \exp\Big(-\int_t^\infty (x-t)\boldsymbol{q}(x)^2\,dx\Big), \tag{2.15}$$

where $\boldsymbol{q}$ is the Hastings–McLeod solution to Painlevé II equation

$$\boldsymbol{q}''(s) = s\boldsymbol{q}(s) + 2\boldsymbol{q}(s)^3, \qquad \boldsymbol{q}(s) \sim \mathrm{Ai}(s) \quad \text{as } s \to \infty. \tag{2.16}$$

Then we have the following.

[2] The conformal map $z(w)$ was introduced below Eq.(5.22) of [126], which in our case is $f(z)$.

Proposition 2.5 (**Free energy expansion for the critical regime**). *For a given fixed parameter $c > 0$, let a be scaled as* (1.20). *Then as $N \to \infty$, we have*

$$\begin{aligned}\log Z_N(a,c) &= -\Big(\frac{3}{4}+\frac{3c}{2}+\frac{c^2}{2}\log c-\frac{(c+1)^2}{2}\log(c+1)-ca^2\Big)N^2 \\ &\quad +\frac{1}{2}N\log N+\Big(\frac{\log(2\pi)}{2}-1\Big)N+\frac{1}{2}\log N \\ &\quad +\frac{\log(2\pi)}{2}+\frac{1}{12}\log\Big(\frac{c}{1+c}\Big)+\log F_{\rm TW}(c^{-2/3}s)+O(N^{-2/3}),\end{aligned} \tag{2.17}$$

where $F_{\rm TW}$ is the Tracy-Widom distribution.

This will be shown in Section 6. Note that the expansion (2.17) is not of the form (1.13). Namely, by the scaling (1.20), we have

$$ca^2N^2 = ca_{\rm cri}^2N^2+\widetilde{C}_1(s)N^{4/3}+\widetilde{C}_2(s)N^{2/3}+\widetilde{C}_3(s)+O(N^{-2/3}) \tag{2.18}$$

for some constants $\widetilde{C}_k$ $(k=1,2,3)$.

The Painlevé II critical asymptotic behaviours of the associated planar orthogonal have been discovered in [16, 84]. However, the asymptotic behaviours presented in [16, 84] are not enough to derive Proposition 2.5, particularly to derive the $O(1)$ term. On the other hand, Proposition 2.5 can be readily derived utilizing the duality relation (Proposition 1.1), the Marchenko-Pastur law (1.7), and the well-established edge universality of the random Hermitian matrix ensemble [51]. This aligns with the probability theoretic intuition of the free energy expansion: the law of large numbers (determining the position of the left edge of the Marchenko-Pastur law) gives rise to the leading order of the free energy, while fluctuations (governed by the Tracy-Widom distribution) contribute to the constant term.

Remark 2.6 (Free energy expansion under the topology transitions). Recall the well-known asymptotic behaviours of the Tracy-Widom distribution: as $x \to +\infty$,

$$F_{\rm TW}(-x) = \frac{2^{1/24}e^{\zeta'(-1)}}{x^{1/8}}e^{-x^3/12}\Big(1+\frac{3}{2^6x^3}+O(x^{-6})\Big), \tag{2.19}$$

$$1-F_{\rm TW}(x) = \frac{1}{32\pi x^{3/2}}e^{-4x^{3/2}/3}\Big(1+O(x^{-3/2})\Big). \tag{2.20}$$

Using this, we have

$$\lim_{s\to+\infty}\log F_{\rm TW}(c^{-2/3}s)=0. \tag{2.21}$$

Thus in this limit, (2.17) matches with Theorem 2.2 for the post-critical regime. On the other hand, in the opposite limit, we have

$$\log F_{\rm TW}(c^{-2/3}s) = \frac{1}{24}\log 2+\zeta'(-1)-\frac{1}{8}\log|s|-\frac{|s|^3}{12}+O(|s|^{-3}), \qquad s\to-\infty. \tag{2.22}$$

Note that by (1.20), at least formally, the proper scaling for the pre-critical regime should be $s = O(N^{2/3})$ in the critical scaling. With this scaling, one can notice the additional term

$$-\frac{1}{12}\log N+\zeta'(-1) \tag{2.23}$$

appearing in Theorem 2.2 for the pre-critical regime when $\chi = 1$.

One can observe such a transition in the opposite direction, from the pre-critical regime. Namely, in the scaling regime (1.20), by (1.18), we have

$$q = 1+\frac{s}{4c^{1/6}(c+1)^{1/6}(\sqrt{c+1}-\sqrt{c})^{2/3}}N^{-2/3}+O(N^{-1}).$$

This gives that $\mathcal{F}^{\rm pre}$ in (2.9) has the asymptotic expansion

$$\mathcal{F}^{\rm pre}(a,c) = \frac{1}{12}\log N+O(1). \tag{2.24}$$

Hence, one can again observe the additional $\frac{1}{12}\log N$ term.

We now turn back to the other formulations of the problem. First of all, as a consequence of Theorem 2.2, we have the asymptotic behaviours of the moments of characteristic polynomial. For this, we shall use the notations in [16, Eqs.(2.17),(4.6)]. Let

$$F(z) = \frac{1}{2R}\Big[z + |\beta| + \sqrt{(z-\beta)(z-\overline{\beta})}\Big] \tag{2.25}$$

be the inverse of f in (1.17), where

$$\beta = f(z_+) = Rq - \frac{\kappa}{q} + 2i\sqrt{\kappa R} \tag{2.26}$$

is a critical value of f.

Theorem 2.7 (**Moments of characteristic polynomial of the GinUE**). *Let $\mathbf{G}_N$ be the complex Ginibre matrix of size N and let $c>0$ be fixed. Then as $N \to \infty$,*

$$\mathbb{E}\Big|\det(\mathbf{G}_N - z)\Big|^{2cN} = N^{\frac{1-\chi}{12}}\, e^{(\chi-1)\zeta'(-1)}\, \mathcal{G}(|z|)\, \exp\Big(\mathcal{H}(|z|)N^2 + \widetilde{\mathcal{E}}_N\Big) \tag{2.27}$$

where $\mathcal{H}(z)$, $\mathcal{G}(z)$ and $\widetilde{\mathcal{E}}_N$ are given as follows.

- *If $|z| < \sqrt{c+1} - \sqrt{c}$, we have $\chi = 0$,*

$$\mathcal{H}(z) = \mathcal{H}^{\rm post}(z) := c\, z^2 - \frac{3c}{2} + \frac{(c+1)^2}{2}\log(c+1) - \frac{c^2}{2}\log c \tag{2.28}$$

and

$$\mathcal{G}(z) = \mathcal{G}^{\rm post}(z) := \Big(\frac{c}{1+c}\Big)^{\frac{1}{12}}. \tag{2.29}$$

Here for any $M > 0$,

$$\widetilde{\mathcal{E}}_N = \sum_{k=1}^{M} \frac{B_{2k+2}}{4k(k+1)}\Big(\frac{1}{(c+1)^{2k}} - \frac{1}{c^{2k}} - 1\Big)\frac{1}{N^{2k}} + O(\frac{1}{N^{2M+2}}). \tag{2.30}$$

- *If $|z| > \sqrt{c+1} - \sqrt{c}$, we have $\chi = 1$,*

$$\begin{aligned} \mathcal{H}(z) = \mathcal{H}^{\rm pre}(z) &:= \frac{3}{8} - \frac{z^2}{8} - \frac{3F(z)^4}{8z^2} + \frac{5F(z)^2}{8} - \Big(\frac{3}{4} + \frac{z^2}{8}\Big)\frac{z^2}{F(z)^2} + \frac{3}{8}\frac{z^4}{F(z)^4} \\ &\quad + \log\Big(\frac{F(z)^{2c^2-1}(F(z)^2+z^2)^{(c+1)^2}}{(2z)^{2c+1}(F(z)^4 + z^2F(z)^2 - 2z^2)^{c^2}}\Big) \end{aligned} \tag{2.31}$$

and

$$\mathcal{G}(z) = \mathcal{G}^{\rm pre}(z) := \Big(\frac{F(z)^4(F(z)^4 + z^2F(z)^2 - 2z^2)^4}{(F(z)^2+z^2)^4(F(z)-1)^3(F(z)^6 - z^4)}\Big)^{\frac{1}{24}}, \tag{2.32}$$

where F is given by (2.25). Here $\widetilde{\mathcal{E}}_N = O(1/N)$.

Note that Theorem 2.7 immediately follows from Proposition 2.1, Theorem 2.2 and Lemma 3.1. Here we have used also the fact that $q = 1/F(a)$.

Remark 2.8 (Comparison with the small insertion). Let us stress again that the case $c = O(1/N)$ was studied in [121] for the bulk case $|z| < 1$ and also in [55] for the edge case $|z| = 1 + O(1/\sqrt{N})$. From the viewpoint of the induced model (1.3), the regime $c = O(1/N)$ represents the case in which the point charge insertion is finite. In this situation, the conditional process does not lead to macroscopic changes, and consequently, the droplet remains the unit disk of the circular law. It particular, it was shown in [121] that for the bulk case $|z| < 1$,

$$\mathbb{E}\Big|\det(\mathbf{G}_N - z)\Big|^{2\gamma} = N^{\frac{\gamma^2}{2}} \exp\Big(\gamma N(|z|^2 - 1)\Big)\frac{(2\pi)^{\gamma/2}}{G(\gamma+1)}(1+o(1)). \tag{2.33}$$

Note that the leading order in the exponent is $O(N)$, contrasting with the order of $O(N^2)$ in (2.27). This difference arises from the fact that the $O(N^2)$ term originates from the energy of the equilibrium measure, which is a macroscopic quantity. Nonetheless, one can observe, at least up to a multiplicative constant, that the asymptotic formula (2.33) coincides with our formula (2.27) by simply setting $c = \gamma/N$:

$$\exp\Big(\mathcal{H}^{\rm post}(|z|)N^2\Big)\Big|_{c=\gamma/N} = N^{\frac{\gamma^2}{2}} \exp\Big(\gamma N(|z|^2-1)\Big)\cdot O(1). \tag{2.34}$$

However, the multiplicative constant term does not match. Nonetheless, if we make use of the Barnes G-function in the asymptotic formula, by (3.16), one can see that the asymptotic behaviour

$$\mathbb{E}\Big|\det(\mathbf{G}_N - z)\Big|^{2\gamma} = N^{-\gamma N}\exp\Big(\gamma N\,|z|^2\Big)\frac{G(\gamma+N+1)}{G(\gamma+1)G(N+1)}(1+o(1)) \tag{2.35}$$

matches both (2.33) and (2.27) for γ fixed and $\gamma = cN$, respectively.

We now discuss the large deviation of the LUE smallest eigenvalue. In general, the large deviation principles of the extremal eigenvalues of random matrices have been extensively studied in the literature. For instance, the statistics of the maximal eigenvalues of the Gaussian ensembles have been studied in [21, 49, 50]. Also, for the Laguerre ensembles, which are closely related to our present case, there has been extensive work on large deviation probabilities for both the smallest and largest eigenvalues [4, 80, 97, 103, 117, 123–125], see Remark 2.10. For such integrable random matrices, the Coulomb gas approach has been mainly implemented, and essentially, the large deviation speed and rate can be derived by computing the energy associated with the potential under the presence of a hard wall. An advantage of this approach is that it can be applied to a general β ensemble [56, 62, 65]. However, it is limited in deriving the leading-order asymptotic behaviours. In addition, within the context of large deviation probabilities of the extremal eigenvalues, there is a universal pulled-to-pushed transition of the third order, see e.g. [48, 96]. We also point out that such large deviation probabilities are closely related to the maximal height of N non-intersecting Brownian excursions [63, 79, 109].

Theorem 2.9 (**Large deviation probabilities of the smallest eigenvalues of the LUE**). *Let λ_1 be the smallest eigenvalue of the LUE in* (1.6)*. Then as $N\to\infty$, we have the following.*

- **(Pulled-regime)** *If $t < \lambda_-$, we have*

$$\log \mathbb{P}\Big[\lambda_1 > t\Big] = O(N^{-\infty}). \tag{2.36}$$

Here, $O(N^{-\infty})$ means $O(N^{-m})$ for any positive integer m.

- **(Pushed-regime)** *If $t > \lambda_-$, we have*

$$\log \mathbb{P}\Big[\lambda_1 > t\Big] = -\Phi(t;\alpha)N^2 - \frac{1}{12}\log(\alpha N) + \zeta'(-1) + \Psi(t;\alpha) + O(\frac{1}{N}), \tag{2.37}$$

where

$$\Phi(t;\alpha) = \alpha^2\Big(\mathcal{I}^{\rm pre}(\sqrt{t/\alpha},1/\alpha) - \mathcal{I}^{\rm post}(\sqrt{t/\alpha},1/\alpha)\Big), \tag{2.38}$$

$$\Psi(t;\alpha) = \mathcal{F}^{\rm pre}(\sqrt{t/\alpha},1/\alpha) - \mathcal{F}^{\rm post}(\sqrt{t/\alpha},1/\alpha). \tag{2.39}$$

Here, $\mathcal{I}^{\rm pre}$, $\mathcal{I}^{\rm post}$, $\mathcal{F}^{\rm pre}$ and $\mathcal{F}^{\rm post}$ are given by (2.2)*,* (2.3) (2.7) *and* (2.9)*.*

Theorem 2.9 follows from Proposition 1.1 and Theorem 2.2 (with $N \mapsto N/c = \alpha N$). Note that the positivity of the rate function $\Phi(t;\alpha) > 0$ follows from the inequality (2.4). See Figure 4 for the graph of Φ.

By the explicit formulas (2.2) and (2.3), one can observe that

$$\Phi(t;\alpha) \sim \frac{\sqrt{\alpha+1}}{12\lambda_-^2}(t-\lambda_-)^3, \qquad t\to\lambda_-, \tag{2.40}$$

which agrees with the tail probability of the Tracy-Widom distribution. This is known as a third-order phase transition appearing in a more general context, and we refer the reader to [96] for a review. Let us also mention that in the opposite limit, we have

$$\Phi(t;\alpha) \sim t - \alpha\log t, \qquad t\to\infty. \tag{2.41}$$

Here, the linear growth t comes from the post-critical energy (2.2), while the logarithmic corrections $\alpha\log t$ comes from the pre-critical energy (2.3), cf. (4.32).

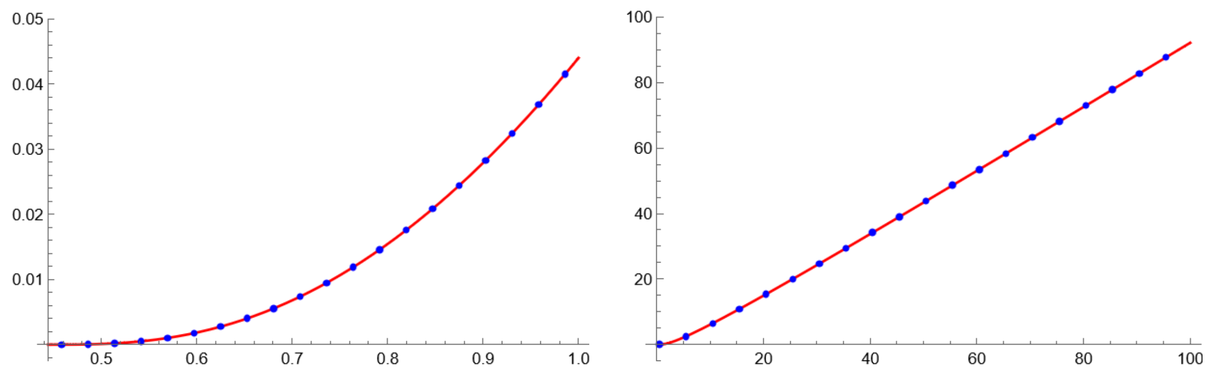


FIGURE 4. The full red line represents the graph of the rate function $t \mapsto \Phi(t;\alpha)$, where $\alpha = 16/9$ and $t \ge \lambda_- = 4/9$. The blue dots indicate the values of the function $t \mapsto S(t) - S(\lambda_-)$. Here, one can also observe the asymptotic behaviours (2.40) and (2.41) in the left and right figures, respectively.

Remark 2.10 (Comparison with the Katzav-Castillo formula from a Coulomb gas approach). In [80], Katzav and Castillo used a Coulomb gas method and derived the leading order asymptotic behaviour

$$\log \mathbb{P}\Big[\lambda_1 > t\Big] = -\Big(S(t) - S(\lambda_-)\Big)N^2 + o(N^2), \qquad (t > \lambda_-) \tag{2.42}$$

where S is given by

$$\begin{aligned} S(t) = \frac{U(t)+t}{2} - \frac{(U(t)-t)^2}{32} + \frac{\alpha}{4}(\sqrt{U(t)} - \sqrt{t})^2 \\ - \log\Big(\frac{U(t)-t}{4}\Big) + \frac{\alpha^2}{4}\log\Big(t\,U(t)\Big) - \alpha(\alpha+2)\log\Big(\frac{\sqrt{U(t)}+\sqrt{t}}{2}\Big). \end{aligned} \tag{2.43}$$

Here,

$$U(t) = \frac{4}{3}\Big(t + 2(\alpha+2)\Big)\cos^2\Big(\frac{\theta+2\pi}{3}\Big), \qquad \theta = \arctan\Big(\sqrt{\frac{(t+2(\alpha+2))^3 - 27\alpha^2 t}{27\alpha^2 t}}\Big). \tag{2.44}$$

Indeed, this value $U(t)$ is the right edge of the *constrained* spectral density (see [117])

$$\frac{\sqrt{U(t)-x}}{2\pi\sqrt{x-t}}\,\frac{x - \alpha\sqrt{t/U(t)}}{x}\,\mathbb{1}_{[t,U(t)]}(x), \tag{2.45}$$

the density of the LUE (1.6) conditioned on $\lambda_1 > t$. By comparing (2.37) and (2.42), we have

$$\Phi(t;\alpha) = S(t) - S(\lambda_-), \tag{2.46}$$

see Figure 4. This identity should follow from the explicit solution of the cubic equation (1.18) (in the variable q^2). We also refer to [85, Section 6.2] for the interpretation of the formula (2.42) from the viewpoint of the recursion scheme. By (2.46), one can observe that our result extends the result (2.42), incorporating not only polynomial but also constant corrections of the large deviation probability $\mathbb{P}[\lambda_1 > t]$. Let us emphasise that while the Coulomb gas approach yields explicit formula for the leading-order asymptotic behaviour, applicable not only to the LUE but also to general Laguerre-β ensembles, this approach is difficult to implement for deriving precise asymptotic behaviours.

In the pulled-regime, the precise order for $O(N^{-\infty})$ in (2.36) is expected to be exponentially small: for a certain $\widetilde{c}(t)$,

$$\mathbb{P}\Big[\lambda_1 > t\Big] = e^{-\widetilde{c}(t)N}(1 + o(1)). \tag{2.47}$$

See [80, Eq.(16)] for the Coulomb gas prediction on the constant $\widetilde{c}(t)$. We also refer to [62, Eq.(1.4)] for the constant $\widetilde{c}(t)$ derived from the tail behaviour of the LUE density. However, capturing such an exponentially decaying behaviour seems challenging based on the Riemann-Hilbert analysis we implement in this work.

Remark 2.11 (Asymptotic expansions of partition functions in one dimension). Compared to the two-dimensional point process, there has been extensive work on the partition functions of log gases in one-dimension.

For instance, in the same spirit as the present work, Riemann-Hilbert analysis proves to be a robust method for the determinantal point process [57], requiring thorough and technical analysis but particularly strong in obtaining the precise form of coefficients and addressing singularities [53, 54]. In this vein, a general result was achieved in [38, 43], where the authors derived the precise asymptotic form of the partition functions associated with Gaussian, Laguerre, and Jacobi type (one-cut regular) weights together with jump and root-type singularities. In particular, Theorem 2.9 can be re-derived by employing [43, Theorem 1.2], alongside additional work involving potential-theoretic computations. Nonetheless, the resulting expressions differ from those in Theorem 2.9, as our formulas stem from the equilibrium measure of the two-dimensional model, thus involving the parameter satisfying a non-trivial cubic equation (1.18). This demonstrates the remarkably powerful nature of the duality formula, allowing us to obtain two-dimensional results from one-dimensional results. It is noteworthy that our method, outlined in the next section, takes the opposite approach; namely, we derive the one-dimensional result (Theorem 2.9) as a consequence of the two-dimensional result (Theorem 2.2). Furthermore, we emphasise that Theorem 3.11, concerning the strong asymptotic behaviour of a planar orthogonal polynomial, which holds its own significance that extends [16], cannot be derived as a consequence of an existing one-dimensional result.

Among different approaches, let us also mention that the topological recursion of Eynard and Orantin [58] provides an efficient method to derive the structural form of the asymptotic expansion of one-dimensional point processes [28, 29]. We also refer to [74, 98] and references therein for the implementation of such approaches for various cases, including Toeplitz determinants.

Organisation of the paper. The rest of this paper is organised as follows. In Section 3, we introduce the overall strategy and complete the proof of Theorem 2.2. However, two main ingredients, Proposition 2.1 for the energy evaluations, and Theorem 3.11 for the fine asymptotic behaviours of the orthogonal polynomials, will be established in the later sections. Section 4 is devoted to proving Proposition 2.1 based on the logarithmic potential theory and conformal mapping method. In Section 5, we prove Theorem 3.11 using Riemann-Hilbert analysis and a partial Schlesinger transform. Section 6 is a separate part, where we provide the derivation of the duality formula of the form (1.9) and also the proof of Proposition 2.5 on the free energy expansion in the critical regime.

3. Proof of Theorem 2.2

This section reaches its culmination with the proof of Theorem 2.2. For reader's convenience, we begin by providing a summary of the overall strategy.

- As a first step, we express the partition function as an integral of its derivative with respect to a deformation parameter. Here, the integration constants are given by the reference partition functions, whose asymptotic behaviours can be computed precisely (Lemma 3.1). With proper choices of reference partition functions, this implies that it suffices to derive the asymptotic expansion of the derivative of free energies (Proposition 3.5), for which we also need the derivatives of the coefficients (Lemma 3.3).
- Using the τ-function, the derivative of the partition functions can be expressed in terms of the solution to the Riemann-Hilbert problem for the associated planar orthogonal polynomial (Lemma 3.8). This allows us to express the derivative of the partition function in terms of the coefficient of the solution to the Riemann-Hilbert problem in the asymptotic expansion (Proposition 3.9).
- As a consequence of the previous step, the free energy expansion can be derived using the asymptotic expansion of the orthogonal polynomial near infinity (Proposition 3.10). Then, by refining the Riemann-Hilbert analysis in [16], we obtain the fine asymptotic behaviour of the orthogonal polynomial outside the motherbody (Theorem 3.11). Consequently, by computing the residue near infinity, we complete the proof of Theorem 2.2.

Let us now be more precise in introducing our strategy. Let p_j be the monic polynomial satisfying

$$\int_{\mathbb{C}} p_j(z)\overline{p_k(z)}e^{-NQ(z)}\,dA(z) = h_j\,\delta_{jk}, \tag{3.1}$$

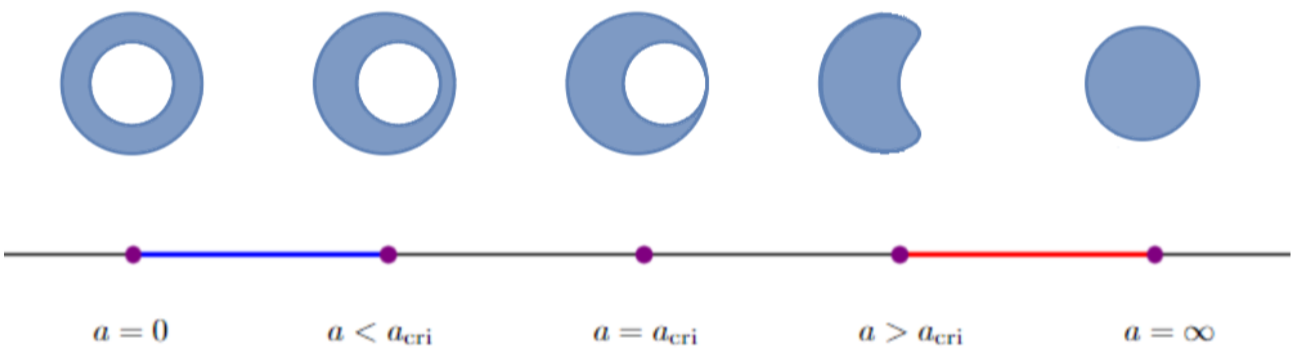


FIGURE 5. The plot shows deformations of the droplet. The leftmost ($a = 0$) and rightmost ($a = \infty$) are the rotationally symmetric cases, for which we use the associated partition functions as reference. The thick lines indicate the integral domains in (3.3) and (3.4), respectively.

where Q is given by (1.5) and δ_{jk} is the Kronecker delta. Then we have

$$Z_N(a,c) = N! \prod_{j=0}^{N-1} h_j, \tag{3.2}$$

see e.g. [32, Chapter 5].

We now begin by explaining the above strategy in more detail. Each step in the above item is given in Subsections 3.1, 3.2, and 3.3, respectively.

3.1. **Deformations of partition functions.** An important idea we employ in the asymptotic analysis of the partition functions is to consider the partition function $a \mapsto Z_N(a,c)$ as a function of the deformation parameter $a \geq 0$. Next, we require reference partition functions for which we can compute explicit formulas, along with their precise asymptotic expansions. The chosen reference partition functions are in the extremal, rotationally symmetric case. Additionally, it is crucial to select a reference partition function whose droplet is topologically equivalent to the droplet of the regime we intend to compute. This choice is essential to simplify the proof by avoiding the critical regime during this deformation. Consequently, we set $a = 0$ and $a = \infty$ for the post- and pre-critical cases, respectively. The concrete statement of the above discussion is provided in the following lemma. We mention that the idea of implementing deformations of partition functions (or structured determinants) has been utilized in Hermitian random matrix theory [45, 52, 83] as well as in the Coulomb gas theory on a Jordan domain [77, 115]. A similar idea can also be found in the context of the transport method developed for general Coulomb gases, see [112, Chapter 4] and [114].

Lemma 3.1 (**Deformations and reference partition functions**). *We have*

$$\log Z_N(a,c) = \log Z_N(0,c) + \int_0^a \frac{d}{dt} \log Z_N(t,c)\,dt \tag{3.3}$$

$$= \log Z_N^{\text{Gin}} + (2c\log a)N^2 - \int_a^\infty \Big(\frac{d}{dt}\log Z_N(t,c) - \frac{2cN^2}{t} \Big)\,dt. \tag{3.4}$$

The reference partition functions are evaluated as

$$Z_N(0,c) = N! \frac{G(N+cN+1)}{G(cN+1)} N^{-(c+\frac12)N^2 - \frac12 N}, \qquad Z_N^{\text{Gin}} = N!\, G(N+1) N^{-\frac12 N^2 - \frac12 N}, \tag{3.5}$$

and they satisfy the asymptotic behaviours as $N \to \infty$:

$$\begin{aligned}\log Z_N(0,c) &= -\Big(\frac{3}{4}+\frac{3c}{2}+\frac{c^2}{2}\log c-\frac{(c+1)^2}{2}\log(c+1)\Big)N^2+\frac{1}{2}N\log N+\Big(\frac{\log(2\pi)}{2}-1\Big)N \\ &\quad +\frac{1}{2}\log N+\frac{\log(2\pi)}{2}+\frac{1}{12}\log\Big(\frac{c}{1+c}\Big) \\ &\quad +\sum_{k=1}^{\infty}\Big(\frac{B_{2k}}{2k(2k-1)}\frac{1}{N^{2k-1}}+\frac{B_{2k+2}}{4k(k+1)}\Big(\frac{1}{(c+1)^{2k}}-\frac{1}{c^{2k}}\Big)\frac{1}{N^{2k}}\Big),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\log Z_N^{\rm Gin} &= -\frac{3}{4}N^2+\frac{1}{2}N\log N+\Big(\frac{\log(2\pi)}{2}-1\Big)N \\ &\quad +\frac{5}{12}\log N+\frac{\log(2\pi)}{2}+\zeta'(-1)+\sum_{k=1}^{\infty}\Big(\frac{B_{2k}}{2k(2k-1)}\frac{1}{N^{2k-1}}+\frac{B_{2k+2}}{4k(k+1)}\frac{1}{N^{2k}}\Big),\end{aligned} \tag{3.7}$$

where B_k is the Bernoulli number.

As previously mentioned, we shall use (3.3) for the post-critical regime, and (3.4) for the pre-critical regime, see Figure 5 for an illustration.

Remark 3.2 (Total integration formula). As an immediate consequence of (3.3), (3.4) and (3.5), it follows that

$$\begin{aligned}\int_0^\infty \Big(\partial_t \log Z_N(t,c)-\frac{2cN^2}{t}\mathbb{1}_{t>a}\Big)\,dt &= (2c\log a)N^2+\log\Big(\frac{Z_N^{\rm Gin}}{Z_N(0,c)}\Big) \\ &= cN^2\log(a^2N)+\log\Big(\frac{G(N+1)G(cN+1)}{G(N+cN+1)}\Big).\end{aligned} \tag{3.8}$$

Proof of Lemma 3.1. The formula (3.3) is obvious. To see (3.4), let us write

$$\widehat{Z}_N(a,c):=\int_{\mathbb{C}^N}\prod_{j>k=1}^{N}|z_j-z_k|^2\prod_{j=1}^{N}e^{-N\widehat{Q}(z_j)}\,dA(z_j),\qquad \widehat{Q}(z)=|z|^2-2c\log\Big|\frac{z-a}{a}\Big|. \tag{3.9}$$

Note that

$$\lim_{a\to\infty}\widehat{Q}(z)=|z|^2,\qquad \widehat{Z}_N(\infty,c)=Z_N^{\rm Gin}. \tag{3.10}$$

Since $e^{-NQ(z)}=e^{-N\widehat{Q}(z)}\,a^{2cN}$, it follows that

$$\log Z_N(a,c)=(2c\log a)N^2+\log\widehat{Z}_N(a,c). \tag{3.11}$$

Then we have

$$\begin{aligned}\log Z_N(a,c) &= (2c\log a)N^2+\log\widehat{Z}_N(a,c) \\ &= (2c\log a)N^2+\log\widehat{Z}_N(\infty,c)-\int_a^\infty \partial_t\log\widehat{Z}_N(t,c)\,dt \\ &= (2c\log a)N^2+\log\widehat{Z}_N(\infty,c)-\int_a^\infty\Big(\partial_t\log Z_N(t,c)-\frac{2cN^2}{t}\Big)\,dt,\end{aligned} \tag{3.12}$$

which leads to (3.4).

If $a=0$, the potential is radially symmetric, and consequently the associated orthogonal polynomial is monomial, i.e. $p_n(z)=z^n$. The orthogonal norm is then computed as

$$h_j=\int_{\mathbb{C}}|z|^{2j}e^{-NQ(z)}\,dA(z)=2\int_0^\infty r^{2j+2cN+1}e^{-Nr^2}\,dr=\frac{\Gamma(j+cN+1)}{N^{j+cN+1}}. \tag{3.13}$$

By using (3.2), we have

$$Z_N(0,c)=N!\prod_{k=0}^{N-1}\frac{\Gamma(k+cN+1)}{N^{k+cN+1}}=N!\frac{G(N+cN+1)}{G(cN+1)}N^{-(c+\frac{1}{2})N^2-\frac{1}{2}N}. \tag{3.14}$$

This also gives (3.5) since $Z_N^{\rm Gin}=Z_N(0,0)$. Now (3.6) and (3.7) follow from the asymptotic behaviours of the gamma function

$$\log N! = \log N + \log\Gamma(N) = \Big(N+\frac12\Big)\log N - N + \frac12\log(2\pi) + \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k-1)N^{2k-1}} \tag{3.15}$$

as $N\to\infty$, and of the Barnes G-function

$$\log G(z+1) = \frac{z^2\log z}{2} - \frac34 z^2 + \frac{\log(2\pi)z}{2} - \frac{\log z}{12} + \zeta'(-1) + \sum_{k=1}^\infty \frac{B_{2k+2}}{4k(k+1)}\frac{1}{z^{2k}}, \tag{3.16}$$

as $z\to\infty$, see e.g. [101, Eqs.(5.11.1), (5.17.5)]. ☐

By Lemma 3.1, we need to derive the asymptotic behaviours of $\frac{d}{dt}\log Z_N(t,c)$. For this purpose, we need to compute the derivatives of the coefficients in the expansion (2.6). In the following, we denote by ∂_a and $\bar\partial_a$ the complex derivatives with respect to a and $\bar a$, respectively. To distinguish the notations further, we also use $\mathfrak{d}_a=\partial_a+\bar\partial_a$ to represent the operator of differentiation with respect to the real variable a. For instance, $\mathfrak{d}_a a^2=2a$, whereas $\bar\partial_a|a|^2=a$.

Lemma 3.3 (**Derivatives of the coefficients in the free energy expansion**). *For the pre-critical case, we have*

$$\mathfrak{d}_a\,\mathcal{I}^{\rm pre}(a,c) = \frac{1}{2a}+a-\frac{1}{aq^2}-\frac{a^3q^4}{2} = -\frac{(1-a^2q^2)(2-q^2-a^2q^4)}{2aq^2} \tag{3.17}$$

and

$$\mathfrak{d}_a\mathcal{F}^{\rm pre}(a,c) = -\frac{q^2(1-a^4q^4)^2}{8a(1-q^2)(1-a^4q^6)^2}. \tag{3.18}$$

Proof. Recall that $q\equiv q(a)$ is a function of a. By (1.18), we have

$$c=\frac{a^2q^2}{2}+\frac{1}{4a^2q^4}-\frac{a^2+2}{4}. \tag{3.19}$$

By differentiating (1.18) with respect to a and using (3.19), we have

$$\frac{q'}{q}=\frac{1}{2a}\frac{1+a^4q^4-2a^4q^6}{a^4q^6-1}. \tag{3.20}$$

Using (3.20), one can express derivatives of (2.3) and (2.9) in terms of q and a. Then the lemma follows from straightforward computations. ☐

Remark 3.4. Note that by (3.17) and (3.19) , we have

$$\mathfrak{d}_a\Big(\mathcal{I}^{\rm pre}(a,c)-\mathcal{I}^{\rm post}(a,c)\Big) = \frac{1}{2a}+a-\frac{1}{aq^2}-\frac{a^3q^4}{2}+2ac = \frac{(1-q^2)^2(1-a^4q^4)}{2aq^4}>0. \tag{3.21}$$

This implies the inequality (2.4).

By combining Proposition 2.1, Lemma 3.1 and Lemma 3.3, Theorem 2.2 can be reduced to the following proposition.

Proposition 3.5 (**Asymptotic expansion of the derivative of free energies**). *As $N\to\infty$, we have the following.*

(i) *For the post-critical case, we have*

$$\mathfrak{d}_a\log Z_N(a,c) = 2caN^2 + O(N^{-\infty}). \tag{3.22}$$

(ii) *For the pre-critical case, we have*

$$\mathfrak{d}_a\log Z_N(a,c) = \frac{(1-a^2q^2)(2-q^2-a^2q^4)}{2aq^2}N^2 - \frac{q^2(1-a^4q^4)^2}{8a(1-q^2)(1-a^4q^6)^2} + O(N^{-1}). \tag{3.23}$$

In the following subsection, we further reduce Proposition 3.5 to certain asymptotic behaviours of the planar orthogonal polynomial (3.1).

3.2. **Riemann-Hilbert problem and τ-function.** We shall implement the τ-function [26], which arises in the context of the Riemann-Hilbert problem, cf. [16, 92, 94]. The first important idea in [16] for the asymptotic analysis of planar orthogonal polynomial p_n is to demonstrate an equivalence between the planar orthogonality (3.1) and a certain orthogonality on a contour. Extensions of such an equivalence can be found in [23, 93]. As a consequence, one can construct a standard Riemann-Hilbert problem for p_n, and we recall it here.

Let us mention that on some occasions, we write both n and N. This distinction has usually been made in Riemann-Hilbert analysis to distinguish between the degree of the orthogonal polynomial and the number of particles. However, in our context, we use $n = N$ throughout the paper.

Definition 3.6 (Riemann-Hilbert problem Y). Let Γ be a simple closed curve enclosing 0 and a. Define the weight function

$$\omega_{n,N}(z) = \frac{(z-a)^{Nc}e^{-Naz}}{z^{Nc+n}}. \tag{3.24}$$

We then consider the following Riemann-Hilbert problem for $Y \equiv Y_n$:

$$\begin{cases} Y(z) \quad \text{is holomorphic}, & z \in \mathbb{C} \setminus \Gamma, \\ Y_+(z) = Y_-(z) \begin{pmatrix} 1 & \omega_{n,N}(z) \\ 0 & 1 \end{pmatrix}, & z \in \Gamma, \\ Y(z) = \big(I + O(z^{-1})\big) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, & z \to \infty. \end{cases} \tag{3.25}$$

Here $Y_\pm(x)$ is the boundary values of Y on Γ.

Let $q_n(z) := q_{n,N}(z)$ be a unique polynomial of degree $n-1$ such that

$$\frac{1}{2\pi i}\int_\Gamma \frac{q_n(w)\omega_{n,N}(w)}{w-z}dw = \frac{1}{z^n}\Big(1 + O\Big(\frac{1}{z}\Big)\Big).$$

Then the matrix function

$$Y(z) \equiv Y_n(z) := \begin{pmatrix} p_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{p_n(w)\omega_{n,N}(w)}{w-z}\,dw \\ q_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{q_n(w)\omega_{n,N}(w)}{w-z}\,dw \end{pmatrix} \tag{3.26}$$

is a unique solution to the Riemann-Hilbert problem (3.25).

We now define

$$\widetilde{Y}(z) := Y(z)T(z), \qquad T(z) \equiv T_n(z) = \begin{pmatrix} \Big(\dfrac{z-a}{z}\Big)^{Nc} e^{-Naz} & 0 \\ 0 & z^n \end{pmatrix}. \tag{3.27}$$

Then $\widetilde{Y}$ satisfies the following Riemann-Hilbert problem:

$$\begin{cases} \widetilde{Y}(z) \quad \text{is holomorphic}, & z \in \mathbb{C} \setminus \Gamma, \\ \widetilde{Y}_+(z) = \widetilde{Y}_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \Gamma, \\ \widetilde{Y}(z) = \big(I + O(z^{-1})\big) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \begin{pmatrix} \Big(\dfrac{z-a}{z}\Big)^{Nc} \dfrac{z^n}{e^{Naz}} & 0 \\ 0 & 1 \end{pmatrix}, & z \to \infty. \end{cases} \tag{3.28}$$

Then the matrices Y_N and T_N can be used to express the derivative of the partition function.

Definition 3.7 (τ-function). Following [26, Definition 3.2], the τ-function is defined by

$$\partial_a \log \tau := \operatorname*{Res}_{z=\infty} \Big[\operatorname{Tr} \Big(Y(z)^{-1} \partial_z Y(z) \partial_a T(z) T(z)^{-1} \Big) \Big]. \tag{3.29}$$

Here $\tau \equiv \tau_n$ depends on the degree n of the orthogonal polynomial p_n.

Lemma 3.8 (**Partition function in terms of the Riemann-Hilbert problem**). *We have*

$$\partial_a \log Z_N(a,c) = \partial_a \log \tau_N. \tag{3.30}$$

Proof. It follows from [26, Theorem 3.4] that

$$\partial_a \log \tau_N = \operatorname*{Res}_{z=\infty} \Big[\operatorname{Tr} \Big(Y_N(z)^{-1} \partial_z Y_N(z) \partial_a T_N(z)\, T_N(z)^{-1} \Big) \Big] = \partial_a \log \Big(\det[\nu_{j+k}]_{j,k=0}^{N-1} \Big), \tag{3.31}$$

where

$$\nu_k = \int_\Gamma z^k \omega_{n,N}(z)\, dz. \tag{3.32}$$

Let us define

$$\mu_{jk} := \int_{\mathbb{C}} z^j \bar{z}^k e^{-NQ(z)}\, dA(z). \tag{3.33}$$

Then by the general theory on determinantal point process, we have

$$Z_N(a,c) = N!\, \det[\mu_{jk}]_{j,k=1}^N. \tag{3.34}$$

Note that by [16, (E.1)], we have

$$\frac{\det[\mu_{jk}]_{j,k=1}^N}{\det[\nu_{j+k}]_{j,k=0}^{N-1}} = \frac{(-1)^{N(N-1)/2}}{\pi^N} \prod_{k=0}^{N-1} \frac{\Gamma(cN+k+1)}{2i\, N^{cN+k+1}}. \tag{3.35}$$

Since the right-hand side of this equation does not depend on a, it follows that

$$\partial_a \det[\mu_{jk}]_{j,k=1}^N = \partial_a \det[\nu_{j+k}]_{j,k=0}^{N-1}. \tag{3.36}$$

Combining all of the above, the lemma follows. ☐

Proposition 3.9 (**Partition functions and fine asymptotics of the orthogonal polynomial**). *Suppose that $Y_N(z)$ satisfies the strong asymptotic behaviour of the form*

$$Y_N(z) = \left(I + \frac{1}{z} \begin{pmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} \\ \mathfrak{A}_{21} & -\mathfrak{A}_{11} \end{pmatrix} + O\Big(\frac{1}{z^2}\Big) \right) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}, \qquad z \to \infty. \tag{3.37}$$

for certain coefficients $\mathfrak{A}_{jk}$'s. Then we have

$$\mathfrak{d}_a Z_N(a,c) = 2N\, \mathfrak{A}_{11}. \tag{3.38}$$

Proof. By (3.27), we have

$$\partial_a T_N(z) T_N(z)^{-1} = -N \begin{pmatrix} z + \dfrac{c}{z-a} & 0 \\ 0 & 0 \end{pmatrix},$$

which gives

$$\operatorname{Tr} \Big(Y_N(z)^{-1} \partial_z Y_N(z) \partial_a T_N(z)\, T_N(z)^{-1} \Big) = -N \Big(z + \frac{c}{z-a} \Big) \Big[Y_N(z)^{-1} \partial_z Y_N(z) \Big]_{11}.$$

Then it follows from the asymptotic behaviour in (3.37) that

$$\operatorname*{Res}_{z=\infty} \Big[\operatorname{Tr} \Big(Y_N(z)^{-1} \partial_z Y_N(z) \partial_a T_N(z)\, T_N(z)^{-1} \Big) \Big] = \operatorname*{Res}_{z=\infty} \Big[-N^2 + \frac{N\mathfrak{A}_{11}}{z} + O\big(\frac{1}{z^2}\big) \Big] = N\mathfrak{A}_{11}.$$

Combining this with (3.30), the proof of the lemma is complete. ☐

Then by (3.26), Proposition 3.5 can be further reduced to the following.

Proposition 3.10 (**Asymptotic behaviour of the coefficients**). *The orthogonal polynomial p_N satisfies the asymptotic behaviour of the form*

$$p_N(z) = z^N + \mathfrak{A}_{11} z^{N-1} + O(z^{N-2}), \qquad z \to \infty. \tag{3.39}$$

Furthermore, as $N \to \infty$, the coefficient $\mathfrak{A}_{11}$ satisfies the following asymptotic behaviour.

- *For the post-critical case,*

$$\mathfrak{A}_{11} = caN + O(\frac{1}{N^\infty}). \tag{3.40}$$

- *For the pre-critical case,*

$$\mathfrak{A}_{11} = \frac{(1-a^2q^2)(2-q^2-a^2q^4)}{4aq^2}N - \frac{q^2(1-a^4q^4)^2}{16a(1-q^2)(1-a^4q^6)^2}\frac{1}{N} + O(\frac{1}{N^2}). \tag{3.41}$$

In Theorem 3.11 below, we shall prove a stronger statement on the fine asymptotic behaviours of p_N.

3.3. **Fine asymptotic behaviour and proof of Theorem 2.2.** We show the fine asymptotic behaviours of the orthogonal polynomial.

Theorem 3.11 (**Fine asymptotic behaviour of the orthogonal polynomial**)**.** *Let $\mathcal{B}$ the curve defined in Definition 4.2 and g be the function defined in Definition 4.3. Then as $N \to \infty$, we have the following.*

- **(Post-critical regime)** *We have*

$$p_N(z) = z^N\Big(\frac{z}{z-a}\Big)^{cN}\Big(1+O(N^{-\infty})\Big), \qquad z \in \operatorname{ext}(\mathcal{B}). \tag{3.42}$$

- **(Pre-critical regime)** *We have*

$$p_N(z) = \Big(\sqrt{RF'(z)}(1+R_{11}(z)) - \frac{\sqrt{\kappa F'(z)}}{F(z)-q}R_{12}(z)\Big)e^{Ng(z)}\Big(1+O(N^{-2})\Big), \qquad z \in \operatorname{ext}(\mathcal{B}) \tag{3.43}$$

where $F(z)$ is given by (2.25)*, and R_{11} and R_{12} are defined in Definition 5.8.*

Here the error terms are uniform over z in compact sets of $\mathbb{C}\setminus\mathcal{B}$.

We stress that Theorem 3.11 extends [16, Theorems 1.3 and 1.4] providing more precise error terms. Let us also mention that the asymptotic behaviour of *orthonormal* polynomials outside the droplet has also been studied by Hedenmalm and Wennman in a more general context [71–73]. In these works, an algorithm computing the subleading correction terms was also introduced, but their explicit evaluations for a given potential require a separate analysis. In particular, Theorem 3.11 is not a direct consequence of the general theory developed by Hedenmalm and Wennman, as we need to compute the *monic* orthogonal polynomial and also derive the asymptotic behaviour inside the droplet.

We are now ready to show Theorem 2.2.

Proof of Theorem 2.2. As explained in the previous subsection, it suffices to show Proposition 3.10.

For the post-critical case, it follows from (3.42) that

$$p_N(z) = \Big(z^N + Nac\,z^{N-1} + O(z^{N-2})\Big)\Big(1+O(N^{-\infty})\Big), \qquad z \to \infty.$$

This gives rise to (3.40).

Next, let us show the pre-critical case. By Lemma 4.4 below, we have

$$e^{Ng(z)} = z^N\Big(1 - \Big(\frac{1}{4a} + \frac{a}{2} - \frac{1}{2aq^2} - \frac{a^3q^4}{4}\Big)\frac{N}{z} + O(\frac{1}{z^2})\Big). \tag{3.44}$$

On the other hand, by using [16, Eq.(1.28)],

$$F(z) = \frac{z}{R} + \frac{\kappa}{Rq} + \frac{\kappa}{z} + O\Big(\frac{1}{z^2}\Big), \qquad z \to \infty. \tag{3.45}$$

Therefore we have

$$RF'(z) = 1 + O(\frac{1}{z^2}), \qquad z \to \infty. \tag{3.46}$$

Furthermore, by definition, we have

$$R_{11}(z) = O(\frac{1}{z}), \qquad R_{12}(z) = O(\frac{1}{z}), \qquad z \to \infty. \tag{3.47}$$

Then the desired identity (3.41) follows from

$$\operatorname*{Res}_{z=\infty}\Big[R_{11}(z)\Big]=-\frac{q^2(1-a^4q^4)^2}{16a(1-q^2)(1-a^4q^6)^2}\frac{1}{N}. \tag{3.48}$$

It remains to show (3.48). For this, note that as $z\to\infty$,

$$R_1(z)R_2(z)=\Big(I+\frac{1}{z-\overline{\beta}}R_2(z)h_{11}R_2(z)^{-1}+\frac{1}{(z-\overline{\beta})^2}R_2(z)h_{12}R_2(z)^{-1}\Big)\Big(I+\frac{h_{21}}{z-\beta}+\frac{h_{22}}{(z-\beta)^2}\Big).$$

Here R_1,R_2, h_{11}, h_{12} and h_{21},h_{22} are given in Definition 5.8. Then

$$\begin{aligned}\operatorname*{Res}_{z=\infty}\Big[R_{11}(z)\Big]&=\frac{1}{N}\frac{1+i}{128\sqrt{2}}\Big(\frac{3}{(R\kappa)^{1/4}}\Big(\frac{1}{\gamma_{11}^{3/2}}-\frac{i}{\gamma_{21}^{3/2}}\Big)+10(R\kappa)^{1/4}\Big(\frac{\gamma_{22}}{\gamma_{21}^{5/2}}-\frac{i\gamma_{12}}{\gamma_{11}^{5/2}}\Big)\Big)\\&=\frac{1+i}{128\sqrt{2}}\frac{1}{(R\kappa)^{1/4}}\Big(\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)-i\,\overline{\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)}\Big),\end{aligned} \tag{3.49}$$

where $\gamma_{11}=\overline{\gamma}_{21}$ and $\gamma_{12}=\overline{\gamma}_{22}$ are given by (5.11) and (5.12). We claim that

$$\begin{aligned}&\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)-i\,\overline{\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)}\\&=-4\sqrt{2}(1-i)(R\kappa)^{1/4}\frac{q^2(1-a^4q^4)^2}{a(1-q^2)(1-a^4q^6)^2},\end{aligned} \tag{3.50}$$

which leads to (3.48). By using (5.11) and (5.12), we have

$$\begin{aligned}\frac{1}{(R\kappa)^{1/4}}\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)&=\frac{2}{(R\kappa)^{1/4}}\frac{(\overline{\beta}-a)\overline{\beta}}{a(\overline{\beta}-R/q)\sqrt{\overline{\beta}-\beta}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)\\&=\frac{2}{(R\kappa)^{1/4}}\frac{(\overline{\beta}-a)\overline{\beta}}{a(\overline{\beta}-R/q)\sqrt{\overline{\beta}-\beta}}\Big(3-8(R\kappa)^{1/2}i\Big(\frac{1}{\overline{\beta}-R/q}+\frac{1}{2(\overline{\beta}-\beta)}-\frac{1}{\overline{\beta}-a}-\frac{1}{\overline{\beta}}\Big)\Big).\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{(R\kappa)^{1/4}}\Big(\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)-i\,\overline{\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)}\Big)\\&=\frac{e^{\frac{\pi i}{4}}}{a(\kappa R)^{1/2}}\frac{(\overline{\beta}-a)\overline{\beta}}{\overline{\beta}-R/q}\Big(3-8(R\kappa)^{1/2}i\Big(\frac{1}{\overline{\beta}-R/q}+\frac{1}{2(\overline{\beta}-\beta)}-\frac{1}{\overline{\beta}-a}-\frac{1}{\overline{\beta}}\Big)\Big)\\&\quad-\frac{e^{\frac{\pi i}{4}}}{a(\kappa R)^{1/2}}\frac{(\beta-a)\beta}{\beta-R/q}\Big(3+8(R\kappa)^{1/2}i\Big(\frac{1}{\beta-R/q}+\frac{1}{2(\beta-\overline{\beta})}-\frac{1}{\beta-a}-\frac{1}{\beta}\Big)\Big).\end{aligned}$$

Note that

$$\begin{aligned}\frac{(\overline{\beta}-a)\overline{\beta}}{\overline{\beta}-R/q}-\frac{(\beta-a)\beta}{\beta-R/q}&=\frac{(\overline{\beta}-a)\overline{\beta}(\beta-R/q)-(\beta-a)\beta(\overline{\beta}-R/q)}{(\overline{\beta}-R/q)(\beta-R/q)}\\&=-q(\beta-\overline{\beta})\frac{q|\beta|^2+(a-\beta-\overline{\beta})R}{q^2|\beta|^2-q(\beta+\overline{\beta})R+R^2}=-4(\kappa R)^{1/2}i\,q\,\frac{q|\beta|^2+(a-\beta-\overline{\beta})R}{q^2|\beta|^2-q(\beta+\overline{\beta})R+R^2}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\frac{(\beta-a)\beta}{\beta-R/q}\Big(\frac{1}{\beta-R/q}+\frac{1}{2(\beta-\overline{\beta})}-\frac{1}{\beta-a}-\frac{1}{\beta}\Big)+\frac{(\overline{\beta}-a)\overline{\beta}}{\overline{\beta}-R/q}\Big(\frac{1}{\overline{\beta}-R/q}+\frac{1}{2(\overline{\beta}-\beta)}-\frac{1}{\overline{\beta}-a}-\frac{1}{\overline{\beta}}\Big)\\&=-\frac{\beta^2+(a-2\beta)R/q}{(\beta-R/q)^2}-\frac{\overline{\beta}^2+(a-2\overline{\beta})R/q}{(\overline{\beta}-R/q)^2}+\frac{q}{2}\frac{q|\beta|^2+(a-\beta-\overline{\beta})R}{q^2|\beta|^2-q(\beta+\overline{\beta})R+R^2}.\end{aligned}$$

Therefore we obtain

$$\frac{1}{(R\kappa)^{1/4}}\bigg(\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)-i\,\overline{\frac{1}{\gamma_{11}^{3/2}}\Big(3-10(R\kappa)^{1/2}i\frac{\gamma_{12}}{\gamma_{11}}\Big)}\bigg)$$
$$=\frac{4\sqrt{2}(1-i)}{a}\bigg(2q\,\frac{q|\beta|^2+(a-\beta-\overline{\beta})R}{q^2|\beta|^2-q(\beta+\overline{\beta})R+R^2}-\frac{\beta^2+(a-2\beta)R/q}{(\beta-R/q)^2}-\frac{\overline{\beta}^2+(a-2\overline{\beta})R/q}{(\overline{\beta}-R/q)^2}\bigg).$$

Then the desired identity (3.50) follows from (2.26) and (1.17). □

For our main theorem, it remains to prove Proposition 2.1 and Theorem 3.11. Before that, we discuss the critical case in a separate subsection.

4. Potential theoretic preliminaries

In this section, we prove Proposition 2.1. Before the proof, let us first consider the radially symmetric case.

Remark 4.1 (Energy for the radially symmetric case). We consider a general radially symmetric potential

$$W(z)=w(|z|),\qquad w:\mathbb{R}_+\to\mathbb{R}. \tag{4.1}$$

Suppose that $\Delta W(z)>0$ in $\mathbb{C}$. Then it follows from a general theory [106] that the associated droplet S_W is given by

$$S_W=\{z\in\mathbb{C}: r_0\le |z|\le r_1\}, \tag{4.2}$$

where (r_0,r_1) is a unique pair of constants satisfying

$$r_0w'(r_0)=0,\qquad r_1w'(r_1)=2. \tag{4.3}$$

Furthermore, the energy $I_V[\sigma_Q]$ is given by

$$I_W[\sigma_W]=w(r_1)-\log r_1-\frac{1}{4}\int_{r_0}^{r_1} rw'(r)^2\,dr. \tag{4.4}$$

Using this, we have that for the potential Q in (1.5) with $a=0$,

$$I_Q[\sigma_Q]\Big|_{a=0}=\frac{3}{4}+\frac{3c}{2}+\frac{c^2}{2}\log c-\frac{(c+1)^2}{2}\log(c+1). \tag{4.5}$$

This coincides with (2.2).

4.1. g-function.

In Riemann-Hilbert analysis, the first step is to normalise the problem so that it is close to the identity near infinity. For this, we need a proper function called the g-function, which indeed gives the logarithmic energy of limiting zeros of the orthogonal polynomial.

We first recall the motherbody defined in [16, Section 1.1].

Definition 4.2 (Motherbody; limiting skeleton). The motherbody $\mathcal{B}$ is defined as follows.

- For the post-critical case, let

$$\beta=\frac{a^2+1-\sqrt{(1-a^2)^2-4a^2c}}{2a},\qquad b=\frac{a^2+1+\sqrt{(1-a^2)^2-4a^2c}}{2a}. \tag{4.6}$$

  Then the simple closed curve $\mathcal{B}$ is defined by the following three conditions.
  - $\mathcal{B}$ is a simple closed curve such that $\beta\in\mathcal{B}$;
  - $\mathcal{B}$ contains 0 and a in its interior;
  - The following inequality is satisfied on the curve $\mathcal{B}$

$$a^2\frac{(z-\beta)^2(z-b)^2}{z^2(z-a)^2}\,dz<0, \tag{4.7}$$

    where dz is the standard differential.
- For the pre-critical case, let β be given by (2.26). Let $b=R/q$, where R and q are given by (1.17) and (1.18).[3] Then $\mathcal{B}$ is defined by the following three conditions.

[3]There is a typo in [16, Eq.(1.12)], where $b:=\alpha/\rho$ should be replaced with $b:=\rho/\alpha$.

– $\mathcal{B}$ has the endpoints at β and $\overline{\beta}$;
– $\mathcal{B}$ intersects the negative real axis;
– The following inequality is satisfied on the curve $\mathcal{B}$

$$\frac{(z-b)^2(z-\beta)(z-\overline{\beta})}{(z-a)^2 z^2}\,dz < 0. \tag{4.8}$$

Here, we use a slight abuse of notation by employing both β and b to denote variables in both post- and pre-critical regimes.

We now recall the g-function defined in [16, Definition 1.2].

Definition 4.3 (g-function). The g-function is defined as follows.

- For the post-critical case,

$$g(z) = \begin{cases} \log z + c\log\Big(\dfrac{z}{z-a}\Big), & z \in \operatorname{ext}(\mathcal{B}), \\ az + \operatorname{Re}\big((1+c)\log\beta - c\log(\beta-a) - \beta a\big), & z \in \operatorname{int}(\mathcal{B}). \end{cases} \tag{4.9}$$

  We also define

$$\phi(z) = \int_\beta^z y(s)\,ds, \qquad y(z) = \pm a\frac{(z-b)(z-\beta)}{z(z-a)}. \tag{4.10}$$

- For the pre-critical case, let

$$V(z) = az - c\log(z-a) + (c+1)\log z. \tag{4.11}$$

  Write

$$y(z) = \frac{a(z-R/q)\sqrt{(z-\beta)(z-\overline{\beta})}}{(z-a)z}, \qquad \phi(z) = \int_\beta^z y(s)\,ds, \qquad (z \in \mathbb{C}\setminus\mathcal{B}). \tag{4.12}$$

  Then the g-function is defined by

$$g(z) = \frac{1}{2}\Big(V(z) - \phi(z) + \ell\Big), \qquad (z \in \mathbb{C}\setminus\mathcal{B}), \tag{4.13}$$

  where the constant $\ell \in \mathbb{R}$ is defined such that

$$\lim_{z\to\infty}(g(z) - \log z) = 0. \tag{4.14}$$

The g-function plays an important role in the asymptotic behaviour of orthogonal polynomials, see Theorem 3.11. This indeed comes from the fact that if we write ζ_j for the zeros of p_n and ν for their limiting distribution, we have formally

$$p_N(z) = \prod_{j=1}^N (z-\zeta_j) = \exp\Big(\sum_{j=1}^N \log(z-\zeta_j)\Big) \sim \exp\Big(N\int \log(z-\zeta)\,d\nu(\zeta)\Big).$$

Thus, the g-function is naturally defined by $\int \log(z-\zeta)\,d\nu(\zeta)$ in a more general context, and Definition 4.3 provides its evaluations in our present context.

For our purposes explained in Section 3, let us compute the asymptotic behaviour of $e^{Ng(z)}$ as $z\to\infty$. In the end, this will provide the leading order asymptotic behaviour of the derivative of the partition function.

Lemma 4.4 (**Asymptotic behaviour of the g-function**). *As $z\to\infty$, we have the following.*

- *For the post-critical case,*

$$e^{Ng(z)} = z^N\Big(1 + ac\frac{N}{z} + O\Big(\frac{1}{z^2}\Big)\Big), \qquad z\to\infty. \tag{4.15}$$

- *For the pre-critical case,*

$$e^{Ng(z)} = z^N\Big(1 - \Big(\frac{1}{4a} + \frac{a}{2} - \frac{1}{2aq^2} - \frac{a^3q^4}{4}\Big)\frac{N}{z} + O\Big(\frac{1}{z^2}\Big)\Big), \qquad z\to\infty. \tag{4.16}$$

Remark 4.5. Notice that the coefficients of the $1/z$ term in Lemma 4.4 coincide with the leading terms in Proposition 3.10.

Proof of Lemma 4.4. This follows from the definition. For instance, for the pre-critical case, we have

$$2g'(z) = a - \frac{c}{z-a} + \frac{c+1}{z} - \frac{a(z-R/q)\sqrt{(z-f(z_+))(z-f(z_-))}}{(z-a)z}. \tag{4.17}$$

Then it follows that

$$g'(z) = \frac{1}{z} + \Big(\frac{1+2a^2}{4a} - \frac{1}{2aq^2} - \frac{a^3q^4}{4}\Big)\frac{1}{z^2} + O\Big(\frac{1}{z^3}\Big). \tag{4.18}$$

This gives

$$g(z) = \log z - \Big(\frac{1}{4a} + \frac{a}{2} - \frac{1}{2aq^2} - \frac{a^3q^4}{4}\Big)\frac{1}{z} + O\Big(\frac{1}{z^2}\Big), \tag{4.19}$$

which completes the proof. □

To compute the energy (2.3), we need to evaluate the g-function at the point a.

Lemma 4.6. *For the pre-critical case, we have*

$$\operatorname{Re} g(a) = \frac{a^2q^2}{2} - \frac{1}{2} + \log\Big(\frac{1+a^2q^2}{2aq^2}\Big) + c\log\Big(\frac{1+a^2q^2}{1+a^2q^2-2a^2q^4}\Big). \tag{4.20}$$

This lemma is shown in the next subsection.

Remark 4.7. We encounter surprising cancellations when differentiating (4.20) with respect to a. Namely,

$$\mathfrak{d}_a \operatorname{Re} g(a) = aq^2, \tag{4.21}$$

where we have used (3.20). We currently lack a good intuition for such a significant simplification.

4.2. **Robin's constant and energy.** Recall that the equilibrium measure μ_Q satisfies the variational equality (also known as the Euler–Lagrange equation):

$$\int \log\frac{1}{|z-w|}\,d\mu_Q(z) + \frac{Q(w)}{2} = C, \qquad \text{if } w \in S_Q \tag{4.22}$$

where $C \equiv C(a)$ is called the modified Robin's constant, see [106, p.27]. Then we have

$$\int_{\mathbb{C}^2} \log\frac{1}{|z-w|}\,d\mu_Q(z)\,d\mu_Q(w) = C(a) - \frac{1}{2}\int Q(z)\,d\mu_Q(z) \tag{4.23}$$

and

$$I_Q[\mu_Q] = C(a) + \frac{1}{2}\int Q(z)\,d\mu_Q(z). \tag{4.24}$$

By Lemma 4.6 and (4.24), it suffices to show the following lemma to complete the proof of Proposition 2.1. Note that the inequality (2.4) was shown in Remark 3.4.

Lemma 4.8 (**Robin's constant and energy**)**.** *We have the following.*

- *For the post-critical case, we have*

$$C(a) = \frac{c+1}{2} - \frac{c+1}{2}\log(c+1) \tag{4.25}$$

and

$$\frac{1}{2}\int Q\,d\mu_Q = c + \frac{1}{4} + \frac{c^2}{2}\log c - \frac{c(1+c)}{2}\log(1+c) - ca^2. \tag{4.26}$$

In particular, we have (2.2).

- *For the pre-critical case, we have*

$$C(a) = c\log q - (c+1)\log\Big(\frac{1+a^2q^2}{2aq}\Big) + \frac{3}{4} + \frac{a^2}{4} + \frac{c}{2} - \Big(\frac{a^2}{4} + c + \frac{3}{4}\Big)a^2q^2 + \frac{a^4q^4}{2} \tag{4.27}$$

and

$$\frac{1}{2}\int Q\,d\mu_Q = \frac{3}{8} + \frac{a^2}{4} + \frac{c}{2} - \Big(\frac{a^2}{4} + c + \frac{1}{2}\Big)a^2q^2 + \frac{3}{8}a^4q^4 - c\operatorname{Re} g(a). \tag{4.28}$$

In particular, we have

$$I_Q[\mu_Q] = c\Big(\log q - \operatorname{Re} g(a)\Big) - (c+1)\log\Big(\frac{1+a^2q^2}{2aq}\Big) + \frac{5}{8} + \frac{a^2}{4} + \frac{1}{4a^2q^4} - \frac{1}{2q^2} + \frac{a^2q^2}{4} - \frac{a^4q^4}{8}. \tag{4.29}$$

Remark 4.9 (Point charge insertion at infinity). It is easy to check from (1.18) that

$$q = \frac{1}{a}\Big(1 - \frac{c}{a^2} + O(\frac{1}{a^4})\Big), \qquad (a\to\infty). \tag{4.30}$$

This in turn gives that

$$R = 1 + O(\frac{1}{a^4}), \qquad \kappa = \frac{c}{a^2} + O(\frac{1}{a^4}), \qquad f(z) = z + O(\frac{1}{a}), \qquad (a\to\infty), \tag{4.31}$$

where R,κ and f are given by (1.17). Therefore the droplet tends to the unit disc when $a\to\infty$, cf. Figure 5. Let us also mention that by (4.30), we have

$$\lim_{a\to\infty}\Big(I_Q[\mu_Q] + 2c\log a\Big) = \frac{3}{4}, \qquad \lim_{a\to\infty}\mathcal{F}(a,c) = 0. \tag{4.32}$$

From the viewpoint of the free energy expansions (1.13), these asymptotic behaviours are consistent with (3.7) and (3.11). Furthermore, by (4.14) and (4.30), we also have

$$\lim_{a\to\infty}\Big(C(a) + c\log a\Big) = \frac{1}{2}, \qquad \lim_{a\to\infty}\Big(\frac{1}{2}\int Q(z)\,d\mu_Q(z) + c\log a\Big) = \frac{1}{4}. \tag{4.33}$$

Note also that by combining Lemma 3.3 and (4.30), one can check that as $a\to\infty$,

$$\partial_a I_Q[\mu_Q] = -\frac{2c}{a} + O(a^{-3}), \qquad \partial_a\mathcal{F}(a,c) = -\frac{2c^2}{a^7} + O(a^{-9}). \tag{4.34}$$

We now prove Lemma 4.8.

Proof of Lemma 4.8. The modified Robin constant was computed in [16, Lemma 7.2]. Let us mention that the Robin constant $\ell_{2\mathrm{D}}$ in [16] corresponds to $\ell_{2\mathrm{D}} = -2\,C(a)$.

It remains to compute $\int Q\,d\mu_Q$. We first show the post-critical case when $c < c_{\rm cri}$. By [30, Lemma 2.4], we have that for $R>0$ and $p\in\mathbb{C}$,

$$\int_{\mathbb{D}(p,R)} \log|z-w|\,dA(z) = \begin{cases} R^2\log|w-p| & \text{if } w\notin\mathbb{D}(p,R), \\ R^2\log R - \dfrac{R^2}{2} + \dfrac{|w-p|^2}{2} & \text{otherwise.} \end{cases} \tag{4.35}$$

Using this, we have

$$\begin{aligned} -2c\int_S \log|w-a|\,dA(w) &= -2c\Big(\int_{\mathbb{D}(0,\sqrt{1+c})}\log|w-a|\,dA(w) - \int_{\mathbb{D}(a,\sqrt{c})}\log|w-a|\,dA(w)\Big) \\ &= c^2(\log c - 1) - 2c\int_{\mathbb{D}(0,\sqrt{1+c})}\log|w-a|\,dA(w) \\ &= c + c^2\log c - c(1+c)\log(1+c) - ca^2. \end{aligned}$$

On the other hand, note that

$$\int_S |w|^2\,dA(w) = \int_{\mathbb{D}(0,\sqrt{1+c})}|w|^2\,dA(w) - \int_{\mathbb{D}(a,\sqrt{c})}|w|^2\,dA(w) = \frac{1}{2}(c+1)^2 - \int_{\mathbb{D}(a,\sqrt{c})}|w|^2\,dA(w).$$

Here by Green's formula, and using the map $z \mapsto w = \sqrt{c}z + a$,

$$\int_{\mathbb{D}(a,\sqrt{c})} |w|^2 \, dA(w) = \frac{1}{2\pi i} \int_{|w-a|=\sqrt{c}} \frac{w\bar{w}^2}{2} \, dw = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\sqrt{c}}{2} (\sqrt{c}z + a)(\sqrt{c}\bar{z} + a)^2 \, dz$$
$$= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\sqrt{c}}{2} (\sqrt{c}z + a)(\sqrt{c}/z + a)^2 \, dz = \frac{1}{2}c^2 + ca^2.$$

This gives

$$\int_S |w|^2 \, dA(w) = c + \frac{1}{2} - ca^2. \tag{4.36}$$

We have shown the post-critical case.

Next, we consider the pre-critical case when $c > c_{\rm cri}$. By Green's formula,

$$\int_S |z|^2 \, dA(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{z\bar{z}^2}{2} \, dz = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(w) f(1/w)^2}{2} \, f'(w) \, dw,$$

where f is given by (1.17). Note that $0 < q < 1$ and

$$f(z) = \frac{R\, z(z - z_0)}{z - q}, \qquad f(1/z) = \frac{R(1 - z_0 z)}{z(1 - qz)}, \qquad z_0 = q + \frac{\kappa}{qR}. \tag{4.37}$$

Then by the residue calculus,

$$\begin{aligned} \int_S |z|^2 \, dA(z) &= \operatorname*{Res}_{w=q} \Big[\frac{f(w) f(1/w)^2}{2} \, f'(w) \Big] + \operatorname*{Res}_{w=0} \Big[\frac{f(w) f(1/w)^2}{2} \, f'(w) \Big] \\ &= \frac{1}{4} \Big(1 + \frac{1}{a^2 q^4} - \frac{2}{q^2} - a^4 q^4 + a^2 (1 + 2q^2) \Big). \end{aligned} \tag{4.38}$$

Here we have used

$$\frac{1}{q^4} = -2a^4 q^2 + (a^2 + 4c + 2) a^2. \tag{4.39}$$

Note that by [16, Lemma 3.7], we have

$$\int_S \log|z - w| \, dA(w) = \operatorname{Re} g(z), \qquad z \notin S. \tag{4.40}$$

This in particular gives

$$\int_S \log|z - a| \, dA(z) = \operatorname{Re} g(a), \tag{4.41}$$

which completes the proof. ◻

We now prove Lemma 4.6, which completes the proof of Proposition 2.1.

Proof of Lemma 4.6. By [16, Eq.(7.9)], it follows that

$$\int_S \log|z - w| \, dA(w) = \operatorname{Re} \int_{z_0}^{z} \mathcal{S}(\zeta) \, d\zeta - c \log|z - a| + \frac{|z_0|^2}{2} - C(a), \tag{4.42}$$

where $\mathcal{S}$ is the Schwarz function associated with the droplet, see [16, Definition 2.1]. Choose $z_0 = f(1)$. Then since

$$\mathcal{S}(f(z)) = f(1/z), \tag{4.43}$$

we have

$$\int_{z_0}^{z} \mathcal{S}(\zeta) \, d\zeta = \int_1^u f(1/w) f'(w) \, dw, \qquad f(u) = z. \tag{4.44}$$

Note that

$$
\begin{aligned}
f(1/z)f'(z) &= \frac{(1+a^2q^2)(1-a^2q^2+2a^2q^4)}{4a^2q^4}\frac{1}{z} + \frac{1-q^2-a^2q^2+a^2q^4}{2q}\frac{1}{(z-q)^2} + \frac{c}{z-1/q} - \frac{c}{z-q} \\
&= \frac{c+1}{z} + \frac{1-q^2-a^2q^2+a^2q^4}{2q}\frac{1}{(z-q)^2} + \frac{c}{z-1/q} - \frac{c}{z-q}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\log|f(1)-a| - \log|z-a| = \log|f(1)-f(1/q)| - \log|f(u)-f(1/q)| \\
&= \int_1^u \frac{1}{z-q} - \frac{1}{z-1/q} - \frac{1+a^2q^2}{(1+a^2q^2)z-2a^2q^3}\,dz.
\end{aligned}
$$

Combining the above, we have

$$
\begin{aligned}
&\operatorname{Re}\int_{z_0}^{z} \mathcal{S}(\zeta)\,d\zeta - c\log|z-a| + c\log|f(1)-a| \\
&= \int_1^u \frac{c+1}{z} + \frac{1-q^2-a^2q^2+a^2q^4}{2q}\frac{1}{(z-q)^2} - \frac{c}{z-2a^2q^3/(1+a^2q^2)}\,dz
\end{aligned}
$$

Note that

$$
\int_1^{1/q} \frac{c+1}{z}\,dz = -(c+1)\log q, \qquad \int_1^{1/q} \frac{1-q^2-a^2q^2+a^2q^4}{2q}\frac{1}{(z-q)^2}\,dz = \frac{1}{2q} - \frac{a^2q}{2},
$$

and

$$
\begin{aligned}
&-\int_1^{1/q} \frac{c}{z-2a^2q^3/(1+a^2q^2)}\,dz - c\log|f(1)-a| \\
&= -c\log\Big(\frac{1+a^2q^2-2a^2q^4}{q(1+a^2q^2)}\Big) + c\log\Big(\frac{1+a^2q^2-2a^2q^3}{1+a^2q^2}\Big) - c\log\Big(\frac{1+a^2q^2-2a^2q^3}{2aq^2}\Big) \\
&= -c\log\Big(\frac{1+a^2q^2-2a^2q^4}{q(1+a^2q^2)}\Big) + c\log\Big(\frac{2aq^2}{1+a^2q^2}\Big) = c\log\Big(\frac{2aq^3}{1+a^2q^2-2a^2q^4}\Big).
\end{aligned}
$$

Therefore it follows that

$$
\begin{aligned}
&\lim_{z\to a}\Big(\operatorname{Re}\int_{z_0}^{z} S(\zeta)\,d\zeta - c\log|z-a|\Big) \\
&= \frac{1}{2q} - \frac{a^2q}{2} - \frac{(1+a^2q^2)(1-a^2q^2+2a^2q^4)}{4a^2q^4}\log q + c\log\Big(\frac{2aq^3}{1+a^2q^2-2a^2q^4}\Big).
\end{aligned}
$$

Then we have shown that

$$
\operatorname{Re} g(a) = \frac{a^2}{8} + \frac{1}{8a^2q^4} - \frac{1}{4q^2} + \frac{a^2q^2}{2} - C(a) - (c+1)\log q + c\log\Big(\frac{2aq^3}{1+a^2q^2-2a^2q^4}\Big).
$$

Now the lemma follows from (4.27). □

5. Riemann-Hilbert analysis and Fine asymptotic behaviours

In this section, we show Theorem 3.11.

Recall that Γ be the contour in the Riemann-Hilbert problem (3.25). We deform the contour Γ so that it matches with the contour $\mathcal{B}$ in Definition 4.2. We choose the contour Γ_+ and Γ_- following the steepest descent paths from β such that $\operatorname{Re}\phi(z) < 0$ on those contours. Here ϕ is given in Definition 4.3. More precisely, we choose the one inside $\operatorname{int}(\mathcal{B})$ to be Γ_+, while the one inside $\operatorname{ext}(\mathcal{B})$ to be Γ_-. The domains $\Omega_\pm$ are defined by the open sets enclosed by $\mathcal{B}$ and $\Gamma_\pm$ respectively.

Let us define the matrix function $A(z)$ by

$$A(z) := \begin{cases} e^{-\frac{N\ell}{2}\sigma_3} Y(z) e^{-N(g(z)-\ell/2)\sigma_3}, & z \in \mathbb{C} \setminus (\Omega_+ \cup \Omega_-), \\ e^{-\frac{N\ell}{2}\sigma_3} Y(z) \begin{pmatrix} 1 & 0 \\ -1/\omega_{n,N}(z) & 1 \end{pmatrix} e^{-N(g(z)-\ell/2)\sigma_3}, & z \in \Omega_+, \\ e^{-\frac{N\ell}{2}\sigma_3} Y(z) \begin{pmatrix} 1 & 0 \\ 1/\omega_{n,N}(z) & 1 \end{pmatrix} e^{-N(g(z)-\ell/2)\sigma_3}, & z \in \Omega_-, \end{cases} \tag{5.1}$$

where Y and $\omega_{n,N}$ are given in Definition 3.6, while ℓ and $g(z)$ are given in Definition 4.3. Here,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.2}$$

is the third Pauli matrix. Then by (3.25), one can check that A satisfies the following Riemann-Hilbert problem:

$$\begin{cases} A_+(z) = A_-(z) \begin{pmatrix} 1 & 0 \\ e^{N\phi(z)} & 1 \end{pmatrix}, & z \in \Gamma_\pm, \\ A_+(z) = A_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathcal{B}, \\ A_+(z) = A_-(z) \begin{pmatrix} 1 & e^{-N\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma \setminus \mathcal{B}, \\ A(z) = I + O(z^{-1}), & z \to \infty, \\ A(z) \quad \text{is holomorphic,} & \text{otherwise.} \end{cases} \tag{5.3}$$

Note that $\mathrm{Re}(\phi(z))$ is negative on $\Gamma_\pm$, while it is positive on $\Gamma \setminus \mathcal{B}$. Therefore, when z is away from $\mathcal{B}$, the jump matrices are close to identity exponentially.

Next, we define the global parametrix, cf. [16, Eqs.(4.14), (5.1)].

Definition 5.1 (Global parametrix). The global parametrix Φ is defined as follows.

- For the post-critical case,

$$\Phi(z) := \begin{cases} I, & z \in \mathrm{ext}(\mathcal{B}), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathrm{int}(\mathcal{B}). \end{cases} \tag{5.4}$$

- For the pre-critical case,

$$\Phi(z) := \sqrt{RF'(z)} \begin{pmatrix} 1 & \dfrac{\sqrt{\kappa R}}{RF(z) - Rq} \\ -\dfrac{\sqrt{\kappa R}}{RF(z) - Rq} & 1 \end{pmatrix} \tag{5.5}$$

  where F is given by (2.25).

By construction, $\Phi(z)$ satisfies the following Riemann-Hilbert problem:

$$\begin{cases} \Phi(z) \quad \text{is holomorphic,} & z \in \mathbb{C} \setminus \mathcal{B}, \\ \Phi_+(z) = \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathcal{B}, \\ \Phi(z) = I + O(z^{-1}), & z \to \infty. \end{cases} \tag{5.6}$$

For the remaining Riemann-Hilbert analysis, we should treat post- and pre-critical cases separately. We shall first discuss the pre-critical case, as it is the more difficult one.

5.1. **Pre-critical case.** We denote D_w as a small disk centered at $w \in \mathbb{C}$ with a finite radius. Let us first define the local coordinates that will be used frequently.

Definition 5.2 (Local coordinates). Recall that ϕ is given by (4.12). The local coordinate ζ is defined by

$$\frac{4}{3}\zeta(z)^{3/2} := N(\phi(z) - \phi(\overline{\beta})), \quad z \in D_{\overline{\beta}}, \tag{5.7}$$

cf. [16, Eq.(4.15)]. Here ζ maps $\Gamma_{b\overline{\beta}}$ into $\mathbb{R}_+$, maps $\mathcal{B}$ into $\mathbb{R}_-$, maps Γ_+ into ray $\gamma_{\overline{\beta}}^+ := [0, e^{2\pi i/3}\infty)$, and maps Γ_- into ray $\gamma_{\overline{\beta}}^- := [0, e^{-2\pi i/3}\infty)$. Similarly, we define the local coordinate ξ by

$$\frac{4}{3}\xi(z)^{3/2} := N\phi(z), \quad z \in D_{\beta}, \tag{5.8}$$

such that ξ maps $\Gamma_{b\beta}$ into $\mathbb{R}_+$, maps $\mathcal{B}$ into $\mathbb{R}_-$, maps Γ_+ into ray $\gamma_{\beta}^+ := [0, e^{-2\pi i/3}\infty)$, and maps Γ_- into ray $\gamma_{\beta}^- := [0, e^{2\pi i/3}\infty)$.

Lemma 5.3 (**Asymptotic of the local coordinates**). *We have*

$$\frac{\zeta(z)}{N^{2/3}} = \gamma_{11}(z-\overline{\beta}) + \frac{\gamma_{12}}{2}(z-\overline{\beta})^2 + O(z-\overline{\beta})^3, \qquad z \to \overline{\beta}, \tag{5.9}$$

$$\frac{\xi(z)}{N^{2/3}} = \overline{\gamma_{11}}(z-\beta) + \frac{\overline{\gamma_{12}}}{2}(z-\beta)^2 + O(z-\beta)^3, \qquad z \to \beta, \tag{5.10}$$

where

$$\gamma_{11} = \frac{1}{2^{2/3}}\Big(\frac{a(\overline{\beta}-R/q)\sqrt{\overline{\beta}-\beta}}{(\overline{\beta}-a)\overline{\beta}}\Big)^{2/3}, \tag{5.11}$$

$$\gamma_{12} = \frac{4}{5}\Big(\frac{1}{\overline{\beta}-R/q} + \frac{1}{2(\overline{\beta}-\beta)} - \frac{1}{\overline{\beta}-a} - \frac{1}{\overline{\beta}}\Big)\gamma_{11}. \tag{5.12}$$

Proof. By definition, we have

$$\zeta'(z) = \frac{2}{3}\Big(\frac{3N}{4}\Big)^{2/3}\Big(\int_{\overline{\beta}}^z y(s)\,ds\Big)^{-1/3} y(z), \qquad \frac{\zeta''(z)}{\zeta'(z)} = -\frac{1}{3}\Big(\int_{\overline{\beta}}^z y(s)\,ds\Big)^{-1} y(z) + \frac{y'(z)}{y(z)}.$$

Let us write

$$y(z) = h(z)\sqrt{z-\overline{\beta}}, \qquad h(z) := \frac{a(z-R/q)\sqrt{z-\beta}}{(z-a)z}.$$

Then we have

$$y(z) = h(\overline{\beta})(z-\overline{\beta})^{1/2}\Big(1 + \frac{h'(\overline{\beta})}{h(\overline{\beta})}(z-\overline{\beta}) + O((z-\overline{\beta})^2)\Big),$$
$$y'(z) = \frac{1}{2}h(\overline{\beta})(z-\overline{\beta})^{-1/2}\Big(1 + 3\frac{h'(\overline{\beta})}{h(\overline{\beta})}(z-\overline{\beta}) + O((z-\overline{\beta})^2)\Big).$$

Using these, direct computations give

$$\Big(\int_{\overline{\beta}}^z y(s)\,ds\Big)^{-1} = \frac{3}{2}\frac{1}{h(\overline{\beta})}(z-\overline{\beta})^{-3/2}\Big(1 - \frac{3}{5}\frac{h'(\overline{\beta})}{h(\overline{\beta})}(z-\overline{\beta}) + O((z-\overline{\beta})^2)\Big),$$
$$\frac{y'(z)}{y(z)} = \frac{1}{2}(z-\overline{\beta})^{-1} + \frac{h'(\overline{\beta})}{h(\overline{\beta})} + O(z-\overline{\beta}),$$

which leads to

$$-\frac{1}{3}\Big(\int_{\overline{\beta}}^z y(s)\,ds\Big)^{-1} y(z) = -\frac{1}{2}(z-\overline{\beta})^{-1} - \frac{1}{5}\frac{h'(\overline{\beta})}{h(\overline{\beta})} + O(z-\overline{\beta}).$$

Then the lemma follows from

$$\frac{h'(z)}{h(z)} = \frac{1}{z-R/q} + \frac{1}{2(z-\beta)} - \frac{1}{z-a} - \frac{1}{z}.$$

□

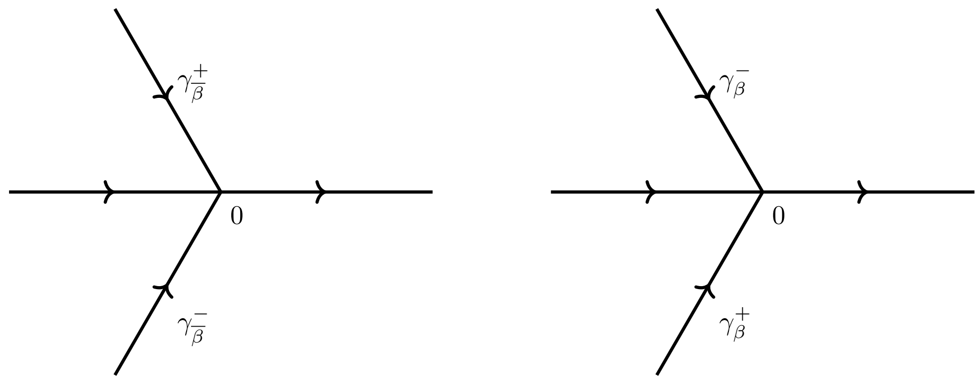


FIGURE 6. The jump contours of $P_{\overline{\beta}}(\zeta)$ in $D_{\overline{\beta}}$ and those of $\widehat{P}_{\beta}(\xi)$ in D_{β}.

We now define the local parametrices.

Definition 5.4 (Riemann-Hilbert problems for local parametrices). We consider the Riemann-Hilbert problems.

- Inside $D_{\overline{\beta}}$, we define the following Riemann-Hilbert problem:

$$\begin{cases} [P_{\overline{\beta}}(\zeta)]_+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} [P_{\overline{\beta}}(\zeta)]_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in \mathbb{R}_-, \\ [P_{\overline{\beta}}(\zeta)]_+ = [P_{\overline{\beta}}(\zeta)]_- \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & z \in \gamma_{\overline{\beta}}^{\pm}, \\ [P_{\overline{\beta}}(\zeta)]_+ = [P_{\overline{\beta}}(\zeta)]_- \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+, \\ P_{\overline{\beta}}(\zeta) = I + O(\zeta^{-3/2}), & \zeta \to \infty, \\ P_{\overline{\beta}}(\zeta) = O(\zeta^{-1/4}), & \zeta \to 0. \end{cases} \tag{5.13}$$

  Then $\Phi(z)P_{\overline{\beta}}(z)$ satisfies the same jump conditions of $A(z)$.
- Inside D_{β}, we define the following Riemann-Hilbert problem:

$$\begin{cases} [\widehat{P}_{\beta}(\xi)]_+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} [\widehat{P}_{\beta}(\xi)]_- \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \xi \in \mathbb{R}_-, \\ [\widehat{P}_{\beta}(\xi)]_+ = [\widehat{P}_{\beta}(\xi)]_- \begin{pmatrix} 1 & 0 \\ -e^{\frac{4}{3}\xi^{3/2}} & 1 \end{pmatrix}, & \xi \in \gamma_{\beta}^{\pm}, \\ [\widehat{P}_{\beta}(\xi)]_+ = [\widehat{P}_{\beta}(\xi)]_- \begin{pmatrix} 1 & -e^{-\frac{4}{3}\xi^{3/2}} \\ 0 & 1 \end{pmatrix}, & \xi \in \mathbb{R}_+, \\ \widehat{P}_{\beta}(\xi) = I + O(\xi^{-3/2}), & \xi \to \infty, \\ \widehat{P}_{\beta}(\xi) = O(\xi^{-1/4}), & \xi \to 0. \end{cases} \tag{5.14}$$

  Then $\Phi(z)\widehat{P}_{\beta}(z)$ satisfies the same jump conditions of $A(z)$.

The jump contours and the corresponding orientations can be seen from Figure 6.

The solution to the above Riemann-Hilbert problems can be constructed using the Airy parametrices, which we now recall.

Definition 5.5 (Airy parametrices). Let

$$y_j(\zeta) = \omega^j \operatorname{Ai}(\omega^j \zeta), \qquad \omega = e^{2\pi i/3}, \quad (j = 0, 1, 2). \tag{5.15}$$

Then we define[4]

$$\mathcal{A}(\zeta) := \sqrt{2\pi}e^{-\frac{i\pi}{4}} \begin{cases} \begin{pmatrix} y_0(\zeta) & -y_2(\zeta) \\ y_0'(\zeta) & -y_2'(\zeta) \end{pmatrix} e^{\frac{2}{3}\zeta^{3/2}\sigma_3}, & \arg\zeta \in (0, 2\pi/3), \\ \begin{pmatrix} -y_1(\zeta) & -y_2(\zeta) \\ -y_1'(\zeta) & -y_2'(\zeta) \end{pmatrix} e^{\frac{2}{3}\zeta^{3/2}\sigma_3}, & \arg\zeta \in (2\pi/3, \pi), \\ \begin{pmatrix} -y_2(\zeta) & y_1(\zeta) \\ -y_2'(\zeta) & y_1'(\zeta) \end{pmatrix} e^{\frac{2}{3}\zeta^{3/2}\sigma_3}, & \arg\zeta \in (\pi, 4\pi/3), \\ \begin{pmatrix} y_0(\zeta) & y_1(\zeta) \\ y_0'(\zeta) & y_1'(\zeta) \end{pmatrix} e^{\frac{2}{3}\zeta^{3/2}\sigma_3}, & \arg\zeta \in (4\pi/3, 2\pi) \end{cases} \tag{5.16}$$

and

$$\widehat{\mathcal{A}}(\xi) := \sqrt{2\pi}e^{-\frac{i\pi}{4}} \begin{cases} \begin{pmatrix} y_0(\xi) & y_2(\xi) \\ y_0'(\xi) & y_2'(\xi) \end{pmatrix} e^{\frac{2}{3}\xi^{3/2}\sigma_3}, & \arg\xi \in (0, 2\pi/3), \\ \begin{pmatrix} -y_1(\xi) & y_2(\xi) \\ -y_1'(\xi) & y_2'(\xi) \end{pmatrix} e^{\frac{2}{3}\xi^{3/2}\sigma_3}, & \arg\xi \in (2\pi/3, \pi), \\ \begin{pmatrix} -y_2(\xi) & -y_1(\xi) \\ -y_2'(\xi) & -y_1'(\xi) \end{pmatrix} e^{\frac{2}{3}\xi^{3/2}\sigma_3}, & \arg\xi \in (\pi, 4\pi/3), \\ \begin{pmatrix} y_0(\xi) & -y_1(\xi) \\ y_0'(\xi) & -y_1'(\xi) \end{pmatrix} e^{\frac{2}{3}\xi^{3/2}\sigma_3}, & \arg\xi \in (4\pi/3, 2\pi). \end{cases} \tag{5.17}$$

Using the standard Airy Riemann-Hilbert problem, one can show that the solution to the above Riemann-Hilbert problems in Definition 5.4 is given by

$$P_{\overline{\beta}}(\zeta) = e^{\frac{i\pi}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \zeta^{\frac{\sigma_3}{4}} \mathcal{A}(\zeta), \tag{5.18}$$

$$\widehat{P}_{\beta}(\xi) = \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & -e^{-\frac{i\pi}{4}} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \xi^{\frac{\sigma_3}{4}} \widehat{\mathcal{A}}(\xi). \tag{5.19}$$

See also [16, Eq.(4.22)].

Lemma 5.6 (**Asymptotics of local parametrices**). *The solutions $P_{\overline{\beta}}$ and $\widehat{P}_\beta$ in* (5.18) *and* (5.19) *satisfy the following asymptotic behaviours.*

- *As $\zeta \to \infty$, we have*

$$P_{\overline{\beta}}(\zeta) = I + \frac{\Pi_{11}}{\zeta^{3/2}} + \frac{\Pi_{12}}{\zeta^3} + O\Big(\frac{1}{\zeta^{9/2}}\Big), \tag{5.20}$$

  where

$$\Pi_{11} = \frac{1}{8} \begin{pmatrix} 1/6 & i \\ i & -1/6 \end{pmatrix}, \qquad \Pi_{12} = \frac{35}{384} \begin{pmatrix} -1/12 & i \\ -i & -1/12 \end{pmatrix}. \tag{5.21}$$

- *As $\xi \to \infty$, we have*

$$\widehat{P}_\beta(\xi) = I + \frac{\Pi_{21}}{\xi^{3/2}} + \frac{\Pi_{22}}{\xi^3} + O\Big(\frac{1}{\xi^{9/2}}\Big), \tag{5.22}$$

  where

$$\Pi_{21} = \frac{1}{8} \begin{pmatrix} 1/6 & -i \\ -i & -1/6 \end{pmatrix}, \qquad \Pi_{22} = \frac{35}{384} \begin{pmatrix} -1/12 & -i \\ i & -1/12 \end{pmatrix}. \tag{5.23}$$

[4]Here, there is a minor typo in [16, Eq.(4.23)], where $5\pi/3$ should be replaced with $4\pi/3$.

Proof. Recall the asymptotic expansion of the Airy function: for $|\arg z| < \pi$,

$$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}\sum_{k=0}^{\infty}(-1)^k\frac{u_k}{(\frac{2}{3}z^{3/2})^k}, \qquad \mathrm{Ai}'(z) \sim -\frac{z^{1/4}e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^k\frac{v_k}{(\frac{2}{3}z^{3/2})^k}, \tag{5.24}$$

where $u_0 = v_0 = 1$ and for $k = 1, 2, \dots$,

$$u_k = \frac{(2k+1)(2k+3)\dots(6k-1)}{(216)^k\, k!}, \qquad v_k = \frac{6k+1}{1-6k}u_k, \tag{5.25}$$

see [101, Eqs.(9.7.5), (9.7.6)]. Using these, the lemma follows from direct computations. ☐

Following the definition of the global parametrix in (5.5) we define

$$\widehat{\mathbf{H}}_\beta(z) := \Phi(z)\widehat{S}(z), \qquad \widehat{S}(z) := \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & -e^{-\frac{i\pi}{4}} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\xi(z)^{\frac{\sigma_3}{4}}, \tag{5.26}$$

$$\mathbf{H}_{\overline{\beta}}(z) := \Phi(z)S(z), \qquad S(z) := e^{\frac{i\pi\sigma_3}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\zeta(z)^{\frac{\sigma_3}{4}}. \tag{5.27}$$

We show that they are holomorphic and find the N-dependence.

Lemma 5.7 (Asymptotics of the global parametrices). *The global parametrices in Definition 5.1 satisfy the following asymptotic behaviours.*

- *As $z \to \overline{\beta}$, we have*

$$\begin{aligned}\mathbf{H}_{\overline{\beta}}(z) &= \frac{(\overline{\beta}-\beta)^{1/4}}{2}\begin{pmatrix} (1+i)\gamma_{11}^{1/4}N^{1/6} & \dfrac{-i}{\sqrt{2}(R\kappa\gamma_{11})^{1/4}N^{1/6}} \\ (1-i)\gamma_{11}^{1/4}N^{1/6} & \dfrac{1}{\sqrt{2}(R\kappa\gamma_{11})^{1/4}N^{1/6}} \end{pmatrix} \\ &\quad - \frac{\gamma_{11}-2i\sqrt{R\kappa}\gamma_{12}}{16(i\sqrt{R\kappa}\gamma_{11})^{3/4}}\begin{pmatrix} iN^{1/6} & \dfrac{1}{2(i\sqrt{R\kappa}\gamma_{11})^{1/2}N^{1/6}} \\ N^{1/6} & \dfrac{i}{2(i\sqrt{R\kappa}\gamma_{11})^{1/2}N^{1/6}} \end{pmatrix}(z-\overline{\beta}) + O\big((z-\overline{\beta})^2\big).\end{aligned} \tag{5.28}$$

- *As $z \to \beta$, we have*

$$\begin{aligned}\widehat{\mathbf{H}}_{\beta}(z) &= \frac{(\beta-\overline{\beta})^{1/4}}{2}\begin{pmatrix} (1+i)\overline{\gamma}_{11}^{1/4}N^{1/6} & \dfrac{-1}{\sqrt{2}(R\kappa\overline{\gamma}_{11})^{1/4}N^{1/6}} \\ -(1-i)\overline{\gamma}_{11}^{1/4}N^{1/6} & \dfrac{i}{\sqrt{2}(R\kappa\overline{\gamma}_{11})^{1/4}N^{1/6}} \end{pmatrix} \\ &\quad - \frac{(i)^{1/4}(\overline{\gamma}_{12}+2i\sqrt{R\kappa}\overline{\gamma}_{11})}{16(R\kappa)^{3/8}\overline{\gamma}_{12}^{3/4}}\begin{pmatrix} N^{1/6} & \dfrac{i}{2(R\kappa)^{1/4}\overline{\gamma}_{12}^{1/2}} \\ iN^{1/6} & \dfrac{1}{2(R\kappa)^{1/4}\overline{\gamma}_{12}^{1/2}} \end{pmatrix}(z-\beta) + O\big((z-\beta)^2\big).\end{aligned} \tag{5.29}$$

Proof. This follows from a long but straightforward computations using Lemma 5.3, (2.26) and (2.25). ☐

We shall apply the partial Schlesinger transform to improve the local parametrix. For this purpose, we shall implement the following steps:

- Construct rational matrix functions $R_1(z)$ with the only pole at $\overline{\beta}$ and $R_2(z)$ with the only pole at β;
- Construct holomorphic matrix functions $H_1(z)$ and $H_2(z)$ such that the modified global parametrix $R_1(z)R_2(z)\Phi(z)$ matches the local parametrix $R_2(z)H_1(z)\Phi(z)P_{\overline{\beta}}(\zeta(z))$ along $\partial D_{\overline{\beta}}$ and matches the local parametrix $R_1(z)H_2(z)\Phi(z)\widehat{P}_\beta(\xi(z))$ along ∂D_β.

We now define the rational functions. At first glance, the definition of these rational functions may appear quite complex, making it challenging to grasp the underlying motivation. However, this definition is crafted with a specific purpose: to ensure that in the proof of the theorem below, the error matrix satisfies a small norm Riemann-Hilbert problem. Notably, this formulation guarantees the elimination of all terms at the order of $O(N^{-1})$ within this norm, resulting in an error term of the order $O(N^{-2})$. This becomes clear in the proof of Theorem 3.11 below.

Definition 5.8 (Rational functions). Let

$$H_1(z) := I - \frac{1}{48}\mathbf{H}_{\overline{\beta}}(z)\begin{pmatrix} 0 & 5/\zeta(z)^2 \\ -7/\zeta(z) & 0 \end{pmatrix}\mathbf{H}_{\overline{\beta}}(z)^{-1} + \frac{h_{11}}{z-\overline{\beta}} + \frac{h_{12}}{(z-\overline{\beta})^2}, \tag{5.30}$$

$$H_2(z) := I - \frac{1}{48}\widehat{\mathbf{H}}_{\beta}(z)\begin{pmatrix} 0 & 5/\xi(z)^2 \\ -7/\xi(z) & 0 \end{pmatrix}\widehat{\mathbf{H}}_{\beta}(z)^{-1} + \frac{h_{21}}{z-\beta} + \frac{h_{22}}{(z-\beta)^2}, \tag{5.31}$$

where

$$\begin{aligned} h_{11} &= \frac{1+i}{128\sqrt{2}(R\kappa)^{1/4}\gamma_{11}^{5/2}N}\begin{pmatrix} 3\gamma_{11} - 10i\sqrt{R\kappa}\gamma_{12} & \dfrac{19\gamma_{11} + 30i\sqrt{R\kappa}\gamma_{12}}{3} \\ \dfrac{19\gamma_{11} + 30i\sqrt{R\kappa}\gamma_{12}}{3} & -3\gamma_{11} + 10i\sqrt{R\kappa}\gamma_{12} \end{pmatrix}, \\ h_{12} &= \frac{5(R\kappa)^{1/4}}{48\sqrt{2}\gamma_{11}^{3/2}N}\begin{pmatrix} -1+i & 1+i \\ 1+i & 1-i \end{pmatrix}, \qquad h_{21} = \overline{h_{11}}, \qquad h_{22} = \overline{h_{12}}. \end{aligned} \tag{5.32}$$

These coefficient h_{jk}'s are defined in a way that the matrix functions H_1 and H_2 are holomorphic at $\overline{\beta}$ and at β respectively.

We further define

$$R_2(z) := I + \frac{h_{21}}{z-\beta} + \frac{h_{22}}{(z-\beta)^2} \tag{5.33}$$

and

$$R_1(z) := I + \frac{1}{z-\overline{\beta}}R_2(z)h_{11}R_2(z)^{-1} + \frac{1}{(z-\overline{\beta})^2}R_2(z)h_{12}R_2(z)^{-1}. \tag{5.34}$$

Here $R_1(z)$ is a rational matrix function with the only pole at $\overline{\beta}$, and $R_2(z)$ is a rational matrix function with the only pole at β. Let us also write

$$R_1(z)R_2(z) := I + \begin{pmatrix} R_{11}(z) & R_{12}(z) \\ R_{21}(z) & R_{22}(z) \end{pmatrix}. \tag{5.35}$$

Definition 5.9 (Strong asymptotics for the pre-critical case). We define the strong asymptotics of $A(z)$ by

$$A^{\infty}(z) = \begin{cases} R_1(z)R_2(z)\Phi(z), & z \notin D_{\overline{\beta}} \cup D_{\beta}, \\ R_2(z)H_1(z)\Phi(z)P_{\overline{\beta}}(\zeta(z)), & z \in D_{\overline{\beta}}, \\ R_1(z)H_2(z)\Phi(z)\widehat{P}_{\beta}(\xi(z)), & z \in D_{\beta}. \end{cases} \tag{5.36}$$

We note that the strong asymptotic in [16] is defined in a similar manner but without the rational functions. Then, the error analysis in [16] can only be applied to derive the leading-order asymptotic behaviour of the orthogonal polynomials. In other words, it is crucial to construct the rational functions in Definition 5.8 to define (5.36), which yields find asymptotic behaviours.

Proof of Theorem 3.11 for the pre-critical case. We aim to show that

$$A(z) = \Big(I + O(N^{-2})\Big)A^{\infty}(z), \qquad z \in \mathrm{ext}(\mathcal{B}). \tag{5.37}$$

For this, let us define the error function

$$\mathcal{E}(z) = A^{\infty}(z)A^{-1}(z) \tag{5.38}$$

and verify that $\mathcal{E}$ satisfies a small-norm Riemann-Hilbert problem with error $O(N^{-2})$.

Note that when z is away from $\partial D_{\overline{\beta}}$ and ∂D_β, the jump of the error matrix is exponentially small in N, see [16, Eq.(4.27)]. On the other hand, by (5.36) and (5.3), we have

$$\mathcal{E}_+(z)\mathcal{E}_-(z)^{-1} = \begin{cases} R_1(z)\Big(H_2(z)\Phi(z)\widehat{P}_\beta(\xi(z))\Phi(z)^{-1}R_2(z)^{-1}\Big)R_1(z)^{-1}, & z\in\partial D_\beta, \\ \Big(R_2(z)H_1(z)\Phi(z)P_{\overline{\beta}}(\zeta(z))\Phi(z)^{-1}R_2(z)^{-1}\Big)R_1(z)^{-1}, & z\in\partial D_{\overline{\beta}}. \end{cases} \tag{5.39}$$

We first discuss the case $z\to\partial D_\beta$. Note that by (5.26) and (5.22), we have

$$\begin{aligned} H_2(z)\Phi(z)\widehat{P}_\beta(\xi(z))\Phi(z)^{-1} &= H_2(z)\widehat{\mathbf{H}}_\beta(z)\Big(S(z)^{-1}\widehat{P}_\beta(\xi(z))S(z)\Big)\widehat{\mathbf{H}}_\beta(z)^{-1} \\ &= H_2(z)\widehat{\mathbf{H}}_\beta(z)\Big(I+\frac{1}{48}\begin{pmatrix}0 & 5/\xi(z)^2\\ -7/\xi(z) & 0\end{pmatrix}+O\Big(\frac{1}{\xi(z)^3}\Big)\Big)\widehat{\mathbf{H}}_\beta(z)^{-1} \\ &= H_2(z)\Big(I+\frac{1}{48}\widehat{\mathbf{H}}_\beta(z)\begin{pmatrix}0 & 5/\xi(z)^2\\ -7/\xi(z) & 0\end{pmatrix}\widehat{\mathbf{H}}_\beta(z)^{-1}+O\Big(\frac{1}{\xi(z)^3}\Big)\Big). \end{aligned} \tag{5.40}$$

Then it follows from the definition (5.31) of H_2 that

$$H_2(z)\Phi(z)\widehat{P}_\beta(\xi(z))\Phi(z)^{-1} = I+\frac{h_{21}}{z-\beta}+\frac{h_{22}}{(z-\beta)^2}+O\Big(\frac{1}{\xi(z)^3},\frac{1}{N^2}\Big). \tag{5.41}$$

Furthermore, by the definition (5.33) of R_2, we have that as $z\to\partial D_\beta$,

$$R_1(z)\Big(H_2(z)\Phi(z)\widehat{P}_\beta(\xi(z))\Phi(z)^{-1}R_2(z)^{-1}\Big)R_1(z)^{-1} = R_1(z)\Big(I+O(N^{-2})\Big)R_1(z)^{-1} = I+O(N^{-2}). \tag{5.42}$$

Here, we have used the fact that $R_1(z)$ is analytic in D_β.

Next, we discuss the case $z\to\partial D_{\overline{\beta}}$. Note that by (5.26),

$$\begin{aligned} &R_2(z)H_1(z)\Big(\Phi(z)P_{\overline{\beta}}(\zeta(z))\Phi(z)^{-1}\Big)R_2(z)^{-1} \\ &= R_2(z)H_1(z)\mathbf{H}_{\overline{\beta}}(z)\Big(\widehat{S}(z)^{-1}P_{\overline{\beta}}(\zeta(z))\widehat{S}(z)\Big)\mathbf{H}_{\overline{\beta}}(z)^{-1}R_2(z)^{-1} \\ &= R_2(z)H_1(z)\Big(I+\frac{1}{48}\mathbf{H}_{\overline{\beta}}(z)\begin{pmatrix}0 & 5/\zeta(z)^2\\ -7/\zeta(z) & 0\end{pmatrix}\mathbf{H}_{\overline{\beta}}(z)^{-1}+O\Big(\frac{1}{\zeta(z)^3}\Big)\Big)R_2(z)^{-1}. \end{aligned} \tag{5.43}$$

Furthermore, by (5.30), we have

$$\begin{aligned} &R_2(z)H_1(z)\Big(\Phi(z)P_{\overline{\beta}}(\zeta(z))\Phi(z)^{-1}\Big)R_2(z)^{-1} \\ &= R_2(z)\Big(I+\frac{h_{11}}{z-\overline{\beta}}+\frac{h_{12}}{(z-\overline{\beta})^2}+O\Big(\frac{1}{\zeta(z)^3}\Big)\Big)R_2(z)^{-1} \\ &= \Big(I+\frac{1}{z-\overline{\beta}}R_2(z)h_{11}R_2(z)^{-1}+\frac{1}{(z-\overline{\beta})^2}R_2(z)h_{12}R_2(z)^{-1}+O\Big(\frac{1}{\zeta(z)^3}\Big)\Big). \end{aligned} \tag{5.44}$$

Then by the definition (5.34) of R_1, one can see that as $z\to D_{\overline{\beta}}$,

$$\Big(R_2(z)H_1(z)\Phi(z)P_{\overline{\beta}}(\zeta(z))\Phi(z)^{-1}R_2(z)^{-1}\Big)R_1(z)^{-1} = I+O(N^{-2}). \tag{5.45}$$

Combining (5.42) and (5.45), we conclude (5.37). As a consequence of (5.37), for $z\in\mathbb{C}\setminus(\Omega_+\cup\Omega_-)$, the explicit transform (5.1) together with the definition (5.36) gives rise to

$$\begin{aligned} Y(z) &= e^{N\ell/2\sigma_3}A(z)e^{tN(g(z)-\ell/2)\sigma_3} = e^{N\ell/2\sigma_3}\Big(I+O(N^{-2})\Big)A^\infty(z)e^{N(g(z)-\ell/2)\sigma_3} \\ &= e^{N\ell/2\sigma_3}\Big(I+O(N^{-2})\Big)R_1(z)R_2(z)\Phi(z)e^{N(g(z)-\ell/2)\sigma_3}. \end{aligned} \tag{5.46}$$

Then by (5.5) and (5.35), this in turn implies that

$$Y(z) = e^{N\ell/2\sigma_3}\Big(I + O(N^{-2})\Big)\begin{pmatrix} 1+R_{11}(z) & R_{12}(z) \\ R_{21}(z) & 1+R_{22}(z)\end{pmatrix}\begin{pmatrix} \sqrt{RF'(z)} & \dfrac{\sqrt{\kappa F'(z)}}{F(z)-q} \\ -\dfrac{\sqrt{\kappa F'(z)}}{F(z)-q} & \sqrt{RF'(z)}\end{pmatrix} e^{N(g(z)-\ell/2)\sigma_3}. \tag{5.47}$$

In particular, since $p_n(z) = [Y(z)]_{11}$, we obtain the desired asymptotic behaviour (3.43). $\square$

5.2. **Post-critical case.** In this subsection, we now discuss the post-critical case, which is much simpler compared to the pre-critical case.

Recall that for the post-critical case, the global parametrix Φ is given by (5.4). Similar to the pre-critical case, we also apply the partial Schlesinger transform to improve the local parametrix. Namely, we derive a rational matrix functions $R(z)$ with the only pole at β, and a holomorphic matrix function $H(z)$ such that the modified global parametrices

$$\Phi(z)R(z), \qquad \Phi(z)\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} R(z)\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \tag{5.48}$$

match with the local parametrix $\Phi(z)P(\zeta(z))$ along ∂D_β. The construction of such $H(z)$ and $R(z)$ with a simple pole at β was described in [16, Section 5].

For the post-critical case, we define the local coordinate ζ inside D_β,

$$\zeta(z)^2 = 2N\begin{cases} \phi(z), & z\in \mathrm{ext}(\mathcal{B}), \\ -\phi(z), & z\in \mathrm{int}(\mathcal{B}). \end{cases} \tag{5.49}$$

Here ϕ is defined in Definition 4.3. Here the sign is $+$ for $z\in\mathrm{ext}(\mathcal{B})$ and $-$ for $z\in\mathrm{int}(\mathcal{B})$. Note that ζ maps $\mathcal{B}$ into rays $\gamma_+\cup\gamma_-$, where $\gamma_+ := [0, e^{\pi i/4\infty})$ and $\gamma_- := [0, e^{-\pi i/4\infty})$, maps Γ_+ into imaginary axis $i\mathbb{R}$.

It can be observed from the definition that as $z\to\beta$, we have the expansion

$$\frac{\zeta(z)}{\sqrt{N}} = \frac{1}{\gamma_1}(z-\beta)(1+O(z-\beta)), \qquad \gamma_1 = \sqrt{\frac{\beta(\beta-a)}{a(b-\beta)}}, \tag{5.50}$$

where β and b are given in (4.6). Inside D_β, we define the following Riemann-Hilbert problem

$$\begin{cases} [P(\zeta)]_+ = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}[P(\zeta)]_-\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, & \zeta\in\gamma_+\cup\gamma_-, \\ [P(\zeta)]_+ = [P(\zeta)]_-\begin{pmatrix} 1 & 0 \\ e^{\zeta^2} & 1\end{pmatrix}, & \zeta\in i\mathbb{R}, \\ P(\zeta) = I + O(\zeta^{-1}), & \zeta\to\infty. \end{cases} \tag{5.51}$$

Then $\Phi(z)P(\zeta(z))$ satisfies the same jump conditions of $A(z)$. Furthermore, the solution to this Riemann-Hilbert problem is given by

$$P(\zeta(z)) = \begin{cases} H(z)F(\zeta(z)), & z\in\mathrm{ext}(\mathcal{B}), \\ \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} H(z)F(\zeta(z))\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, & z\in\mathrm{int}(\mathcal{B}), \end{cases} \tag{5.52}$$

where

$$F(\zeta(z)) := \begin{pmatrix} 1 & -\dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty}\frac{e^{s^2/2}}{s-\zeta(z)}ds \\ 0 & 1\end{pmatrix} \tag{5.53}$$

and

$$H(z) := R(z)\Big(I - \frac{1}{\sqrt{2\pi}\zeta(z)}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\Big), \qquad R(z) := I + \frac{\gamma_1}{\sqrt{2\pi N}}\frac{1}{z-\beta}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}. \tag{5.54}$$

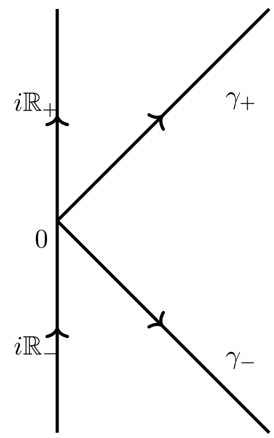


FIGURE 7. The jump contours of $P(\zeta)$ in D_β.

Definition 5.10 (Strong asymptotics for the post-critical case). We define the strong asymptotics of $A(z)$ by

$$A^\infty(z) = \begin{cases} \Phi(z)R(z), & z \in \operatorname{ext}(\mathcal{B}) \setminus D_\beta, \\ \Phi(z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} R(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \operatorname{int}(\mathcal{B}) \setminus D_\beta, \\ \Phi(z)P(\zeta(z)), & z \in D_\beta. \end{cases} \tag{5.55}$$

We are now ready to complete the proof of Theorem 3.11.

Proof of Theorem 3.11 for the post-critical case. As before, we define the error function

$$\mathcal{E}(z) = A^\infty(z)A^{-1}(z). \tag{5.56}$$

Note that as $z \to D_\beta$,

$$F(\zeta(z)) = I + \frac{1}{\sqrt{2\pi}\zeta(z)}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + O\Big(\frac{1}{\zeta(z)^3}\Big). \tag{5.57}$$

Therefore, when z is on the boundary of $\partial D_\beta \cap \operatorname{ext}(\mathcal{B})$, we have

$$\begin{aligned} \mathcal{E}_+(z)\mathcal{E}_-(z)^{-1} &= \Phi(z)P(\zeta(z))R(z)^{-1}\Phi(z)^{-1} = H(z)F(\zeta(z))R(z)^{-1} \\ &= R(z)\Big(I - \frac{1}{\sqrt{2\pi}\zeta}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Big)\Big(I + \frac{1}{\sqrt{2\pi}\zeta}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + O\Big(\frac{1}{\zeta^3}\Big)\Big)R(z)^{-1} = I + O(N^{-3/2}). \end{aligned}$$

Similarly, one can check that the same error bound holds for $z \in \partial D_\beta \cap \operatorname{int}(\mathcal{B})$. One can also check that the error bounds are exponentially small for z in other regions. Using the small norm theorem, we obtain

$$A(z) = \Big(I + O(N^{-3/2})\Big)A^\infty(z).$$

Then for $z \in \operatorname{ext}(\mathcal{B}) \setminus D_\beta$, it follows that

$$\begin{aligned} Y(z) &= e^{N\ell/2\sigma_3}A(z)e^{N(g(z)-\ell/2)\sigma_3} = e^{N\ell/2\sigma_3}\Big(I + O(N^{-3/2})\Big)A^\infty(z)e^{N(g(z)-\ell/2)\sigma_3} \\ &= e^{N\ell/2\sigma_3}\Big(I + O(N^{-3/2})\Big)\Phi(z)R(z)e^{N(g(z)-\ell/2)\sigma_3}. \end{aligned}$$

This gives

$$Y(z) = e^{N\ell/2\sigma_3}\Big(I + O(N^{-3/2})\Big)\begin{pmatrix} 1 & \frac{1}{\sqrt{2\pi N}}\frac{\gamma_1}{z-\beta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{z^{Nc+N}}{(z-a)^{Nc}} & 0 \\ 0 & \frac{(z-a)^{Nc}}{z^{Nc+N}} \end{pmatrix} e^{-N\ell/2\sigma_3}, \tag{5.58}$$

which in particular yields

$$p_N(z) = [Y(z)]_{11} = z^N \Big(\frac{z}{z-a}\Big)^{cN} \Big(1 + O(N^{-3/2})\Big).$$

Note that the above partial Schlesinger transform only updates the $(1,2)$-entry in $Y(z)$, but not the $(1,1)$-entry. Thus, by continuing to use the partial Schlesinger transform, we can improve the error bound so that it becomes $O(N^{-m})$ for any $m > 0$. Therefore, we conclude (3.42). □

6. Duality and critical case

In this section, we establish the free energy expansion for the critical case. Let us first discuss the duality formula (1.9).

Proof of Proposition 1.1. We consider the LUE of size $(cN) \times (cN)$ with joint probability distribution proportional to

$$\prod_{j>k=1}^{cN} |\widetilde{\lambda}_j - \widetilde{\lambda}_k|^2 \prod_{j=1}^{cN} \widetilde{\lambda}_j^N e^{-N\widetilde{\lambda}_j}, \qquad (\widetilde{\lambda}_{cN} > \cdots > \widetilde{\lambda}_1 > 0). \tag{6.1}$$

Then by [55, Proposition 3.1] with $k = cN$,

$$\mathbb{P}\Big[\widetilde{\lambda}_1 > x^2\Big] = \mathbb{E}\Big|\det(\mathbf{G}_N - x)\Big|^{2cN} N^{cN^2} e^{-cN^2x^2} \frac{G(cN+1)G(N+1)}{G(cN+N+1)}. \tag{6.2}$$

Furthermore by (3.5) and (1.8), it can be rewritten as

$$\mathbb{P}\Big[\widetilde{\lambda}_1 > x^2\Big] = \mathbb{E}\Big|\det(\mathbf{G}_N - x)\Big|^{2cN} e^{-cN^2x^2} \frac{Z_N^{\mathrm{Gin}}}{Z_N(0,c)} = e^{-cN^2x^2} \frac{Z_N(x,c)}{Z_N(0,c)}. \tag{6.3}$$

We make a change of variable $\widetilde{\lambda}_j = c\,\widehat{\lambda}_j$. Then $\widehat{\lambda}_j$ follows the distribution proportional to

$$\prod_{j>k=1}^{cN} |\widehat{\lambda}_j - \widehat{\lambda}_k|^2 \prod_{j=1}^{cN} \widehat{\lambda}_j^N e^{-cN\widehat{\lambda}_j}, \qquad (\widehat{\lambda}_{cN} > \cdots > \widehat{\lambda}_1 > 0). \tag{6.4}$$

If we relabelling $cN \mapsto N$, this follows the LUE (1.6) with $\alpha = 1/c$. Then the duality formula (1.9) follows from

$$\mathbb{P}\Big[\lambda_1 > \frac{x^2}{c}\Big] = \mathbb{P}\Big[\widehat{\lambda}_1 > \frac{x^2}{c}\Big]\Big|_{N \to N/c}, \qquad \mathbb{P}\Big[\widehat{\lambda}_1 > \frac{x^2}{c}\Big] = \mathbb{P}\Big[\widetilde{\lambda}_1 > x^2\Big] = e^{-cN^2x^2} \frac{Z_N(x,c)}{Z_N(0,c)}. \tag{6.5}$$

This completes the proof. □

We now prove Proposition 2.5.

Proof of Proposition 2.5. Note that by (1.20),

$$\frac{a^2}{c} = \lambda_- - \frac{(\sqrt{c+1}-\sqrt{c})^{4/3}}{c^{7/6}(c+1)^{1/6}}\, s\, N^{-2/3} + O(N^{-4/3}).$$

Notice here that the Marchenko-Pastur law (1.7) satisfies the behaviour

$$\frac{1}{2\pi} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{x} \sim \frac{\delta}{\pi}\sqrt{x - \lambda_-}, \qquad x \to \lambda_-, \tag{6.6}$$

where

$$\delta = \frac{1}{2}\frac{\sqrt{\lambda_+ - \lambda_-}}{\lambda_-} = \frac{(\alpha+1)^{1/4}}{(\sqrt{\alpha+1}-1)^2} = \frac{c^{3/4}(c+1)^{1/4}}{(\sqrt{c+1}-\sqrt{c})^2}. \tag{6.7}$$

Then by the edge universality of the Hermitian unitary ensembles [51], it follows that

$$\mathbb{P}\Big[(\lambda_1 - \lambda_-)(\delta N)^{2/3} > -y\Big] \to F_{\mathrm{TW}}(y). \tag{6.8}$$

(In our present case, this also follows from the classical Plancherel-Rotach asymptotic formula for the Laguerre polynomials.) To be more precise, we have

$$\begin{aligned}\mathbb{P}\Big[\lambda_1 > \frac{a^2}{c}\Big] &= \mathbb{P}\Big[\lambda_1 > \lambda_- - \frac{(\sqrt{c+1}-\sqrt{c})^{4/3}}{c^{7/6}(c+1)^{1/6}}\, s\, N^{-2/3} + O(N^{-4/3})\Big] \\ &= \mathbb{P}\Big[(\lambda_1-\lambda_-)(\delta N)^{2/3} > -\delta^{2/3}\frac{(\sqrt{c+1}-\sqrt{c})^{4/3}}{c^{7/6}(c+1)^{1/6}} s\Big] + O(N^{-2/3}) \\ &= \mathbb{P}\Big[(\lambda_1-\lambda_-)(\delta N)^{2/3} > -\frac{s}{c^{2/3}}\Big] + O(N^{-2/3}).\end{aligned}$$

Therefore we obtain

$$\mathbb{P}\Big[\lambda_1 > \frac{a^2}{c}\Big] = F_{\mathrm{TW}}(c^{-2/3}s) + O(N^{-2/3}). \tag{6.9}$$

Then the proposition follows from Proposition 1.1 and (3.6). ☐

While the proof of Proposition 2.5 follows easily from the duality relation as well as the well-established Hermitian random matrix theory, our overall strategy presented in Subsections 3.1, 3.2 and 3.3 also works for the critical case. For this case, again by using the partial Schlesinger transform, one can obtain the fine asymptotic behaviours of the orthogonal polynomial, which can be written in terms of the solution to the Painlevé II equation

$$\mathfrak{q}''(s) = s\mathfrak{q}(s) + 2\mathfrak{q}(s)^3 \tag{6.10}$$

with the asymptotic behaviour

$$\mathfrak{q}(s) = \begin{cases} \mathrm{Ai}(s) + O\Big(\dfrac{e^{-\frac{4}{3}s^{3/2}}}{s^{1/4}}\Big), & s \to +\infty, \\ \sqrt{-\frac{1}{2}s}\Big(1 + O\big(s^{-3}\big)\Big), & s \to -\infty. \end{cases} \tag{6.11}$$

An additional difficulty arises from the fact that, regardless of the deformation (3.3) or (3.4) we employ, in the critical case, we also need to address either the post-critical or pre-critical regimes. Nonetheless, our approach leads to the free energy expansion expressed in terms of the Painlevé II solution, which should also coincide with the Tracy-Widom distribution. This would lead to an alternative representation of the Tracy-Widom distribution, see also [19] for a recently found another representation.

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