ZERO DISPERSION LIMIT OF THE CALOGERO–MOSER DERIVATIVE NLS EQUATION

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Abstract. We study the zero–dispersion limit of the Calogero–Moser derivative NLS equation

$$i\partial_t u + \partial_x^2 u \pm 2D\Pi(|u|^2)u = 0\,, \qquad x \in \mathbb{R}\,,$$

starting from an initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where $D = -i\partial_x$, and Π is the Szegő projector defined as $\widehat{\Pi u}(\xi) = \mathbb{1}_{[0,+\infty)}(\xi)\widehat{u}(\xi)$. We characterize the zero–dispersion limit solution by an explicit formula. Moreover, we identify it, in terms of the branches of the multivalued solution of the inviscid Burgers–Hopf equation. Finally, we infer that it satisfies a maximum principle.

Contents

1. Introduction

We consider a nonlocal nonlinear Schrödinger equation called *the Calogero–Moser derivative nonlinear Schrödinger equation*

$$i\partial_t u + \partial_x^2 u \pm 2D\Pi(|u|^2)u = 0\,, \qquad x \in \mathbb{R}\,, \tag{CM}$$

where $D = -i\partial_x$, and $\Pi \equiv \Pi_+$ is the Szegő projector

$$\Pi u(x) = \int_{\mathbb{R}} \frac{u(y)}{y-x}\frac{\mathrm{d}y}{2\pi i}\,,$$

which is an orthogonal projector from $L^2(\mathbb{R})$ into the Hardy space

$$\begin{aligned} L^2_+(\mathbb{R}) &:= \{u \in L^2(\mathbb{R})\,, \operatorname{supp}\widehat{u} \subseteq [0,+\infty[\,\} \\ &\cong \{u \in \mathrm{Hol}(\mathbb{C}_+)\,, \sup_{y>0}\int_{\mathbb{R}} |u(x+iy)|^2 dx < +\infty\}\,, \end{aligned}$$

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with $\mathbb{C}_+ := \{z \in \mathbb{C}\,;\, \mathrm{Im}(z) > 0\}$. Typically, in Fourier transform, Π can be read as

$$\widehat{\Pi u}(\xi) = \mathbb{1}_{[0,+\infty)}(\xi)\,\widehat{u}(\xi)\,. \tag{1.1}$$

This equation comes in two versions: one with the "+" sign in front of the nonlinearity, referring to the focusing equation, and the other with the "−" sign, referring to the defocusing equation. Through this paper, the $\pm$ and $\mp$ symbols will be interchanged based on the following rule: the upper sign will correspond to the focusing case and the lower sign to the defocusing case.

It is known since the work of [GL22] that the focusing (CM)–equation is globally well–posedness in $H^k_+(\mathbb{R}) := H^k(\mathbb{R}) \cap L^2_+(\mathbb{R})$, $k \in \mathbb{N}_{\geq 1}$ for small initial data ($\|u_0\|_{L^2} < \sqrt{2\pi}$). This was achieved by establishing a uniform H^k–bound of the solution $u(t)$ over time. The same line of arguments enable the global well–posedness of the defocusing equation in $H^k_+(\mathbb{R})$, $k \in \mathbb{N}_{\geq 1}$ for any initial data u_0. Subsequently, the case on the torus ($x \in \mathbb{T}$) has been investigated by the author under the name of *Calogero–Sutherland DNLS equation* [Ba23a, Ba23b], where the GWP has been established in $H^s_+(\mathbb{T})$, $s \geq 0$ in the focusing and defocusing cases, with small initial data in the focusing case. Later, [KLV23] extended the GWP results on $\mathbb{R}$, from the high regularity spaces [GL22] up to the scaling–critical space $L^2_+(\mathbb{R})$. More recently, [HK24] established the blow–up of the H^s–norm's solution, in a time $T \in (0, +\infty]$ for initial data $u_0 \in H^\infty_+(\mathbb{R})$ satisfying $\|u_0\|^2_{L^2} = 2\pi + \varepsilon$, for any $\varepsilon > 0$.

From a physical standpoint, the scenarios described by the Calogero–Moser DNLS equation share notable similarities with the **Benjamin–Ono equation**. In both cases, they characterize weakly nonlinear dispersive internal waves located at the interface between two fluid layers of different densities, with the lower layer having infinite depth [BLS08, Sa19, Pe95]. In the context of the Benjamin–Ono equation, the solution delineates the progression of these internal waves. On the other hand, concerning the (CM)–equation, it illustrates a model for the envelope of approximately monochromatic waves within the aforementioned settings.

Recently, Gérard [Ge23] studied the Benjamin–Ono equation with small dispersion $\varepsilon > 0$, described as

$$\begin{cases} \partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2) = \varepsilon |D| \partial_x u^\varepsilon \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \tag{BO-eps}$$

where he established that the weak limit in L^2 of u^ε, as ε approaches 0, is characterized in terms of the branches of the multivalued solution of the **inviscid Burgers–Hopf equation**. Novelty, this characterization holds for any $u_0 \in L^2(\mathbb{R})$, with u_0 is a C^1 function tending to 0 at infinity as well as its first derivative [Ge23]. Observe that one can discern the emergence of the Burgers equation by formally taking the limit in (BO-eps) as $\varepsilon \to 0$. The act of neglecting the dispersion component in the equation is commonly acknowledged in the literature as the "zero-dispersion limit" or "semiclassical limit". In the following, we will use the terminology of 'zero-dispersion limit" . Additionally, we will refer to the weak L^2–limit of u^ε when $\varepsilon \to 0$, as "the weak zero–dispersion limit solution". It's important to note that the selection of initial data, represented by u_0, remains independent of ε.

In this paper, we propose to investigate the zero–dispersion limit problem for the Calogero–Moser DNLS equation. Thus, we consider the rescaled version of (CM) with small dispersion $\varepsilon > 0$,

$$\begin{cases} i\partial_t u^\varepsilon + \varepsilon\, \partial_x^2 u^\varepsilon \pm 2D\Pi(|u^\varepsilon|^2)u^\varepsilon = 0 \\ u^\varepsilon|_{t=0} = u_0 \end{cases} . \tag{CM-eps}$$

The aim is to write the weak limit in L^2 of the solution u^ε of (CM-eps), as $\varepsilon \to 0$, in terms of the branches of the multivalued solution of the Burgers equation. However, here, it is less evident compared to the Benjamin–Ono case, why the Burgers equation emerges in this context. For this purpose, observe that when formally taking $\varepsilon \to 0$, the (CM-eps) becomes

$$i\partial_t u \pm 2D\Pi(|u|^2)u = 0\,. \tag{CM–zero}$$

Consequently, if u solves the previous equation, then $\boldsymbol{v} = |u|^2$ solves the Burgers equation

$$\partial_t \boldsymbol{v} = \pm 2\boldsymbol{v}\, \partial_x \boldsymbol{v} \tag{1.2}$$

as

$$\begin{aligned} \partial_t \boldsymbol{v} =& 2\,\mathrm{Re}(\partial_t u \bar{u}) \\ =& \pm 4\,\mathrm{Re}(\partial_x \Pi(|u|^2)|u|^2) \\ =& \pm 2\left(\partial_x \Pi(|u|^2) + \overline{\partial_x \Pi(|u|^2)}\right)|u|^2 \\ =& \pm \partial_x |u|^4 \ = \pm\partial_x \boldsymbol{v}^2 \\ =& \pm 2\boldsymbol{v}\, \partial_x \boldsymbol{v}\,. \end{aligned}$$

But before proceeding, it is essential to prove the existence of a weak zero dispersion limit for (CM). The upcoming theorem seeks to establish this existence and even to characterize this L^2–weak limit explicitly as an element of the Hardy space. The notation $ZD_+[u_0]$ represents the weak zero-dispersion limit solution in the focusing case for (CM), and $ZD_-[u_0]$ corresponds to the one in the defocusing case.

Theorem 1.1. *Given an initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$* [1] *in the focusing case), the weak (in L^2–space) zero-dispersion limit solution $ZD_\pm[u_0]$ of* (CM) *exists, and is characterized via the following explicit formula*

$$ZD_\pm[u_0](t,z) = \left(\mathrm{Id} \mp 2tT_{u_0}T_{\bar{u}_0}(X^* - z)^{-1}\right)^{-1} u_0(z)\,, \qquad t \in \mathbb{R}\,,\, z \in \mathbb{C}_+\,, \tag{1.3}$$

[1] The constant $\sqrt{2\pi}$ is to ensure the GWP of (CM) in the focusing case.

where the operators T_v and X^* are defined respectively at (2.2) *and* (2.3)*. In addition, we have*

$$\|ZD_{\pm}[u_0](t)\|_{L^2} \leq \|u_0\|_{L^2} .$$

Furthermore, if $u_0^n \to u_0$ strongly in L^2 as $n \to \infty$, with $\sup_n \|u_0^n\|_{L^\infty} < +\infty$, then for all $T > 0$,

$$\sup_{t\in[-T,T]} |ZD_{\pm}[u_0^n](t) - ZD_{\pm}[u_0](t)| \underset{n\to\infty}{\rightharpoonup} 0 \text{ in } L^2(\mathbb{R}) .$$

Usually, when considering the scenario of zero-dispersion limit, the emergence of shocks can be observed. These shocks manifest as we begin to neglect dispersive effects, allowing the nonlinear term to dominate. With the existence of the weak zero-dispersion limit established in the previous theorem, our objective in the following theorem is to highlight these shocks, by addressing the connection between this zero-dispersion limit solution of (CM) and the branches of the multivalued solution of the inviscid Burgers equation, which is known for its tendency to exhibit shock formations.

Theorem 1.2. *Let $u_0 \in L^2_+(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), such that u_0 is a C^1 function tending to 0 at infinity, with a bounded derivative in $L^\infty(\mathbb{R})$.*[2] *Then, for every time $t \in \mathbb{R}$, and for almost every $x \in \mathbb{R}$, the algebraic equation*

$$y \mp 2t|u_0(y)|^2 = x \tag{1.4}$$

has an odd number of simple real solutions $y_0 := y_0(t,x) < \ldots < y_{2\ell} := y_{2\ell}(t,x)$, and the zero–dispersion limit of (CM) *is given by*

$$ZD_{\pm}[u_0](t,x) = \mathrm{e}^{i\varphi(t,x)} \left(\mp i\, \frac{|t|}{t}\right)^{\ell} \prod_{k=0}^{2\ell} |u_0(y_k)|^{(-1)^k}, \tag{1.5}$$

where

$$\varphi(t,x) = \arg(u_0(x)) + \frac{1}{2\pi}\int_0^{+\infty} \frac{1}{s} \log\left(\frac{s \mp 2t|u_0(x+s)|^2}{-s \mp 2t|u_0(x-s)|^2} \frac{\prod_{k=0}^{2\ell}(x-s-y_k)}{\prod_{k=0}^{2\ell}(x+s-y_k)} \right) \mathrm{d}s\,.$$

Remark 1.1. When $\ell > 0$, then any solution $y_k := y_k(t,x)$, $k \in \{0, \ldots, 2\ell\}$ satisfying the algebraic equation (1.4), represents a branch of the multivalued solution of the Burgers equation (1.2) at a time t beyond the shock time, and at a position x.

Remark 1.2. Observe that if we start from a rational initial data u_0, then in view of identity (1.5), the weak zero–dispersion limit is also a rational function. This outcome does not seem to be evident by solely examining the identity (1.3) obtained in Theorem 1.1.

[2] Note that any function in $H^s_+(\mathbb{R}) := H^s(\mathbb{R}) \cap L^2_+(\mathbb{R})$, $s > \frac{3}{2}$ satisfies these conditions.

Remark 1.3. By taking the modulus of (1.5), we deduce

$$\log|ZD_{\pm}[u_0](t,x)| = \sum_{k=0}^{2\ell}(-1)^k \log|u_0(y_k)|\,.$$

This result should be compared to the one obtained by [Ge23] for the (BO)–equation, where he found : for all $t\in\mathbb{R}$, for almost every $x\in\mathbb{R}$, and under the same condition of smoothness on the initial data of Theorem 1.2, the zero–dispersion limit of (BO) is given as

$$ZD_{(\mathrm{BO})}[u_0](t,x) = \sum_{k=0}^{2\ell}(-1)^k u_0(y_k^{BO})\,, \tag{1.6}$$

where the $(y_k^{BO})_{0,\dots 2\ell}$, $\ell=\ell(x)\in\mathbb{N}_{\geq 0}$, are real solutions for the algebraic equation

$$y + 2tu_0(y) = x\,.$$

A consequence of the previous Theorem, is the existence of a maximum Principle.

Corollary 1.3 (A Maximum Principle). *Let $u_0\in L^2_+(\mathbb{R})\cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2}<\sqrt{2\pi}$ in the focusing case). For all $t\in\mathbb{R}$,*

$$\|ZD_{\pm}[u_0]\|_{L^\infty} \leq \|u_0\|_{L^\infty}\,.$$

Related works. *Zero dispersion limit of the KdV equation.* The problem of zero dispersion limit was first investigated by Lax and Levermore [LL83] for the KdV equation on the real line

$$\partial_t u - 3\partial_x(u^2) + \varepsilon^2\partial_x^3 u = 0\,, \qquad u^\varepsilon(0,x) = u_0(x)\,, \tag{KdV}$$

describing, thus, the weak zero dispersion limit for nonpositive initial data decaying sufficiently fast at infinity. In contrast with the Benjamin–Ono equation [Ge23] and the Calogero–Moser DNLS equation, the zero dispersion limit for the KdV equation is expressed implicitly, as it is characterized by a quadratic minimum problem with constraints. Lax–Levermore's work initiated a series of papers. One can cite [Ve87, Ve91, GK07, CG09, CG10a, CG10b], where in all these works the inverse scattering theory, the spectral theory of the Lax operator and the associated Riemann–Hilbert problem are the main keys.

Zero dispersion limit of the Benjamin–Ono equation. We have previously referenced the research by [Ge23], which characterized the zero-dispersion limit of the Benjamin-Ono equation as an alternative sum (1.6). However, this formula traces back to the work of [MX11, MW16] and [Ga23a, Ga23b] who had already derived this sum (1.6) for specific examples of initial data and by using scattering theory or the spectral theory.

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2. The explicit formula of the zero dispersion limit of (CM)

This section aims to establish the existence of a weak (in L^2) zero–dispersion limit solution to the Calogero–Moser DNLS equation (CM). Additionally, it seeks to properly characterize this weak limit for all time t, through an explicit formula. However, before proceeding, it is necessary to revisit some properties regarding the (CM) equation.

It is a completely integrable PDE in the following two senses : First, it possesses a Lax Pair structure (L_u, B_u) that satisfies the Lax equation

$$\frac{dL_u}{dt} = [B_u\,, L_u]\,, \qquad [B_u\,, L_u] := B_uL_u - L_uB_u$$

for enough regular u satisfying the (CM)–equation [GL22]. The Lax operators for (CM) are given by

$$L_u = D \mp T_uT_{\bar{u}}\,, \qquad B_u = \pm T_uT_{\partial_x\bar{u}} \mp T_{\partial_x u}T_{\bar{u}} + i(T_uT_{\bar{u}})^2\,, \tag{2.1}$$

where T_v is the Toeplitz operator of symbol v defined as

$$T_vf = \Pi(vf)\,, \qquad f \in L^2_+(\mathbb{R})\,, \tag{2.2}$$

and Π denotes the Szegő projector (1.1). Second, the complete integrability manifests through the finding of an explicit formula of the solution of the (CM)–equation [KLV23]. To introduce this formula, specific notation needs to be presented. Thus, we consider on $L^2_+(\mathbb{R})$, the contraction semigroup

$$S(\eta)h(x) = \Pi(\mathrm{e}^{ix\eta}\,h(x))\,, \qquad \eta > 0\,.$$

And we denote by X its infinitesimal generator

$$Xh(x) \;=\; -i\frac{d}{d\eta}\Big|_{\eta=0}(S(\eta)h(x)) \;=\; xh(x)\,,$$

of domain

$$\begin{aligned}\mathrm{Dom}(X) &= \{h \in L^2_+(\mathbb{R})\,;\, xh \in L^2(\mathbb{R})\}\\ &= \{h \in L^2_+(\mathbb{R})\,;\, \widehat{h} \in H^1\big([0,+\infty)\big)\,,\, \widehat{h}(0) = 0\}\,.\end{aligned}$$

Its adjoint X^* has the following domain

$$\begin{aligned}\mathrm{Dom}(X^*) &= \{f \in L^2_+(\mathbb{R})\,;\, \exists\, c > 0\,,\, \forall h \in \mathrm{Dom}(X)\,,\, |\langle f\,|\,Xh\rangle| \le c\,\|h\|_{L^2}\}\\ &= \{f \in L^2_+(\mathbb{R})\,;\, \widehat{f}\,|_{(0,+\infty)} \in H^1\big((0,+\infty)\big)\}\,,\end{aligned}$$

and is defined for all $\xi > 0$ as

$$\widehat{X^*f}(\xi) = i\partial_\xi\widehat{f}(\xi)\,.$$

That is, for all $f \in \mathrm{Dom}(X^*)$,

$$X^*f(x) = xf + \frac{1}{2\pi i}\widehat{f}(0^+)\,. \tag{2.3}$$

The following theorem aims to recall the explicit formula of (CM) defined for any $u_0 \in L^2_+(\mathbb{R})$ [KLV23], and to extend the global well–posedness result in $H^k_+(\mathbb{R}) := H^k(\mathbb{R}) \cap L^2_+(\mathbb{R})$, $k \in \mathbb{N}_{\geq 1}$ obtained by [GL22], to $L^2_+(\mathbb{R})$ [KLV23].

Theorem 2.1 ([KLV23])**.** *Let $u_0 \in L^2_+(\mathbb{R})$ (such that $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case). Then there exists a unique global solution $u \in \mathcal{C}_t L^2_+(\mathbb{R})$ such that for any $(u_n^0) \subseteq H^\infty_+(\mathbb{R})$, $(xu_n^0) \subseteq L^2$, $u_n^0 \to u_0$ in L^2, we have for all $T > 0$,*

$$u_n \to u \qquad \text{in } \ \mathcal{C}_t L^2_+([-T,T], \mathbb{R}).$$

Additionally, for all $z \in \mathbb{C}_+ := \{z \in \mathbb{C}, \operatorname{Im}(z) > 0\}$,

$$u(t,z) = \frac{1}{2\pi i} I_+((X^* + 2tL_{u_0} - z)^{-1} u_0), \tag{2.4}$$

where I_+ denotes

$$I_+(f) := \widehat{f}(0^+), \qquad \forall f \in \operatorname{Dom}(X^*). \tag{2.5}$$

As a consequence, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$.

Remark 2.1. The explicit formula on $L^2_+(\mathbb{T})$ was earlier derived in [Ba23a][Proposition 3.4]. This is not the first instance of discovering an explicit formula for a completely integrable PDE; for previous examples, we refer to [GG15, Ge22, GP23].

In what follows, we consider the rescaled version of the (CM)–equation with initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$: For all $\varepsilon > 0$,

$$\begin{cases} i\partial_t u^\varepsilon + \varepsilon\, \partial_x^2 u^\varepsilon \pm 2D\Pi(|u^\varepsilon|^2)u^\varepsilon = 0, \\ u^\varepsilon|_{t=0} = u_0 . \end{cases} \tag{CM–ε}$$

Our primary focus is on establishing the existence of a weak zero dispersion limit for (CM); that is, determining whether the (CM–ε) equation has a weak limit in L^2 as ε tends to 0. The following theorem addresses this question. We recall that the considered initial data u_0 is independent of the parameter ε, and that $ZD_+[u_0]$ represents the weak zero-dispersion limit solution in the focusing case, and $ZD_-[u_0]$ corresponds to the one in the defocusing case.

Theorem. 1.1*. Given an initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$, (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), the weak (in L^2–space) zero-dispersion limit solution $ZD_\pm[u_0]$ of (CM) exists and is characterized via the following explicit formula*

$$ZD_\pm[u_0](t,z) = \left(\operatorname{Id} \mp 2tT_{u_0}T_{\bar{u}_0}(X^* - z)^{-1}\right)^{-1} u_0(z), \qquad t \in \mathbb{R},\, z \in \mathbb{C}_+ . \tag{2.6}$$

In addition, we have

$$\|ZD_\pm[u_0](t)\|_{L^2} \leq \|u_0\|_{L^2} . \tag{2.7}$$

Furthermore, if $u_0^n \to u_0$ strongly in L^2 as $n \to \infty$ with $\sup_n \|u_0^n\|_{L^\infty} < +\infty$, then for all $T > 0$,

$$\sup_{t\in[-T,T]} |ZD_\pm[u_0^n](t) - ZD_\pm[u_0](t)| \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}). \tag{2.8}$$

Proof. In view of Theorem 2.1 we have for all $\varepsilon > 0$,

$$\|u^\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2}\,,$$

where $u^\varepsilon(t)$ is the solution of (CM–ε). Hence, by Banach's theorem, we deduce that there exists $ZD_\pm[u_0] \in L^2_+(\mathbb{R})$, such that, up to a sequence, $u^\varepsilon(t) \rightharpoonup ZD[u_0](t)$ in L^2 as $\varepsilon \to 0$, and

$$\|ZD_\pm[u_0](t)\|_{L^2} \leq \liminf_{\varepsilon\to 0} \|u^\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2}\,.$$

To characterize $ZD_\pm[u_0]$, we will use the explicit formula of Theorem 2.1. However, first observe that when u is a solution of (CM) for an initial data u_0, then $\sqrt{\varepsilon}u(\varepsilon t,\cdot) \equiv \sqrt{\varepsilon}\mathcal{S}(\varepsilon t)[u_0]$ is a solution to (CM–ε) for an initial data $\sqrt{\varepsilon}u_0$, where $\mathcal{S}(t)$ denotes the flow of (CM). That is

$$u^\varepsilon(t,z) := \sqrt{\varepsilon}\mathcal{S}(\varepsilon t)\left[\frac{u_0}{\sqrt{\varepsilon}}\right]$$

is a solution to (CM–ε) for an initial data u_0. Therefore, starting from an initial data u_0, one deduces by (2.4) and (2.1) that the solution of (CM–ε) in the focusing and defocusing case is explicitly given, for all $\varepsilon > 0$, by

$$u^\varepsilon(t,z) := \frac{1}{2\pi i} I_+((X^* + 2t\varepsilon D \mp 2tT_{u_0}T_{\overline{u_0}} - z)^{-1}u_0)\,, \qquad z \in \mathbb{C}_+\,. \tag{2.9}$$

The next step is to pass to the limit $\varepsilon \to 0$ in the above formula. For this purpose, we rewrite (2.9) as follows

$$u^\varepsilon(t,z) = \frac{1}{2\pi i}\Big(\mathrm{Id} \mp 2t\mathrm{e}^{-i\varepsilon tD^2}T_{u_0}T_{\bar{u}_0}\mathrm{e}^{i\varepsilon tD^2}(X^*-z)^{-1}\Big)^{-1}u_0\,.$$

Indeed, by using the Fourier transform, for all $\xi > 0$,

$$(\widehat{X^* + 2t\varepsilon D)f}(\xi) = \mathrm{e}^{it\xi^2}\, i\partial_\xi(\mathrm{e}^{-it\xi^2}\,\widehat{f}(\xi))\,,$$

(2.9) becomes

$$\begin{aligned}
u^\varepsilon(t,z) &= \frac{1}{2\pi i} I_+\Big((\mathrm{e}^{i\varepsilon tD^2}\,X^*\,\mathrm{e}^{-i\varepsilon tD^2} \mp 2tT_{u_0}T_{\overline{u_0}} - z)^{-1}u_0\Big)\\
&= \frac{1}{2\pi i} I_+\Big(\,\mathrm{e}^{i\varepsilon tD^2}(X^* \mp 2t\,\mathrm{e}^{-i\varepsilon tD^2}\,T_{u_0}T_{\overline{u_0}}\,\mathrm{e}^{i\varepsilon tD^2} - z)^{-1}\,\mathrm{e}^{-i\varepsilon tD^2}\,u_0\Big)
\end{aligned}$$

Thus, by definition of I_+ in (2.5), we deduce

$$\begin{aligned}
u^\varepsilon(t,z) &= \frac{1}{2\pi i} I_+\Big((X^* \mp 2t\,\mathrm{e}^{-i\varepsilon tD^2}\,T_{u_0}T_{\overline{u_0}}\,\mathrm{e}^{i\varepsilon tD^2} - z)^{-1}\,\mathrm{e}^{-i\varepsilon tD^2}\,u_0\Big)\\
&= \frac{1}{2\pi i} I_+\Big(\big(X^*-z\big)^{-1}\cdot\big(\,\mathrm{Id} \mp 2t\,\mathrm{e}^{-i\varepsilon tD^2}\,T_{u_0}T_{\overline{u_0}}\,\mathrm{e}^{i\varepsilon tD^2}(X^*-z)^{-1}\big)^{-1}\,\mathrm{e}^{-i\varepsilon tD^2}\,u_0\Big)
\end{aligned}$$

Now, using the fact that

$$\begin{aligned}
I_+\left((X^*-z)^{-1}f\right) &= \lim_{\varepsilon\to 0}\left\langle (X^*-z)^{-1}f\,,\,\frac{1}{1-i\varepsilon x}\right\rangle = \lim_{\varepsilon\to 0}\left\langle f\,,\,(X-\bar{z})^{-1}\left(\frac{1}{1-i\varepsilon x}\right)\right\rangle\\
&= \lim_{\varepsilon\to 0}\left\langle f,(x-z)^{-1}\left(\frac{1}{1-i\varepsilon x}\right)\right\rangle = 2\pi i f(z),
\end{aligned} \tag{2.10}$$

we infer,

$$u^\varepsilon(t,z)=\Big(\mathrm{Id}\mp 2t\,\mathrm{e}^{-i\varepsilon tD^2}\,T_{u_0}T_{\overline{u_0}}\,\mathrm{e}^{i\varepsilon tD^2}(X^*-z)^{-1}\Big)^{-1}\mathrm{e}^{-i\varepsilon tD^2}\,u_0(z)\,. \tag{2.11}$$

Observing first that $\|\,\mathrm{e}^{-i\varepsilon tD^2}\,u_0\|_{L^2}=\|u_0\|_{L^2}$, and second, $\mathrm{e}^{-i\varepsilon tD^2}\,T_{u_0}T_{\overline{u_0}}\,\mathrm{e}^{i\varepsilon tD^2}(X^*-z)^{-1}$ is a bounded operator as $u_0\in L^\infty$, we infer by passing to the limit $\varepsilon\to 0$ in (2.11),

$$ZD_\pm[u_0](t,z):=\Big(\mathrm{Id}\mp 2tT_{u_0}T_{\overline{u_0}}(X^*-z)^{-1}\Big)^{-1}u_0(z)\,.$$

□

3. Link with the multivalued solution of the Burgers equation

The aim of this section is to prove Theorem 1.2, which describes the weak zero dispersion limit solution of (CM) starting from an initial data $u_0\in L^2_+(\mathbb{R})\cap\mathcal{C}^1$ tending to 0 at infinity and satisfying $u_0'\in L^\infty(\mathbb{R})$, in terms of the branches of the multivalued solution for the Burgers equation (1.2). However, before proving this theorem for such initial data u_0, we will first focus on proving it for rational initial data in the Hardy space

$$u_0(y)=\frac{P(y)}{Q(y)}\,,\qquad Q(y):=(y+\overline{p}_0)\dots(y+\overline{p}_{N-1})\,,\qquad p_k\neq p_j\,,\ k\neq j\,. \tag{3.1}$$

where $p_k\in\mathbb{C}$, $\mathrm{Im}(p_k)<0$ for all $k=0,\cdots,N-1$, and $P(y)=\sum_{n=0}^{N-1}a_ny^n$, $a_n\in\mathbb{C}$.

Proposition 3.1. *Let u_0 be a rational function defined in (3.1). Then for every time $t\in\mathbb{R}$, and for almost every $x\in\mathbb{R}$, the algebraic equation*

$$y\mp 2t|u_0(y)|^2=x \tag{3.2}$$

has an odd number of simple real solutions $y_0:=y_0(t,x)<\ldots<y_{2\ell}:=y_{2\ell}(t,x)$, and the zero–disperion limit of (CM) is given, for almost every $x\in\mathbb{R}$, by

$$ZD_\pm[u_0](t,x)=\mathrm{e}^{i\varphi(t,x)}\left(\mp i\,\frac{|t|}{t}\right)^\ell\prod_{k=0}^{2\ell}|u_0(y_k)|^{(-1)^k} \tag{3.3}$$

where

$$\varphi(t,x)=\arg(u_0(x))+\frac{1}{2\pi}\int_0^{+\infty}\frac{1}{s}\log\left(\frac{s\mp 2t|u_0(x+s)|^2}{-s\mp 2t|u_0(x-s)|^2}\frac{\prod_{k=0}^{2\ell}(x-s-y_k)}{\prod_{k=0}^{2\ell}(x+s-y_k)}\right)\mathrm{d}s\,.$$

To prove the previous proposition, we split the proof into the following lemmas.

Lemma 3.2. *Let $u_0(y)=\frac{P(y)}{Q(y)}$ be a rational function defined as in (3.1). Then, for all $t\in\mathbb{R}$, $x\in\mathbb{R}\backslash K_t$, where K_t is a finite set in $\mathbb{R}$, the algebraic equation (3.2) admits an odd number of simple real solutions*

$$y_0:=y_0(t,x)<\ldots<y_{2\ell}:=y_{2\ell}(t,x)\,.$$

Furthermore, the function $\gamma_t(y) := y \mp 2t|u_0(y)|^2$ *is increasing near* y_{2k} , $k = 0, \cdots, \ell$, *and decreasing near* y_{2k+1} , $k = 0, \cdots, \ell - 1$.

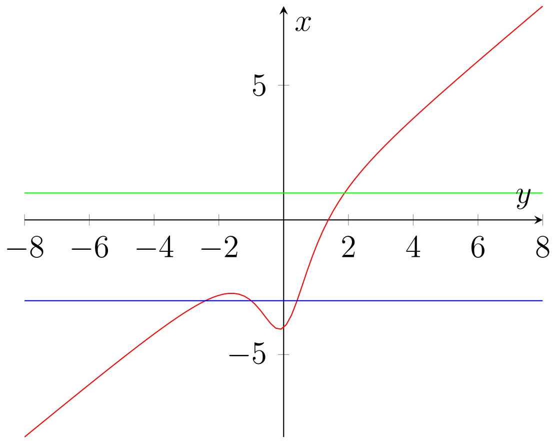

FIGURE 1. For an initial data $u_0(y) := \frac{1}{y+i}$, we have in red the graph of $\gamma_2(y) := y - 2 \cdot 2|u_0(y)|^2 = y - \frac{2 \cdot 2}{y^2+1}$. In green, we have the graph of $x = 1$. The abscissa of the intersection of the axe $x = 1$ with the graph of $\gamma_2(y)$ corresponds to the unique real solution y_0 of $\gamma_2(y) = 1$. In blue, we have the graph of $x = -3$. The abscissas of the intersection of the graph $\gamma_2(y)$ with $x = -3$ correspond to the points $y_0 < y_1 < y_2$ solutions to the algebraic equation $\gamma_2(y) = -3$.

Proof. Given $u_0(y) = \frac{P(y)}{Q(y)}$ as in (3.1) , we introduce [3]

$$y \in \mathbb{C} \longmapsto v_0(y) := \frac{P(y)\overline{P}(y)}{Q(y)\overline{Q}(y)} .$$

Our goal is to study the *real* solutions of the algebraic equation (3.2) . Observe that, y is a real solution of (3.2) , if and only if, y is a real solution of

$$y \mp 2tv_0(y) = x , \qquad t \in \mathbb{R} ,$$

that is, if and only if, y is a real solution of the polynomial equation of degree $2N+1$,

$$(y - x)Q(y)\overline{Q}(y) \mp 2tP(y)\bar{P}(y) = 0 . \tag{3.4}$$

Now, focusing on (3.4), one notices $y \in \mathbb{C}\backslash\mathbb{R}$ is a solution to (3.4) , if and only if, its complex conjugate $\bar{y}$ is a solution to (3.4) . Therefore, the polynomial equation (3.4) of degree $2N + 1$ admits an odd number of real solutions

$$y_0 := y_0(t, x) \le \ldots \le y_{2\ell} := y_{2\ell}(t, x) .$$

Discarding a finite set of critical values[4] x of the function $\gamma_t(y) := y \mp 2t|u_0(y)|^2$ for a given t , one may assume that these real solutions y_k are simple and that the

[3] Observe that when $y \in \mathbb{R}$, $v_0(y) = |u_0(y)|^2$.

[4] We mean by critical values of a function γ_t, the γ_t–image of the critical points of γ_t , i.e. the γ_t–images of the points where $\gamma_t'(y) = 0$.

function γ_t is increasing in a neighborhood of the points y_{2k} , $k = 0, \dots, \ell$, and decreasing in a neighbor of the points y_{2k+1} , $k = 0 , \dots , , \ell - 1$, as $y \longmapsto \gamma_t(y)$ is a continuous function behaving like

$$\gamma_t(y) \underset{y\to\pm\infty}{\sim} y\,.$$

□

Lemma 3.3. *Let* $u_0(y) = \frac{P(y)}{Q(y)}$ *be the rational function and* $(p_k)_{k=0,\cdots,N-1}$ *be the complex constants defined in* (3.1)*. Moreover denote, for almost every* $x \in \mathbb{R}$, $y_0 := y_0(t,x)$, $\cdots$, $y_{2N} := y_{2N}(t,x)$ *the solutions of the equation*

$$y \mp 2t\,\frac{P(y)\overline{P}(y)}{Q(y)\overline{Q}(y)} = x\,, \qquad t \in \mathbb{R}\,. \tag{3.5}$$

Then, the zero–dispersion limit of (CM) *is given by*

$$ZD_{\pm}[u_0](t,x) = \frac{u_0(y_0)u_0(y_2)\dots u_0(y_{2N})\begin{vmatrix} 1 & \frac{1}{y_0+p_0} & \frac{1}{y_0+p_1} & \dots & \frac{1}{y_0+p_{N-1}} \\ 1 & \frac{1}{y_2+p_0} & \frac{1}{y_2+p_1} & \dots & \frac{1}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{1}{y_{2N}+p_0} & \frac{1}{y_{2N}+p_1} & \dots & \frac{1}{y_{2N}+p_{N-1}} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{u_0(y_0)}{y_0+p_0} & \frac{u_0(y_0)}{y_0+p_1} & \dots & \frac{u_0(y_0)}{y_0+p_{N-1}} \\ 1 & \frac{u_0(y_2)}{y_2+p_0} & \frac{u_0(y_2)}{y_2+p_1} & \dots & \frac{u_0(y_2)}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{u_0(y_{2N})}{y_{2N}+p_0} & \frac{u_0(y_{2N})}{y_{2N}+p_1} & \dots & \frac{u_0(y_{2N})}{y_{2N}+p_{N-1}} \end{vmatrix}}\,. \tag{3.6}$$

Proof. The main component is to use the explicit formula (1.3) of $ZD_{\pm}[u_0]$, which can be be reexpressed using (2.10) as

$$ZD_{\pm}[u_0](t,z) = \frac{1}{2\pi i} I_+\big[(X^* \mp 2tT_{u_0}T_{\overline{u_0}} - z)^{-1}u_0\big]\,, \qquad z \in \mathbb{C}_+\,, \tag{3.7}$$

where I_+ is defined in (2.5) . The goal is to transform (3.7) into (3.6) . For that, we decompose u_0 in terms of its partial fractional decomposition $u_0(y) = \sum_{k=0}^{N-1} \frac{c_k}{y+\overline{p_k}}$, $c_k \in \mathbb{C}$, to infer by (2.2) and (2.3) ,

$$\begin{aligned}(X^* \mp 2tT_{u_0}T_{\overline{u_0}} - z)f(y) = (y \mp 2t|u_0(y)|^2 - z)f(y) + \frac{1}{2\pi i}I_+(f)\\ \pm 2tu_0(y)\sum_{k\geq 0}^{N-1}\frac{\overline{c_k}}{y+p_k}f(-p_k)\,.\end{aligned} \tag{3.8}$$

Indeed, for all $f \in L^2_+(\mathbb{R})$,

$$T_{\overline{u_0}}f(y) = \sum_{k=0}^{N-1} \Pi_+\left(\frac{\overline{c_k}}{y+p_k}f(y)\right) = \overline{u_0}(y)f(y) - \sum_{k=0}^{N-1}\frac{\overline{c_k}}{y+p_k}f(-p_k)\,.$$

Thus, for all $f \in L^2_+(\mathbb{R})$,

$$T_{u_0}T_{\overline{u_0}}f(y) = |u_0(y)|^2 f(y) - u_0(y)\sum_{k\geq 0}^{N-1}\frac{\overline{c_k}}{y+p_k}f(-p_k)\,.$$

Now, observe since formula (3.8) is valide for any $f \in L^2_+(\mathbb{R})$, then one can extended it to any holomorphic function f in $\mathbb{C}_+$ whose trace on $\mathbb{R}$ is in $L^2(\mathbb{R})$. That is, if we denote by

$$y \in \mathbb{C} \longmapsto v_0(y) := \frac{P(y)\bar{P}(y)}{Q(y)\overline{Q}(y)}\,,$$

then the following identity holds

$$\begin{aligned}(X^* \mp 2tT_{u_0}T_{\overline{u_0}} - z)f(y) = (y \mp 2tv_0(y) - z)f(y) + \frac{1}{2\pi i}I_+(f)\\ \pm 2tu_0(y)\sum_{k\geq 0}^{N-1}\frac{\overline{c_k}}{y+p_k}f(-p_k)\,,\end{aligned} \tag{3.9}$$

for all $y \in \mathbb{C}_+$, and for all holomorphic function f on $\mathbb{C}_+$ whose trace is in L^2 . In particular, for $f(y) = f_{t,z}(y) := (X \mp 2tT_{u_0}T_{\overline{u_0}} - z)^{-1}u_0(y) \in L^2_+(\mathbb{R})$, we infer by (3.7) ,

$$u_0(y) = (y \mp 2tv_0(y) - z)f_{t,z}(y) + ZD_\pm[u_0](t,z) \pm 2tu_0(y)\sum_{k\geq 0}^{N-1}\frac{\overline{c_k}}{y+p_k}f_{t,z}(-p_k)\,,$$

or

$$f_{t,z}(y) = \frac{u_0(y) - ZD_\pm[u_0](t,z) \mp 2tu_0(y)\displaystyle\sum_{k\geq 0}^{N-1}\frac{\overline{c_k}}{y+p_k}f_{t,z}(-p_k)}{y \mp 2tv_0(y) - z}\,. \tag{3.10}$$

However, recall $y \mapsto f_{t,z}(y)$ is a holomorphic function in the upper–half complex plane. This means that the zeros in $\mathbb{C}_+$ of the denominator of $f_{t,z}$ must cancel its numerator. Therefore, the next step is to find the zeros of the algebraic equation $y \mp 2tv_0(y) = z$ on $\mathbb{C}_+$, with the note that, at the end of the day, $z \in \mathbb{C}_+$ will be replaced by $x \in \mathbb{R}$ almost everywhere, as $ZD_\pm[u_0]$ belongs to the Hardy space $L^2_+(\mathbb{R})$ thanks to (2.7) .

Let $x \in \mathbb{R}$. In view of Lemma 3.2 , the algebraic equation $y \mp 2tv_0(y) = x$ admits an odd number of real solutions [5]

$$y_0 := y_0(t,x) < \ldots < y_{2\ell} := y_{2\ell}(t,x)\,.$$

Moreover, we denote by

$$y_{2\ell+1} := y_{2\ell+1}(t,x)\,,\ \ldots\ ,\ y_{2N} := y_{2N}(t,x)\,,$$

[5] We recall that the real solutions y of the equation $y \mp 2tv_0(y) = x$ are the same real solutions of $y \mp 2t|u_0(y)|^2 = x$ as $v_0(y) = |u_0(y)|^2$ when y is real.

the remaining solutions of $y \mp 2tv_0(y) = x$ belonging to the complex plane, where thanks to (3.4) we notice $y_{2p-1} = \overline{y_{2p}}$ for all $p = \ell+1\,, \cdots, N$; and in what follows, we suppose $\mathrm{Im}(y_{2p}) > 0$ for all $p = \ell+1\,, \cdots, N$.

By moving $x = z$ slightly up to the upper half–complex plane, one proves by using the implicit function theorem in its holomorphic version, that $z \mapsto y_k(t,z)$ is a holomorphic function and thus satisfies the Cauchy–Riemann equations

$$\frac{\partial\,\mathrm{Im}(y_k)}{\partial\,\mathrm{Im}(z)} = \frac{\partial\,\mathrm{Re}(y_k)}{\partial\,\mathrm{Re}(z)}\,.$$

Besides, recall for all $k = 0\,, \dots \ell\,,\ j = 1\,, \dots, \ell\,,$

$$\frac{\partial\,\mathrm{Re}(y_{2k})}{\partial\,\mathrm{Re}(z)} > 0\,, \qquad \frac{\partial\,\mathrm{Re}(y_{2j-1})}{\partial\,\mathrm{Re}(z)} < 0\,,$$

since by Lemma 3.2 the function $\gamma_t(y) := y \mp 2tv_0(y)$ is increasing near y_{2k}, $k = 0\,, \cdots, \ell$, and decreasing near y_{2k+1}, $k = 0\,, \cdots, \ell-1$. As a result, for all $k = 0\,, \dots N\,,\ j = 1\,, \dots, N\,,$

$$\frac{\partial\,\mathrm{Im}(y_{2k})}{\partial\,\mathrm{Im}(z)} > 0\,, \qquad \frac{\partial\,\mathrm{Im}(y_{2j-1})}{\partial\,\mathrm{Im}(z)} < 0\,.$$

That is $(y_{2k})_{k=0,\dots N} \subseteq \mathbb{C}_+$, and thus, at these points, the numerator of (3.10) must vanish. Consequently, one deduces the following linear system of unknowns $ZD[u_0](t,z)$, and $(f_{t,z}(-p_j))_{j=1,\dots,N-1}$,

$$u_0(y_{2k}) = ZD_\pm[u_0](t,z) \pm 2tu_0(y_{2k}) \sum_{j\geq 0}^{N-1} \frac{\overline{c_j}}{y_{2k}+p_j} f_{t,z}(-p_j)\,, \qquad k = 0, \cdots, N\,.$$

Applying the Cramer rule, one finds for all $z \in \mathbb{C}_+$,

$$ZD_\pm[u_0](t,z) = \frac{u_0(y_0)u_0(y_2)\dots u_0(y_{2N}) \begin{vmatrix} 1 & \frac{1}{y_0+p_0} & \frac{1}{y_0+p_1} & \cdots & \frac{1}{y_0+p_{N-1}} \\ 1 & \frac{1}{y_2+p_0} & \frac{1}{y_2+p_1} & \cdots & \frac{1}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{1}{y_{2N}+p_0} & \frac{1}{y_{2N}+p_1} & \cdots & \frac{1}{y_{2N}+p_{N-1}} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{u_0(y_0)}{y_0+p_0} & \frac{u_0(y_0)}{y_0+p_1} & \cdots & \frac{u_0(y_0)}{y_0+p_{N-1}} \\ 1 & \frac{u_0(y_2)}{y_2+p_0} & \frac{u_0(y_2)}{y_2+p_1} & \cdots & \frac{u_0(y_2)}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{u_0(y_{2N})}{y_{2N}+p_0} & \frac{u_0(y_{2N})}{y_{2N}+p_1} & \cdots & \frac{u_0(y_{2N})}{y_{2N}+p_{N-1}} \end{vmatrix}}\,. \tag{3.11}$$

Hence, for almost every $x \in \mathbb{R}$, identity (3.6) holds, as $ZD_\pm[u_0](t)$ belongs to the Hardy space $L^2_+(\mathbb{T})$ for all t, since it is a holomorphic function, exhibiting a finite trace in $L^2(\mathbb{R})$ thanks to inequality (2.7).

□

Lemma 3.4 (Solving the determinants of Lemma 3.3)**.** *Under the same conditions and notations as in Lemma 3.3, the zero–dispersion limit of* (CM) *associated with* $u_0 = \frac{P(y)}{Q(y)}$ *defined in* (3.1)*, is given for almost every* $x \in \mathbb{R}$ *by*

$$ZD_{\pm}[u_0](t,x) \;=\; \frac{P(x)}{\displaystyle\prod_{k=1}^{N}(x - y_{2k-1})}\,. \tag{3.12}$$

Proof. We recall from Lemma 3.3, for almost every $x \in \mathbb{R}$,

$$ZD_{\pm}[u_0](t,x) = \frac{u_0(y_0)u_0(y_2)\dots u_0(y_{2N}) \begin{vmatrix} 1 & \frac{1}{y_0+p_0} & \frac{1}{y_0+p_1} & \dots & \frac{1}{y_0+p_{N-1}} \\ 1 & \frac{1}{y_2+p_0} & \frac{1}{y_2+p_1} & \dots & \frac{1}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{1}{y_{2N}+p_0} & \frac{1}{y_{2N}+p_1} & \dots & \frac{1}{y_{2N}+p_{N-1}} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{u_0(y_0)}{y_0+p_0} & \frac{u_0(y_0)}{y_0+p_1} & \dots & \frac{u_0(y_0)}{y_0+p_{N-1}} \\ 1 & \frac{u_0(y_2)}{y_2+p_0} & \frac{u_0(y_2)}{y_2+p_1} & \dots & \frac{u_0(y_2)}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{u_0(y_{2N})}{y_{2N}+p_0} & \frac{u_0(y_{2N})}{y_{2N}+p_1} & \dots & \frac{u_0(y_{2N})}{y_{2N}+p_{N-1}} \end{vmatrix}}\,.$$

By expanding the determinants, one finds

$$ZD_{\pm}[u_0](t,x) = \prod_{k=0}^{2N} u_0(y_{2k})\, \frac{\displaystyle\sum_{k=0}^{N}(-1)^k\,\Delta_k}{\displaystyle\sum_{k=0}^{N}(-1)^k\,\frac{1}{u_0(y_{2k})}\,\Delta_k}$$

where Δ_k is the minor obtained after removing the first column and the k^{th}–row of the matrix in the numerator of (3.12). Therefore, observing that Δ_k is a Cauchy determinant, one infers that

$$ZD_{\pm}[u_0](t,x) = \frac{\displaystyle\sum_{k=0}^{N}(-1)^k \prod_{j=0}^{N-1}(y_{2k}+p_j) \prod_{\substack{0\le j<j'\le N\\ j,j'\neq k}}(y_{2j'}-y_{2j})}{\displaystyle\sum_{k=0}^{N}(-1)^k \frac{1}{u_0(y_{2k})} \prod_{j=0}^{N-1}(y_{2k}+p_j) \prod_{\substack{0\le s<s'\le N\\ s,s'\neq k}}(y_{2s'}-y_{2s})} =: \frac{R}{D}\,. \tag{3.13}$$

First, by using the Vandermonde determinant, one can rewrite R in (3.13) as

$$R := \sum_{k=0}^{N}(-1)^k \prod_{j=0}^{N-1}(y_{2k}+p_j) \prod_{\substack{0\le j<j'\le N\\ j,j'\neq k}}(y_{2j'}-y_{2j})$$

$$= \begin{vmatrix} \prod_{j=0}^{N-1}(y_0+p_j) & 1 & y_0 & \dots & y_0^{N-1} \\ \prod_{j=0}^{N-1}(y_2+p_j) & 1 & y_2 & \dots & y_2^{N-1} \\ \vdots & & & & \\ \prod_{j=0}^{N-1}(y_{2N}+p_j) & 1 & y_{2N} & \dots & y_{2N}^{N-1} \end{vmatrix}$$

which is equivalent to

$$R = \begin{vmatrix} y_0^N & 1 & y_0 & \dots & y_0^{N-1} \\ y_2^N & 1 & y_2 & \dots & y_2^{N-1} \\ \vdots & & & & \\ y_{N-1}^N & 1 & y_{2N} & \dots & y_{2N}^{N-1} \end{vmatrix}$$

thanks to the property of the determinant. Hence, applying once more the Vandermonde determinant, we conclude that

$$R = (-1)^N \prod_{0\le m<n\le N}(y_{2n}-y_{2m})\,. \tag{3.14}$$

Second, moving to the expression of D in (3.13). Observe that by definition of the polynomial Q in (3.1), we have

$$\frac{1}{u_0(y_{2k})}\prod_{j=0}^{N-1}(y_{2k}+p_j) \equiv \frac{\overline{Q}(y_{2k})}{u_0(y_{2k})}\,. \tag{3.15}$$

In addition, recall that the $(y_{2k})_{k=0,\dots N}$ are solutions of the algebraic equation (3.4). Hence, for all $k=0\,,\cdots,N\,$,

$$\frac{\overline{Q}(y_{2k})}{u_0(y_{2k})} = \mp 2t\frac{\overline{P}(y_{2k})}{x-y_{2k}}\,,$$

which can be rewritten as

$$\frac{\overline{Q}(y_{2k})}{u_0(y_{2k})} = \mp 2t\left(\frac{\overline{P}(y_{2k})-\overline{P}(x)}{x-y_{2k}}+\frac{\overline{P}(x)}{x-y_{2k}}\right)\,,$$

to infer via (3.15)

$$\frac{1}{u_0(y_{2k})}\prod_{j=0}^{N-1}(y_{2k}+p_j) = \mp 2t\left(\frac{\overline{P}(y_{2k})-\overline{P}(x)}{x-y_{2k}}+\frac{\overline{P}(x)}{x-y_{2k}}\right)\,. \tag{3.16}$$

Therefore, by observing that $\frac{\overline{P}(y_{2k})-\overline{P}(x)}{x-y_{2k}}$ is a polynomial in y_{2k} of degree less strictly than $N-1$, one finds by (3.16), that D defined in (3.13) is equal to,

$$D = \mp 2t\,\overline{P}(x) \begin{vmatrix} \dfrac{1}{x-y_0} & 1 & y_0 & \dots & y_0^{N-1} \\ \dfrac{1}{x-y_2} & 1 & y_2 & \dots & y_2^{N-1} \\ \vdots & & & & \\ \dfrac{1}{x-y_{2N}} & 1 & y_{2N} & \dots & y_{2N}^{N-1} \end{vmatrix}.$$

Consequently, by using the Vandermonde determinant,

$$D = \mp 2t\overline{P}(x) \sum_{k=0}^{N} \frac{(-1)^k}{x-y_{2k}} \prod_{\substack{0\le m<n\le N \\ m,n\neq k}} (y_{2n}-y_{2m}),$$

which can be rewritten as

$$D = \mp 2t\,(-1)^N\,\overline{P}(x)\,\frac{\displaystyle\prod_{0\le m<n\le N}(y_{2n}-y_{2m})}{\displaystyle\prod_{k=0}^{N}(x-y_{2k})}, \tag{3.17}$$

thanks to the partial fractional decomposition of

$$\frac{\displaystyle\prod_{0\le m<n\le N}(y_{2n}-y_{2m})}{\displaystyle\prod_{k=0}^{N}(x-y_{2k})} = \sum_{k=0}^{N} \frac{\dfrac{\displaystyle\prod_{0\le m<n\le N}(y_{2n}-y_{2m})}{\displaystyle\prod_{j=0,j\neq k}^{N}(y_{2k}-y_{2j})}}{x-y_{2k}}$$

$$= \sum_{k=0}^{N} \frac{\dfrac{\displaystyle\prod_{0\le m<n\le N}(y_{2n}-y_{2m})}{(-1)^{N-k}\displaystyle\prod_{0\le j<k}(y_{2k}-y_{2j})\prod_{k<j\le N}(y_{2j}-y_{2k})}}{x-y_{2k}}$$

$$= \sum_{k=0}^{N} \frac{(-1)^{N-k}}{x-y_{2k}} \prod_{\substack{0\le m<n\le N \\ m,n\neq k}} (y_{2n}-y_{2m}).$$

Consequently, substituting (3.14), (3.17) in (3.13), one infers that

$$ZD_{\pm}[u_0](t,x) = \frac{\displaystyle\prod_{k=0}^{N}(x-y_{2k})}{\mp 2t\overline{P}(x)}. \tag{3.18}$$

Furthermore, if we take into account that $(y_k)_{k=0,\cdots,2N}$ are solutions to (3.5), thus also to the polynomial equation (3.4), we can write

$$\prod_{k=0}^{2N}(y-y_k)=(y-x)Q(y)\overline{Q}(y)\mp 2tP(y)\overline{P}(y)\,, \tag{3.19}$$

and when $y=x$ in the above equation, we obtain $\displaystyle\prod_{k=0}^{2N}(x-y_k)=\mp 2tP(x)\overline{P}(x)$, which implies that (3.18) can be replaced by

$$ZD_{\pm}[u_0](t,x)=\frac{P(x)}{\prod_{k=1}^{N}(x-y_{2k-1})}\,. \tag{3.20}$$

□

Now, equipped with Lemma 3.4, let us prove Proposition 3.1.

Proof of Proposition 3.1. We recall from Lemma 3.4, that for almost every $x\in\mathbb{R}$,

$$ZD_{\pm}[u_0](t,x)=\frac{P(x)}{\prod_{k=1}^{N}(x-y_{2k-1})}\,, \tag{3.21}$$

where the $(y_k)_{k=0,\cdots,2\ell}$ are the real solutions of the algebraic equation (3.2), and the $(y_p)_{p=2\ell+1,\cdots,2N}$ are the complex solutions of (3.5) with $y_{2p-1}=\overline{y_{2p}}$ for all $p=\ell+1\,,\cdots,N$. We rewrite (3.21) as

$$\begin{aligned} ZD_{\pm}[u_0](t,x)&=u_0(x)\,\frac{Q(x)}{\prod_{k=1}^{N}(x-y_{2k-1})} \\ &=\frac{u_0(x)}{\prod_{k=1}^{\ell}(x-y_{2k-1})}\,\frac{Q(x)}{\prod_{p=\ell+1}^{N}(x-y_{2p-1})}\,. \end{aligned} \tag{3.22}$$

The goal is to get rid of $\frac{Q(x)}{\prod_{k=\ell+1}^{N}(x-y_{2k-1})}$, in order to express $ZD_{\pm}[u_0]$ only in terms of $y_0\,,\cdots,y_{2\ell}$, thereby ensuring that $ZD_{\pm}[u_0]$ can be expressed exclusively in terms of the branches of the multivalued solution of the Burgers equation (1.2).

For that, we recall from (3.19), for all $y\in\mathbb{C}$,

$$\frac{\prod_{k=0}^{2N}(y-y_k)}{Q(y)\overline{Q}(y)}=y-x\mp 2t\frac{P(y)\overline{P}(y)}{Q(y)\overline{Q}(y)}\,.$$

In particular, for all $y\in\mathbb{R}$,

$$\frac{y\mp 2t|u_0(y)|^2-x}{\prod_{k=0}^{2\ell}(y-y_k)}=\frac{\prod_{p=2\ell+1}^{2N}(y-y_p)}{|Q(y)|^2}\,,$$

or, since $y_{2p-1}=\overline{y_{2p}}$ for all $p=\ell+1,\dots,N$, and as $Q(y):=(y+\overline{p}_0)\cdots(y+\overline{p}_{N-1})$ by (3.1),

$$\frac{y\mp 2t|u_0(y)|^2-x}{\displaystyle\prod_{k=0}^{2\ell}(y-y_k)}=\frac{\displaystyle\prod_{p=\ell+1}^{N}|y-y_{2p-1}|^2}{\displaystyle\prod_{j=0}^{N-1}|y-p_j|^2}\,. \tag{3.23}$$

Let $a>0$, the next step is to prove that the term we need to get ride of is equal to

$$\frac{Q(x)}{\displaystyle\prod_{p=\ell+1}^{N}(x-y_{2p-1})}=\frac{(x+ia)^\ell}{\exp\Big(\Pi_+\Big(\log\big((y^2+a^2)^\ell g_{t,x}(y)\big)\Big)\Big)\Big|_{y=x}}\,, \tag{3.24}$$

where

$$g_{t,x}(y):=\frac{\displaystyle\prod_{p=\ell+1}^{N}|y-y_{2p-1}|^2}{\displaystyle\prod_{j=0}^{N-1}|y-p_j|^2}\,.$$

Indeed, by definition of $g_{t,x}$, [6]

$$\log\Big((y^2+a^2)^\ell g_{t,x}(y)\Big)=\log\left(\frac{(y+ia)^\ell\prod_{p=\ell+1}^{N}(y-y_{2p-1})}{\prod_{j=0}^{N-1}(y-p_j)}\right)+\log\left(\frac{(y-ia)^\ell\prod_{p=\ell+1}^{N}(y-\overline{y_{2p-1}})}{\prod_{j=0}^{N-1}(y-\overline{p_j})}\right),$$

where one observes the first term of the right–hand side belongs to $L^2_+(\mathbb{R})$ as $\mathrm{Im}(p_k)<0$, while the second term belongs to $L^2_-(\mathbb{R})$ [7]. Therefore, by the uniqueness of the decomposition of any L^2–function in $L^2_+(\mathbb{R})\oplus L^2_-(\mathbb{R})$, we infer for a fixed $x\in\mathbb{R}$,

$$\Pi_+\Big(\log\big((y^2+a^2)^\ell g_{t,x}(y)\big)\Big)=\log\left(\frac{(y+ia)^\ell\,\prod_{k=\ell+1}^{N}(y-y_{2k-1})}{Q(y)}\right).$$

Applying the exponential function to both sides of the previous identity and setting $y=x$, one deduces (3.24). Now, the only task left is to compute the right–hand side of (3.24). To do so, we need the following classical lemma, the proof of which will be presented later for the convenience of the reader,

Lemma 3.5. *For any $h\in L^2(\mathbb{R})$ of class C^1 satisfying $\|h'\|_{L^\infty}<\infty$,*

$$\Pi h(x)=\frac{h(x)}{2}-\frac{i}{2\pi}\int_0^{+\infty}\frac{h(x+s)-h(x-s)}{s}\,\mathrm{d}s\,,\qquad x\in\mathbb{R}\,.$$

[6] The multiplication of $g_{t,x}$ by $(y^2+a^2)^\ell$ aims to ensure that each term on the right-hand side of the following identity is in $L^2(\mathbb{R})$

[7] i.e. the space of functions having a trace in $L^2(\mathbb{R})$, such that they can be holomorphically extended to $\mathbb{C}_-$)

Thus, applying this lemma with $h(y)=\log\big((y^2+a^2)^\ell g(y)\big)$, one obtains

$$\begin{aligned}\Pi_+\Big(\log\big((y^2+a^2)^\ell g_{t,x}(y)\big)\Big) &= \frac{1}{2}\log\Big((y^2+a^2)^\ell g_{t,x}(y)\Big) - \frac{i\,\ell}{2\pi}\int_0^{+\infty}\frac{1}{s}\log\left(\frac{(y+s)^2+a^2}{(y-s)^2+a^2}\right)\mathrm{d}s\\ &\quad - \frac{i}{2\pi}\int_0^{+\infty}\frac{\log(g_{t,x}(y+s))-\log(g_{t,x}(y-s))}{s}\,\mathrm{d}s\,.\end{aligned} \tag{3.25}$$

where after some computation, one finds

$$\frac{1}{2\pi}\int_0^{+\infty}\frac{1}{s}\log\left(\frac{(y+s)^2+a^2}{(y-s)^2+a^2}\right)\mathrm{d}s = \frac{\pi}{2}-\arctan\left(\frac{a}{y}\right)\,. \tag{3.26}$$

Indeed, let $f(y):=\int_0^{+\infty}\frac{1}{s}\log\left(\frac{(y+s)^2+a^2}{(y-s)^2+a^2}\right)\mathrm{d}s$. By deriving f, one finds

$$\begin{aligned}f'(y) &= \int_0^{+\infty}\frac{4(a^2+s^2-x^2)}{\big((y+s)^2+a^2\big)\big((y-s)^2+a^2\big)}\mathrm{d}s\\ &= \int_{-\infty}^{+\infty}\frac{2(a^2+s^2-x^2)}{\big((y+s)^2+a^2\big)\big((y-s)^2+a^2\big)}\mathrm{d}s\\ &= \lim_{R\to\infty}\int_{\Gamma_R}\frac{2(a^2+z^2-y^2)}{\big((y+z)^2+a^2\big)\big((y-z)^2+a^2\big)}\mathrm{d}z\end{aligned}$$

where Γ_R is the contour composed of the real axis from $-R$ to R and the upper semicircle. Using Cauchy's residue theorem, we infer $f'(y)=\frac{2\pi a}{y^2+a^2}$. Therefore, by integrating the previous expression and observing that $f(0)=0$, and using that $\arctan(\theta)+\arctan(\frac{1}{\theta})=\frac{\pi}{2}$, one obtains (3.26). Consequently, (3.25) becomes

$$\begin{aligned}\Pi_+\Big(\log\big((y^2+a^2)^\ell g_{t,x}(y)\big)\Big) &= \frac{1}{2}\log\Big((y^2+a^2)^\ell g_{t,x}(y)\Big) - \frac{i\ell\,\pi}{2} + i\ell\,\mathrm{artan}\left(\frac{a}{y}\right)\\ &\quad - \frac{i}{2\pi}\int_0^{+\infty}\frac{\log(g_{t,x}(y+s))-\log(g_{t,x}(y-s))}{s}\,\mathrm{d}s\,,\end{aligned} \tag{3.27}$$

and hence,

$$\begin{aligned}\exp\Big(\Pi_+\Big(\log\big((y^2+a^2)^\ell g_{t,x}(y)\big)\Big)\Big)\Big|_{y=x} &= \mathrm{e}^{\frac{-i\ell\,\pi}{2}}\sqrt{(x^2+a^2)^\ell\,g_{t,x}(x)}\,\mathrm{e}^{i\ell\arctan\left(\frac{a}{x}\right)}\\ &\quad \exp\left(-\frac{i}{2\pi}\int_0^{+\infty}\frac{\log(g_{t,x}(x+s))-\log(g_{t,x}(x-s))}{s}\,\mathrm{d}s\right)\,.\end{aligned} \tag{3.28}$$

Thus, combining (3.22) , (3.24) and (3.28)

$$\begin{aligned}ZD_{\pm}[u_0](t,x) &= \frac{u_0(x)}{\prod_{k=1}^{\ell}(x-y_{2k-1})}\,\frac{(x+ia)^{\ell}}{\sqrt{(x^2+a^2)^{\ell}}\,\mathrm{e}^{i\ell\arctan(a/x)}}\,\frac{\mathrm{e}^{i\frac{\pi\ell}{2}}}{\sqrt{g_{t,x}(x)}}\\ &\quad\cdot\exp\left(\frac{i}{2\pi}\int_0^{+\infty}\frac{\log(g_{t,x}(x+s))-\log(g_{t,x}(x-s))}{s}\,\mathrm{d}s\right)\\ &= \frac{|u_0(x)|}{\prod_{k=1}^{\ell}(x-y_{2k-1})}\,\frac{(i)^{\ell}}{\sqrt{g_{t,x}(x)}}\,\mathrm{e}^{i\varphi(t,x)}\,,\end{aligned}\tag{3.29}$$

where

$$\varphi(t,x)=\arg(u_0(x))+\frac{1}{2\pi}\int_0^{+\infty}\frac{\log(g_{t,x}(x+s))-\log(g_{t,x}(x-s))}{s}\,\mathrm{d}s\,.\tag{3.30}$$

Substituting $g_{t,x}$ in (3.29) by its value in (3.23) with $y=x$,

$$ZD_{\pm}[u_0](t,x)=\frac{|u_0(x)|}{\prod_{k=1}^{\ell}(x-y_{2k-1})}\,\frac{(i)^{\ell}}{\sqrt{\dfrac{\mp 2t|u_0(x)|^2}{\displaystyle\prod_{k=0}^{2\ell}(x-y_k)}}}\,\mathrm{e}^{i\varphi(t,x)}\,,$$

and using the fact that the y_k are solutions of the algebraic equation $y_k \mp 2t|u_0(y_k)|^2 = x$, for all $k=0\,,\ldots\,,2\ell$, we conclude

$$\begin{aligned}ZD_{\pm}[u_0](t,x) &= \frac{1}{(\mp 2t)^{\ell}\prod_{k=1}^{\ell}|u_0(y_{2k-1})|^2}\,\frac{(i)^{\ell}}{\sqrt{\dfrac{\mp 2t}{(\mp 2t)^{2\ell+1}\displaystyle\prod_{k=0}^{2\ell}|u_0(y_k)|^2}}}\,\mathrm{e}^{i\varphi(t,x)}\\ &= \left(\mp i\,\frac{|t|}{t}\right)^{\ell}\frac{\displaystyle\prod_{k=0}^{\ell}|u_0(y_{2k})|}{\displaystyle\prod_{k=1}^{\ell}|u_0(y_{2k-1})|}\,\mathrm{e}^{i\varphi(t,x)}\,,\end{aligned}$$

where by (3.30) and (3.23) ,

$$\varphi(t,x)=\arg(u_0(x))+\frac{1}{2\pi}\int_0^{+\infty}\frac{1}{s}\log\left(\frac{s\mp 2t|u_0(x+s)|^2}{-s\mp 2t|u_0(x-s)|^2}\,\frac{\prod_{k=0}^{2\ell}(x-s-y_k)}{\prod_{k=0}^{2\ell}(x+s-y_k)}\right)\mathrm{d}s\,.$$

□

Proof of Lemma 3.5. For all $x\in\mathbb{R}$,

$$\begin{aligned}\lim_{\delta\to 0^+}\Pi h(x+i\delta) &= \lim_{\delta\to 0^+}\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{h(y)}{y-x-i\delta}\,\mathrm{d}y\\ &= \lim_{\delta\to 0^+}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\delta\,h(y)}{(y-x)^2+\delta^2}\,\mathrm{d}y+\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{h(y)(y-x)}{(y-x)^2+\delta^2}\,\mathrm{d}y\end{aligned}$$

Using the Poisson integral formula on the upper half-plane of $\mathbb{C}$, we infer for all $\delta>0$,

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\delta\, h(y)}{(y-x)^2+\delta^2}\,\mathrm{d}y=\frac{h(x)}{2}\,.$$

For the second term, observe that,

$$\begin{aligned}\lim_{\delta\to 0^+}\int_{\mathbb{R}}\frac{h(y)(y-x)}{(y-x)^2+\delta^2}\,\mathrm{d}y &= \lim_{\delta\to 0^+}\lim_{\varepsilon\to 0}\int_{-\infty}^{x-\varepsilon}\frac{h(y)(y-x)}{(y-x)^2+\delta^2}\,\mathrm{d}y+\lim_{\delta\to 0^+}\lim_{\varepsilon\to 0}\int_{x+\varepsilon}^{+\infty}\frac{h(y)(y-x)}{(y-x)^2+\delta^2}\,\mathrm{d}y\\ &=\lim_{\delta\to 0^+}\lim_{\varepsilon\to 0}\left(-\int_{\varepsilon}^{+\infty}\frac{h(x-s)s}{s^2+\delta^2}\,\mathrm{d}s+\int_{\varepsilon}^{+\infty}\frac{h(x+s)s}{s^2+\delta^2}\,\mathrm{d}s\right)\\ &=\lim_{\delta\to 0^+}\int_0^{+\infty}\frac{\left(h(x+s)-h(x-s)\right)s}{s^2+\delta^2}\,\mathrm{d}y\,,\end{aligned}$$

where we can switch lim and integral, as $\int_0^{+\infty}\frac{h(x+s)-h(x-s)}{s}\,\mathrm{d}s$ is well–defined as the principal value of $\frac{h(x)}{x}$ since $\|h'\|_{L^\infty}<\infty$. □

Now, armed with Proposition 3.1, let us proceed to prove the extended version of the statement of this proposition to Theorem 1.2.

Proof of Theorem 1.2. Let $u_0\in L^2_+(\mathbb{R})\cap C^1(\mathbb{R})$, such that u_0 is tending to 0 at infinity and $u_0'\in L^\infty(\mathbb{R})$. For such u_0, the C^1–function $\gamma_t(y):=y\mp 2t|u_0(y)|^2$ is asymptotically equivalent to y at $\pm\infty$. Moreover, since u_0' is bounded in $L^\infty(\mathbb{R})$, then for all $t\in\mathbb{R}$, and for any $x\in\mathbb{R}$ that is not a critical value of γ_t, the equation $\gamma_t(y)=x$ has a finite number of real solutions

$$y_0(t,x)<\ldots<y_{2\ell}(t,x)\,. \tag{3.31}$$

Besides, note that by the Sard theorem, the set of critical values of γ_t has zero Lebesgue measure. Moreover, since

$$\gamma_t'(y)=1+2t\big(|u_0(y)|^2\big)'\longrightarrow 1\,,\qquad y\to\pm\infty\,,$$

then the set of critical points $\{y\,;\,\gamma_t'(y)=0\}$ is compact for a given t, so that its image –the set of critical values of γ_t– is compact, and hence in particular closed. Thus, let Ω be any open connected set (for the x variable) where (3.31) is satisfied. The idea, at this stage of the proof, is to deduce the result from Proposition 3.1. Thus, by using a standard mollifier, we approximate u_0 in $L^2(\mathbb{R})\cap C^1(\mathbb{R})$ by a sequence of rational functions (u_0^δ) belonging to the Hardy space. Now, take ω to be any arbitrary open subset of Ω such that $\overline{\omega}$ is compact. Therefore, for δ small enough,

$$y+2t|u_0^\delta(y)|^2=x\,,\qquad x\in\omega\,,$$

has $2\ell+1$ solutions

$$y_0^\delta(t,x)<\ldots<y_{2\ell}^\delta(t,x)\,.$$

Since u_0^δ is a rational function in the Hardy space, then by Proposition 3.1,

$$ZD_\pm[u_0^\delta](t,x) = \mathrm{e}^{i\varphi_\delta(x)}\left(\mp i\,\frac{|t|}{t}\right)^\ell \prod_{k=0}^{2\ell} |u_0^\delta(y_k^\delta)|^{(-1)^k},$$

where

$$\varphi_\delta(x) = \arg(u_0^\delta(x)) + \frac{1}{2\pi}\int_0^{+\infty} \frac{1}{s}\log\left(\frac{s \mp 2t|u_0^\delta(x+s)|^2}{-s \mp 2t|u_0^\delta(x-s)|^2}\,\frac{\prod_{k=0}^{2\ell}(x-s-y_k^\delta)}{\prod_{k=0}^{2\ell}(x+s-y_k^\delta)}\right)\mathrm{d}s\,.$$

By passing to the limit as $\delta \to 0$, and using that $u_0^\delta \to u_0$ in $L^2 \cap C^1$ so that $y_k^\delta(t,x) \to y_k(t,x)$, and by the weak limit of (2.8), we deduce for every $x \in \omega$, formula (1.5). Now, since ω is chosen arbitrarily in Ω, this achieves the proof. □

An immediate consequence of Theorem 1.2 is the following corollary.

Corollary 3.6. *Let $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), then for all $t \in \mathbb{R}$,*

$$\|ZD_\pm[u_0](t)\|_{L^\infty} \le \|u_0\|_{L^\infty}\,.$$

Proof. In view of Theorem 1.2, we have for any $u_0 \in L^2_+(\mathbb{R}) \cap C^1(\mathbb{R})$, satisfying the property that u_0 is tending to 0 at infinity,

$$|ZD_\pm[u_0](t,x)| = \frac{\prod_{k=0}^{\ell} |u_0(y_{2k})|}{\prod_{k=1}^{\ell} |u_0(y_{2k-1})|}\,,$$

where $y_0 < \ldots < y_{2\ell}$ are solutions for the algebraic equation

$$x - y_k = \mp 2t|u_0(y_k)|^2\,.$$

Therefore, by the monotonicity of $k \mapsto y_k$ we infer the monotonicity of $k \mapsto |u_0(y_k)|^2$, so that we can deduce

$$|ZD_\pm[u_0](t,x)| \le \max\{|u_0(y_0)|\,, |u_0(y_{2\ell})|\} \le \|u_0\|_{L^\infty}\,.$$

The general case of $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ follows by applying a standard mollifier to u_0 like the one described to prove Theorem 1.2 and by using property (2.8) in Theorem 1.1. □

References

[Ba23a] R. Badreddine. On the global well-posedness of the Calogero-Sutherland derivative nonlinear Schrödinger equation. *arXiv preprint arXiv:2303.01087 (2023). To appear in Pure and Applied analysis.* https://doi.org/10.48550/arXiv.2303.01087

[Ba23b] R. Badreddine. Traveling waves & finite gap potentials for the Calogero-Sutherland Derivative nonlinear Schrödinger equation. *arXiv preprint arXiv:2307.01592 (2023). To appear in Annales IHP C, Analyse non linéaire.* https://doi.org/10.48550/arXiv.2307.01592

[CG09] T. Claeys and T. Grava. Universality of the break-up profile for the KdV equation in the small dispersion limit using the Riemann-Hilbert approach. *Communications in Mathematical Physics, 286(3):979–1009, (2009).*

[CG10a] T. Claeys and T. Grava. Painlevé II asymptotics near the leading edge of the oscillatory zone for the Korteweg—de Vries equation in the small-dispersion limit. *Communications on Pure and Applied Mathematics, 63(2):203–232, (2010).*

[CG10b] T. Claeys and T. Grava. Solitonic asymptotics for the Korteweg–de Vries equation in the small dispersion limit. *SIAM journal on mathematical analysis, 42(5):2132–2154, (2010).*

[BLS08] JL. Bona , D. Lannes and J-C. Saut. *Asymptotic models for internal waves. Journal de mathématiques pures et appliquées 89.6 (2008): 538-566.*

[Ga23a] L. Gassot. Zero-dispersion limit for the Benjamin-Ono equation on the torus with single well initial data. *arXiv:2111.06800, Communications in mathematical Physics 401, pp.2793–2843* (2023). https://doi.org/10.1007/s00220-023-04701-0

[Ga23b] L. Gassot. Lax eigenvalues in the zero-dispersion limit for the Benjamin-Ono equation on the torus. *arXiv:2301.03919, To appear in SIAM J. Math. Analysis.* (2023). https://doi.org/10.1137/23M154635X

[Ge22] P. Gérard. An explicit formula for the Benjamin–Ono equation. *Tunisian Journal of Mathematics 5.3 (2023): 593-603.* https://doi.org/10.2140/tunis.2023.5.593

[Ge23] P. Gérard. The zero dispersion limit for the Benjamin–Ono equation on the line. *ArXiv preprint arXiv:2307.12768* (2023). To appear in Compte Rendus-Série Mathématique. https://doi.org/10.48550/arXiv.2307.12768

[GG15] P. Gérard and S. Grellier. An explicit formula for the cubic Szegő equation. *Transactions of the American Mathematical Society, 367(4), 2979-2995.* (2015). https://doi.org/10.48550/arXiv.1304.2619

[GL22] P. Gérard, and E. Lenzmann. The Calogero–Moser Derivative Nonlinear Schrödinger Equation. *arXiv preprint arXiv:2208.04105* (2022). To appear in Comm. Pure Appl. Math. https://doi.org/10.48550/arXiv.2208.04105

[GP23] P. Gérard, and A. Pushnitski. The cubic Szegő equation on the real line: explicit formula and well-posedness on the Hardy class. *arXiv preprint arXiv:2307.06734* (2023).

[GK07] T. Grava, C. Klein. Numerical solution of the small dispersion limit of Korteweg–de Vries and Whitham equations. *Comm. Pure Appl. Math. 60, 1623–1664.* (2007).

[HK24] J. Hogan, and M. Kowalski. Turbulent Threshold for Continuum Calogero-Moser Models. *arXiv preprint arXiv:2401.16609.* (2024). https://doi.org/10.48550/arXiv.2401.16609

[KLV23] R. Killip, T. Laurens, and M. Visan. Scaling-critical well-posedness for continuum Calogero-Moser models. *arXiv preprint arXiv:2311.12334* (2023). https://doi.org/10.48550/arXiv.2311.12334

[LL83] P. Lax and C.D. Levermore. The small dispersion limit of the Korteweg-de Vries equation. I. *Communications on Pure and Applied Mathematics 36, no. 3, pp.253-290* (1983).

[MW16] P. Miller, A. Wetzel. The scattering transform for the Benjamin–Ono equation in the small dispersion limit *Physica D 333, p. 185–199* (2016).

[MX11] P. Miller and Z. Xu. On the zero–dispersion limit of the Benjamin–Ono Cauchy problem for positive initial data. *Communications on pure and applied mathematics 64 no. 2, pp.205-270* (2011).

[Ve87] S. Venakides. The zero dispersion limit of the Korteweg-de Vries equation with periodic initial data. *Transactions of the American Mathematical Society 301, no. 1 pp. 189-226* (1987).

[Ve91] S. Venakides. The Korteweg-de Vries equation with small dispersion limit: higher order LAx–Levermore theory. *In Applied and Industrial Mathematics, p. 255–262. Springer,* (1991).

[Pe95] D.E. Pelinovsky. Intermediate nonlinear Schrödinger equation for internal waves in a fluid of finite depth, *Phys. Lett. A 197 (1995), no. 5–6, 401–406.*

[Sa19] J-C. Saut. Benjamin-Ono and intermediate long wave equations: Modeling, IST and PDE. *Nonlinear dispersive partial differential equations and inverse scattering (2019): 95-160.*

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