General Construction of Bra-Ket Formalism for Identical Particle Systems in Rigged Hilbert Space Approach

S. Ohmori

National Institute of Technology, Gunma College,

580 Toribamachi, Maebashi-shi, Gunma 371-8530, Japan. and

Waseda Research Institute for Science and Engineering,

Waseda University, Shinjuku, Tokyo 169-8555, Japan.

J. Takahashi

Faculty of Science and Technology, Waseda University, Shinjuku, Tokyo 169-8555, Japan.

(Dated: March 4, 2024)

This study discussed Dirac's bra-ket formalism for the identical particles system to extend the rigged Hilbert space reformulated by R. Madrid [J. Phys. A:Math. Gen. 37, 8129 (2004)]. The bra and ket vectors for a composite system that form the basis of an identical particle system were reconstructed using the tensor product of rigged Hilbert space. They were found to be characterized in the dual spaces of the tensor product of nuclear spaces. The proofs utilized in this paper adopt a format similar to that used in physics, yet they will be mathematically rigorous. This formulation lays the foundation for modern quantum theories based on perturbation theory, such as quantum statistical mechanics and quantum field theory.

I. INTRODUCTION

The mathematical formalism of Dirac's bra-ket notations is considered insufficient in case of von Neumann's Hilbert space theory. Consequently, a mathematical approach utilizing rigged Hilbert space (RHS) has been developed to handle the bra-ket notations precisely[1–17]. RHS comprises the following triplet of topological vectors spaces[18, 19],

$$\Phi \subset \mathcal{H} \subset \Phi', \tag{1}$$

where $\mathcal{H} = (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a complex Hilbert space and $\Phi = (\Phi, \tau_\Phi)$ is a nuclear space that is a dense linear subspace of $\mathcal{H}$. The inner product $\langle \cdot, \cdot \rangle_\Phi$ on Φ becomes separately continuous on (Φ, τ_Φ), where $\langle \phi, \psi \rangle_\Phi \equiv \langle \phi, \psi \rangle_{\mathcal{H}}$ for $\phi, \psi \in \Phi$. Φ' is a family of continuous linear functional on (Φ, τ_Φ). In case of the RHS approach, the nuclear spectral theorem for a self-adjoint operator (observable) in $\mathcal{H}$ assures the existence of generalized eigenvectors that individually endows the eigenequations for the bra and ket vectors. This theorem also provides the spectral expansions based on which the spectral decomposition for discrete and continuous spectrum, specified by the Dirac's' δ-function (distributions) found in literature, can be constructed. Hence, the Dirac's bra-ket formalism is complete and currently RHS is considered as the underlying space of quantum mechanics. Several studies have employed RHS to construct accurate and aesthetic formulations to address problems found in quantum theory; for example, harmonic oscillators [7], resonance state (Gamow vectors)[8], and scattering problem[14].

Recently, the mathematical treatment using RHS has garnered attention and has been applied to modern quantum physics[20–24]. For instance, the resonance state found in open quantum system and non-Hermitian operator with the characteristic symmetry for the non-Hermite quantum system have been examined using RHS. The physical phenomena observed in these systems cannot be handled by only the Hilbert space, such as L^2-space. Consequently, a broader and more general space such as RHS is required. For instance, in the problem of a quantum damped system, the given Hamiltonian contains complex eigenvalues in RHS that can be interpreted as the resonant state[20, 21]. Thus, the RHS is indispensable for addressing complex eigenvalues beyond the L^2-space theory. As evident from this example, we believe that the development of an RHS theory is crucial for mathematical foundations and the elucidation of the quantum phenomena.

For constructing the bra and ket vectors using RHS, a more elegant and simple approach, proposed by Madrid[14], has been developed. This approach adapts the RHS (1) including the dual space $\Phi^{\times}$ of Φ,

$$\Phi \subset \mathcal{H} \subset \Phi', \Phi^{\times}, \tag{2}$$

where $\Phi^{\times}$ is a family of continuous *anti-linear* functionals on (Φ, τ_{Φ}). (A function $f \in \Phi^{\times}$ is anti-linear if it satisfies $f(a\varphi + b\phi) = a^* f(\varphi) + b^* f(\phi)$ where a and b are complex numbers with complex conjugates a^* and b^* and $\varphi, \phi \in \Phi$.) Using (2), bra and ket vectors are established as the elements of Φ' and $\Phi^{\times}$, in the following procedure. Let $\varphi \in \Phi$, and we define a map $|\varphi\rangle_{\mathcal{H}} : \Phi \to \mathbb{C}^1$ using $|\varphi\rangle_{\mathcal{H}}(\phi) \equiv \langle \phi, \varphi \rangle_{\mathcal{H}}$ for $\phi \in \Phi$; this map is called a ket of φ. The bra vector of φ is defined as the complex conjugate of $|\varphi\rangle_{\mathcal{H}}$, namely, the the map $\langle \varphi|_{\mathcal{H}} : \Phi \to \mathbb{C}^1$ where $\langle \varphi|_{\mathcal{H}}(\phi) = |\varphi\rangle^*_{\mathcal{H}}(\phi) = (|\varphi\rangle_{\mathcal{H}}(\phi))^* = \langle \varphi, \phi \rangle_{\mathcal{H}}$. Clearly, $\langle \varphi|_{\mathcal{H}}$ and $|\varphi\rangle_{\mathcal{H}}$ belong to Φ' and $\Phi^{\times}$ of Φ, respectively. The combination of of dual and anti-dual spaces, Φ' and $\Phi^{\times}$ is referred to as dual spaces, hereafter. Furthermore, it is shown that the generalized eigenvectors of the observable derived from the nuclear spectral theorem belong to $\Phi' \cup \Phi^{\times}$. In the approach, the bra $\langle \varphi|_{\mathcal{H}}$ and ket $|\varphi\rangle_{\mathcal{H}}$ are assigned to Φ' and $\Phi^{\times}$. Ttheir spectral expansions can be performed as the elements of the dual spaces. All calculations in terms of the bra and ket vectors are conducted in the dual spaces. This approach has been applied to problems such as 1D rectangular barrier [14] and non-Hermite system [24], where mathematical rigorous treatment of bra-ket notations is used to solve these physical problems.

Such a Madrid approach exactly supplies the rigorous formalism of bra-ket notation. Till date, these studies have predominantly focused on single-particle systems. However, studies on the RHS focusing on composite systems containing identical particle systems remain insufficient compared to single-particle systems. Thus, considering the issues in modern quantum physics, such as non-Hermitian systems, open quantum systems, and problems in quantum statistical mechanics, the mathematical formalism is incomplete. Thus, this study aimed to obtain this formalization wherein the underlying bra-ket formalism that describes composite systems was established using the RHS (2). In addition, we aimed to develop the construction of the bra-ket space in the dual spaces for identical particles.

The remainder of this paper is organized as follows. In Section II, we construct the bra-ket vectors for the tensor product of RHS (2) on the dual spaces and show the relation to the single bra-ket vectors obtained from an RHS. In addition, the permutation operator is introduced on the dual spaces, which endows the symmetric properties of the bra-ket vectors derived from the identical RHS. Using the nuclear spectral theorem for the tensor product of the RHS, we present the formulation of the spectral expansions of the bra-ket vectors by the generalized eigenvectors for a self-adjoint operator in Secion III. The generalized eigenvectors exhibit the complete orthonormality in the dual spaces. Furthermore, the permutation operator obtained in Section II aided in the generalization of the eigenvectors, thus preserving the symmetric structure. Finally, Section IV presents the conclusions.

II. CONSTRUCTION OF THE BRA-KET VECTORS IN THE DUAL SPACES FOR THE TENSOR PRODUCT

Nuclear spectral theorem for the tensor product of RHS (1) was presented by Maurin [19]. In this section, we introduce the nuclear spectral theorem for the tensor product of (2) in the bra-ket notation.

A. General formulation

When establishing the state space that describes a composite system without interactions using the Hilbert space theory, the tensor product of given Hilbert spaces is introduced[25, 26]. In the RHS context, the tensor product of RHS is necessary for constructing the bra and ket vectors related to a composite system. For simplicity, we focused on a two-particles system. Let $\Phi_i \subset \mathcal{H}_i \subset \Phi_i^\times, \Phi_i'$ $(i = 1, 2)$ be a RHS (2), where each $(\mathcal{H}_i, \langle\cdot,\cdot\rangle_{\mathcal{H}_i})$ is a complex Hilbert space, $\Phi_i = (\Phi_i, \tau_{\Phi_i})$ is a subspace of $\mathcal{H}_i$ with the nuclear topology τ_{Φ_i}, and $\Phi_i^\times and \Phi_i'$ are the dual and anti-dual spaces of (Φ_i, τ_{Φ_i}), respectively. From each RHS, the bra and ket obtained is expressed as the maps $\langle\varphi|_{\mathcal{H}_i}$ and $|\varphi\rangle_{\mathcal{H}_i}$ in $\Phi_i^\times$ and Φ_i', respectively $(i = 1, 2)$. Now we introduce the algebraic tensor product for the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as an inner product space $\mathcal{H}_1 \otimes \mathcal{H}_2 = (\mathcal{H}_1 \otimes \mathcal{H}_2, \langle\cdot,\cdot\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2})$ where $\mathcal{H}_1 \otimes \mathcal{H}_2 = \left\{ \sum_{j=1}^{m} \varphi_{1j} \otimes \varphi_{2j} \mid \varphi_{1j} \in \mathcal{H}_1, \varphi_{2j} \in \mathcal{H}_2, m \in \mathbb{N} \right\}$. Its inner product satisfies

$\langle \varphi_1 \otimes \varphi_2,\, \phi_1 \otimes \phi_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle \varphi_1\, \phi_1 \rangle_{\mathcal{H}_1} \langle \varphi_2\, \phi_2 \rangle_{\mathcal{H}_2}$. The completion of the algebraic tensor product with respect to the topology induced by $\langle \cdot,\, \cdot \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2}$ is denoted by $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2 = (\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2, \langle \cdot,\, \cdot \rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2})$. The algebraic tensor product of the nuclear spaces (Φ_1, τ_{Φ_1}) and (Φ_2, τ_{Φ_2}) is also expressed as a locally convex space $\Phi_1 \otimes \Phi_2 = \left\{ \sum_{j=1}^{m} \varphi_{1j} \otimes \varphi_{2j} \mid \varphi_{1j} \in \Phi_1, \varphi_{2j} \in \Phi_2, m \in \mathbb{N} \right\}$ equipping the locally convex topology τ_p with the local base $\mathcal{B}_p = \{\Gamma(V_1 \otimes V_2) \mid V_i \in \mathcal{B}_i, i = 1, 2\}$ where each $\mathcal{B}_i$ is a local base of τ_{Φ_i} and $\Gamma(X)$ stands for the convex circled hull of a set X[27]. As is well-known, the completion of $(\Phi_1 \otimes \Phi_2, \tau_p)$ is the nuclear space, denoted by $(\Phi_1 \hat{\otimes} \Phi_2, \widehat{\tau_p})$. Therefore, considering the dual and anti-dual spaces of $\Phi_1 \hat{\otimes} \Phi_2$, it is verified that the following triplet comprises an RHS[19],

$$\Phi_1 \hat{\otimes} \Phi_2 \subset \mathcal{H}_1 \bar{\otimes} \mathcal{H}_2 \subset (\Phi_1 \hat{\otimes} \Phi_2)',\ (\Phi_1 \hat{\otimes} \Phi_2)^{\times}. \tag{3}$$

Using the RHS (3) the bra and ket vectors corresponding to $\varphi \in \Phi_1 \hat{\otimes} \Phi_2$ are defined by

$$\langle \varphi|_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2} : \Phi_1 \hat{\otimes} \Phi_2 \to \mathbb{C},\ \ \langle \varphi|_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}(\phi) = \langle \varphi,\, \phi \rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}, \tag{4}$$

$$|\varphi\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2} : \Phi_1 \hat{\otimes} \Phi_2 \to \mathbb{C},\ \ |\varphi\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}(\phi) = \langle \phi,\, \varphi \rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}. \tag{5}$$

Consequently, the relations $|\varphi\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2} = \langle \varphi|^{*}_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}$, $\langle \varphi|_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2} \in (\Phi_1 \hat{\otimes} \Phi_2)'$, and $|\varphi\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2} \in (\Phi_1 \hat{\otimes} \Phi_2)^{\times}$, are satisfied.

To observe a connection between the ket $|\varphi\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}$ for the tensor product of the RHS and ket $|\varphi\rangle_{\mathcal{H}_i}$ for the single RHS, we considered $\varphi = \varphi_1 \otimes \varphi_2 \in \Phi_1 \otimes \Phi_2 \subset \Phi_1 \hat{\otimes} \Phi_2$. The ket $|\varphi\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}$ becomes $|\varphi\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2} = |\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}$ in $(\Phi_1 \hat{\otimes} \Phi_2)^{\times}$. We introduce a map $|\varphi\rangle_{\mathcal{H}_1} |\varphi\rangle_{\mathcal{H}_2} : \Phi_1 \times \Phi_2 \to \mathbb{C}$ where $|\varphi\rangle_{\mathcal{H}_1} |\varphi\rangle_{\mathcal{H}_2}(\phi_1, \phi_2) = \langle \phi_1, \varphi_1 \rangle_{\mathcal{H}_1} \langle \phi_2, \varphi_2 \rangle_{\mathcal{H}_2}$ for $(\phi_1, \phi_2) \in \Phi_1 \times \Phi_2$. Thus, we obtain the relation,

$$|\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}(\phi) = \langle \phi_1, \varphi_1 \rangle_{\mathcal{H}_1} \langle \phi_2, \varphi_2 \rangle_{\mathcal{H}_2} = |\varphi\rangle_{\mathcal{H}_1} |\varphi\rangle_{\mathcal{H}_2}(\phi_1, \phi_2), \tag{6}$$

for $\phi = \phi_1 \otimes \phi_2 \in \Phi_1 \otimes \Phi_2$. As $|\varphi\rangle_{\mathcal{H}_1} |\varphi\rangle_{\mathcal{H}_2}$ is anti-linear continuous on $\Phi_1 \times \Phi_2$, there exists the unique element v of $(\Phi_1 \otimes \Phi_2)^{\times}$ satisfying $|\varphi\rangle_{\mathcal{H}_1} |\varphi\rangle_{\mathcal{H}_2} = v \circ \chi$, namely, $v \circ \chi(\phi_1, \phi_2) = v(\phi_1 \otimes \phi_2) = |\varphi\rangle_{\mathcal{H}_1} |\varphi\rangle_{\mathcal{H}_2}(\phi_1, \phi_2)$ for any $(\phi_1, \phi_2) \in \Phi_1 \times \Phi_2$, where $\chi : (\phi_1, \phi_2) \mapsto \phi_1 \otimes \phi_2$ is the canonical bilinear map on $\Phi_1 \times \Phi_2$ into $\Phi_1 \otimes \Phi_2$[27]. Notably, the mapping $H : v \mapsto v \circ \chi$ becomes an isomorphism between $(\Phi_1 \otimes \Phi_2)^{\times}$ and $\mathcal{B}^{\times}(\Phi_1, \Phi_2)$ where $\mathcal{B}^{\times}(\Phi_1, \Phi_2)$ is the family of continuous antilinear functionals on $(\Phi_1 \times \Phi_2, \tau_{\Phi_1 \times \Phi_2})$. From (6), the uniqueness of v shows $v = |\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}|_{(\Phi_1 \otimes \Phi_2)}$. ($f|_A$ denotes the restriction of the map f on A.) Here, we set $|\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2}|_{(\Phi_1 \otimes \Phi_2)} \equiv |\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2}$.

Based on the isomorphism H, we identify

$$|\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = |\varphi\rangle_{\mathcal{H}_1} |\varphi\rangle_{\mathcal{H}_2} . \tag{7}$$

We set an isomorphic mapping $\hat{L} : \Phi_1^\times \otimes \Phi_2^\times \to \hat{L}(\Phi_1^\times \otimes \Phi_2^\times) \subset \mathcal{B}^\times(\Phi_1, \Phi_2)$ where $\hat{L}(f \otimes g)(\varphi, \phi) = f(\varphi)g(\phi)$ for $f \otimes g \in \Phi_1^\times \otimes \Phi_2^\times$ and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$. Considering the kets $|\varphi_1\rangle_{\mathcal{H}_1}$ and $|\varphi_2\rangle_{\mathcal{H}_2}$ as f and g in $\Phi_1^\times$ and $\Phi_2^\times$, respectively, we have $\hat{L}(|\varphi_1\rangle_{\mathcal{H}_1} \otimes |\varphi_2\rangle_{\mathcal{H}_2}) = |\varphi_1\rangle_{\mathcal{H}_1} |\varphi_2\rangle_{\mathcal{H}_2}$. By considering the isomorphism $\hat{L}$ as an identification, we obtain

$$|\varphi_1\rangle_{\mathcal{H}_1} \otimes |\varphi_2\rangle_{\mathcal{H}_2} = |\varphi_1\rangle_{\mathcal{H}_1} |\varphi_2\rangle_{\mathcal{H}_2} . \tag{8}$$

Thus, using (7) and (8), we obtain

$$|\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = |\varphi_1\rangle_{\mathcal{H}_1} \otimes |\varphi_2\rangle_{\mathcal{H}_2} \tag{9}$$

for any $\varphi_1 \in \Phi_1$ and $\varphi_2 \in \Phi_2$. (9) shows that in $(\Phi_1 \otimes \Phi_2)^\times$ the ket (5) of the type $\varphi = \varphi_1 \otimes \varphi_2$ can be represented by the tensor product of kets, with each ket being obtained in an RHS. In the literature, the connection (9) has been assumed where the ket, $|\varphi_1 \otimes \varphi_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}$, describing a state of a composite system, is represented by the tensor product of the ket vectors $|\varphi_i\rangle_{\mathcal{H}_i}$ $(i = 1, 2)$. Here, each ket describes the state of the single particle. Notably, if we consider φ as a linear combination of $\varphi_1 \otimes \varphi_2$ in $\Phi_1 \otimes \Phi_2$ and $\varphi = \sum_{j=1}^m \varphi_{1j} \otimes \varphi_{2j}$, the following relation holds as the generalization of (9),

$$\left| \sum_{j=1}^m \varphi_{1j} \otimes \varphi_{2j} \right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = \sum_{j=1}^m |\varphi_{1j}\rangle_{\mathcal{H}_1} \otimes |\varphi_{2j}\rangle_{\mathcal{H}_2} . \tag{10}$$

The obtained relations can be applied to a N-particles system $(N < \infty)$. The RHS comprises the N-multiple tensor product of RHS, represented by,

$$\widehat{\otimes}_{j=1}^N \Phi_j \subset \overline{\otimes}_{j=1}^N \mathcal{H}_j \subset (\widehat{\otimes}_{j=1}^N \Phi_j)',\ (\widehat{\otimes}_{j=1}^N \Phi_j)^\times, \tag{11}$$

where $\widehat{\otimes}_{j=1}^N \Phi_j = (\widehat{\otimes}_{j=1}^N \Phi_j, \widehat{\tau}_p)$ is the tensor product obtained by completion of the algebraic tensor product $(\otimes_{j=1}^N \Phi_j, \tau_p)$ of the nuclear spaces (Φ_j, τ_{Φ_j}) $(j = 1, \cdots, N)$. $\overline{\otimes}_{j=1}^N \mathcal{H}_j = (\overline{\otimes}_{j=1}^N \mathcal{H}_j, \langle\cdot, \cdot\rangle_{\overline{\otimes}_{j=1}^N \mathcal{H}_j})$ is the tensor product space of Hilbert space whose inner product represents $\langle\cdot, \cdot\rangle_{\overline{\otimes}_{j=1}^N \mathcal{H}_j}$. The spaces $(\widehat{\otimes}_{j=1}^N \Phi_j)'$ and $(\widehat{\otimes}_{j=1}^N \Phi_j)^\times$ are the dual and anti-dual spaces of $\widehat{\otimes}_{j=1}^N \Phi_j$, respectively. Note that

$\widehat{\otimes}_{j=1}^{N}\Phi_j$ becomes a nuclear space. Using (11), the bra and ket vectors are defined as

$$\langle\varphi|_{\overline{\otimes}_{j=1}^{N}\mathcal{H}_j} : \widehat{\otimes}_{j=1}^{N}\Phi_j \to \mathbb{C}, \ \ \langle\varphi|_{\overline{\otimes}_{j=1}^{N}\mathcal{H}_j}(\phi) = \langle\varphi, \phi\rangle_{\overline{\otimes}_{j=1}^{N}\mathcal{H}_j}, \tag{12}$$

$$|\varphi\rangle_{\overline{\otimes}_{j=1}^{N}\mathcal{H}_j} : \widehat{\otimes}_{j=1}^{N}\Phi_j \to \mathbb{C}, \ \ |\varphi\rangle_{\overline{\otimes}_{j=1}^{N}\mathcal{H}_j}(\phi) = \langle\phi, \varphi\rangle_{\overline{\otimes}_{j=1}^{N}\mathcal{H}_j}, \tag{13}$$

for $\varphi \in \widehat{\otimes}_{j=1}^{N}\Phi_j$. Further, using (9), we obtain in $(\widehat{\otimes}_{j=1}^{N}\Phi_j)^{\times}$

$$|\varphi_1 \otimes \cdots \otimes \varphi_n\rangle_{\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n} = |\varphi_1\rangle_{\mathcal{H}_1} \otimes \cdots \otimes |\varphi_n\rangle_{\mathcal{H}_n} \tag{14}$$

for $\varphi_j \in \Phi_j, \ j = 1, \ldots, N$. Similar to (14), the following relation of the bra vectors in $(\widehat{\otimes}_{j=1}^{N}\Phi_j)'$ is derived:

$$\langle\varphi_1 \otimes \cdots \otimes \varphi_n|_{\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n} = \langle\varphi_1|_{\mathcal{H}_1} \otimes \cdots \otimes \langle\varphi_n|_{\mathcal{H}_n}. \tag{15}$$

B. Symmetry

The symmetry of identical particles in the Hilbert space theory can be introduced by using the permutation operator[25, 26]. Now we focus on the case where $\mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathcal{H}_N \equiv \mathcal{H}$ and $\Phi_1 = \Phi_2 = \cdots = \Phi_N \equiv \Phi$. Let $\mathfrak{S}_N$ be the symmetry group of degree N. We fix $\sigma \in \mathfrak{S}_N$ and define the permutation, $U_\sigma : \otimes^N\mathcal{H} \to \overline{\otimes}^N\mathcal{H}$, on the algebraic tensor product $\otimes^N\mathcal{H}$ where

$$U_\sigma(\phi) = \sum_{j=1}^{m} \phi_{\sigma(1)j} \otimes \cdots \otimes \phi_{\sigma(N)j} \ \ \text{for} \ \ \phi = \sum_{j=1}^{m} \phi_{1j} \otimes \cdots \otimes \phi_{Nj} \in \otimes^N\mathcal{H}. \tag{16}$$

The permutation has the unique extension to the completion $(\overline{\otimes}^N\mathcal{H}, \langle\cdot, \cdot\rangle_{\overline{\otimes}^N\mathcal{H}})$ of the inner product space $(\otimes^N\mathcal{H}, \langle\cdot, \cdot\rangle_{\otimes^N\mathcal{H}})$. We denote this extension by U_σ. Corresponding to this case, the following triplet of the N-tensor product space of RHS is adapted, similar to that of (11),

$$\widehat{\otimes}^N\Phi \subset \overline{\otimes}^N\mathcal{H} \subset (\widehat{\otimes}^N\Phi)^{\times}, \ (\widehat{\otimes}^N\Phi)'. \tag{17}$$

The permutation of the nuclear space $\widehat{\otimes}^N\Phi$ can be established as follows. Let the permutation U_σ on $\otimes^N\mathcal{H}$ be restricted to the algebraic tensor product $\otimes^N\Phi$. Consequently, the restriction $U_\sigma|_{\otimes^N\Phi}$ becomes an isomorphism of $\otimes^N\Phi$ onto itself, with respect to the nuclear topology τ_p [19]. Therefore, for the nuclear space $(\widehat{\otimes}^N\Phi, \widehat{\tau_p})$, there exists the unique extension $U_\sigma^{\widehat{\otimes}^N\Phi}$ of $U_\sigma|_{\otimes^N\Phi}$. The uniqueness of $U_\sigma^{\widehat{\otimes}^N\Phi}$ shows $U_\sigma^{\widehat{\otimes}^N\Phi} = U_\sigma|_{\widehat{\otimes}^N\Phi}$. Consequently, we obtain the permutation on the nuclear space

$\widehat{\otimes}^N\Phi$ in the form of

$$U_\sigma^{\widehat{\otimes}^N\Phi}: (\widehat{\otimes}^N\Phi, \widehat{\tau_p}) \to (\widehat{\otimes}^N\Phi, \widehat{\tau_p}),\ \phi \mapsto U_\sigma(\phi). \tag{18}$$

Note that $U_\sigma^{\widehat{\otimes}^N\Phi}$ is an isomorphism and the relation

$$i \circ U_\sigma^{\widehat{\otimes}^N\Phi} = U_\sigma \circ i \tag{19}$$

is satisfied where i is the canonical embedding that characterizes the RHS (17).

The symmetric structure for the tensor product of the Hilbert space, $\overline{\otimes}^N\mathcal{H}$, is characterized by the following projection, referred to as the permutation operator[25],

$$P_c = \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N} c(\sigma)U_\sigma. \tag{20}$$

Similarly, the projection for the nuclear space $\widehat{\otimes}^N\Phi$ is introduced by using $U_\sigma^{\widehat{\otimes}^N\Phi}$ as

$$P_c^{\widehat{\otimes}^N\Phi} = \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N} c(\sigma)U_\sigma^{\widehat{\otimes}^N\Phi}. \tag{21}$$

The bra and ket vectors derived using the tensor product of RHS belong to its dual spaces, as shown in the previous section. This necessitates the extension of the permutation operator (21) to the dual spaces. As the operator (21) is continuous on $\widehat{\otimes}^N\Phi$ and maps onto $\widehat{\otimes}^N\Phi$, the extension of (21) can be easily constructed as follows. We set a operator $\widetilde{P_c^{\widehat{\otimes}^N\Phi}}$ on $(\widehat{\otimes}^N\Phi)^\times \cup (\widehat{\otimes}^N\Phi)'$, where

$$\widetilde{P_c^{\widehat{\otimes}^N\Phi}}(f)(\phi) = f(P_c^{\widehat{\otimes}^N\Phi}(\phi)), \tag{22}$$

for $f \in (\widehat{\otimes}^N\Phi)^\times \cup (\widehat{\otimes}^N\Phi)'$, $\phi \in \widehat{\otimes}^N\Phi$. This operator (22) endows the symmetric structure for the bra and ket vectors satisfying (14) and (15). To show this fact, we fixed $N = 2$ in short. In the nuclear space $(\widehat{\otimes}^2\Phi, \widehat{\tau_p})$, each $\phi \in \widehat{\otimes}^2\Phi$ can be represented as the form of the sum of an absolutely convergent series, $\phi = \sum_{i=1}^\infty \lambda_i\phi_i^1 \otimes \phi_i^2$, where $\sum_i |\lambda_i| \le 1$ and $\{\phi_i^1\}$ and $\{\phi_i^2\}$ are null sequences in Φ[27]. As $P_c^{\widehat{\otimes}^2\Phi}$ is continuous linear on $(\widehat{\otimes}^2\Phi, \widehat{\tau_p})$, we have

$$\begin{aligned} P_c^{\widehat{\otimes}^2\Phi}(\phi) &= \sum_{i=1}^\infty \lambda_i P_c^{\widehat{\otimes}^2\Phi}(\phi_i^1 \otimes \phi_i^2) \\ &= \begin{cases} \displaystyle\sum_{i=1}^\infty \frac{\lambda_i}{2}(\phi_i^1 \otimes \phi_i^2 + \phi_i^2 \otimes \phi_i^1) & (c = c_1) \\ \displaystyle\sum_{i=1}^\infty \frac{\lambda_i}{2}(\phi_i^1 \otimes \phi_i^2 - \phi_i^2 \otimes \phi_i^1) & (c = sgn) \end{cases} \end{aligned} \tag{23}$$

Here, we focused on the symmetry case, $c = c_1$. (in the same manner, the anti-symmetric case is also obtained.) We set $\varphi = \varphi_1 \otimes \varphi_2 \in \otimes^2\Phi \subset \widehat{\otimes}^2\Phi$. Using (9), we have

$$|\varphi\rangle_{\overline{\otimes}^2\mathcal{H}} = |\varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}} = |\varphi_1\rangle_{\mathcal{H}} \otimes |\varphi_2\rangle_{\mathcal{H}}. \tag{24}$$

Using (23), considering the continuity of kets in $(\widehat{\otimes}^2\Phi, \widehat{\tau}_p)$, we have

$$\begin{aligned}
\widetilde{P_c^{\widehat{\otimes}^2\Phi}}(|\varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}})(\phi) &= |\varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}}(P_{c_1}^{\widehat{\otimes}^2\Phi}(\phi)) \\
&= |\varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}}\Big\{\sum_{i=1}^{\infty}\frac{\lambda_i}{2}(\phi_i^1 \otimes \phi_i^2 + \phi_i^2 \otimes \phi_i^1)\Big\} \\
&= \sum_{i=1}^{\infty}\frac{\lambda_i}{2}\Big\{\,|\varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}}(\phi_i^1 \otimes \phi_i^2) + |\varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}}(\phi_i^2 \otimes \phi_i^1)\Big\} \\
&= \sum_{i=1}^{\infty}\frac{\lambda_i}{2}\Big\{\langle\phi_i^1 \otimes \phi_i^2, \varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}} + \langle\phi_i^2 \otimes \phi_i^1, \varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}}\Big\} \\
&= \sum_{i=1}^{\infty}\frac{\lambda_i}{2}\Big\{\langle\phi_i^1, \varphi_1\rangle_{\mathcal{H}}\langle\phi_i^2, \varphi_2\rangle_{\mathcal{H}} + \langle\phi_i^2, \varphi_1\rangle_{\mathcal{H}}\langle\phi_i^1, \varphi_2\rangle_{\mathcal{H}}\Big\} \\
&= \sum_{i=1}^{\infty}\frac{\lambda_i}{2}\Big\{\langle\phi_i^1 \otimes \phi_i^2, \varphi_1 \otimes \varphi_2\rangle_{\overline{\otimes}^2\mathcal{H}} + \langle\phi_i^1 \otimes \phi_i^2, \varphi_2 \otimes \varphi_1\rangle_{\overline{\otimes}^2\mathcal{H}}\Big\} \\
&= \frac{1}{2}(|\varphi_1\rangle_{\mathcal{H}} \otimes |\varphi_2\rangle_{\mathcal{H}} + |\varphi_2\rangle_{\mathcal{H}} \otimes |\varphi_1\rangle_{\mathcal{H}})(\sum_{i=1}^{\infty}\lambda_i\phi_i^1 \otimes \phi_i^2) \\
&= \frac{1}{2}(|\varphi_1\rangle_{\mathcal{H}} \otimes |\varphi_2\rangle_{\mathcal{H}} + |\varphi_2\rangle_{\mathcal{H}} \otimes |\varphi_1\rangle_{\mathcal{H}})(\phi).
\end{aligned} \tag{25}$$

Because the relation (25) holds for any $\phi \in \widehat{\otimes}^2\Phi$, we obtain the following relation in $(\widehat{\otimes}^2\Phi)^\times$ using (24),

$$\widetilde{P_c^{\widehat{\otimes}^2\Phi}}(|\varphi_1\rangle_{\mathcal{H}} \otimes |\varphi_2\rangle_{\mathcal{H}}) = \frac{1}{2}(|\varphi_1\rangle_{\mathcal{H}} \otimes |\varphi_2\rangle_{\mathcal{H}} + |\varphi_2\rangle_{\mathcal{H}} \otimes |\varphi_1\rangle_{\mathcal{H}}). \tag{26}$$

This relation can be generalized to the N-tensor product case : for $|\varphi_1, \otimes \cdots \otimes \varphi_N\rangle_{\overline{\otimes}^N\mathcal{H}} = |\varphi_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\varphi_N\rangle_{\mathcal{H}}$ in $(\widehat{\otimes}^N\Phi)^\times$ where $\varphi_1 \otimes \cdots \otimes \varphi_N \in \widehat{\otimes}^N\Phi$,

$$\widetilde{P_c^{\widehat{\otimes}^2\Phi}}(|\varphi_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\varphi_N\rangle_{\mathcal{H}}) = \begin{cases} \dfrac{1}{N!}\displaystyle\sum_{\sigma\in\mathfrak{S}_n}\left|\varphi_{\sigma(1)}\right\rangle_{\mathcal{H}} \otimes \cdots \otimes \left|\varphi_{\sigma(N)}\right\rangle_{\mathcal{H}} & (c = c_1) \\ \dfrac{1}{N!}\displaystyle\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\left|\varphi_{\sigma(1)}\right\rangle_{\mathcal{H}} \otimes \cdots \otimes \left|\varphi_{\sigma(N)}\right\rangle_{\mathcal{H}} & (c = sgn). \end{cases} \tag{27}$$

Here, (27) presents the symmetry and anti-symmetry for only the ket vectors of in the space $(\widehat{\otimes}^2\Phi)^\times$. Related to (27), We set the spaces

$$(\widehat{\otimes}^N\Phi)_s^\times = \widetilde{P_{c_1}^{\widehat{\otimes}^2\Phi}}((\widehat{\otimes}^N\Phi)^\times), \tag{28}$$

$$(\widehat{\otimes}^N\Phi)_a^\times = \widetilde{P_{sgn}^{\widehat{\otimes}^2\Phi}}((\widehat{\otimes}^N\Phi)^\times), \tag{29}$$

and refer to them as the symmetric and anti-symmetric ket spaces, respectively. In terms of $(\widehat{\otimes}^2\Phi)'$, the symmetric structure for the bra vector is expressed using the permutation operator (22), as follows,

$$\widetilde{P_c^{\widehat{\otimes}^2\Phi}}(\langle\varphi_1|_{\mathcal{H}} \otimes \cdots \otimes \langle\varphi_N|_{\mathcal{H}}) = \begin{cases} \dfrac{1}{N!}\displaystyle\sum_{\sigma\in\mathfrak{S}_n} \langle\varphi_{\sigma(1)}\big|_{\mathcal{H}} \otimes \cdots \otimes \langle\varphi_{\sigma(N)}\big|_{\mathcal{H}} & (c = c_1) \\ \dfrac{1}{N!}\displaystyle\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\, \langle\varphi_{\sigma(1)}\big|_{\mathcal{H}} \otimes \cdots \otimes \langle\varphi_{\sigma(N)}\big|_{\mathcal{H}} & (c = sgn). \end{cases} \tag{30}$$

Further, the symmetric and the anti-symmetric bra spaces are expressed as the following sets, respectively :

$$(\widehat{\otimes}^N\Phi)'_s = \widetilde{P_{c_1}^{\widehat{\otimes}^2\Phi}}((\widehat{\otimes}^N\Phi)'), \tag{31}$$

$$(\widehat{\otimes}^N\Phi)'_a = \widetilde{P_{sgn}^{\widehat{\otimes}^2\Phi}}((\widehat{\otimes}^N\Phi)'). \tag{32}$$

Thus, in the RHS formalism characterizing the identical particles system, the symmetric structure can be individually assigned to the bra and ket vectors. When we combine the dual spaces $(\widehat{\otimes}^N\Phi)^\times$ and $(\widehat{\otimes}^N\Phi)'$ as $(\widehat{\otimes}^N\Phi)^\times \cup (\widehat{\otimes}^N\Phi)'$, the symmetric and anti-symmetric spaces of $(\widehat{\otimes}^N\Phi)^\times \cup (\widehat{\otimes}^N\Phi)'$ become

$$\begin{aligned} \left[(\widehat{\otimes}^N\Phi)^\times \cup (\widehat{\otimes}^N\Phi)'\right]_s &= \widetilde{P_{c_1}^{\widehat{\otimes}^2\Phi}}((\widehat{\otimes}^N\Phi)^\times \cup (\widehat{\otimes}^N\Phi)') = \widetilde{P_{c_1}^{\widehat{\otimes}^2\Phi}}((\widehat{\otimes}^N\Phi)^\times) \cup \widetilde{P_{c_1}^{\widehat{\otimes}^2\Phi}}((\widehat{\otimes}^N\Phi)') \\ &= (\widehat{\otimes}^N\Phi)^\times_s \cup (\widehat{\otimes}^N\Phi)'_s. \end{aligned} \tag{33}$$

and

$$\left[(\widehat{\otimes}^N\Phi)^\times \cup (\widehat{\otimes}^N\Phi)'\right]_a = (\widehat{\otimes}^N\Phi)^\times_a \cup (\widehat{\otimes}^N\Phi)'_a, \tag{34}$$

respectively.

III. OBSERVABLE

A. Spectral expansion for a RHS

A spectral expansion of each bra and ket vector for a given self-adjoint operator A can be performed using the generalized eigenvectors of A[15]. Considering RHS (2), we set $A : \mathcal{D}(A) \to \mathcal{H}$

as a self-adjoint operator in the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and assume that A is continuous on (Φ, τ_Φ) and $A\Phi \subset \Phi$. The nuclear spectral theorem for A yields[19]

$$\langle\varphi,\psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \sum_{k=1}^{dim\hat{\mathcal{H}}(\lambda)} e_k(\lambda)^*(\varphi)e_k(\lambda)(\psi)d\mu_\lambda, \tag{35}$$

$$\langle\varphi,A\psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \sum_{k=1}^{dim\hat{\mathcal{H}}(\lambda)} \lambda\, e_k(\lambda)^*(\varphi)e_k(\lambda)(\psi)d\mu_\lambda, \tag{36}$$

for any $\varphi,\psi \in \Phi$, where $\hat{\mathcal{H}}(\lambda)$ is a Hilbert space constituting the direct integral that realizes $\mathcal{H}$ (λ goes through the spectra $\mathrm{Sp}(A) \subset \mathbb{R}$), μ_λ is a Borel measure on the spectrum of A, and for each λ, $e_{\lambda,k}(k = 1 \sim dim\hat{\mathcal{H}}(\lambda))$ is the generalized eigenvectors of A corresponding to λ. Here a linear functional F on Φ is referred to as a generalized eigenvector of A corresponding to the eigenvalue λ when F satisfies $F(A\phi) = \lambda F(\phi)$ for every $\phi \in \Phi$[18]. For simplicity, we assume $dim\,\hat{\mathcal{H}}(\lambda) \equiv 1$ for any λ hereafter.

Now, we denote the generalized eigenvector e_λ (e^*_λ) by $\langle\lambda|_{\mathcal{H}}$ ($|\lambda\rangle_{\mathcal{H}}$) and its value $e_\lambda(\varphi)$ ($e^*_\lambda(\varphi)$) by $\langle\lambda|\varphi\rangle_{\mathcal{H}}$ ($\langle\varphi|\lambda\rangle_{\mathcal{H}}$). These notations transform the relations (35)-(36) into

$$\langle\phi,\varphi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \langle\phi|\lambda\rangle_{\mathcal{H}}\,\langle\lambda|\varphi\rangle_{\mathcal{H}}\,d\mu_\lambda, \tag{37}$$

$$\langle\phi,A\varphi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \lambda\,\langle\phi|\lambda\rangle_{\mathcal{H}}\,\langle\lambda|\varphi\rangle_{\mathcal{H}}\,d\mu_\lambda. \tag{38}$$

using these, the spectral expansions for the bra $\langle\varphi|_{\mathcal{H}}$ and ket $|\varphi\rangle_{\mathcal{H}}$ vectors in Φ' and $\Phi^\times$ by the generalized eigenvectors $\{\langle\lambda|_{\mathcal{H}}\}$ and $\{|\lambda\rangle_{\mathcal{H}}\}$ of A are derived. For $\varphi \in \Phi$,

$$|\varphi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \langle\lambda|\varphi\rangle_{\mathcal{H}}\,|\lambda\rangle_{\mathcal{H}}\,d\mu_\lambda, \tag{39}$$

$$|A\varphi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \lambda\,\langle\lambda|\varphi\rangle_{\mathcal{H}}\,|\lambda\rangle_{\mathcal{H}}\,d\mu_\lambda \tag{40}$$

and

$$\langle\varphi|_{\mathcal{H}} = \int_{\mathbb{R}} \langle\varphi|\lambda\rangle_{\mathcal{H}}\,\langle\lambda|_{\mathcal{H}}\,d\mu_\lambda \tag{41}$$

$$\langle A\varphi|_{\mathcal{H}} = \int_{\mathbb{R}} \lambda\,\langle\varphi|\lambda\rangle_{\mathcal{H}}\,\langle\lambda|_{\mathcal{H}}\,d\mu_\lambda. \tag{42}$$

They are can be divided into the continuous and discrete spectrum parts. For instance,

$$|\varphi\rangle_{\mathcal{H}} = \sum_{\lambda_n\in Sp(A)} \langle\lambda_n|\varphi\rangle_{\mathcal{H}}\,|\lambda_n\rangle_{\mathcal{H}} + \int_{\lambda\in Sp(A)} \langle\lambda|\varphi\rangle_{\mathcal{H}}\,|\lambda\rangle_{\mathcal{H}}\,d\mu_\lambda, \tag{43}$$

$$\langle\varphi|_{\mathcal{H}} = \sum_{\lambda_n\in Sp(A)} \langle\varphi|\lambda_n\rangle_{\mathcal{H}}\,\langle\lambda_n|_{\mathcal{H}} + \int_{\lambda\in Sp(A)} \langle\varphi|\lambda\rangle_{\mathcal{H}}\,\langle\lambda|_{\mathcal{H}}\,d\mu_\lambda, \tag{44}$$

where the sum is taken over the discrete spectrum and the integral is over the continuous spectrum.

Similar to the permutation operator presented in Sec. II B, we can consider the extension of A on the dual spaces. $\hat{A}$ on $\Phi' \cup \Phi^{\times}$ is defined as

$$(\hat{A}(f))(\phi) := f(A(\phi)), \tag{45}$$

for any $\phi \in \Phi$ and $f \in \Phi' \cup \Phi^{\times}$. Based on the definition of the generalized eigenvectors, $\hat{A}$ satisfies the eigenequations for $\{\langle\lambda|_{\mathcal{H}}\}$ and $\{|\lambda\rangle_{\mathcal{H}}\}$ of A, namely,

$$\langle\lambda|_{\mathcal{H}} \hat{A} = \lambda \langle\lambda|_{\mathcal{H}}\,,\ \hat{A}\,|\lambda\rangle_{\mathcal{H}} = \lambda\,|\lambda\rangle_{\mathcal{H}}\,, \tag{46}$$

where we denote $\hat{A}(\langle\lambda|_{\mathcal{H}})$ by $\langle\lambda|_{\mathcal{H}}\,\hat{A}$.

B. Spectral expansion for tensor product of RHS

Here, we focus on an self-adjoint operator with respect to the tensor product of RHS (3) and attempt to derive its spectral expansion form. We consider the case $N = 2$. Let $A_i : D(A_i) \to \mathcal{H}_i$ be self-adjoint in $\mathcal{H}_i$ where $D(A_i)$ indicates the domain of A_i $(i = 1, 2)$. Each A_i is assumed to be continuous on Φ_i, satisfying $A_i(\Phi_i) \subset \Phi_i$. Now, we focus on a self-adjoint operator defined in the tensor product $\mathcal{H}_1\overline{\otimes}\mathcal{H}_2$, $A = \overline{A_1 \otimes I_2 + I_1 \otimes A_2} : D(A) \to \mathcal{H}_1\overline{\otimes}\mathcal{H}_2$ [], which is given by the self-adjoint extension of the operator $A_1 \otimes I_2 + I_1 \otimes A_2$ in $\mathcal{H}_1\overline{\otimes}\mathcal{H}_2$ where I_i is the identity map for $\mathcal{H}_i$ $(i = 1, 2)$. Notably, this form of A is generally utilized as the Hamiltonian of a composite system[25, 28]. As evident, A has the spectrum $Sp(A) = Cl(Sp(A_1) + Sp(A_2))$ lying on the real line (ClX is the closure of a set X in the real line) and it is continuous on the nuclear space $\Phi_1\widehat{\otimes}\Phi_2$ as per the relation $A(\Phi_1\widehat{\otimes}\Phi_2) \subset \Phi_1\widehat{\otimes}\Phi_2$. Closure is important in proving the nuclear spectral theorem.

Regarding RHS (3), the nuclear spectral theorem for A provides the following relation [19] : for any $\varphi, \psi \in \Phi_1\widehat{\otimes}\Phi_2$,

$$\langle\varphi,\,\psi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} = \int_{\lambda\in Sp(A)} \left\langle\hat{\varphi}\middle|\hat{\psi}\right\rangle_{\lambda} d\mu_{\lambda}, \tag{47}$$

$$\langle\varphi,\,A\psi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} = \int_{\lambda\in Sp(A)} \lambda\left\langle\hat{\varphi}\middle|\hat{\psi}\right\rangle_{\lambda} d\mu_{\lambda}, \tag{48}$$

with

$$\left\langle \hat{\varphi} \middle| \hat{\psi} \right\rangle_\lambda = \int_{\lambda=\lambda_1+\lambda_2} \sum_{k=1}^{dim\hat{\mathcal{H}}_1(\lambda_1)} \sum_{l=1}^{dim\hat{\mathcal{H}}_2(\lambda_2)} (e^1_{\lambda_1,k} \otimes e^2_{\lambda_2,k})^*(\varphi)(e^1_{\lambda_1,k} \otimes e^2_{\lambda_2,l})(\psi) d\sigma^\lambda_{\lambda_1,\lambda_2}, \tag{49}$$

where μ_λ is the Borel measure given in III A, $\sigma^\lambda_{\lambda_1,\lambda_2}$ is also a Borel measure on $\mathbb{R}^2$ whose support is contained in the set $\{(\lambda_1, \lambda_2) \in \mathbb{R}^2; \lambda = \lambda_1 + \lambda_2, \lambda_i \in Sp(A_i)(i = 1, 2)\}$. $e^1_{\lambda_1,k}(k = 1, 2, \cdots, dim\hat{\mathcal{H}}_1(\lambda_1))$ and $e^2_{\lambda_2,l}$ $(l = 1, 2, \cdots, dim\hat{\mathcal{H}}_2(\lambda_2))$ are the generalized eigenvectors of A_1 and A_2, respectively, corresponding to λ_1 and λ_2 obtained by the nuclear spectral theorem, respectively. When $dim\hat{\mathcal{H}}_1(\lambda_1)) = dim\hat{\mathcal{H}}_2(\lambda_2) = 1$, the relation (47) and (48) are expressed as

$$\begin{aligned} \langle \varphi,\, \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} &= \int_{\lambda \in Sp(A)} \Big\{ \int_{\lambda=\lambda_1+\lambda_2} (e^1_{\lambda_1} \otimes e^2_{\lambda_2})^*(\varphi)(e^1_{\lambda_1} \otimes e^2_{\lambda_2})(\psi) d\sigma^\lambda_{\lambda_1,\lambda_2} \Big\} d\mu_\lambda, & (50) \\ \langle \varphi,\, A\psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} &= \int_{\lambda \in Sp(A)} \lambda \Big\{ \int_{\lambda=\lambda_1+\lambda_2} (e^1_{\lambda_1} \otimes e^2_{\lambda_2})^*(\varphi)(e^1_{\lambda_1} \otimes e^2_{\lambda_2})(\psi) d\sigma^\lambda_{\lambda_1,\lambda_2} \Big\} d\mu_\lambda. & (51) \end{aligned}$$

Here, we introduce the notations,

$$e^i_{\lambda_i} \to \langle \lambda_i |_{\mathcal{H}_i}\,, \quad (e^i_{\lambda_i})^* \to | \lambda_i \rangle_{\mathcal{H}_i}\,, \quad (i = 1, 2) \tag{52}$$

and

$$\begin{aligned} e^1_{\lambda_1} \otimes e^2_{\lambda_2}(\varphi) &\to \langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2} | \varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\,, \\ (e^1_{\lambda_1} \otimes e^2_{\lambda_2})^*(\varphi) &\to \langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2}\,. \end{aligned} \tag{53}$$

These notations induce from (50) and (51) the relations,

$$\begin{aligned} \langle \varphi,\, \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \Big\{ \int_{\lambda=\lambda_1+\lambda_2} & \langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2} \\ & \langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2} | \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\, d\sigma^\lambda_{\lambda_1,\lambda_2} \Big\} d\mu_\lambda, \end{aligned} \tag{54}$$

and

$$\begin{aligned} \langle \varphi,\, A\psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \lambda \Big\{ \int_{\lambda=\lambda_1+\lambda_2} & \langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2} \\ & \langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2} | \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\, d\sigma^\lambda_{\lambda_1,\lambda_2} \Big\} d\mu_\lambda, \end{aligned} \tag{55}$$

for any $\varphi, \psi \in \Phi_1 \widehat{\otimes} \Phi_2$. Note that both the bra vactor $\langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2}$ and the ket vector $| \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2}$ belong to $(\Phi_1 \widehat{\otimes} \Phi_2)'$ and $(\Phi_1 \widehat{\otimes} \Phi_2)^\times$, Respectively. Further, they satisfy the eigenequations (Appendix A):

$$\begin{aligned} \langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2} (A\varphi) &= (\lambda_1 + \lambda_2) \langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2} (\varphi), & (56) \\ | \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2} (A\varphi) &= (\lambda_1 + \lambda_2) | \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2} (\varphi). & (57) \end{aligned}$$

However,, using (37)-(38), $\langle \varphi, \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$ can be represented using the generalized eigenvector $\{|\lambda\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\}_{\lambda \in Sp(A)}$ of A, as follows :

$$\langle \varphi, \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \langle \varphi | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \langle \lambda | \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \, d\mu, \tag{58}$$

$$\langle \varphi, A\varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \lambda \langle \varphi | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \langle \lambda | \psi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \, d\mu. \tag{59}$$

Noting the relation $\langle \varphi | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} (\varphi)$ and $\langle \lambda | \varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} (\varphi)$, from (54) and (58), the following relation is satisfied :

$$\langle \lambda | \varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda = \lambda_1 + \lambda_2} \langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2} | \varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2} \, d\sigma^{\lambda}_{\lambda_1, \lambda_2}, \tag{60}$$

for any $\varphi \in \Phi_1 \widehat{\otimes} \Phi_2$. Thus, using (60), the generalized eigenvector of the ket $|\lambda\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$ derived from the nuclear spectral theorem for A can be represented based on the tensor product of the generalized eigenvectors $|\lambda_1\rangle_{\mathcal{H}_1}$ and $|\lambda_2\rangle_{\mathcal{H}_2}$ corresponding to A_1 and A_2, respectively. Similarly, for the generalized eigenvector of the bra $\langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$, the relation,

$$\langle \varphi | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda = \lambda_1 + \lambda_2} \langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda_1 \rangle_{\mathcal{H}_1} \otimes | \lambda_2 \rangle_{\mathcal{H}_2} \langle \lambda_1 |_{\mathcal{H}_1} \otimes \langle \lambda_2 |_{\mathcal{H}_2} \, d\sigma^{\lambda}_{\lambda_1, \lambda_2}, \tag{61}$$

can be obtained, with (61)) exhibiting a representation of $|\lambda\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$ based on the tensor product of $|\lambda_1\rangle_{\mathcal{H}_1}$ and $|\lambda_2\rangle_{\mathcal{H}_2}$.

To elucidate the spectral expansion of the bra $\langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$ and ket $|\varphi\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$ for the self-adjoint operator A, we utilize the relations (54) and (55). Subsequently, the following expansion can be expressed as well as (39)-(42). For any $\varphi \in \Phi_1 \widehat{\otimes} \Phi_2$,

$$|\varphi\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \, d\mu_\lambda, \tag{62}$$

$$|A\varphi\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \lambda \langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \varphi \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \, d\mu_\lambda, \tag{63}$$

and

$$\langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \, d\mu_\lambda, \tag{64}$$

$$\langle A\varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \int_{\lambda \in Sp(A)} \lambda \langle \varphi |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} | \lambda \rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \langle \lambda |_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \, d\mu_\lambda. \tag{65}$$

Furthermore, (60) and (61) indicate the other beneficial expansions that are performed based on the tensor products of generalized eigenvectors $\{\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}\}$ and $\{|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\}$, as follows,

$$|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{\lambda\in Sp(A)}\int_{\lambda=\lambda_1+\lambda_2}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\,d\sigma^{\lambda}_{\lambda_1,\lambda_2}d\mu_\lambda, \tag{66}$$

$$|A\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{\lambda\in Sp(A)}\lambda\int_{\lambda=\lambda_1+\lambda_2}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\,d\sigma^{\lambda}_{\lambda_1,\lambda_2}d\mu_\lambda, \tag{67}$$

$$\langle\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{\lambda\in Sp(A)}\int_{\lambda=\lambda_1+\lambda_2}\langle\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}\,d\sigma^{\lambda}_{\lambda_1,\lambda_2}d\mu_\lambda, \tag{68}$$

$$\langle A\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{\lambda\in Sp(A)}\lambda\int_{\lambda=\lambda_1+\lambda_2}\langle\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}\,d\sigma^{\lambda}_{\lambda_1,\lambda_2}d\mu_\lambda. \tag{69}$$

Hereafter, we adopt the sign $\int_{Sp(A)}d\nu$ in stead of $\int_{\lambda\in Sp(A)d}d\sigma^{\lambda}_{\lambda_1,\lambda_2}\int_{\lambda=\lambda_1+\lambda_2}d\mu_\lambda$. Consequently, the relations obtained till now can be represented simply by using the abbreviation,

$$\int_{\lambda\in Sp(A)}\int_{\lambda=\lambda_1+\lambda_2}\to\int_{Sp(A)}\quad\text{and}\quad d\sigma^{\lambda}_{\lambda_1,\lambda_2}d\mu_\lambda\to d\nu. \tag{70}$$

For instance, the spectral expansions of $|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}$ and $|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}$ of (66)- (69) convert into

$$|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{Sp(A)}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\,d\nu, \tag{71}$$

$$|A\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{Sp(A)}\lambda\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}|\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\,d\nu, \tag{72}$$

and

$$\langle\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{Sp(A)}\langle\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}\,d\nu, \tag{73}$$

$$\langle A\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}=\int_{Sp(A)}\lambda\langle\varphi|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}\,d\nu. \tag{74}$$

When $\varphi=\varphi_1\otimes\varphi_2\in\Phi_1\otimes\Phi_2$, the relation, $\langle\lambda_1|_{\mathcal{H}}\otimes\langle\lambda_2|_{\mathcal{H}}(\varphi_1\otimes\varphi_2)=\langle\lambda_1|_{\mathcal{H}}(\varphi_1)\otimes\langle\lambda_2|_{\mathcal{H}}(\varphi_2)=\langle\lambda_1|\varphi_1\rangle_{\mathcal{H}_1}\langle\lambda_2|\varphi_2\rangle_{\mathcal{H}_2}$, can be utilized to obtain the spectral expansions of $|\varphi_1\rangle_{\mathcal{H}_1}\otimes|\varphi_2\rangle_{\mathcal{H}_2}$:

$$\begin{aligned}|\varphi_1\rangle_{\mathcal{H}_1}\otimes|\varphi_2\rangle_{\mathcal{H}_2}&=|\varphi_1\otimes\varphi_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}=|\varphi_1\otimes\varphi_2\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}\\&=\int_{Sp(A)}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}|\varphi_1\otimes\varphi_2\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\,d\nu\\&=\int_{Sp(A)}\langle\lambda_1|_{\mathcal{H}_1}\otimes\langle\lambda_2|_{\mathcal{H}_2}(\varphi_1\otimes\varphi_2)|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\,d\nu\\&=\int_{Sp(A)}\langle\lambda_1|\varphi_1\rangle_{\mathcal{H}_1}\langle\lambda_2|\varphi_2\rangle_{\mathcal{H}_2}|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\,d\nu.\end{aligned} \tag{75}$$

From (75), we obtain the expansion coefficients of $|\varphi_1\rangle_{\mathcal{H}_1}\otimes|\varphi_2\rangle_{\mathcal{H}_2}$ using the set of the (generalized) eigenvectors $\{|\lambda_1\rangle_{\mathcal{H}_1}\otimes|\lambda_2\rangle_{\mathcal{H}_2}\}$ of A are given $\langle\lambda_1|\varphi_1\rangle_{\mathcal{H}_1}\langle\lambda_2|\varphi_2\rangle_{\mathcal{H}_2}$ where $\lambda=\lambda_1+\lambda_2$ goes through

$Sp(A)$. In the literature, the coefficient has been obtained from the expansion of the ket state of a composite system without interaction using the complete base spanned by eigenvectors of a observable. Moreover, each eigenvector comprises the tensor product of ket states of a single system. Therefore, the proposed framework manifested the exact derivation of the coefficients by expanding the eigenvectors. The complete orthonormality for the eigenvectors, $\{|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}\}$, will be focused on in the next subsection. The expansion for the bra, $\langle\varphi_1|_{\mathcal{H}_1} \otimes \langle\varphi_2|_{\mathcal{H}_2}$, is also obtained as

$$\langle\varphi_1|_{\mathcal{H}_1} \otimes \langle\varphi_2|_{\mathcal{H}_2} = \int_{Sp(A)} \langle\varphi_1|\lambda_1\rangle_{\mathcal{H}_1} \langle\varphi_2|\lambda_2\rangle_{\mathcal{H}_2} \langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} d\nu, \tag{76}$$

whose expansion coefficients are $\langle\varphi_1|\lambda_1\rangle_{\mathcal{H}_1} \langle\varphi_2|\lambda_2\rangle_{\mathcal{H}_2}$.

C. Complete orthonormal system

From (58), the completion form is obtained as

$$I = \int_{\lambda\in Sp(A)} |\lambda\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} \langle\lambda|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} d\mu_\lambda, \tag{77}$$

where I is the identity for the bra $\langle\cdot|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}$ and ket $|\cdot\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}$. In addition, we set $\varphi(\lambda) \equiv \langle\lambda|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}(\varphi) = \langle\lambda|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} |\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}$ $(\varphi \in \Phi_1\widehat{\otimes}\Phi_2)$,

$$\begin{aligned}
&\int_{\lambda\in Sp(A)} \langle\lambda'|\lambda\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} \varphi(\lambda) d\mu_\lambda \\
&= \int_{\lambda\in Sp(A)} \langle\lambda'|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} |\lambda\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} \langle\lambda|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} |\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} d\mu_\lambda \\
&= \langle\lambda'|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} |\varphi\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} \\
&= \varphi(\lambda'),
\end{aligned} \tag{78}$$

This provides the orthonormal relation,

$$\langle\lambda'|\lambda\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} = \langle\lambda'|_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} |\lambda\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2} = \delta(\lambda' - \lambda), \tag{79}$$

where δ is Dirac's δ-function as the normalization factor of the eigenvectors of A. (77) and (79) show that $\{|\lambda\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}\}$ creates the complete orthonormal system in terms of the self-adjoint operator A in RHS (3). Using (60) and (61), it was found that each eigenvector of $\{|\lambda\rangle_{\mathcal{H}_1\overline{\otimes}\mathcal{H}_2}\}$ is associated with $\{|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}\}$ where $\lambda = \lambda_1 + \lambda_2$. Thus, the complete orthonormal form is also established using $\{|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}\}$. Actually, by using (60) and (61), the completion form (77) is converted into

$$\begin{aligned} I &= \int_{\lambda \in Sp(A)} \int_{\lambda=\lambda_1+\lambda_2} |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} \langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} d\sigma^{\lambda}_{\lambda_1,\lambda_2} d\mu_\lambda \\ &= \int_{Sp(A)} |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} \langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} d\nu. \end{aligned} \tag{80}$$

Here, the notation (70) is adapted. To consider the orthonormality, putting $\varphi(\lambda_1, \lambda_2) \equiv \langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2}(\varphi) = \langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} |\varphi\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$ for $\varphi \in \Phi_1 \widehat{\otimes} \Phi_2$, we have

$$\begin{aligned} &\int_{\lambda \in Sp(A)} \int_{\lambda=\lambda_1+\lambda_2} \langle\lambda'_1|_{\mathcal{H}_1} \otimes \langle\lambda'_2|_{\mathcal{H}_2} |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} \varphi(\lambda_1,\lambda_2) d\sigma^{\lambda}_{\lambda_1,\lambda_2} d\mu_\lambda \\ &= \int_{\lambda \in Sp(A)} \int_{\lambda=\lambda_1+\lambda_2} \langle\lambda'_1|_{\mathcal{H}_1} \otimes \langle\lambda'_2|_{\mathcal{H}_2} |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} \langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} |\varphi\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} d\sigma^{\lambda}_{\lambda_1,\lambda_2} d\mu_\lambda \\ &= \langle\lambda'_1|_{\mathcal{H}_1} \otimes \langle\lambda'_2|_{\mathcal{H}_2} |\varphi\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \\ &= \varphi(\lambda'_1, \lambda'_2). \end{aligned} \tag{81}$$

(81) implies that the combination, $\langle\lambda'_1|_{\mathcal{H}_1} \otimes \langle\lambda'_2|_{\mathcal{H}_2} |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}$, of $\langle\lambda'_1|_{\mathcal{H}_1} \otimes \langle\lambda'_2|_{\mathcal{H}_2}$ and $|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}$ can be represented by the product of δ-functions :

$$\langle\lambda'_1|_{\mathcal{H}_1} \otimes \langle\lambda'_2|_{\mathcal{H}_2} |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} = \check{\delta}(\lambda'_1 - \lambda_1)\check{\delta}(\lambda'_2 - \lambda_2), \tag{82}$$

Here, $\check{\delta}$ is preformed as

$$\begin{aligned} f(\lambda'_1, \lambda'_2) &= \int_{\lambda \in Sp(A)} \int_{\lambda=\lambda_1+\lambda_2} f(\lambda_1,\lambda_2)\check{\delta}(\lambda'_1 - \lambda_1)\check{\delta}(\lambda'_2 - \lambda_2) d\sigma^{\lambda}_{\lambda_1,\lambda_2} d\mu_\lambda \\ &= \int_{Sp(A)} f(\lambda_1,\lambda_2)\check{\delta}(\lambda'_1 - \lambda_1)\check{\delta}(\lambda'_2 - \lambda_2) d\nu \end{aligned}$$

for any function $f(\lambda_1, \lambda_2)$. Thus, the complete orthonormal form given by $\{|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}\}$ is constructed as the relations (80) and (82).

D. Extension

The self-adjoint operator $A = \overline{A_1 \otimes I_2 + I_1 \otimes A_2}$ can be extended to the dual spaces as follows. As A is continuous on $(\Phi_1 \hat{\otimes} \Phi_2, \widehat{\tau_p})$ with $A(\Phi_1 \hat{\otimes} \Phi_2) \subset \Phi_1 \hat{\otimes} \Phi_2$, an operator $\hat{A}$ on $(\Phi_1 \widehat{\otimes} \Phi_2)' \cup (\Phi_1 \widehat{\otimes} \Phi_2)^\times$ can be expressed as

$$(\hat{A}(f))(\varphi) := f(A(\varphi)), \tag{83}$$

for any $\varphi \in \Phi_1 \widehat{\otimes} \Phi_2$ and $f \in (\Phi_1 \widehat{\otimes} \Phi_2)' \cup (\Phi_1 \widehat{\otimes} \Phi_2)^\times$. $\hat{A}$ satisfies the following eigenequations for the generalized eigenvectors $\{\langle\lambda|_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\}$ and $\{|\lambda\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\}$ of A:

$$\langle\lambda|_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} \hat{A} = \lambda \langle\lambda|_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\,,\ \hat{A}\,|\lambda\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2} = \lambda\,|\lambda\rangle_{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}\,, \tag{84}$$

as well as (46). Further, from (56)-(57), we have the eigenequations with respect to $\{\langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2}\}$ and $\{|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}\}$,

$$\langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} \hat{A} \;=\; (\lambda_1 + \lambda_2)\,\langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} \tag{85}$$

$$\hat{A}\,|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} \;=\; (\lambda_1 + \lambda_2)\,|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}\,. \tag{86}$$

Note that $\langle\lambda_1|_{\mathcal{H}_1} \otimes \langle\lambda_2|_{\mathcal{H}_2} \in \Phi_1' \otimes \Phi_2'$ and $|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} \in \Phi_1^\times \otimes \Phi_2^\times$.

To each self-adjoint operator A_i in RHS $\Phi_i \subset \mathcal{H}_i \subset \Phi_i^\times, \Phi_i'$ $(i = 1, 2)$, there corresponds the extension $\hat{A}_i$ on $\Phi_i' \cup \Phi_i^\times$ when defined as (46). Consequently, we obtain the relation

$$\hat{A} = \hat{A}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{A}_2 \tag{87}$$

on the subset $(\Phi_1' \otimes \Phi_2') \cup (\Phi_1^\times \otimes \Phi_2^\times)$ of $(\Phi_1 \widehat{\otimes} \Phi_2)' \cup (\Phi_1 \widehat{\otimes} \Phi_2)^\times$, where $\hat{I}_i$ is the identity on $\Phi_i' \cup \Phi_i^\times$. (87) shows the connection of the self-adjoint observable $\hat{A}_i (i = 1, 2)$ of the isolated systems with $\hat{A}$ of their composite system in the bra-ket formalism.

Notably, the relations obtained in SectionsIII B-III D can be easily generalized to the N-tensor product of RHS (11) using the self-adjpont operator $A = \overline{\sum_{i=1}^N \check{A}_i}$ where $\check{A}_i = I \otimes I \otimes \cdots \otimes I \otimes A_i \otimes I \otimes \cdots \otimes I$.

E. Relation to the permutation operator

In this subsection, we assume $\mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathcal{H}_N \equiv \mathcal{H}$ and $\Phi_1 = \Phi_2 = \cdots = \Phi_N \equiv \Phi$. We set the RHS comprising the N-tensor product of RHS (17) and focus on a self-adjoint operator in the RHS, $A = \overline{\sum_{i=1}^N \check{A}_i}$, with $\check{A}_i = I \otimes I \otimes \cdots \otimes I \otimes A_i \otimes I \otimes \cdots \otimes I$. Its extension, expressed as (83), to $(\widehat{\otimes}^N \Phi)^\times \cup (\widehat{\otimes}^N \Phi)'$ is denoted by $\hat{A}$. Subsequently, A has the complete orthonormal system composed of the eigenvectors $\{|\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}}\}$. Here, each $|\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}}$ belongs to the bra-ket space $(\widehat{\otimes}^N \Phi)^\times \cup (\widehat{\otimes}^N \Phi)'$ and satisfies the eigenequations (85) and (86). Using (27) we found that the symmetry of $|\varphi_1, \otimes \cdots \otimes \varphi_N\rangle_{\overline{\otimes}^N \mathcal{H}} = |\varphi_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\varphi_N\rangle_{\mathcal{H}}$ in $(\widehat{\otimes}^N \Phi)^\times \cup (\widehat{\otimes}^N \Phi)'$ is determined by the permutation operator $\widetilde{P_c^{\widehat{\otimes}^2\Phi}}$ on it. When $\widetilde{P_c^{\widehat{\otimes}^2\Phi}}$ acts on $|\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}}$, for any $\phi = \phi_1 \otimes \cdots \otimes \phi \in \otimes^N \Phi_N \subset \widehat{\otimes}^N \Phi$, we have

$$\begin{aligned}
\widetilde{P_c^{\widehat{\otimes}^2\Phi}}(|\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}})(\phi) &= |\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}} (P_{c_1}^{\widehat{\otimes}^2\Phi}(\phi)) \\
&= |\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}} \Big\{ \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) \left|\phi_{\sigma(1)}\right\rangle_{\mathcal{H}} \otimes \cdots \otimes \left|\phi_{\sigma(N)}\right\rangle_{\mathcal{H}} \Big\} \\
&= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) \, |\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}} \left|\phi_{\sigma(1)}\right\rangle_{\mathcal{H}} \otimes \cdots \otimes \left|\phi_{\sigma(N)}\right\rangle_{\mathcal{H}} \\
&= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) \left\langle \phi_{\sigma(1)} \middle| \lambda_1 \right\rangle \dots \left\langle \phi_{\sigma(N)} \middle| \lambda_N \right\rangle \\
&= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) \left\langle \phi_1 \middle| \lambda_{\sigma(1)} \right\rangle \dots \left\langle \phi_N \middle| \lambda_{\sigma(N)} \right\rangle \\
&= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) \left|\lambda_{\sigma(1)}\right\rangle_{\mathcal{H}} \otimes \cdots \otimes \left|\lambda_{\sigma(N)}\right\rangle_{\mathcal{H}} (\phi). \qquad (88)
\end{aligned}$$

From (88), it is confirmed that

$$\widetilde{P_c^{\widehat{\otimes}^2\Phi}} \, |\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}} = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) \left|\lambda_{\sigma(1)}\right\rangle_{\mathcal{H}} \otimes \cdots \otimes \left|\lambda_{\sigma(N)}\right\rangle_{\mathcal{H}}. \qquad (89)$$

(89) shows that the permutation operator $\widetilde{P_c^{\widehat{\otimes}^2\Phi}}$ determines the symmetric structure of the eigenvectors $|\lambda_1\rangle_{\mathcal{H}} \otimes \cdots \otimes |\lambda_N\rangle_{\mathcal{H}}$ of A via the relation (27). Similarly, we obtain

$$\langle\lambda_1|_{\mathcal{H}} \otimes \cdots \otimes \langle\lambda_N|_{\mathcal{H}} \, \widetilde{P_c^{\widehat{\otimes}^2\Phi}} = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) \left\langle\lambda_{\sigma(1)}\right|_{\mathcal{H}} \otimes \cdots \otimes \left\langle\lambda_{\sigma(N)}\right|_{\mathcal{H}}. \qquad (90)$$

In Quantum mechanics, the commutative relation between an observable and the permutation operator is considered the fundamental condition for proving that the symmetric and anti-symmetric states of an identical particles become the eigenvectors of the observable[28]. According to the proposed RHS framework, the operators (89) and (90) become the generalized eigenvectors of A when A and $P_c^{\widehat{\otimes}^2\Phi}$ are commutative on $\widehat{\otimes}^2\Phi$, namely,

$$[A, P_c^{\widehat{\otimes}^2\Phi}] = 0 \text{ on } \widehat{\otimes}^2\Phi. \qquad (91)$$

To verify this fact, the following proposition is applicable.

[Proposition] *Let $\Phi \subset \mathcal{H} \subset \Phi', \Phi^\times$ be an RHS and let $A : D(A) \to \mathcal{H}$ and $B : D(B) \to \mathcal{H}$ be self-adjoint operators in $\mathcal{H}$ such that they are continuous on Φ and the $A\Phi \subset \Phi$ and $B\Phi \subset \Phi$ are satisfied. If A and B are commutative on Φ, for each generalized eigen bra $\langle\lambda|$ and eigen ket $|\lambda\rangle$*

corresponding to the eigen value λ, the elements $\langle\lambda|\,\hat{B}\in\Phi'$ and $\hat{B}\,|\lambda\rangle\in\Phi^{\times}$ are the generalized eigen bra and ket, corresponding to λ. Consequently, the relations

$$\langle\lambda|\,\hat{B}\hat{A}=\lambda\,\langle\lambda|\,\hat{B},\quad \hat{A}\hat{B}\,|\lambda\rangle=\lambda\hat{B}\,|\lambda\rangle \tag{92}$$

are satisfied, where $\hat{A}$ and $\hat{B}$ are the extensions on $\Phi'\cup\Phi^{\times}$ induced by (45). (Appendix B presents the proof.)

Based on this proposition, it can be easily shown that under the condition (91), the elements of (89) and (90) are the generalized eigenvectors of A, satisfying

$$\hat{A}\widetilde{P_c^{\widehat{\otimes}^2\Phi}}(|\lambda_1\rangle_{\mathcal{H}}\otimes\cdots\otimes|\lambda_N\rangle_{\mathcal{H}})=(\lambda_1+\cdots+\lambda_N)\widetilde{P_c^{\widehat{\otimes}^2\Phi}}(|\lambda_1\rangle_{\mathcal{H}}\otimes\cdots\otimes|\lambda_N\rangle_{\mathcal{H}}), \tag{93}$$

and

$$\langle\lambda_1|_{\mathcal{H}}\otimes\cdots\otimes\langle\lambda_N|_{\mathcal{H}}\,\widetilde{P_c^{\widehat{\otimes}^2\Phi}}\hat{A}=(\lambda_1+\cdots+\lambda_N)\,\langle\lambda_1|_{\mathcal{H}}\otimes\cdots\otimes\langle\lambda_N|_{\mathcal{H}}\,\widetilde{P_c^{\widehat{\otimes}^2\Phi}}. \tag{94}$$

IV. CONCLUSION

This study discussed the mathematical treatment of Dirac bracket formalism for composite systems in addition to identical particle system based on the RHS. The tensor product of an RHS facilitates the precise construction of bra and ket vectors under the dual space. For identical particles systems, the symmetric structure of the bra-ket vectors can be introduced by extending the permutation operator to the dual space. The spectral expansions of bra-ket vectors for a self-adjoint operator corresponding to an observable in composite systems via its generalized eigenvectors was established. These generalized eigenvectors were associated with the eigenvectors of self-adjoint operators for single particles and established the complete orthonormal system in the dual space. Furthermore, we found a mathematical condition between the extended self-adjoint operator with the permutation operator for preserving the symmetric structure of the bra and ket vectors via the action of the observable. The RHS formalism for quantum many-body systems proposed in this paper is limited to non-interacting systems, yet it forms the foundation for modern quantum theory based on the perturbation theory, such as the quantum statistical mechanics and the quantum field theory. In future work, we aim to apply our RHS formulation to these areas to facilitate more precise discussions of established studies[11, 12, 29].

Appendix A: The eigenequations (56)-(57)

Let $\varphi \in \Phi_1 \widehat{\otimes} \Phi_2$. Here, φ can be represented as the form of the sum of an absolutely convergence series, $\varphi = \sum_{i=1}^{\infty} r_i \varphi_i^1 \otimes \varphi_i^2$, where $\sum_i |r_i| \leq 1$ and $\{\varphi_i^1\}$ and $\{\varphi_i^2\}$ are null sequencea in Φ[27]. As $A = \overline{A_1 \otimes I_2 + I_1 \otimes A_2}$ is continuous linear on $\Phi_1 \widehat{\otimes} \Phi_2$, we obtain $A\varphi = \sum_{i=1}^{\infty} r_i (A_1 \varphi_i^1 \otimes \varphi_i^2 + \varphi_i^1 \otimes A_2 \varphi_i^2)$. Therefore, the continuous linearity of $|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2}$ on $\Phi_1 \widehat{\otimes} \Phi_2$ provides

$$\begin{aligned}
|\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} (A\varphi) &= \sum_{i=1}^{\infty} r_i \, |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} (A_1 \varphi_i^1 \otimes \varphi_i^2 + \varphi_i^1 \otimes A_2 \varphi_i^2) \\
&= \sum_{i=1}^{\infty} r_i \Big(\lambda_1 \, |\lambda_1\rangle_{\mathcal{H}_1} \varphi_i^1 \otimes |\lambda_2\rangle_{\mathcal{H}_2} \varphi_i^2 + \lambda_2 \, |\lambda_1\rangle_{\mathcal{H}_1} \varphi_i^1 \otimes |\lambda_2\rangle_{\mathcal{H}_2} \varphi_i^2 \Big) \\
&= (\lambda_1 + \lambda_2) \, |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} (\sum_{i=1}^{\infty} r_i \varphi_i^1 \otimes \varphi_i^2) \\
&= (\lambda_1 + \lambda_2) \, |\lambda_1\rangle_{\mathcal{H}_1} \otimes |\lambda_2\rangle_{\mathcal{H}_2} (\varphi). \qquad \text{(A.1)}
\end{aligned}$$

Similarly, we obtain $\langle \lambda_1|_{\mathcal{H}_1} \otimes \langle \lambda_2|_{\mathcal{H}_2} (A\varphi) = (\lambda_1 + \lambda_2) \, \langle \lambda_1|_{\mathcal{H}_1} \otimes \langle \lambda_2|_{\mathcal{H}_2} (\varphi)$.

Appendix B: Proof of Proposition

The proposition is proven as follows. As $|\lambda\rangle$ is a generalized eigenvector of A corresponding to λ, $\hat{A} \, |\lambda\rangle (\varphi) = \lambda \, |\lambda\rangle (\varphi)$ for any $\varphi \in \Phi$ is satisfied. Let $\varphi \in \Phi$. Noting the anti-linearity of $|\lambda\rangle$ and $\hat{B} \, |\lambda\rangle$, we have,

$$\begin{aligned}
\hat{A}(\hat{B} \, |\lambda\rangle)(\varphi) &= \hat{B} \, |\lambda\rangle (A\varphi) = |\lambda\rangle (B(A\varphi)) = |\lambda\rangle (A(B\varphi)) \\
&= \lambda \, |\lambda\rangle (B\varphi) = |\lambda\rangle (B\lambda^* \varphi) = \hat{B}(|\lambda\rangle)(\lambda^* \varphi) = \lambda \hat{B} \, |\lambda\rangle (\varphi). \qquad \text{(B.1)}
\end{aligned}$$

Similarly, $\hat{A}(\hat{B} \, \langle \lambda|)(\varphi) = \lambda \, \langle \lambda| \, \hat{B}(\varphi)$. Thus, the desired assertion is complete.

Acknowledgement

The authors are grateful to Prof. Y. Yamazaki, Prof. T. Yamamoto, Prof. Y. Yamanaka, Prof. K. Iida, Prof. H. Ujino, Prof. I. Sasaki, Prof. H. Saigo, Prof. F. Hiroshima, Prof. S. Matsutani, and Emeritus A. Kitada for their useful comments and encouragement. This work was supported by the Sasakawa Scientific Research Grant from The Japan Science Society and JSPS KAKENHI Grant Number 22K13976. We would like to thank Editage (www.editage.jp) for English language

editing.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

---

[1] J. E. Roberts, J. Math. Phys. **7**, 1097 (1966).

[2] J. E. Roberts, Commun. Math. Phys. **3**, 98 (1966).

[3] J. P. Antoine, J. Math. Phys. **10**, 53 (1969).

[4] J. P. Antoine J. Math. Phys. **10**, 2276 (1969).

[5] O. Melsheimer, J. Math. Phys. **15**, 902 (1974).

[6] O. Melsheimer, J. Math. Phys. **15**, 917 (1974).

[7] A. Bohm, *The Rigged Hilbert Space and Quantum Mechanics*, Lecture Notes in Physics, vol. 78, Springer-Verlag, Berlin Heidelberg, New York, (1978).

[8] A. Bohm, J. Math. Phys. **22**, 2813 (1981).

[9] I. Prigogine and S. A. Rice, eds., *Advances in Chemical Physics, Resonances, Instability, and Irreversibility, Vol 99* (John Wiley & Sons, NewYork, 1996).

[10] A. Bohm, ed., *Irreversibility and Causality, Semigroups and Rigged Hilbert Spaces* (Springer-Verlag, Berlin Heidelberg, 1998).

[11] I. Antoniou, M. Gadella, Z. Suchanecki, *Some General Properties of the Liouville Operator*, Lecture Notes in Physics, vol. 54, pp. 38–56 (1998).

[12] I. Antoniou and M. Gadella, *Irreversibility, resonances and rigged Hilbert spaces*, In Irreversible Quantum Dynamics; F. Benatti and R. Floreanini, Eds., Lecture Notes in Physics vol. 622, Springer, Berlin, Germany, pp. 245–302 (2003).

[13] M. Gadella and F. Gomez, Int. J. Theor. Phys. **42**, 2225 (2003).

[14] R. Madrid, J. Phys A:Math. Gen. **37**, 8129 (2004).

[15] R. Madrid, Eur. J. Phys. **26**, 287 (2005).

[16] J. P. Antoine, R. Bishop, A. Bohm, S. Wickramasekara, *Rigged Hilbert Spaces in Quantum Physics*, In A Compendium of Quantum Physics—Concepts, Experiments, History and Philosophy; Weinert, F., Hentschel, K., Greenberger, D., Eds.; Springer: Heidelberg/Berlin, Germany; New York, NY, USA, pp. 640–651 (2009).

[17] J. P. Antoine, Entropy, **23**, 124 (2021).

[18] I. M. Gelfand and and N. Y. Vilenkin, *Generalized Functions vol IV* (New York Academic 1964).

[19] K. Maurin *Generalized Eigenfunction Expansions and Unitary Representations of Topological Groups* (Polish Scientific Publishers, Warsaw, 1968).

[20] D. Chruściński, J. Math. Phys. **44**, 3718 (2003).

[21] D. Chruściński, J. Math. Phys. **45**, 841 (2004).

[22] L. Laan, Bachelor Thesis:Mathematics and Physics, University of Groningen FSE (2019).

[23] V. Fernández, R. Ramírez, and M. Reboiro, J. Phys. A: Math. Theor. **55** 015303 (2022).

[24] S. Ohmori and J. Takahashi, J. Math. Phys. **63**, 123503 (2022).

[25] M. Reed and B. Simon *Functional Analysis*, Volume I: Functional Analysis, 2nd edn (Academic, San Diego, 1980).

[26] B. C. Hall *Quantum Theory for Mathematicians* (Springer, New York, 2013)

[27] H. H. Schaefer *Topological Vector Spaces*, 2nd edn (Springer-Verlag, New York, 1999).

[28] A. Messiah, *Quantum Mechanics*, (North Holland, 1966).

[29] W. Liu and Z. Huang, Int. J. Theor. Phys. **52**, 4323 (2013).