# QUANTITATIVE HOMOGENIZATION FOR LOG-NORMAL COEFFICIENTS 

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#### Abstract

We establish quantitative homogenization results for the popular log-normal coefficients. Since the coefficients are neither bounded nor uniformly elliptic, standard proofs do not apply directly. Instead, we take inspiration from the approach developed for the nonlinear setting by the first two authors and capitalize on large-scale regularity results by Bella, Fehrmann, and Otto for degenerate coefficients in order to leverage an optimal control (in terms of scaling and stochastic integrability) of oscillations and fluctuations.


MSC: 35R60, 35B27, 35B65, 60F05, 60H07
Keywords: quantitative stochastic homogenization, log-normal coefficients, regularity theory, fluctuations

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## 1. Main Results

1.1. Degenerate coefficients in stochastic homogenization. Quantitative stochastic homogenization for linear elliptic PDEs with bounded and uniformly elliptic coefficients is by now well-established, either based on functional calculus and nonlinear concentration of measure $[29,30,31,25,24,15,14,26,27,18,17,19,13]$ or based on renormalization and linear concentration of measure $[6,5,3,28,4,2]$. In both cases, the theory assumes non-degeneracy of the coefficients. Whereas there is no doubt the theory should extend to mildly degenerate coefficients, only quantitative homogenization on the percolation cluster [ 1,12 ] and for rigid inclusions [16] has been established so far.

In geology (and more precisely for applications to oil or water recovery, and $\mathrm{CO}_{2}$ storage), permeability can range over more than 8 orders of magnitude. Such permeability fields are often modelled using log-normal distributions (that is, they are obtained as the exponential of a Gaussian random field - failing boundedness and uniform ellipticity). When one is interested in the behavior of the system at the scale of the correlation-length, this is the realm of expansions à la Karhunen-Loève, see e. g. [7, 11]. When one is interested in the large-scale behavior of the system, this is the realm of (numerical) homogenization. It is not a coincidence that the very first article on numerical homogenization [34] in the applied mathematics community (which triggered a long-lasting activity in multiscale modeling and simulations, e. g. [20]) deals with log-normal coefficients. Although the numerical analysis of such methods for random coefficients was the main motivation to develop a quantitative stochastic homogenization theory in $[29,30,22,25,23]$, the case of degenerate coefficients was not covered. The aim of this article is to fill this gap, and extend the quantitative homogenization theory based on functional calculus and nonlinear concentration of measure to log-normal coefficients. The results of this work were announced in the plenary talk of the second author at the SIAM conference on Mathematical Aspects of Materials Science (MS21), and extend [32] to higher dimensions.

Quantitative homogenization starts with regularity theory - like the perturbative Meyers' estimate (which provides a higher integrability result). For bounded and uniformly elliptic coefficients, such estimates are standard and deterministic. For degenerate coefficients however, one cannot hope for a deterministic version of these estimates, and we only expect large-scale versions - that is, such estimates only hold at a random scale onwards. Our strategy to get around this problem is inspired by the work [10] of the first two authors on the homogenization of genuinely nonlinear monotone operators. In this setting the linearized equation is an elliptic equation with coefficients that involve the nonlinear corrector gradient, and therefore are unbounded (and thus degenerate in the usual sense of the word in homogenization). In [10] we established large-scale Meyers' estimates using quantitative homogenization itself (and in particular the decay of spatial averages of correctors). In the present work, the problem is somehow simpler because the degeneracy of the coefficients is a given datum rather than an unknown of the problem (as opposed to the integrability of the nonlinear correctors in [10]). The main insight of the present work are the large-scale Meyers estimates we establish in Theorem 1.3 and the control of correctors in Theorem 1.8, that we prove following the strategy laid out in [10] for a nonlinear problem. From there, we capitalize on [8] and on by-now standard methods to leverage a complete quantitative homogenization theory, that we state for completeness with precise references to the literature. In particular, we give a complete description of fluctuations of observables of the (random) solution, in the spirit of uncertainty quantification.

For simplicity of the exposition, we consider the coefficient field $a(x)$ Id where $a(x)=$ $\exp (G(x))$ is a scalar field obtained as the exponential of a Gaussian field $G$ (this specific form is not essential, and encapsulates the main difficulty of unboundedness and degeneracy). We further assume that this Gaussian field has integrable covariance (this is essential for the upcoming proof). Since our strategy is based on functional calculus and nonlinear concentration, one can also treat Poissonian models. A typical example would be as follows: consider a Poisson point process, and the associated Voronoi tessellation of $\mathbb{R}^{d}$. Define the random coefficients as $a(x) \mathrm{Id}$, where $a(x)$ denotes the diameter of the Voronoi cell containing $x$ - we have $\inf a=\inf a^{-1}=0$. Since $a$ satisfies a multiscale spectral gap inequality (with exponential weight) in the sense of [15, 14], our strategy still holds with minor modifications (see e.g. [25, 24] where both Gaussian and Poissonian fields are considered).

To conclude, let us comment on the interest of multiscale functional inequalities in the sense of $[15,14]$ in this context. Quantitative stochastic homogenization requires two ingredients on the coefficient field: a rate for the convergence of spatial averages to their expectation, and concentration of measures (that is, the stochastic integrability). Whereas (nonlinear) concentration depends little on the covariance function when the coefficient field satisfies a functional inequality, (linear) concentration degrades very fast with correlations when it is obtained by (linear) mixing conditions (such as alpha-mixing). In particular, as soon as one is interested in coefficient fields whose correlations (in a suitable mixing sense) do not decay at a super-algebraic rate, functional inequalities seem unavoidable.

In this article we shall consider the following class of coefficient fields.
Hypothesis 1.1. Let $G: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a Gaussian field with integrable covariance function $\mathcal{C}$ which is $2 \gamma$-Hölder continuous at 0 for some $0<\gamma<\frac{1}{2}$. We set $a(x):=\exp (G(x))$. $\diamond$

All the upcoming results assume Hypothesis 1.1.
1.2. Perturbative large-scale regularity. We start by defining a scale at which the log-normal field behaves like a uniformly and elliptic coefficient field.
Proposition 1.2. Let $a$ be as in Hypothesis 1.1. Set $p_{\diamond}=d+1$. There exists a stationary $\frac{1}{8}$-Lipschitz field $r_{\diamond}$ such that for all $x \in \mathbb{R}^{d}$ and $r \geq r_{\diamond}(x)$

$$
\frac{1}{2} \mathbb{E}\left[a^{p_{\diamond}}+a^{-p_{\diamond}}\right] \leq f_{B_{r}(x)} a^{p_{\diamond}}+a^{-p_{\diamond}} \leq 2 \mathbb{E}\left[a^{p_{\diamond}}+a^{-p_{\diamond}}\right],
$$

and which satisfies

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(\frac{1}{C} \log ^{2}\left(1+r_{\diamond}\right)\right)\right] \leq 2, \tag{1.1}
\end{equation*}
$$

for some $C>0$. In what follows, for all $x \in \mathbb{R}^{d}$ we set $B_{\diamond}(x):=B_{r_{\diamond}(x)}(x)$.
Let $0<\varepsilon \leq 1$, we define for all $R \geq 1$

$$
r_{\boldsymbol{\omega}}(R):=\left(R^{-\frac{\varepsilon}{2}} \sup _{B_{R}}\left(a+a^{-1}\right)\right)^{2} .
$$

There exists $C>0$ (depending on $\varepsilon$ ) such that

$$
\begin{equation*}
\sup _{R \geq 1} \mathbb{E}\left[\exp \left(\frac{1}{C} \log ^{2}\left(1+r_{k}(R)\right)\right)\right] \leq 2 \tag{1.2}
\end{equation*}
$$

Based on this (which we prove in the appendix), we shall establish the following largescale Meyers estimate.

Theorem 1.3 (Quenched Meyers' estimates). There exists $\kappa>0$ (depending on d) such that for all $2 \leq p \leq 2+\kappa$, and all functions $u$, $h$ satisfying in the weak sense in $\mathbb{R}^{d}$

$$
-\nabla \cdot a \nabla u=\nabla \cdot \sqrt{a} h,
$$

we have

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left(f_{B_{\odot}(x)} a|\nabla u|^{2}\right)^{\frac{p}{2}} d x \lesssim_{p} \int_{\mathbb{R}^{d}}\left(f_{B_{\odot}(x)}|h|^{2}\right)^{\frac{p}{2}} d x . \tag{1.3}
\end{equation*}
$$

A consequence of the proof of the Meyers estimates is the following large-scale hole-filling estimate.

Corollary 1.4 (Hole-filling estimate in the large). There exists $0<\beta \leq d$ with the following property. Let $R \geq r_{\diamond}$, and let $u$ satisfy

$$
-\nabla \cdot a \nabla u=0 \text { in } B_{R}
$$

Then for all $r_{\diamond} \leq r \leq R$, we have

$$
\begin{equation*}
f_{B_{r}} a|\nabla u|^{2} \lesssim\left(\frac{R}{r}\right)^{d-\beta} f_{B_{R}} a|\nabla u|^{2} . \tag{1.4}
\end{equation*}
$$

Since by Proposition 1.2 the stationary field $r_{\diamond}$ has finite super-algebraic moments, one may upgrade the above into annealed estimates, following an argument of [19].

Theorem 1.5 (Annealed Meyers' estimate). There exists $\kappa>0$ such that for all $\mid p-$ $2\left|,|m-2| \leq \kappa\right.$ and $0<\delta \leq \frac{1}{2}$, and all functions $u, h$ satisfying in the weak sense in $\mathbb{R}^{d}$

$$
-\nabla \cdot a \nabla u=\nabla \cdot \sqrt{a} h,
$$

we have

$$
\int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(f_{B(x)} a|\nabla u|^{2}\right)^{\frac{p}{2}}\right]^{\frac{m}{p}} d x \lesssim_{p} \delta^{-\frac{1}{4}}|\log \delta|^{\frac{1}{2}} \int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(f_{B(x)}|h|^{2}\right)^{\frac{p(1+\delta)}{2}}\right]^{\frac{m}{p(1+\delta)}} d x
$$

These results are proved in Section 2.
1.3. Minimal radius and bounds on correctors. Based on these perturbative largescale regularity estimates and a buckling argument making use of the CLT scaling, we establish bounds on correctors, which are defined in this degenerate context in [8, Lemma 1].
Lemma 1.6. There exist two tensor fields $\left\{\phi_{i}\right\}_{1 \leq i \leq d}$ and $\left\{\sigma_{i j k}\right\}_{1 \leq i, j, k \leq d}$ with the following properties. The gradient fields are stationary, have bounded moments, and are of vanishing expectation: $\mathbb{E}\left[\nabla \phi_{i}\right]=\mathbb{E}\left[\nabla \sigma_{i j k}\right]=0$ and for all $1 \leq p<\infty$,

$$
\sum_{i=1}^{d} \mathbb{E}\left[a\left|\nabla \phi_{i}\right|^{2}\right]+\sum_{i=1}^{d} \mathbb{E}\left[\left|\nabla \phi_{i}\right|^{\frac{2 p}{p+1}}\right]^{\frac{p+1}{2 p}}+\sum_{i, j, k=1}^{d} \mathbb{E}\left[\left|\nabla \sigma_{i j k}\right|^{\frac{2 p}{p+1}}\right]^{\frac{p+1}{2 p}} \lesssim \mathbb{E}\left[a^{p}+a^{-p}\right]
$$

The field $\sigma$ is skew-symmetric in its last two indices, that is,

$$
\sigma_{i j k}=-\sigma_{i k j} .
$$

Furthermore, for $\mathbb{E}[\cdot]-$ a.e. a we have

$$
\begin{equation*}
-\nabla \cdot a\left(\nabla \phi_{i}+e_{i}\right)=0, \tag{1.5}
\end{equation*}
$$

$a n d^{1}$

$$
q_{i}:=a\left(\nabla \phi_{i}+e_{i}\right)=\mathbb{E}\left[q_{i}\right] e_{i}+\nabla \cdot \sigma_{i},
$$

together with the gauge equation

$$
\begin{equation*}
-\triangle \sigma_{i j k}=\left[\nabla \times q_{i}\right]_{j k}:=\partial_{j} q_{i k}-\partial_{k} q_{i j} . \tag{1.6}
\end{equation*}
$$

Finally, the homogenized coefficient $\bar{a} e_{i}:=\mathbb{E}\left[a\left(\nabla \phi_{i}+e_{i}\right)\right]$ is uniformly elliptic in the sense that for $K:=\mathbb{E}\left[a^{d+1}+a^{-(d+1)}\right]$ and for all $\xi \in \mathbb{R}^{d}$,

$$
\frac{1}{K}|\xi|^{2} \leq \xi \cdot \bar{a} \xi \quad \text { and } \quad|\bar{a} \xi| \leq K|\xi| .
$$

An output of [8, Lemma 1] is that $\nabla \phi_{i}$ and $\nabla \sigma_{i j k}$ are uniquely defined. We are in position to introduce the minimal radius.
Definition 1.7. Recall $p_{\diamond}=d+1$. We define the stationary field $\tilde{r_{\star}}: \mathbb{R} \rightarrow \mathbb{R}_{+}$via

$$
\begin{equation*}
\tilde{r_{\star}}: x \mapsto \inf _{r \geq r_{\rho}(x)}\left\{\forall \rho \geq r, \frac{1}{\rho}\left(f_{B_{\rho}}\left|(\phi, \sigma)-f_{B_{\rho}}(\phi, \sigma)\right|^{\frac{2 p_{\rho}-1}{p_{\varnothing}-1}}\right)^{\frac{p_{\rho}-1}{2 p_{\diamond}}} \leq \frac{1}{C}\right\}, \tag{1.7}
\end{equation*}
$$

and we denote by $r_{\star}$ the smallest $\frac{1}{8}$-Lipschitz field larger or equal to $\tilde{r_{\star}}$ (which is also stationary). In what follows, for all $x \in \mathbb{R}^{d}$ we set $B_{\star}(x):=B_{r_{\star}(x)}(x)$.

Our first quantitative result is the following bound on the growth of the correctors.
Theorem 1.8. For all $x \in \mathbb{R}^{d}$,

$$
\begin{equation*}
\left(f_{B(x)}\left|(\phi, \sigma)-f_{B(0)}(\phi, \sigma)\right|^{\frac{2 p_{o}-1}{p_{\rho}-1}}\right)^{\frac{p_{0}-1}{2 p_{\phi}}} \leq C_{x} \mu_{d}(|x|), \tag{1.8}
\end{equation*}
$$

where

$$
\mu_{d}(t):=\left\{\begin{array}{rll}
\sqrt{t+1} & : & d=1,  \tag{1.9}\\
\log (t+2)^{\frac{1}{2}} & : & d=2, \\
1 & : & d>2,
\end{array}\right.
$$

and where $C_{x}$ is a random variable that satisfies for some $0<C<\infty$

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(\frac{1}{C} \log ^{2}\left(1+C_{x}\right)\right)\right] \leq 2 \tag{1.10}
\end{equation*}
$$

We also have the following control of the minimal radius.
Theorem 1.9. There exists $0<C<\infty$ such that the minimal radius $r_{\star}$ satisfies

$$
\mathbb{E}\left[\exp \left(\frac{1}{C} \log ^{2}\left(1+r_{\star}\right)\right)\right] \leq 2
$$

[^0]These results are proved in Section 3. As shown in dimension $d=1$ in [32], the stochastic integrability in the above results is optimal.
1.4. Non-perturbative large-scale regularity. The following non-perturbative largescale regularity results are post-processings of [8] using Theorem 1.9. (See Section 4 for the necessary adaptations.)
Proposition 1.10 (Large-scale Schauder estimates). If $u, f, g$ satisfy in the weak sense in $B_{R}$ for some $R \geq r_{\star}$

$$
-\nabla \cdot a(\nabla u+g)=\nabla \cdot h
$$

then we have for all $0<\alpha<1$

$$
\begin{aligned}
& \sup _{r \in\left[r_{\star}, R\right]} f_{B_{r}}(\nabla u+g) \cdot a(\nabla u+g) \lesssim f_{B_{R}}(\nabla u+g) \cdot a(\nabla u+g) \\
& \quad+\sup _{r \in\left[r_{\star}, R\right]}\left(\frac{R}{r}\right)^{2 \alpha} f_{B_{r}}\left(\left(g-f_{B_{r}} g\right) \cdot a\left(g-f_{B_{r}} g\right)+\left(h-f_{B_{r}} h\right) \cdot a^{-1}\left(h-f_{B_{r}} h\right)\right) .
\end{aligned}
$$

In particular, for $g=h=0$, we have the following mean-value property for a-harmonic functions (or large-scale Lipschitz property): For all $r_{\star} \leq r \leq R$,

$$
f_{B_{r}} \nabla u \cdot a \nabla u \lesssim f_{B_{R}} \nabla u \cdot a \nabla u .
$$

Based on the above Lipschitz estimate, [26, Corollary 4] directly yields large-scale Calderón-Zygmund estimates.
Theorem 1.11 (Quenched Calderón-Zygmund estimates). For all $1<p<\infty$ and for all functions $u, g$, $h$ satisfying in the weak sense in $\mathbb{R}^{d}$

$$
-\nabla \cdot a(\nabla u+g)=\nabla \cdot h,
$$

we have

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left(f_{B_{\star}(x)} \nabla u \cdot a \nabla u\right)^{\frac{p}{2}} d x \lesssim_{p} \int_{\mathbb{R}^{d}}\left(f_{B_{\star}(x)} g \cdot a g+h \cdot a^{-1} h\right)^{\frac{p}{2}} d x . \tag{1.11}
\end{equation*}
$$

As in [19], one can turn the quenched CZ estimates into the following annealed CZ estimates.
Theorem 1.12 (Annealed Calderón-Zygmund estimates). For all $1<p, m<\infty$ and $0<\delta \leq \frac{1}{2}$, and for all functions $u, g$, $h$ satisfying in the weak sense in $\mathbb{R}^{d}$

$$
-\nabla \cdot a(\nabla u+g)=\nabla \cdot h
$$

we have

$$
\begin{align*}
& \int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(f_{B(x)} a|\nabla u|^{2}\right)^{\frac{p}{2}}\right]^{\frac{m}{p}} d x \\
& \lesssim p, m  \tag{1.12}\\
& \delta^{-\frac{1}{4}}|\log \delta|^{\frac{1}{2}} \int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(f_{B(x)} a|g|^{2}+a^{-1}|h|^{2}\right)^{\frac{p(1+\delta)}{2}}\right]^{\frac{m}{p(1+\delta)}} d x
\end{align*}
$$

1.5. Quantitative homogenization. We now have all the necessary tools to make homogenization quantitative. Let $f \in L^{2}\left(\mathbb{R}^{d}\right)^{d}$, and for all $\varepsilon>0$, consider the unique Lax-Milgram solution ${ }^{2} u_{\varepsilon} \in \dot{H}^{1}\left(\mathbb{R}^{d}\right)$ of

$$
\begin{equation*}
-\nabla \cdot a(\dot{\bar{\varepsilon}}) \nabla u_{\varepsilon}=\nabla \cdot f \tag{1.13}
\end{equation*}
$$

as well as the unique Lax-Milgram solution $\bar{u} \in \dot{H}^{1}\left(\mathbb{R}^{d}\right)$ of the homogenized problem

$$
\begin{equation*}
-\nabla \cdot \bar{a} \nabla \bar{u}=\nabla \cdot f \tag{1.14}
\end{equation*}
$$

Our first result characterizes oscillations of $u_{\varepsilon}$ in a strong norm in form of a quantitative two-scale expansion result.

Theorem 1.13. For all $\varepsilon>0, v \in L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$, and $x \in \mathbb{R}^{d}$, set $S_{\varepsilon}(v)(x)=f_{B_{\varepsilon}(x)} v$, and define the two-scale expansion $\bar{u}_{\varepsilon}^{2 s}$ of $u_{\varepsilon}$ as

$$
\bar{u}_{\varepsilon}^{2 s}:=S_{\varepsilon}(\bar{u})+\varepsilon \phi_{i}(\dot{\bar{\varepsilon}}) S_{\varepsilon}\left(\partial_{i} \bar{u}\right) .
$$

Then for all $p \geq 1$ and $q \geq 1$ we have

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_{\mathbb{R}^{d}}\left(f_{B_{\varepsilon}(x)} a\left|\nabla\left(u_{\varepsilon}-\bar{u}_{\varepsilon}^{2 s}\right)\right|^{2}\right)^{\frac{p}{2}} d x\right)^{q}\right]^{\frac{1}{p q}} \\
& \lesssim \varepsilon \mu_{d}\left(\frac{1}{\varepsilon}\right)\left(\int_{\mathbb{R}^{d}} \mu_{d}(|x|)^{p}\left(f_{B_{\varepsilon}(x)}|f(x)|^{2}\right)^{\frac{p}{2}} d x\right)^{\frac{1}{p}}
\end{aligned}
$$

where $\mu_{d}$ is defined in (1.9).
The proof is the same as for [27, Proposition 1] based on Theorem 1.12 rather than [26, Corollary 4].

The second set of results quantifies fluctuations of $u_{\varepsilon}$. More precisely, we study the scaling limits of observables of the field $\nabla u_{\varepsilon}$ and flux $a(\dot{\bar{\varepsilon}}) \nabla u_{\varepsilon}$ of the solution. To this aim, we recall the definition of the standard homogenization commutator $\Xi$ and of the commutator of the solution $\Xi_{\varepsilon}(f)$, as introduced in $[18,17]$.

Definition 1.14. The standard homogenization commutator is the second-order tensor defined for all $1 \leq i \leq d$ by

$$
[\Xi]_{i}:=(a-\bar{a})\left(\nabla \phi_{i}+e_{i}\right)
$$

The homogenization commutator of the solution $u_{\varepsilon}$ of (1.13) is defined by

$$
\Xi_{\varepsilon}(f):=(a(\dot{\bar{\varepsilon}})-\bar{a}) \nabla u_{\varepsilon} .
$$

Our first result is a quantitative two-scale expansion at the level of observables of commutators in the fluctuation scaling (the so-called pathwise theory of fluctuations), where we understand an observable as a local average with a test function.

[^1]Theorem 1.15. Let $g \in L^{2}\left(\mathbb{R}^{d}\right)^{d}$, and denote by $\bar{v}$ the solution of (1.14) with $f$ replaced by $g$ (and $\bar{a}$ by its transpose matrix, here $\bar{a}^{\prime}=\bar{a}$ ). On the one hand, $\operatorname{Var}\left[\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^{d}} g \cdot \Xi_{\varepsilon}(f)\right] \lesssim$ 1. On the other hand, for all $p \geq 1$,

$$
\begin{aligned}
& \mathbb{E}\left[\left(\varepsilon^{-\frac{d}{2}}\left|\int_{\mathbb{R}^{d}} g \cdot\left(\Xi_{\varepsilon}(f)-\mathbb{E}\left[\Xi_{\varepsilon}(f)\right]\right)-\nabla \bar{v} \cdot \Xi(\dot{\bar{\varepsilon}}) \nabla \bar{u}\right|\right)^{p}\right]^{\frac{1}{p}} \\
& \lesssim \varepsilon \mu_{d}\left(\frac{1}{\varepsilon}\right)\|f\|_{L^{4}\left(\mathbb{R}^{d}, \mu_{d}^{2}\right)}\|g\|_{L^{4}\left(\mathbb{R}^{d}, \mu_{d}^{2}\right)}
\end{aligned}
$$

where $\mu_{d}$ is defined in (1.9), and $L^{4}\left(\mathbb{R}^{d}, \mu_{d}^{2}\right)$ is the weighted space with measure $\mu_{d}^{2}(x) d x$.
This result follows from [19, 17] using Theorems 1.8 and 1.12. A direct post-processing of Theorem 1.15 allows to recover corresponding results for the field and the flux of the solution (the main two quantities of interest from a physical viewpoint), see $[17,(1.10) \&$ (1.11)].

To complete the analysis of fluctuations of observables, it remains to investigate the scaling limit of the homogenization commutator. In the case of integrable covariance, the limit is a colored noise, which follows from the proofs of [19, 13] using Theorems 1.8 and 1.12 , on top of Malliavin calculus. We start by a convenient strengthening of the integrability of the covariance function in Hypothesis 1.1:

Hypothesis 1.16. The covariance function $\mathcal{C}$ can be decomposed as $\mathcal{C}=\mathcal{C}_{0} * \mathcal{C}_{0}$ where $\mathcal{C}_{0}$ satisfies for some $\beta>d^{3}$

$$
\left|\mathcal{C}_{0}(x)\right| \leq C_{0}(1+|x|)^{-\frac{1}{2}(d+\beta)}
$$

and its Fourier transform $\hat{\mathcal{C}}$ is positive almost everywhere.
The following quantitative central limit theorem combines the results of [13, Proposition 3.1, Corollary 4.5, and Theorem 1(ii)].

Theorem 1.17. Assume Hypotheses 1.1 and 1.16. For all functions $F \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)^{d \times d}$, set $I_{\varepsilon}(F):=\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^{d}} \Xi(\dot{\bar{\varepsilon}}): F$. On the one hand, there exists a non-degenerate constant tensor $\mathcal{Q}$ of order 4 such that for all $F, F^{\prime} \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)^{d \times d}$,

$$
\begin{aligned}
& \operatorname{Cov}\left[I_{\varepsilon}(F) ; I_{\varepsilon}\left(F^{\prime}\right)\right]-\int_{\mathbb{R}^{d}} F(x): \mathcal{Q}: F^{\prime}(x) d x \mid \\
& \quad{ }_{F, F^{\prime}} \begin{cases}\varepsilon & : d>2, \beta \geq d+1 \\
\varepsilon|\log \varepsilon|^{\frac{1}{2}} & : \\
\varepsilon^{\beta-d} & \\
& d=2, \beta \geq d+1 \\
& d<\beta<d+1\end{cases}
\end{aligned}
$$

On the other hand, for all $F \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)^{d \times d}$ and $\varepsilon>0$,

$$
\begin{aligned}
\mathrm{W}_{2}\left(\frac{I_{\varepsilon}(F)}{\operatorname{Var}\left[I_{\varepsilon}(F)\right]^{\frac{1}{2}}}, \mathcal{N}\right)+\mathrm{d}_{\mathrm{TV}}\left(\frac{I_{\varepsilon}(F)}{\operatorname{Var}\left[I_{\varepsilon}(F)\right]^{\frac{1}{2}}}, \mathcal{N}\right) & \\
& \lesssim_{F} \frac{1}{\operatorname{Var}\left[I_{\varepsilon}(F)\right]^{2}} \varepsilon^{\frac{d}{2}} \exp \left(C|\log \varepsilon|^{\frac{1}{2}}\right)
\end{aligned}
$$

[^2]where $\mathrm{W}_{2}(\cdot, \mathcal{N})$ and $\mathrm{d}_{\mathrm{TV}}(\cdot, \mathcal{N})$ denote the 2 -Wasserstein (see e.g. [37]) and the total variation distance to a standard Gaussian law, respectively. In particular, with $\sigma^{2}(F):=$ $\int_{\mathbb{R}^{d}} F(x): \mathcal{Q}: F(x) d x$, these two estimates combine to
\[

$$
\begin{align*}
& \mathrm{W}_{2}\left(\frac{I_{\varepsilon}(F)}{\sigma(F)}, \mathcal{N}\right)+\mathrm{d}_{\mathrm{TV}}\left(\frac{I_{\varepsilon}(F)}{\sigma(F)}, \mathcal{N}\right) \lesssim_{F} \varepsilon^{\frac{d}{2}} \exp \left(C|\log \varepsilon|^{\frac{1}{2}}\right) \\
&+\left\{\begin{array}{lll}
\varepsilon & : & d>2, \beta \geq d+1, \\
\varepsilon|\log \varepsilon|^{\frac{1}{2}} & : & d=2, \beta \geq d+1 \\
\varepsilon^{\beta-d} & : & d<\beta<d+1 .
\end{array}\right.
\end{align*}
$$
\]

The only difference with the case of uniformly elliptic and bounded coefficients is the error estimate for the asymptotic normality: In [13, Theorem 1(ii)], $\exp \left(C|\log \varepsilon|^{\frac{1}{2}}\right)$ is replaced by $|\log \varepsilon|$. This comes from the optimization argument in Step 6 Proof of Theorem 1(ii) in [13] which involves the stochastic integrability of the corrector gradient a similar optimization in the context of log-normal coefficients is worked out in formula (5.20) in Step 2 of the proof of Proposition 5.4 in [32], to which we refer the interested reader.

## 2. Large-scale Meyers estimates

2.1. Proof of Theorem 1.3. We follow the standard proof based on a reverse Hölder's inequality (using Caccioppoli's inequality and the Sobolev embedding) and Gehring's inequality.

We start with the reverse Hölder inequality.
Lemma 2.1 (Reverse Hölder). Recall that $p_{\diamond}=d+1$. For all $r \geq r_{\diamond}$ and all $u, h$ related in $B_{2 r}$ by

$$
-\nabla \cdot a \nabla u=\nabla \cdot \sqrt{a} h,
$$

we have

$$
f_{B_{r}}(\sqrt{a}|\nabla u|)^{2} \lesssim f_{B_{2 r}}|h|^{2}+\left(f_{B_{2 r}}(\sqrt{a}|\nabla u|)^{\alpha}\right)^{\frac{2}{\alpha}},
$$

where $1 \leq \alpha:=\frac{2 d(d+1)}{d^{2}+d+2}<2$.
Proof of Lemma 2.1. We start by establishing a Caccioppoli inequality. Let $\eta$ be a smooth cut-off for $B_{r}$ in $B_{2 r}$ with $\sup |\nabla \eta| \lesssim r^{-1}$. Set $\bar{u}:=\frac{\int \eta^{2} a u}{\int \eta^{2} a}$, and notice that $\frac{1}{C} \int \eta^{2} \leq$ $\int \eta^{2} a \leq C \int \eta^{2}$ since $r \geq r_{\diamond}$. Testing the equation for $u$ with $\eta^{2}(u-\bar{u})$, and integrating by parts we obtain

$$
\begin{aligned}
\int \eta^{2} a|\nabla(u-\bar{u})|^{2}+2 \int \eta a(u-\bar{u}) \nabla \eta & \cdot \nabla(u-\bar{u}) \\
& =\int \eta^{2} \sqrt{a} \nabla(u-\bar{u}) \cdot h+2 \int \eta \sqrt{a}(u-\bar{u}) \nabla \eta \cdot h .
\end{aligned}
$$

Using Cauchy-Schwarz' and Young's inequalities, this entails

$$
\begin{equation*}
\int \eta^{2} a|\nabla(u-\bar{u})|^{2} \lesssim \int \eta^{2}|h|^{2}+\int|\nabla \eta|^{2} a(u-\bar{u})^{2} . \tag{2.1}
\end{equation*}
$$

It remains to deal with the weights on the right-hand side. We start with the control of $\bar{u}$. Set $\bar{U}:=f_{B_{2 r}} u$. We have, since $\int_{B_{2 r}}\left(\frac{1}{\left|B_{2 r}\right|}-\frac{a \eta^{2}}{\int a \eta^{2}}\right)=0$,

$$
\bar{U}-\bar{u}:=\int_{B_{2 r}}\left(\frac{1}{\left|B_{2 r}\right|}-\frac{a \eta^{2}}{\int a \eta^{2}}\right) u=\int_{B_{2 r}}\left(\frac{1}{\left|B_{2 r}\right|}-\frac{a \eta^{2}}{\int a \eta^{2}}\right)(u-\bar{U}) .
$$

Hence, by Hölder's inequality with exponents ( $d+1, \frac{d+1}{d}$ ) and Poincaré-Sobolev' inequality, and since $r \geq r_{\diamond}$

$$
\begin{equation*}
|\bar{U}-\bar{u}| \lesssim\left(f_{B_{2 r}}(u-\bar{U})^{\frac{d+1}{d}}\right)^{\frac{d}{d+1}} \lesssim r\left(f_{B_{2 r}}|\nabla u|^{\frac{d(d+1)}{d^{2}+d+1}}\right)^{\frac{d^{2}+d+1}{d(d+1)}} \tag{2.2}
\end{equation*}
$$

Inserting (2.2) into (2.1), using Hölder's inequality with exponents ( $d+1, \frac{d+1}{d}$ ), and using that $r \geq r_{\diamond}$, the Poincaré-Sobolev inequality, and Jensen's inequality, we obtain

$$
\begin{aligned}
f_{B_{r}} a|\nabla u|^{2} & \lesssim f_{B_{2 r}} \eta^{2}|h|^{2}+\left(f_{B_{2 r}}|\nabla u|^{\frac{d(d+1)}{d^{2}+d+1}}\right)^{\frac{d^{2}+d+1}{d(d+1)}}+\int|\nabla \eta|^{2} a(u-\bar{U})^{2} \\
& \lesssim f_{B_{2 r}} \eta^{2}|h|^{2}+\left(f_{B_{2 r}}|\nabla u|^{\frac{d(d+1)}{d^{2}+d+1}}\right)^{2 \frac{d^{2}+d+1}{d(d+1)}}+r^{-2}\left(f_{B_{2 r}}|u-\bar{U}|^{\frac{2(d+1)}{d}}\right)^{\frac{d+1}{d}} \\
& \lesssim f_{B_{2 r}} \eta^{2}|h|^{2}+\left(f_{B_{2 r}}|\nabla u|^{\frac{2 d(d+1)}{d^{2}+2(d+1)}}\right)^{\frac{d^{2}+2(d+1)}{d(d+1)}}
\end{aligned}
$$

It remains to reintroduce $a$. By Hölder's inequality with exponents $\left(\frac{d^{2}+2(d+1)}{d}, \frac{d^{2}+2(d+1)}{d^{2}+d+2}\right)$,

$$
\begin{aligned}
\left(f_{B_{2 r}}|\nabla u|^{\frac{2 d(d+1)}{d^{2}+2(d+1)}}\right)^{\frac{d^{2}+2(d+1)}{d(d+1)}} & =\left(f_{B_{2 r}} a^{-\frac{d(d+1)}{d^{2}+2(d+1)}}|\sqrt{a} \nabla u|^{\frac{2 d(d+1)}{d^{2}+2(d+1)}}\right)^{\frac{d^{2}+2(d+1)}{d(d+1)}} \\
& \leq\left(f_{B_{2 r}} a^{-(d+1)}\right)^{\frac{1}{d+1}}\left(f_{B_{2 r}}|\sqrt{a} \nabla u|^{\frac{2 d(d+1)}{d^{2}+d+2}}\right)^{\frac{d^{2}+d+2}{d(d+1)}} \\
& \stackrel{r \geq r_{0}}{\lesssim}\left(f_{B_{2 r}}|\sqrt{a} \nabla u|^{\frac{2 d(d+1)}{d^{2}+d+2}}\right)^{\frac{d^{2}+d+2}{d(d+1)}},
\end{aligned}
$$

from which the claim follows by our choice $\alpha=\frac{2 d(d+1)}{d^{2}+d+2} \in[1,2)$.
We now recall Gehring's inequality in a form which is convenient for our purposes (see for instance [21, Theorem 6.38]):
Lemma 2.2 (Gehring's lemma). Let $s>1$, and let $U$ and $V$ be two non-negative measurable functions in $L_{\mathrm{loc}}^{q}\left(\mathbb{R}^{d}\right)$ such that there exists $C>0$ for which for all $r>0$ and $x \in \mathbb{R}^{d}$

$$
\left(f_{B_{r}(x)} U^{s}\right)^{\frac{1}{s}} \leq C\left(f_{B_{2 r}(x)} U+\left(f_{B_{2 r}(x)} V^{s}\right)^{\frac{1}{s}}\right)
$$

Then, there exists $\bar{s}>s$ depending on $q$ and $C$ such that for all $r>0$ and $x \in \mathbb{R}^{d}$, we have

$$
\left(f_{B_{r}(x)} U^{\bar{s}}\right)^{\frac{1}{\bar{s}}} \lesssim f_{B_{2 r}(x)} U+\left(f_{B_{2 r}(x)} V^{\bar{s}}\right)^{\frac{1}{\bar{s}}}
$$

We conclude this paragraph with the proof of Theorem 1.3.

Proof of Theorem 1.3. We first prove that for all $r>0$ we have

$$
\begin{equation*}
f_{B_{r}}\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right) d x \lesssim\left(f_{B_{2 r}}\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right)^{\frac{\alpha}{2}} d x\right)^{\frac{2}{\alpha}}+f_{B_{2 r}} f_{B_{\diamond}(x)}|h|^{2} d x \tag{2.3}
\end{equation*}
$$

and notice that the origin plays no role in this estimate.
If $r \leq 3 r_{\diamond}(0)$, the first right-hand side term controls the left-hand side by a covering argument. It remains to address the case $r>3 r_{\diamond}(0)$. Since $r_{\diamond}$ is $\frac{1}{8}$-Lipschitz we have

$$
f_{B_{r}}\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right) d x \lesssim f_{B_{\frac{67}{48} r}} a|\nabla u|^{2}
$$

see $[10,(\mathrm{C} 7)]$. We now appeal to the reverse Hölder inequality in form of

$$
f_{B_{\frac{67}{48} r}} a|\nabla u|^{2} \lesssim f_{B_{\frac{17}{12} r}}|h|^{2}+\left(f_{B_{\frac{17}{12} r}}(\sqrt{a}|\nabla u|)^{\alpha}\right)^{\frac{2}{\alpha}}
$$

which, by a covering argument (see $[10,(\mathrm{C} 7)]$ again), yields

$$
f_{B_{\frac{67}{48} r}} a|\nabla u|^{2} \lesssim f_{B_{2 r}} f_{B_{\diamond}(x)}|h|^{2} d x+\left(f_{B_{2 r}} f_{B_{\diamond}(x)}(\sqrt{a}|\nabla u|)^{\alpha} d x\right)^{\frac{2}{\alpha}}
$$

Using Jensen's inequality on the second right-hand side (since $\alpha<2$ ), this proves (2.3). By Lemma 2.2 applied to

$$
U(x):=\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right)^{\frac{\alpha}{2}}, \quad V(x):=\left(f_{B_{\diamond}(x)}|h|^{2}\right)^{\frac{\alpha}{2}}, \quad s:=\frac{2}{\alpha}>1
$$

one obtains for all $r>0$ and some $\gamma>1$

$$
\begin{equation*}
\left(\int_{B_{r}}\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right)^{\gamma} d x\right)^{\frac{1}{\gamma}} \lesssim r^{d\left(\frac{1}{\gamma}-1\right)} \int_{B_{2 r}} f_{B_{\diamond}(x)} a|\nabla u|^{2} d x+\left(\int_{B_{2 r}}\left(f_{B_{\diamond}(x)}|h|^{2}\right)^{\gamma}\right)^{\frac{1}{\gamma}} \tag{2.4}
\end{equation*}
$$

The desired Meyers' estimate follows by monotone convergence in the limit $r \uparrow+\infty$ combined with the $L^{2}$-energy estimate

$$
\int_{\mathbb{R}^{d}} f_{B_{\diamond}(x)} a|\nabla u|^{2} d x \lesssim \int_{\mathbb{R}^{d}} a|\nabla u|^{2} \lesssim \int_{\mathbb{R}^{d}}|h|^{2}
$$

2.2. Proof of Corollary 1.4. Let $\gamma$ be as in (2.4). Without loss of generality we can assume $r \geq 3 r_{\diamond}$ so that by [10, Lemma C.2] we have for all non-negative functions $h$

$$
\begin{equation*}
\int_{B_{r}} h \lesssim \int_{B_{2 r}} f_{B_{\diamond}(x)} h d x, \quad \int_{B_{r}} f_{B_{\diamond}(x)} h d x \lesssim \int_{B_{2 r}} h \tag{2.5}
\end{equation*}
$$

We may also assume $R \geq 4 r$. By (2.5) we have

$$
\begin{aligned}
\int_{B_{r}} a|\nabla u|^{2} & \lesssim r^{d} f_{B_{2 r}} f_{B_{\diamond}(x)} a|\nabla u|^{2} d x \\
& \leq r^{d}\left(f_{B_{2 r}}\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right)^{\gamma} d x\right)^{\frac{1}{\gamma}} \\
& \lesssim r^{d\left(1-\frac{1}{\gamma}\right)}\left(\int_{B_{\frac{R}{2}}}\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right)^{\gamma} d x\right)^{\frac{1}{\gamma}}
\end{aligned}
$$

By (2.4) (used with $h \equiv 0$ ), this entails

$$
\int_{B_{r}} a|\nabla u|^{2} \lesssim r^{d\left(1-\frac{1}{\gamma}\right)} R^{-d\left(1-\frac{1}{\gamma}\right)} \int_{B_{\frac{R}{2}}}\left(f_{B_{\diamond}(x)} a|\nabla u|^{2}\right) d x
$$

and the claim follows from (2.5) with $0<\beta=d\left(1-\frac{1}{\gamma}\right) \leq d$.
2.3. Proof of Theorem 1.5. The proof is based on the quenched Meyers estimate in the large of Theorem 1.3, on the moment bounds on $r_{\diamond}$ of Proposition 1.2 (which allows us to use duality at the price of a loss of stochastic integrability), real interpolation, and a refined dual version of the Calderón-Zygmund lemma due to Shen [38, Theorem 3.2], based on ideas by Caffarelli and Peral [9]. Since it follows the proof of [19] almost line by line and is identical to the proof of [10, Theorem 4.11], we leave the details to the reader.

## 3. Control of correctors and of the minimal Radius

In this section we prove Theorems 1.8 and 1.9. We proceed using an approximation argument, and for all $M \geq 1$, we set $a_{M}:=(a \wedge M) \vee \frac{1}{M}$, which is uniformly elliptic and bounded. By [25, 24], Theorems 1.8 and 1.9 hold true for all $M<\infty$ (with bounds that depend on $M$ a priori). By uniqueness of correctors, it is an exercise to show that $\mathbb{E}\left[\left|\left(\nabla \phi\left(a_{M}\right), \nabla \sigma\left(a_{M}\right)\right)-(\nabla \phi(a), \nabla \sigma(a))\right|^{2 \gamma}\right] \rightarrow 0$ for all $\gamma<1$ as $M \uparrow+\infty$, so that uniform moment bounds on $\left(\nabla \phi\left(a_{M}\right), \nabla \sigma\left(a_{M}\right)\right)$ are retained by $(\nabla \phi(a), \nabla \sigma(a))$ in the limit $M \uparrow+\infty$. The control of $(\phi(a), \sigma(a))$ will follow similarly from the uniform control of $\left(\phi\left(a_{M}\right), \sigma\left(a_{M}\right)\right)$, which in turn allows to bound $r_{\star}$. Since $r_{\star}$ is defined using $L^{\frac{2 p_{\diamond}}{p_{\diamond}-1}}$ instead of $L^{2}$, we display the proofs for completeness.

Before we turn to the proofs, notice that Theorems 1.3 and 1.5 hold uniformly with $a$ replaced by $a_{M}$ for all $M \geq 1$ (with the same constants, exponents, and random radii). In what follows, it is therefore enough to assume

- that $a$ is bounded and uniformly elliptic,
- that Theorems 1.8 and 1.9 hold true for some constants,
and to obtain bounds which only depend on Theorems 1.3 and 1.5 - and are therefore uniform wrt $M$.
3.1. General strategy. Our main result is as follows.

Theorem 3.1 (Decay of averages of corrector gradient). There exists $c>0$ depending on $\mathcal{C}(0)$ such that for all $g \in L^{2}\left(\mathbb{R}^{d}\right)$ and unit vectors $e \in \mathbb{R}^{d}$ the random field $F:=$ $\int_{\mathbb{R}^{d}}(\nabla \phi, \nabla \sigma) \cdot g$ satisfies for all $q \geq 1$

$$
\mathbb{E}\left[|F|^{2 q}\right]^{\frac{1}{q}} \leq e^{c q} \int_{\mathbb{R}^{d}}|g|^{2}
$$

Notice that this encodes the CLT scaling. Indeed, for $g_{R}(x)=\left|B_{R}\right|^{-d} \mathbb{1}_{B_{R}}$ this entails $\mathbb{E}\left[\left|f_{B_{R}}(\nabla \phi, \nabla \sigma)\right|^{2 q}\right]^{\frac{1}{q}} \lesssim_{q} R^{-d}$. Passing from this result on averages of corrector gradients to correctors is routine, see e.g. [24]. Passing from correctors to $r_{\star}$ is routine too, see e.g. [10]. We thus only focus on the proof of Theorem 3.1, and follow the strategy of [10, Section 6].

We start with a control of averages of correctors that involves the energy density of the corrector.

Proposition 3.2. For all $0<\theta<1$, there exists a constant $c>0$ depending on $\mathcal{C}(0)$ such that for all $g \in L^{2}\left(\mathbb{R}^{d}\right)$ and unit vectors $e \in \mathbb{R}^{d}$ the random field $F:=\int_{\mathbb{R}^{d}}(\nabla \phi, \nabla \sigma) \cdot g$ satisfies for all $q \geq 1$ such that $2 q^{\prime} \leq 2+\kappa$ (where $\kappa$ is as in Theorem 1.5)

$$
\mathbb{E}\left[|F|^{2 q}\right]^{\frac{1}{q}} \leq e^{c q} \mathbb{E}\left[\left(\int_{B(0)} a\left(|\nabla \phi|^{2}+1\right)\right)^{q(1+\theta)}\right]^{\frac{1}{q(1+\theta)}} \int_{\mathbb{R}^{d}}|g|^{2} .
$$

Based on Proposition 3.2 itself, we shall prove moment bounds on the energy density of the corrector.

Proposition 3.3. There exists a constant $c>0$ depending on $\mathcal{C}(0)$ such that for all $q \geq 1$,

$$
\begin{equation*}
\mathbb{E}\left[\left(\int_{B(0)} a\left(|\nabla \phi|^{2}+1\right)\right)^{q}\right]^{\frac{1}{q}} \leq e^{c q} \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}\left[\left(\int_{B(0)}|\nabla \sigma|^{2}\right)^{q}\right]^{\frac{1}{q}} \leq e^{c q} . \tag{3.2}
\end{equation*}
$$

The combination of Propositions 3.2 and 3.3 directly yields Theorem 3.1. For the proof of Proposition 3.3, we introduce a third minimal radius, which quantifies at which scale the energy density of the corrector behaves like a uniformly bounded function of $a$.

Definition 3.4. We define the minimal radius $r_{\boldsymbol{c}}$ as

$$
r_{\bullet}:=\max _{|e|=1} \inf _{r=2^{k} \geq r_{\diamond}, k \in \mathbb{N}}\left\{\forall R \geq r \text { dyadic, } \quad f_{B_{R}} a\left|\nabla \phi_{e}\right|^{2} \leq C f_{B_{2 R}} a\right\},
$$

where $\phi_{e}$ denotes the corrector in the unit direction $e$, and $C>0$ will be chosen in the proof of Proposition 3.3.

The core of the proof of Proposition 3.3 is a control of the level-sets of $r_{\boldsymbol{\omega}}$ using averages of the corrector gradient.
3.2. Proof of Proposition 3.2. Let the direction $e \in \mathbb{R}^{d}$ with $|e|=1$, and $i j$ be fixed, and consider $F_{1}:=\int_{\mathbb{R}^{d}} \nabla \phi \cdot g$, and $F_{2}:=\int_{\mathbb{R}^{d}} \nabla \sigma_{i j} \cdot g$. For background on Malliavin calculus and functional calculus, we refer the reader to [32, Section 2] (dimension 1) and [13, Section 2.1] (any dimension).
Step 1. Malliavin derivatives of $F_{1}$ and $F_{2}$.
We claim that

$$
\begin{align*}
D F_{1} & =\int_{\mathbb{R}^{d}} \nabla u \cdot D a(\nabla \phi+e),  \tag{3.3}\\
D F_{2} & =\int_{\mathbb{R}^{d}}\left(\partial_{i} v e_{j}-\partial_{j} v e_{i}+\nabla w\right) \cdot D a(\nabla \phi+e), \tag{3.4}
\end{align*}
$$

where $u, v, w$ solve

$$
\begin{align*}
-\nabla \cdot a \nabla u & =\nabla \cdot g,  \tag{3.5}\\
-\Delta v & =\nabla \cdot g,  \tag{3.6}\\
-\nabla \cdot a \nabla w & =\nabla \cdot a\left(\partial_{i} v e_{j}-\partial_{j} v e_{i}\right) . \tag{3.7}
\end{align*}
$$

(Notice that these equations are all well-posed in $\dot{H}^{1}\left(\mathbb{R}^{d}\right)$ since $a$ is uniformly elliptic and bounded by assumption.) Indeed $D F_{1}=\int_{\mathbb{R}^{d}} \nabla D \phi \cdot g$ and, using the corrector equation (1.5), $D \phi$ solves

$$
-\nabla \cdot a \nabla D \phi=\nabla \cdot D a(\nabla \phi+e) .
$$

Using (3.5), we then obtain the representation formula (3.3). We now turn to $F_{2}$. As for $F_{1}$ we have $D F_{2}=\int_{\mathbb{R}^{d}} \nabla D \sigma_{i j} \cdot g$ and, using (1.6), $D \sigma_{i j}$ solves

$$
-\triangle D \sigma_{i j}=\partial_{i} D\left(a(\nabla \phi+e) \cdot e_{j}\right)-\partial_{j} D\left(a(\nabla \phi+e) \cdot e_{i}\right) .
$$

Using first (3.6), we have

$$
\begin{aligned}
D F_{2} & =\int\left(\partial_{i} v e_{j}-\partial_{j} v e_{i}\right) \cdot D(a(\nabla \phi+e)) \\
& =\int\left(\partial_{i} v e_{j}-\partial_{j} v e_{i}\right) \cdot D a(\nabla \phi+e)+\int\left(\partial_{i} v e_{j}-\partial_{j} v e_{i}\right) \cdot a \nabla D \phi,
\end{aligned}
$$

and we reformulate the last term using (3.7) to obtain (3.4).
Notice that our choice $a(G(x))=\exp (G(x))$ yields $D_{z} a=a \delta(\cdot-z)$ for all $z \in \mathbb{R}^{d}$, which we shall use in the estimates (again, the specific form is convenient but not essential).

Step 2. Application of the logarithmic-Sobolev inequality.
By LSI, Step 1, and the identity $D_{z} a=a \delta(\cdot-z)$, we have for all $q \geq 1$,

$$
\mathbb{E}\left[|F|^{2 q}\right]^{\frac{1}{q}} \lesssim q \mathbb{E}\left[\left(\int_{\mathbb{R}^{d}}\left(\int_{B(x)}(|\nabla u|+|\nabla v|+|\nabla w|) a(|\nabla \phi|+1)\right)^{2} d x\right)^{q}\right]^{\frac{1}{q}} .
$$

We only focus on the term involving $\nabla w$, which is more involved since $w$ is obtained by solving two equations in a row. By Cauchy-Schwarz',

$$
\left(\int_{B(x)}|\nabla w| a(|\nabla \phi|+1)\right)^{2} \leq \int_{B(x)} a|\nabla w|^{2} \int_{B(x)} a\left(|\nabla \phi|^{2}+1\right) .
$$

By duality in probability, this yields

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_{\mathbb{R}^{d}}\left(\int_{B(x)}|\nabla w| a(|\nabla \phi|+1)\right)^{2} d x\right)^{q}\right]^{\frac{1}{q}} \\
& \leq \sup _{X} \mathbb{E}\left[\int_{\mathbb{R}^{d}}\left(\int_{B(x)} a\left(|\nabla \phi|^{2}+1\right)\right)\left(\int_{B(x)} a|\nabla X w|^{2}\right) d x\right],
\end{aligned}
$$

where the supremum runs over random variables $X$ (which are thus independent of the space variable) such that $\mathbb{E}\left[|X|^{2 q^{\prime}}\right]=1$. We then set $\eta_{0}:=\frac{\theta}{(1+\theta)(q-1)}$, to the effect
that $q^{\prime}>1+\eta_{0}$ and $\frac{q^{\prime}}{q^{\prime}-\left(1+\eta_{0}\right)}=q(1+\theta)$, and use Hölder's inequality with exponents $\left(\frac{q^{\prime}}{q^{\prime}-\left(1+\eta_{0}\right)}, \frac{q^{\prime}}{1+\eta_{0}}\right)$, so that the above turns into

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_{\mathbb{R}^{d}}\left(\int_{B(x)}|\nabla w| a(|\nabla \phi|+1)\right)^{2} d x\right)^{q}\right]^{\frac{1}{q}} \\
& \quad \leq \mathbb{E}\left[\left(\int_{B(0)} a\left(|\nabla \phi|^{2}+1\right)\right)^{q(1+\theta)}\right]^{\frac{1}{q(1+\theta)}} \sup _{X} \int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(\int_{B(x)} a|\nabla X w|^{2}\right)^{\frac{q^{\prime}}{1+\eta_{0}}}\right]^{\frac{1+\eta_{0}}{q^{\prime}}} d x,
\end{aligned}
$$

where we used the stationarity of $x \mapsto \int_{B(x)} a\left(|\nabla \phi|^{2}+1\right)$. For convenience, we rewrite $1+\eta_{0}$ as $(1+\eta)^{2}$, and apply Theorem 1.5 to (3.7), which yields provided $2 q^{\prime} \leq 2+\kappa$,

$$
\int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(\int_{B(x)} a|\nabla X w|^{2}\right)^{\frac{q^{\prime}}{(1+\eta)^{2}}}\right]^{\frac{(1+\eta)^{2}}{q^{\prime}}} d x \lesssim \zeta\left(\eta_{0}\right) \int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(\int_{B(x)} a|\nabla X v|^{2}\right)^{\frac{q^{\prime}}{1+\eta}}\right]^{\frac{1+\eta}{q^{\prime}}} d x
$$

where $\zeta: t \mapsto t^{-\frac{1}{4}}|\log t|^{\frac{1}{2}}$ (since for $\left.0<\eta_{\circ}<\frac{1}{2}, \zeta(\eta)=\zeta\left(\sqrt{1+\eta_{0}}-1\right) \lesssim \zeta\left(\eta_{0}\right)\right)$. By Hölder's inequality with exponents $\left(\frac{1+\eta}{\eta}, 1+\eta\right)$, followed by the version of Theorem 1.5 for the Laplacian (see for instance [10, Theorem 1.14]) applied to (3.6) (with exponent $q^{\prime} \lesssim 1$ ) we further have

$$
\begin{aligned}
& \int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(\int_{B(x)} a|\nabla X v|^{2}\right)^{\frac{q^{\prime}}{1+\eta}}\right]^{\frac{1+\eta}{q^{\prime}}} d x \leq \mathbb{E}\left[a^{\frac{q^{\frac{q}{\eta}}}{}}\right]^{\frac{\eta}{q^{\eta}}} \int_{\mathbb{R}^{d}} \mathbb{E}\left[\left(\int_{B(x)}|\nabla X v|^{2}\right)^{q^{\prime}}\right]^{\frac{1}{q^{\prime}}} d x \\
& \lesssim \mathbb{E}\left[a^{\frac{q}{}^{\frac{1}{\eta}}}\right]^{\frac{\eta}{q^{\eta}}} \mathbb{E}\left[|X|^{2 q^{\prime}}\right]^{\frac{1}{q^{\prime}}} \int_{\mathbb{R}^{d}}|g|^{2} \\
& =\mathbb{E}\left[a^{\frac{q^{\prime}}{\eta}}\right]^{\frac{\eta}{q^{\eta}}} \int_{\mathbb{R}^{d}}|g|^{2},
\end{aligned}
$$

where we used that $g$ is deterministic and $\mathbb{E}\left[|X|^{2 q^{\prime}}\right]=1$. The desired stochastic integrability comes from a direct calculation. On the one hand, if $q$ is large enough, $2 \eta \sim \eta_{0} \sim \frac{\theta}{1+\theta} q^{-1}$. On the other hand, using the Taylor expansion of the exponential and Gaussianity, we have

$$
\begin{equation*}
\mathbb{E}\left[a^{r}\right]^{\frac{1}{r}}=\exp \left(\frac{\mathcal{C}(0)}{2} r\right) \quad \text { for any } r \geq 1 \tag{3.8}
\end{equation*}
$$

Finally, noticing in addition that $q^{\prime}<2$, there is a $c>0$ depending on $\theta$ and $\mathcal{C}(0)$ such that

$$
\zeta\left(\eta_{0}\right) \mathbb{E}\left[a^{\frac{q^{\prime}}{\eta}}\right]^{\frac{\eta}{q^{\prime}}}=\eta_{0}^{-\frac{1}{4}}\left|\log \eta_{0}\right|^{\frac{1}{2}} e^{\frac{q^{\prime}}{2 \eta}} \leq e^{c q} .
$$

This concludes the proof of the proposition.
3.3. Proof of Proposition 3.3. Wlog we may fix a direction $e$. We split the proof into four steps. For the proof of (3.1), the core of the argument is to control the moments of $r_{\boldsymbol{\infty}}$ in form of

$$
\begin{equation*}
\mathbb{E}\left[r_{\bullet}^{q}\right] \leq e^{c q^{2}}, \tag{3.9}
\end{equation*}
$$

for some $c>0$. We then deduce (3.1) using that the definition of $r_{\boldsymbol{\omega}}$ in form of

$$
\int_{B(0)} a\left(|\nabla \phi|^{2}+1\right) \lesssim r_{\bullet}^{d} f_{B_{r_{\bullet}}(0)} a \lesssim r_{\star}^{d}
$$

where the latter is a consequence of $r_{\boldsymbol{\wedge}} \geq r_{\diamond}$. We show (3.9) in the first three steps. In the last step, we prove that (3.2) is a consequence of (3.1).

Since $r_{\boldsymbol{\wedge}}=r_{\diamond} \mathbb{1}_{r_{\star}=r_{\circ}}+r_{\boldsymbol{\wedge}} \mathbb{1}_{r_{\bullet}}>r_{\diamond}$ and $r_{\diamond}$ is well-controlled by Proposition 1.2, it is enough to focus on $r_{\boldsymbol{\omega}} \mathbb{1}_{r_{\boldsymbol{*}}}>r_{0}$. In what follows we use $r_{\boldsymbol{\omega}}$ as a short-hand notation for $r_{\boldsymbol{\omega}} \mathbb{1}_{r_{\boldsymbol{\infty}}}>r_{o}$, and recall that we have set $r_{\boldsymbol{L}}(R):=\left(R^{-\frac{\varepsilon}{2}} \sup _{B_{R}}\left(a+a^{-1}\right)\right)^{2}$.
Step 1. Control of the level-sets of $r$
We claim that for all $0<\mu<1$ there exists $c>0$ such that for all dyadic $R$ and all $q \geq 1$, we have

$$
\begin{align*}
& \mathbb{E}\left[\mathbb{1}\left(r_{\boldsymbol{\wedge}}=R\right)\right] \leq c^{q} R^{-q(d-\beta+2(1-\mu)-\varepsilon)} \mathbb{E}\left[r_{\boldsymbol{\omega}}(R)^{q} r_{\boldsymbol{*}}^{q(d-\beta)}\right] \\
&+c^{q} R^{q \varepsilon} \mathbb{E}\left[r_{\boldsymbol{\omega}}(R)^{q}\left(f_{B_{R}}\left|f_{B_{R^{\mu}}(x)} \nabla \phi\right|^{2} d x\right)^{q}\right] . \tag{3.10}
\end{align*}
$$

Assume that $r_{\boldsymbol{\wedge}}=R>r_{\diamond}$. Then, by definition, we both have

$$
\begin{aligned}
& f_{B_{R}} a|\nabla \phi|^{2} \leq C f_{B_{2 R}} a, \\
& f_{B_{\frac{R}{2}}} a|\nabla \phi|^{2} \geq C f_{B_{R}} a .
\end{aligned}
$$

We now appeal to the Caccioppoli inequality (2.1) applied to the corrector equation $-\nabla$. $a \nabla \phi=\nabla \cdot a e$, which we rewrite in the form,

$$
f_{B_{\frac{R}{2}}} a|\nabla \phi|^{2} \leq C^{\prime}\left(f_{B_{R}} a+\inf _{c} \frac{1}{R^{2}} f_{B_{R}} a|\phi-c|^{2}\right),
$$

for some universal constant $C^{\prime}$. Provided we choose $C=2 C^{\prime}$ (which completes Definition 3.4), this entails

$$
\inf _{c} \frac{1}{R^{2}} f_{B_{R}} a|\phi-c|^{2} \gtrsim f_{B_{R}} a \gtrsim 1,
$$

using that $R \geq r_{\diamond}$.
Let $0<\mu<1$, and set $c_{R}:=f_{B_{R}} f_{B_{R}(x)} \phi$. By definition of $r_{\boldsymbol{\ell}}$, we have

$$
1 \lesssim \frac{1}{R^{2}} f_{B_{R}} a\left|\phi-c_{R}\right|^{2} \leq \sqrt{R^{\varepsilon} r_{\boldsymbol{k}}(R)} \frac{1}{R^{2}} f_{B_{R}}\left|\phi-c_{R}\right|^{2} .
$$

By the triangle inequality,

$$
\frac{1}{R^{2}} f_{B_{R}}\left|\phi-c_{R}\right|^{2} \lesssim \frac{1}{R^{2}} f_{B_{R}}\left|\phi-f_{B_{R^{\mu}}(x)} \phi\right|^{2} d x+\frac{1}{R^{2}} f_{B_{R}}\left|f_{B_{R^{\mu}}(x)} \phi-c_{R}\right|^{2} .
$$

Using Poincaré's inequality (on domains of size $R^{\mu}$ for the first term, and on $B_{R}$ for the second term), this yields

$$
\frac{1}{R^{2}} f_{B_{R}}\left|\phi-c_{R}\right|^{2} \lesssim R^{2(\mu-1)} f_{B_{R}}|\nabla \phi|^{2}+f_{B_{R}}\left|f_{B_{R^{\mu}}(x)} \nabla \phi\right|^{2} .
$$

Whereas the second right-hand side term already has the correct form, we need to put back the weight on the first right-hand side term. For that we pay again $\sqrt{r_{\boldsymbol{d}}(R) R^{\varepsilon}}$. We have thus proved

$$
\begin{aligned}
1 & \lesssim r_{\boldsymbol{\omega}}(R) R^{\varepsilon}\left(R^{2(\mu-1)}+f_{B_{R}}\left|f_{B_{R^{\mu}}(x)} \nabla \phi\right|^{2}\right) \\
& \stackrel{r_{\star}=R}{=} r_{\boldsymbol{\omega}}(R) R^{\varepsilon}\left(R^{2(\mu-1)-d+\beta} r_{\star}^{d-\beta}+f_{B_{R}}\left|f_{B_{R^{\mu}}(x)} \nabla \phi\right|^{2}\right) .
\end{aligned}
$$

By Markov' inequality, this implies the claim.
Step 2. Hole-filling and control of the right-hand side of (3.10).
We claim that (3.10) entails for all $0<\theta<1$, all dyadic $R$ and all $q \geq 1$,

$$
\begin{equation*}
\mathbb{E}\left[\mathbb{1}\left(r_{\bullet}=R\right)\right] \leq e^{c_{\theta} q^{2}}\left(R^{-q(d-\beta+2(1-\mu)-\varepsilon)}+R^{-q(d \mu-\varepsilon)}\right) \mathbb{E}\left[r_{\star}^{q(d-\beta)(1+\theta)^{3}}\right]^{\frac{1}{(1+\theta)^{3}}}, \tag{3.11}
\end{equation*}
$$

for some $c_{\theta}$ depending on $\theta$. The first right-hand side term of (3.11) comes from the first right-hand side term of (3.10) using Hölder's inequality and the moment bound, for some $c>0$ and any $q \geq 1$,

$$
\begin{equation*}
\mathbb{E}\left[r_{\boldsymbol{e}}(R)^{q}\right] \leq e^{c q^{2}}, \tag{3.12}
\end{equation*}
$$

that we deduce from (1.2). The second right-hand side term is more subtle. By Hölder's inequality with exponents $\left(\frac{1+\theta}{\theta}, 1+\theta\right)$, Jensen's inequality, and stationarity of $x \mapsto f_{B_{R^{\mu}}(x)} \nabla \phi$ we have

$$
\begin{aligned}
& \mathbb{E}\left[r_{\boldsymbol{k}}(R)^{q}\left(f_{B_{R}}\left|f_{B_{R^{\mu}}(x)} \nabla \phi\right|^{2} d x\right)^{q}\right] \leq \mathbb{E}\left[r_{\boldsymbol{\alpha}}(R)^{q^{\frac{1+\theta}{\theta}}}\right]^{\frac{\theta}{1+\theta}} \mathbb{E}\left[\left|f_{B_{R^{\mu}}} \nabla \phi\right|^{2 q(1+\theta)}\right]^{\frac{1}{1+\theta}} \\
& \stackrel{(3.12)}{\leq} e^{c_{\theta, 1} q^{2}} \mathbb{E}\left[\left|f_{B_{R^{\mu}}} \nabla\right|^{2 q(1+\theta)}\right]^{\frac{1}{1+\theta}}
\end{aligned}
$$

for a constant $c_{\theta, 1}$. Then, by Proposition 3.2 applied to $g=\left|B_{R^{\mu}}\right|^{-1} \mathbb{1}_{B_{R^{\mu}}}$, we get

$$
\begin{aligned}
\mathbb{E}\left[\left|f_{B_{R^{\mu}}} \nabla \phi\right|^{2 q(1+\theta)}\right]^{\frac{1}{q(1+\theta)}} & \leq e^{c_{\theta, 2}^{\prime} q} \mathbb{E}\left[\left(\int_{B} a\left(|\nabla \phi|^{2}+1\right)\right)^{q(1+\theta)^{2}}\right]^{\frac{1}{q(1+\theta)^{2}}} \int_{\mathbb{R}^{d}}|g|^{2} \\
& =e^{c_{\theta, 2}^{\prime} q} \mathbb{E}\left[\left(\int_{B} a\left(|\nabla \phi|^{2}+1\right)\right)^{q(1+\theta)^{2}}\right]^{\frac{1}{q(1+\theta)^{2}}} R^{-d \mu},
\end{aligned}
$$

for some constant $c_{\theta, 2}^{\prime}$. It remains to control $\int_{B} a\left(|\nabla \phi|^{2}+1\right)$ by $r_{A}$, and we claim that $\int_{B} a\left(|\nabla \phi|^{2}+1\right) \lesssim r_{\diamond}^{d}+r_{\star}^{d-\beta} r_{\diamond}^{\beta}$. Indeed, if $r_{\star}<r_{\diamond}$, then

$$
\int_{B} a\left(|\nabla \phi|^{2}+1\right) \leq \int_{B_{\star}} a\left(|\nabla \phi|^{2}+1\right) \lesssim \int_{B_{\star}} a \lesssim r_{\diamond}^{d} .
$$

Otherwise, $r_{\boldsymbol{\wedge}} \geq r_{\diamond}$, and by the hole-filling estimate (1.4) in Corollary 1.4,

$$
\int_{B} a\left(|\nabla \phi|^{2}+1\right) \lesssim r_{\diamond}^{d} f_{B_{\diamond}} a\left(|\nabla \phi|^{2}+1\right) \lesssim r_{\diamond}^{d}\left(\frac{r_{\star}}{r_{\diamond}}\right)^{d-\beta} f_{B_{\star}} a\left(|\nabla \phi|^{2}+1\right) \lesssim r_{\star}^{d-\beta} r_{\diamond}^{\beta} .
$$

All in all, this entails by Hölder's inequality with exponents $\left(\frac{1+\theta}{\theta}, 1+\theta\right)$ and the moment bound (1.1) on $r_{\diamond}$,

$$
\mathbb{E}\left[\left|f_{B_{R^{\mu}}} \nabla \phi\right|^{2 q(1+\theta)}\right]^{\frac{1}{q(1+\theta)}} \leq e^{c_{\theta, 2} q} R^{-d \mu} \mathbb{E}\left[r_{\phi}^{q(d-\beta)(1+\theta)^{3}}\right]^{\frac{1}{q(1+\theta)^{3}}}
$$

for a constant $c_{\theta, 2}$ and (3.11) follows in combination with (3.10).
Step 3. Buckling argument.
By expressing the moments of $r_{\boldsymbol{\omega}}$ using its level-sets we have for all $q \geq 1$, and all $K \geq 1$,

$$
\begin{aligned}
& \mathbb{E}\left[r_{\boldsymbol{\alpha}}^{q\left(d-\frac{\beta}{K}\right)}\right] \quad \leq 1+\sum_{n=1}^{\infty} 2^{n q\left(d-\frac{\beta}{K}\right)} \mathbb{E}\left[\mathbb{1}\left(r_{\boldsymbol{\omega}}=2^{n}\right)\right] \\
& \stackrel{(3.11)}{\leq} 1+e^{c_{\theta} q^{2}} \mathbb{E}\left[r_{\boldsymbol{\alpha}}^{q(d-\beta)(1+\theta)^{3}}\right]^{\frac{1}{(1+\theta)^{3}}} \\
& \times \sum_{n=1}^{\infty}\left(2^{-n q\left(\frac{\beta}{K}+2(1-\mu)-\beta-\varepsilon\right)}+2^{-n q\left(d(1-\mu)+\frac{\beta}{K}-\varepsilon\right)}\right)
\end{aligned}
$$

We now choose our exponents. We first fix $0<\mu<1$ so that $d(1-\mu)=\frac{\beta}{2}$, then set $\varepsilon:=\frac{\beta}{5 d}$ and $\frac{1}{K}:=1-\frac{1}{5 d}$ to the effect that

$$
\frac{1}{2}\left(2^{-n q\left(\frac{\beta}{K}+2(1-\mu)-\beta-\varepsilon\right)}+2^{-n q\left(d(1-\mu)+\frac{\beta}{K}-\varepsilon\right)}\right) \leq 2^{-n q \frac{\beta}{5 d}}
$$

With this choice, the series is summable and the above turns into

$$
\mathbb{E}\left[r_{\boldsymbol{\omega}}^{q\left(d-\frac{\beta}{K}\right)}\right] \leq 1+e^{c q^{2}} \mathbb{E}\left[r_{\boldsymbol{\phi}}^{(d-\beta)(1+\theta)^{3} q}\right]^{\frac{1}{(1+\theta)^{3}}} .
$$

We may then absorb part of the right-hand side into the left-hand side by Young's inequality upon choosing $0<\theta<1$ so small but independent of $q$ that $(d-\beta)(1+\theta)^{3}<d-\frac{\beta}{K}$ (which is possible since $K>1$ ), and the claimed moment bound (3.1) follows.
Step 4. Control of $\nabla \sigma$.
We could control moment bounds of $\nabla \sigma$ by using Malliavin calculus and moment bounds on $\nabla \phi$. Here we directly apply standard Calderón-Zygmund estimates for the Laplacian (the scaling of the multiplicative constants follow from the Marcinkiewicz interpolation theorem [33, Theorem 1.3.2], [36]) to (1.6): For all $R \geq 1$ and $q \geq 1$

$$
\left(f_{B_{R}}\left(f_{B(x)}|\nabla \sigma|^{2}\right)^{q}\right)^{\frac{1}{q}} \lesssim\left(q+q^{\prime}\right)\left(\left(f_{B_{2 R}} f_{B(x)}|\nabla \sigma|^{2}\right)^{q}+f_{B_{2 R}}\left(f_{B(x)}|a(\nabla \phi+e)|^{2}\right)^{q}\right)^{\frac{1}{q}}
$$

By ergodicity and stationarity, as $R \uparrow+\infty$, each spatial average converges almost surely to the associated expectation, so that we derive

$$
\mathbb{E}\left[\left(f_{B(0)}|\nabla \sigma|^{2}\right)^{q}\right]^{\frac{1}{q}} \lesssim\left(q+q^{\prime}\right)\left(\mathbb{E}\left[|\nabla \sigma|^{2}\right]^{q}+\mathbb{E}\left[\left(\int_{B(0)} a^{2}\left(|\nabla \phi|^{2}+1\right)\right)^{q}\right]\right)^{\frac{1}{q}}
$$

Finally, using

$$
\int_{B(0)} a^{2}\left(|\nabla \phi|^{2}+1\right) \leq\left(\sup _{B(0)} a\right) \int_{B(0)} a\left(|\nabla \phi|^{2}+1\right)
$$

we obtain (3.2) from (3.1) and the local boundedness of $a$ (see (A.5)).
3.4. Arguments for Theorems 1.8 and 1.9. We split the proof into two steps and we show separately Theorem 1.8 and Theorem 1.9 using the standard strategy from [26, 35].

Step 1. Proof of Theorem 1.8.
Since $p_{\diamond}=d+1$, we have $\frac{p_{\diamond}-1}{2 p_{\diamond}}>\frac{p_{\diamond}+1}{2 p_{\diamond}}-\frac{1}{d}$, so that by Poincaré-Sobolev' inequality,

$$
\begin{aligned}
& \left(f_{B(x)}\left|(\phi, \sigma)-f_{B(0)}(\phi, \sigma)\right|^{\frac{2 p_{\diamond}}{p_{\diamond}-1}}\right)^{\frac{p_{\diamond}-1}{2 p_{\diamond}}} \\
& \leq\left|f_{B(x)}(\phi, \sigma)-f_{B(0)}(\phi, \sigma)\right|+\left(f_{B(x)} \left\lvert\,(\phi, \sigma)-f_{B(x)}(\phi, \sigma)^{\frac{2 p_{\diamond}}{p_{\diamond}-1}}\right.\right)^{\frac{p_{\diamond}-1}{2 p_{\diamond}}} \\
& \lesssim\left|f_{B(x)}(\phi, \sigma)-f_{B(0)}(\phi, \sigma)\right|+\left(f_{B(x)}|\nabla(\phi, \sigma)|^{\frac{2 p_{\diamond}}{p_{\diamond}+1}}\right)^{\frac{p_{\diamond}+1}{2 p_{\diamond}}}
\end{aligned}
$$

Using Hölder's inequality, we have

$$
\left(f_{B(x)}|\nabla(\phi, \sigma)|^{\frac{2 p_{\diamond}}{p_{\diamond}+1}}\right)^{\frac{p_{\Omega}+1}{2 p_{\diamond}}} \leq\left(f_{B(x)} a^{-p_{\diamond}}\right)^{\frac{1}{2 p_{\diamond}}}\left(f_{B(x)} a\left(|\nabla \phi|^{2}+1\right)\right)^{\frac{1}{2}}+\left(f_{B(x)}|\nabla \sigma|^{2}\right)^{\frac{1}{2}}
$$

that has the stochastic integrability (1.10) from Proposition 3.3 and (3.8). We then follow the standard proof to control $\left|f_{B(x)}(\phi, \sigma)-f_{B(0)}(\phi, \sigma)\right|$ in [35], and we show the argument for $\phi$ only (the proof for $\sigma$ follows the same way). We have the representation formula

$$
\begin{equation*}
f_{B(x)} \phi-f_{B(0)} \phi=\int_{\mathbb{R}^{d}} \nabla w \cdot \nabla \phi \tag{3.13}
\end{equation*}
$$

where $w$ denotes the decaying solution of

$$
-\Delta w=\frac{1}{|B(0)|}\left(\mathbb{1}_{B(x)}-\mathbb{1}_{B(0)}\right)
$$

By classical potential theory, it holds

$$
\left(\int_{\mathbb{R}^{d}}|\nabla w|^{2}\right)^{\frac{1}{2}} \lesssim \mu_{d}(|x|)=\left\{\begin{aligned}
\sqrt{|x|+1} & : \quad d=1 \\
\log (|x|+2)^{\frac{1}{2}} & : \quad d=2 \\
1 & : \quad d>2
\end{aligned}\right.
$$

Since $w$ is deterministic, as a consequence of (3.13) and (3.1), we finally obtain (1.8).
Step 2. Proof of Theorem 1.9.
By the layer-cake formula and since $r_{\diamond}$ is controlled in (1.1), it is sufficient to show that for any $\lambda>0$

$$
\begin{equation*}
\mathbb{P}\left(\left\{r_{\star} \geq \lambda\right\} \cap\left\{\lambda \geq r_{\diamond}\right\}\right) \leq e^{c q^{2}} \frac{\left(\mu_{d}(\lambda)\right)^{q}}{\lambda^{q}} \tag{3.14}
\end{equation*}
$$

for some $c>0$. The estimate (3.14) is simply obtained using the Definition 1.7 of $r_{\star}$ in form of

$$
\mathbb{P}\left(\left\{r_{\star} \geq \lambda\right\} \cap\left\{\lambda \geq r_{\diamond}\right\}\right) \leq \mathbb{P}\left(\left(f_{B_{\frac{\lambda}{2}}} \left\lvert\,(\phi, \sigma)-f_{B_{\frac{\lambda}{2}}}(\phi, \sigma)^{\frac{2 p_{\diamond}}{p_{\diamond}-1}}\right.\right)^{\frac{p_{\diamond}-1}{2 p_{\diamond}}}>\frac{\lambda}{2 C}\right)
$$

which, together with Markov's inequality and Theorem 1.8, gives (3.14).

## 4. LARGE-SCALE REGULARITY THEORY

4.1. Large-scale Lipschitz estimates. The proof is based on a post-processing of $[8$, Theorem 2]. Since the proof is very similar to that of [26, Corollary 3], we only highlight the main steps of the argument. The starting point is

$$
\begin{align*}
\sup _{r \in\left[r_{\star}, R\right]} & \frac{1}{r^{2 \alpha}} \operatorname{Exc}(\nabla u+g, r) \lesssim \frac{1}{R^{2 \alpha}} \operatorname{Exc}(\nabla u+g, R) \\
& +\sup _{r \in\left[r_{\star}, R\right]} \frac{1}{r^{2 \alpha}} f_{B_{r}}\left(\left(g-f_{B_{r}} g\right) \cdot a\left(g-f_{B_{r}} g\right)+\left(h-f_{B_{r}} h\right) \cdot a^{-1}\left(h-f_{B_{r}} h\right)\right), \tag{4.1}
\end{align*}
$$

where the excess Exc is defined for any $\rho>0$ by

$$
\operatorname{Exc}(\nabla u+g, \rho):=\inf _{\xi \in \mathbb{R}^{d}} f_{B_{\rho}}\left(\nabla u-\left(\xi+\nabla \phi_{\xi}\right)\right) \cdot a\left(\nabla u-\left(\xi+\nabla \phi_{\xi}\right)\right)
$$

The estimate (4.1) can be obtained following the lines of [26, Step 1 p. 135] that is based on energy estimates and the excess decay [8, Theorem 2]. We can then post-process (4.1) following the lines of [26, Step 2 p. 136] where the additional main ingredient is the non-degeneracy of the correctors which reads : for any $\rho \geq r_{\star}$

$$
\begin{equation*}
f_{B_{\rho}}\left(\nabla \phi_{\xi}+\xi\right) \cdot a\left(\nabla \phi_{\xi}+\xi\right) \gtrsim|\xi|^{2} \tag{4.2}
\end{equation*}
$$

To see (4.2), first note that from the Definition 1.2 of $r_{\diamond}$ we have by Hölder and Poincaré's inequalities

$$
\left(f_{B_{\rho}}\left(\nabla \phi_{\xi}+\xi\right) \cdot a\left(\nabla \phi_{\xi}+\xi\right)\right)^{\frac{1}{2}} \gtrsim \frac{1}{\rho}\left(f_{B_{\rho}}\left|\phi_{\xi}+\xi \cdot x-f_{B_{\rho}} \phi_{\xi}\right|^{\frac{2 p_{\diamond}}{p_{\diamond}+1}}\right)^{\frac{p_{\diamond}+1}{2 p_{\diamond}}}
$$

Then, using the triangle inequality in form of

$$
\frac{1}{\rho}\left(f_{B_{\rho}}\left|\phi_{\xi}+\xi \cdot x-f_{B_{\rho}} \phi_{\xi}\right|^{\frac{2 p_{\diamond}}{p_{\diamond}+1}}\right)^{\frac{p_{\diamond}+1}{2 p_{\diamond}}} \geq\left|C_{d}\right| \xi\left|-\frac{1}{\rho}\left(f_{B_{\rho}}\left|\phi_{\xi}-f_{B_{\rho}} \phi_{\xi}\right|^{\frac{2 p_{\diamond}}{p_{\diamond}+1}}\right)^{\frac{p_{\diamond}+1}{2 p_{\diamond}}}\right|
$$

for some $C_{d}>0$, together with the definition of $r_{\star}$ in form of

$$
\frac{1}{\rho}\left(f_{B_{\rho}}\left|\phi_{\xi}-f_{B_{\rho}} \phi_{\xi}\right|^{\frac{2 p_{\diamond}}{p_{\diamond}+1}}\right)^{\frac{p_{\diamond}+1}{2 p_{\diamond}}} \leq \frac{1}{\rho}\left(f_{B_{\rho}}\left|\phi_{\xi}-f_{B_{\rho}} \phi_{\xi}\right|^{\frac{2 p_{\diamond}}{p_{\diamond}-1}}\right)^{\frac{p_{\diamond}-1}{2 p_{\diamond}}} \leq \frac{|\xi|}{C}
$$

we obtain (4.2) (up to increasing the value of $C$ ).
4.2. Quenched and annealed Calderón-Zygmund estimates. The proof of Theorem 1.11 is based on the Schauder regularity theory in Proposition 1.10 and a refined dual version of the Calderón-Zygmund lemma due to Shen [38, Theorem 3.2]. We then obtain Theorem 1.12 as a post-processing of Theorem 1.11 using the moment bound on $r_{\star}$ in Theorem 1.9. Since the proofs of Theorem 1.11 and Theorem 1.12 follow almost line by line the proofs of [19, Proposition 6.4] and [19, Theorem 6.1] respectively, we leave the details to the reader.

## Appendix A. Proof of Proposition 1.2

A.1. A result on stochastic integrability. We start by showing the following lemma on the characterization of super-algebraic stochastic integrability appearing in Proposition 1.2:

Lemma A.1. Let $\alpha>1$. Given a positive random variable $X$, the following statements are equivalent:
(1) There exists a constant $C>0$ such that

$$
\mathbb{E}\left[\exp \left(\frac{1}{C} \log ^{\alpha}(1+X)\right)\right]<+\infty
$$

(2) There exists a constant $C>0$ such that for $x$ large enough,

$$
\mathbb{P}(X \geq x) \leq \exp \left(-\frac{1}{C} \log ^{\alpha}(1+x)\right)
$$

(3) There exists a constant $C>0$ such that for $p$ large enough,

$$
\mathbb{E}\left[X^{p}\right] \leq \exp \left(C p^{\frac{\alpha}{\alpha-1}}\right)
$$

Here the constants $C$ might be different. In particular, if $\alpha=2$, all the exponents are 2 .
We prove $(1) \Rightarrow(2) \Rightarrow(3) \Rightarrow(1)$.
$(1) \Rightarrow(2)$ : Direct application of Chebychev's inequality.
$(2) \Rightarrow(3)$ : For $A>0$ large enough to be chosen later, we have by a change of variable

$$
\begin{aligned}
\mathbb{E}\left[X^{p}\right] & \leq A^{p}+p \int_{A}^{+\infty} \mathbb{P}(X>x) x^{p-1} d x \\
& \lesssim A^{p}+p \int_{A}^{+\infty} \exp \left(-\frac{1}{C} \log ^{\alpha}(1+x)\right) x^{p-1} d x \\
& \lesssim A^{p}+p \int_{\log ^{\alpha}(1+A)}^{+\infty} e^{-\frac{1}{C} t} e^{p t^{\frac{1}{\alpha}}} d t
\end{aligned}
$$

We then choose $A$ so large that for any $t>A$ one has $-\frac{1}{C} t+p t^{\frac{1}{\alpha}}<-\frac{1}{2 C} t$. Since $\alpha>1$, we can take $A=e^{(2 C p)^{\frac{1}{\alpha-1}}}$. Hence by changing the constant $C$,

$$
\mathbb{E}\left[X^{p}\right] \lesssim A^{p}+p C \leq e^{C p^{\frac{\alpha}{\alpha-1}}}
$$

$(3) \Rightarrow(1)$ : Denote by $C^{\prime}$ the constant in (3). By Chebychev's inequality with power $p_{n}$ yet to be chosen,

$$
\begin{aligned}
\mathbb{E}\left[\exp \left(\frac{1}{C} \log ^{\alpha}(1+X)\right)\right] & =\int_{0}^{+\infty} \mathbb{P}\left(\exp \left(\frac{1}{C} \log ^{\alpha}(1+X)\right)>t\right) d t \\
& =\int_{0}^{1} 1 d t+\left(\sum_{n \in \mathbb{N}} \int_{e^{n}}^{e^{n+1}}\right) \mathbb{P}\left(X>\exp \left((C \log t)^{\frac{1}{\alpha}}\right)\right) d t \\
& \leq 1+\sum_{n \in \mathbb{N}} \int_{e^{n}}^{e^{n+1}} \mathbb{P}\left(X>\exp \left((C \log t)^{\frac{1}{\alpha}}\right)\right) d t \\
& \leq 1+\sum_{n \in \mathbb{N}} e^{n+1} \exp \left(-C^{\frac{1}{\alpha}} p_{n} n^{\frac{1}{\alpha}}\right) \exp \left(C^{\prime} p_{n}^{\frac{\alpha}{\alpha-1}}\right)
\end{aligned}
$$

With $p_{n}=n^{1-\frac{1}{\alpha}}$ and $C$ large enough, the sum converges, and the claim follows.
A.2. Proof of Proposition 1.2. We split the proof into two steps. We first estimate $r_{\odot}$, and then turn to $r_{s}$.

Step 1. Estimate of $r_{\diamond}$.
We first introduce a stationary random radius $\tilde{r}_{\diamond}$ defined by

$$
\begin{aligned}
\tilde{r}_{\diamond}(x):= & \inf _{r \geq 1 \text { dyadic }}\{\forall \rho \geq r \text { dyadic, } \\
& \left.\frac{1}{C_{d}} \mathbb{E}\left[a^{p_{\diamond}}+a^{-p_{\diamond}}\right] \leq f_{B_{r}(x)} a^{p_{\diamond}}+a^{-p_{\diamond}} \leq C_{d} \mathbb{E}\left[a^{p_{\diamond}}+a^{-p_{\diamond}}\right]\right\},
\end{aligned}
$$

where $C_{d}$ will be chosen later. For all $r>0$, we set

$$
X_{r}:=f_{B_{r}} a^{p_{\diamond}}+a^{-p_{\diamond}} .
$$

The Malliavin derivative of $X_{r}$ is $D X_{r}=p_{\diamond}\left|B_{r}\right|^{-1}\left(a^{p_{\diamond}}+a^{-p_{\diamond}}\right) \mathbb{1}_{B_{r}}$. Hence the moment bound for all $q \geq 1$

$$
\begin{equation*}
\mathbb{E}\left[|X-\mathbb{E} X|^{q}\right]^{\frac{1}{q}} \lesssim \sqrt{q} \mathbb{E}\left[\|D X\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{q}\right]^{\frac{1}{q}} \tag{A.1}
\end{equation*}
$$

(which is a consequence of the logarithmic-Sobolev inequality) combined with Minkowski's inequality implies

$$
\left.\mathbb{E}\left[\left|X_{r}-\mathbb{E} X_{r}\right|^{q}\right]^{\frac{1}{q}} \lesssim \sqrt{q} \mathbb{E}\left[\|D X\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{q}\right)\right]^{\frac{1}{q}} \lesssim \sqrt{q}\left|B_{r}\right|^{-\frac{1}{2}} \mathbb{E}\left[a^{p_{\circ} q}\right]^{\frac{1}{q}} .
$$

By definition of the random field $a, \mathbb{E}\left[a^{p}\right]=\exp \left(\frac{\mathcal{C}(0)}{2} p^{2}\right)$, so for suitable positive constants $c$ and $C$ (depending on $p_{\diamond}$ ),

$$
\mathbb{E}\left[\left|X_{r}-\mathbb{E} X_{r}\right|^{q}\right] \leq C^{q} q^{q}\left|B_{r}\right|^{-\frac{q}{2}} \exp \left(\frac{1}{2} p_{\curvearrowright}^{2} q^{2}\right) \leq r^{-\frac{d}{2} q} \exp \left(c q^{2}\right) .
$$

Since the definition of $\tilde{r}_{\diamond}$ only involves dyadic radii, we have

$$
\mathbb{P}\left(\tilde{r}_{\diamond}>x\right) \leq \sum_{r=2^{n}>x} \mathbb{P}\left(\left|X_{r}-\mathbb{E} X_{r}\right|>\left(1-C_{d}^{-1}\right) \mathbb{E} X_{r}\right) .
$$

By applying Chebychev's inequality with power $q_{n}=\frac{d \log 2}{4 c} n$ to each term in the sum, we obtain

$$
\mathbb{P}\left(\tilde{r}_{\diamond}>x\right) \lesssim \sum_{n>\log _{2} x} 2^{-\frac{d}{2} n q_{n}} \exp \left(c q_{n}^{2}\right) \lesssim \sum_{n>\log _{2} x} \exp \left(-c n^{2}\right) \leq e^{-c \log ^{2} x},
$$

here the constant $c$ varies but only depends on $d$ and $p_{\diamond}$. This implies the claimed stochastic integrability of $\tilde{r}_{\diamond}$ by Lemma A.1.

To pass from dyadic radii to general radii, it suffices to multiply $\tilde{r}_{\diamond}$ by 2 . In fact, for a radius $r \in\left[r_{0}, 2 r_{0}\right]$,

$$
f_{B_{r}}\left|a_{\diamond}^{p}+a^{-p_{\diamond}}-\mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right| \leq 2^{d} f_{B_{r_{0}}}\left|a_{\diamond}^{p}+a^{-p_{\diamond}}-\mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right|
$$

As a result, for any $r>2 \tilde{r}_{\diamond} \geq 2^{\left\lceil\log _{2} \tilde{r}_{\diamond}\right\rceil}$, by comparing the integral with the average on the ball with the nearest dyadic radius (larger than $r_{\diamond}$ ),

$$
\begin{equation*}
f_{B_{r}}\left|a_{\diamond}^{p}+a^{-p_{\diamond}}-\mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right| \leq\left(1-C_{d}^{-1}\right) 2^{d} f_{B_{r}}\left|a_{\diamond}^{p}+a^{-p_{\diamond}}-\mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right| \tag{A.2}
\end{equation*}
$$

We then pick

$$
\underline{r}_{\diamond}(x, \epsilon)=\inf _{y} \tilde{r}_{\diamond}(y, \epsilon)+\frac{1}{8}|x-y|,
$$

with

$$
\tilde{r}_{\diamond}(x, \epsilon):=\inf \left\{r: \forall \rho>r, f_{B_{\rho}(x)}\left|\left(a_{\diamond}^{p}+a^{-p_{\diamond}}\right)-\mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right| \leq \epsilon \mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right\}
$$

By construction, $\underline{r}_{\diamond}$ is the maximal $\frac{1}{8}$-Lipschitz field smaller than $\tilde{r}_{\diamond}$, so $\underline{r}_{\diamond}(x, \epsilon) \leq \tilde{r}_{\diamond}(x, \epsilon)$. If $\underline{r}_{\diamond}(x, \epsilon) \leq R$, by definition there is a $y \in \mathbb{R}^{d}$ such that $|y-x| \leq 8 R$ and $\tilde{r}_{\diamond}(y, \epsilon) \leq R$. This implies that for any $\rho>R, B_{\rho}(x) \subset B_{9 \rho}(y)$. Thus

$$
\begin{aligned}
f_{B_{\rho}(x)}\left|\left(a_{\diamond}^{p}+a^{-p_{\diamond}}\right)-\mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right| & \leq 9^{d} f_{B_{9 \rho}(x)}\left|\left(a_{\diamond}^{p}+a^{-p_{\diamond}}\right)-\mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]\right| \\
& \leq 9^{d} \in \mathbb{E}\left[a_{\diamond}^{p}+a^{-p_{\diamond}}\right]
\end{aligned}
$$

Again by definition, $\tilde{r}_{\diamond}\left(x, 9^{d} \epsilon\right) \leq R$, which yields $\tilde{r}_{\diamond}\left(x, 9^{d} \epsilon\right) \leq \underline{r}_{\diamond}(x, \epsilon)$. So if we define $r_{\diamond}$ to be the minimal $\frac{1}{8}$-Lipschitz random field with the desired large scale regularity property, since $\tilde{r}_{\diamond}\left(x, \frac{1}{2}\right) \leq \underline{r}_{\diamond}\left(x, \frac{1}{2} 9^{-d}\right)$ and the latter one is $\frac{1}{8}$-Lipschitz, by minimality

$$
r_{\diamond}(x) \leq \underline{r}_{\diamond}\left(x, \frac{1}{2} 9^{-d}\right) \leq \tilde{r}_{\diamond}\left(x, \frac{1}{2} 9^{-d}\right)
$$

As a result, in view of (A.2), if we pick $C_{d}$ such that $\left(1-C_{d}^{-1}\right) 2^{d}=\frac{1}{2} 9^{-d}$, $r_{\diamond}$ will have the desired stochastic integrability.

Step 2. Estimate of $r_{\boldsymbol{\mu}}$.
Since the laws of $a$ and $a^{-1}$ are the same, it is enough to treat positive powers of $a$. We first claim that it suffices to control $r_{\boldsymbol{\&}}(1)$. In fact, if

$$
\mathbb{E}\left[\exp \left(\frac{1}{C} \log ^{2}\left(1+r_{\boldsymbol{Q}}(1)\right)\right)\right]<+\infty
$$

holds, then since one can cover $B_{R}$ with $c_{d} R^{d}$ balls $B_{i}$ of radius 1 , by stationarity of $a$

$$
\begin{aligned}
\mathbb{P}\left(r_{\boldsymbol{\bullet}}(R)>r\right) & \leq \mathbb{P}\left(\exists 1 \leq i \leq c_{d} R^{d} \text { s.t. } \sup _{B_{i}} a>R^{\varepsilon} r\right) \\
& \leq \sum_{i=1}^{c_{d} R^{d}} \mathbb{P}\left(\sup _{B_{i}} a>R^{\varepsilon} r\right) \\
& \lesssim c_{d} R^{d} \exp \left(-\frac{1}{C} \log ^{2}\left(1+R^{\varepsilon} r\right)\right) \\
& \lesssim c_{d} R^{d} \exp \left(-\frac{1}{C} \log ^{2}\left(R^{\varepsilon}\right)\right) \exp \left(-\frac{1}{C} \log ^{2}(1+r)\right) \\
& \lesssim \exp \left(-\frac{1}{C} \log ^{2}(1+r)\right)
\end{aligned}
$$

since $R \mapsto R^{d} \exp \left(-\frac{1}{C} \log ^{2}\left(R^{\varepsilon}\right)\right)$ is bounded on $\mathbb{R}_{+}$. This entails the claimed stochastic integrability of $r_{\boldsymbol{\ell}}(R)$ by Lemma A.1.

We now estimate $r_{\&}(1)$. Under the hypothesis that the covariance function $\mathcal{C}$ is $2 \gamma$ Hölder continuous at 0 , we have the following regularity estimate: for a constant $c$, and all $q \geq 1$,

$$
\begin{equation*}
\mathbb{E}\left[|a(x)-a(y)|^{q}\right] \lesssim|x-y|^{\gamma q} e^{c q^{2}} \tag{A.3}
\end{equation*}
$$

Indeed, by the triangle inequality and the growth of moments of Gaussian variables,

$$
\begin{aligned}
\mathbb{E}\left[|a(x)-a(y)|^{q}\right]^{\frac{1}{q}} & =\mathbb{E}\left[e^{q G(y)}\left|1-e^{G(x)-G(y)}\right|^{q}\right]^{\frac{1}{q}} \\
& \leq \mathbb{E}\left[e^{2 q G(y)}\right]^{\frac{1}{2 q}}\left(\sum_{n=1}^{+\infty} \frac{1}{n!} \mathbb{E}\left[|G(x)-G(y)|^{2 n q}\right]^{\frac{1}{2 q}}\right) \\
& \lesssim e^{q} \sum_{n=1}^{+\infty} \frac{1}{n!}(2 n q)^{\frac{n}{2}} \mathbb{E}\left[|G(x)-G(y)|^{2}\right]^{\frac{n}{2}} \\
& \leq e^{q} \sum_{n=1}^{+\infty}\left(\left[\frac{n}{2}\right]!\right)^{-1}(4 q)^{\frac{n}{2}}(2-2 \mathcal{C}(x-y))^{\frac{n}{2}} \\
& \lesssim e^{q} \sum_{n=1}^{+\infty}(n!)^{-1}(c q)^{n}|x-y|^{n \gamma} \leq e^{c q}|x-y|^{\gamma}
\end{aligned}
$$

where $c$ is a constant only depending on the Hölder continuity of $\mathcal{C}$.
It remains to control $r_{\&}(1)$ by a Kolmogorov criterion-type argument. We now work in the unit cube $Q_{1}$ for convenience. Denote $\left(\left(Q_{k}^{n}\right)_{, 1 \leq k \leq 2^{n d}}\right)_{n \in \mathbb{N}}$ the family of coverings of $Q_{1}$ by disjoint dyadic cubes of side length $2^{-n}$ such that $Q_{1}=Q_{1}^{0}$. For $Q_{1}, Q_{2} \in\left(Q_{k}^{n}\right)_{n \in \mathbb{N}, 1 \leq k \leq 2^{n}}$ with $Q_{1} \subset Q_{2}$, by (A.3) and the Minkowski inequality we have

$$
\mathbb{E}\left[\left|f_{Q_{1}} a-f_{Q_{2}} a\right|^{q}\right]^{\frac{1}{q}} \leq \mathbb{E}\left[\left|f_{Q_{1}} f_{Q_{2}}\right| a(x)-a(y)|d x d y|^{q}\right]^{\frac{1}{q}} \lesssim e^{c q}\left(\operatorname{diam} Q_{2}\right)^{\gamma}
$$

which implies, by replacing the first sup by a sum,

$$
\begin{equation*}
\mathbb{E}\left[\sup _{Q^{n}} \sup _{k: Q_{k}^{n+1} \subset Q^{n}}\left|f_{Q^{n}} a-f_{Q_{k}^{n+1}} a\right|^{q}\right]^{\frac{1}{q}} \lesssim e^{c q} 2^{-n\left(\gamma-\frac{d}{q}\right)} . \tag{A.4}
\end{equation*}
$$

For all $n$, approximate $a$ by its local averages $a_{n}$ at scale $2^{-n}$, that is,

$$
a_{n}:=\sum_{k=1}^{2^{n}} \mathbb{1}_{Q_{k}^{n}} f_{Q_{k}^{n}} a
$$

For $q$ such that $\gamma-\frac{d}{q}>0$, (A.4) implies that $a_{n}$ is a Cauchy sequence in $L^{q}\left(d \mathbb{P}, L^{\infty}\left(Q_{1}\right)\right)$. Since $a_{n}$ converges to $a$ almost surely almost everywhere by the Lebesgue differentiation
theorem, we obtain

$$
\begin{align*}
\mathbb{E}\left[\left(\sup _{Q_{1}} a\right)^{q}\right]^{\frac{1}{q}} & \leq \sum_{n=1}^{\infty} \mathbb{E}\left[\sup _{Q_{1}}\left|a_{n}-a_{n+1}\right|^{q}\right]^{\frac{1}{q}}+\mathbb{E}\left[\sup _{Q_{1}}\left|a_{0}\right|^{q}\right]^{\frac{1}{q}} \\
& \lesssim \sum_{n=0}^{+\infty} e^{c q} 2^{-n\left(\gamma-\frac{d}{q}\right)}+\mathbb{E}\left[\left|f_{Q_{1}} a\right|^{q}\right]^{\frac{1}{q}} \leq e^{c q} . \tag{A.5}
\end{align*}
$$

For all $q>\frac{2 d}{\gamma}$, this yields a moment bound, which, according to Lemma A.1, entails the claimed stochastic integrability of $r_{\boldsymbol{c}}(1)$.

## Acknowledgements

NC has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No 948819). The two last authors acknowledge support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant Agreement No 864066).

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[^0]:    ${ }^{1}$ the divergence is taken wrt the last index

[^1]:    ${ }^{2}$ Since $a$ is degenerate, there is a small approximation argument needed to establish existence - see e.g. Section 3.

[^2]:    ${ }^{3}$ Note that this decay assumption for $\mathcal{C}_{0}$ implies the integrability of $\mathcal{C}$.

