CLOSED GEODESICS ON HYPERBOLIC SURFACES WITH FEW INTERSECTIONS

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Abstract. We prove that, if a closed geodesic Γ on a complete finite-type hyperbolic surface has at least 2 self-intersections, then the length of Γ has an lower bound $2\log(5+2\sqrt{6})$, and the lower bound is sharp, attained on a corkscrew geodesic on a thrice punctured sphere.

1. Introduction

The study of nonsimple closed geodesics on hyperbolic surfaces is an interesting question in 2-dimensional hyperbolic geometry. An oriented finite type hyperbolic surface is an oriented surface $\Sigma = \Sigma_{g,n}$ without boundary, with genus g and n points removed(puncture), and a complete metric of constant curvature -1 with finite volume. The hyperbolic structure near each removed point is a cusp tends to infinity. For positive integer k, let M_k be the infimum of lengths of closed geodesics of self-intersection number at least k among all oriented finite-type hyperbolic surfaces. When $k \geqslant 1$, we call the closed geodesic is *nonsimple*, otherwise we call it *simple*.

There has been lots of works on the problem till now. When $k = 1$, Hempel showed in [7] that a nonsimple closed geodesic has a universal lower bound $2\log(1+\sqrt{2})$ and Yamada showed in [9] that $2\cosh^{-1}(3) = 4\log(1+\sqrt{2})$ is the best possible lower bound and is attained on a pair of pants with ideal punctures. Basmajian showed in [3] that a nonsimple closed geodesic has a similar *stable neighborhood*, and the length of a closed geodesic gets arbitrarily large as its self-intersection number gets large ([3, Corollary 1.2]). In [2], Baribaud computed the minimal length of geodesics with given self-intersection number or given homotopy types on pairs of pants.

It is interesting to study the minimal length of non-simple closed geodesics with respect to its self-intersection number k . Let ω be a closed geodesic or a geodesic segment on a hyperbolic surface Σ which can be expressed as a local isometry ϕ from S^1 or segment I to Σ. We denote the length as $\ell(\omega)$.

Definition 1.1. Its self-intersection number is denoted by $|\omega \cap \omega|$. $|\omega \cap \omega|$ counts the intersection points of ω with multiplicity that an intersection point with n preimages of ϕ contribute $\binom{n}{2}$ to $|\omega \cap \omega|$.

First, when $k \to \infty$, Basmajian showed ([4, Corollary 1.4]) that

$$\tfrac{1}{2}\log\frac{k}{2} \leqslant M_k \leqslant 2\cosh^{-1}(2k+1) \asymp 2\log k \tag{1}$$

The notation $f(k) \asymp g(k)$ means that $f(k)/g(k)$ is bounded from above and below by positive constants.

Conjecture 1.2. *When $k \geqslant 1$,*

$$M_k = 2\cosh^{-1}(1+2k) = 2\log(1+2k+2\sqrt{k^2+k}) \tag{2}$$

and the equality holds when Γ is a corkscrew geodesic(See definition below) on a thrice-punctured sphere. In other words, any nonsimple closed geodesic of self-intersection number at least k on any finite-type hyperbolic surface has length no less than $2\cosh^{-1}(1+2k)$ and the bound is sharp.

In [10, Theorem 1.1] Shen-Wang improved the lower bound of M_k, that M_k has explicit growth rate $2\log k$, and for a closed geodesic of length L, the self intersection number is no more than $9L^2e^{\frac{L}{2}}$. The exact value for M_k for sufficiently large k is computed in [11, Theorem 1.1], proved Conjecture 1.2 holds when $k > 10^{13350}$. In [13] the lower bound $k > 10^{13350}$ is refined to $k > 1750$ using another method.

However, to the best of the author's knowledge, for small k, even $k = 2$ we cannot compute the exact value of M_k. In the present paper we give an answer:

Theorem 1.3. *When $k = 2$, $M_2 = 2\log(5+2\sqrt{6})$, and the lower bound is sharp and attained on a corkscrew geodesic on a thrice punctured sphere.*

Note that Theorem 1.3 can be generalized to general orientable finite-type hyperbolic surfaces, possibly with holes and geodesic boundaries, since they can be doubled to get a surface as in Theorem 1.3.

Plan of the paper. The idea of the proof is based on the conclusion of [8], that is $4\log(1+\sqrt{2})$ is the best possible lower bound of nonsimple closed geodesic on hyperbolic surfaces, i.e. self-intersection number $k = 1$.

Section 2 to section 4 is the preparation of the proof consisting of 3 parts. First is the collar lemma, a basic lemma in 2-dimensional hyperbolic geometry, and we will give a generalization, the collar neighborhood is not equal on the different side, which will be used in section 5. Second is about geodesics on pair of pants, we will state the result in [2] and as a simple corollary, we prove Theorem 1.3 for the case that geodesic Γ is contained in a pair of pants(or ideal pair of pants) of given homotopy type. Third is about winding number of a geodesic segment in a collar and we can compute the length based on the winding number and the collar.

Section 5 is the main part of the proof. If $\ell(\Gamma) < 2\log(5+2\sqrt{6})$, then choose the shortest closed loop γ in Γ, we prove the theorem in 2 cases, by whether $\ell(\Gamma \setminus \gamma) < 4\log(1+\sqrt{2})$. If it holds then both γ and $\Gamma \setminus \gamma$ are freely homotopic to a multiple of a simple closed geodesic(or cusp), hence Γ contained in a pair of pants(or ideal pair of pants). Otherwise $\ell(\gamma)$ is small, hence contained in a collar in section 2 where the generalized collar lemma is used. Then we complete the proof by contradiction since either the length of Γ is big or Γ contained in a pair of pants.

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2. A generalization of the collar lemma

The collar lemma ([6, Lemma 13.6]) is as follows:

Lemma 2.1. *for any simple closed geodesic c in Σ,*

$$N(c) = \{p \in \Sigma \,:\, d(p,c) < w(\ell(c))\}$$

is an embedded annulus, where $w(x)$ is defined by

$$w(x) := \sinh^{-1}\left(\frac{1}{\sinh(x/2)}\right)$$

Moreover if c_1, c_2 are two disjoint simple closed geodesics in Σ, $N(c_1) \cap N(c_2) = \emptyset$.

Let Σ' be the surface with boundary components c', c'' constructed by cutting along curve $c \subseteq \Sigma$, together with gluing map $\phi' : \Sigma' \to \Sigma$ that gluing c', c'' to the curve $c \subseteq \Sigma$. There is a *maximal collar* $N'(c')$ in Σ' that $\delta(c') > 0$ is the maximum real number satisfies the collar

$$\widetilde{N}'(c') = \left\{p \in \Sigma' \,:\, d(p,c') < \delta(c')\right\}$$

is an embedded annulus in Σ'. $N'(c) = \phi'(N'(c'))$ is an embedded annulus(a half collar) with c as one of its boundary components. We define

$$w_1(x) = \log\left(\frac{e^{x/4}+1}{e^{x/4}-1}\right)$$

for $x > 0$. Note that for $x \leqslant \log(5+2\sqrt{6}) < 2.3$ we have $w_1(x) < 2w(x)$. For simple closed geodesic $c \in \Sigma$ we have the generalized collar lemma:

Lemma 2.2. *If $\ell(c') < 2.3$, then we have $\delta(c') \geqslant w_1(\ell(c'))$. In other words, the collar*

$$N'(c') = \{x \in \Sigma \,:\, d(x,c) < w_1(\ell(c))\}$$

is an embedded annulus.

Proof. We know $\widetilde{N}'(c')$ has two boundary components. If $\widetilde{N}'(c') \cap c'' = \emptyset$, then the one is c and the other is a curve tangent to itself by the maximality of $\delta(c')$, assume $q \in \partial N'(c)$ is one of the tangent points. (Note that if $\widetilde{N}'(c') \cap c'' \neq \emptyset$, then since $w_1(\ell(c')) < 2w(\ell(c'))$, we have $N'(c') \cap c'' = \emptyset$, the lemma holds.) Hence there exists a shortest simple geodesic γ which contained in $\overline{N'(c)} \subseteq \Sigma$, contains two different shortest geodesic segments of length $\delta(c')$ connecting q and c in $N'(c)$. γ has endpoints $r_1, r_2 \in c$, and perpendicular to c at r_1, r_2.

Suppose $c \setminus \{r_1, r_2\}$ contains two arcs c_1' and c_2', and $c_1' \cup \gamma$, $c_2' \cup \gamma$ are simple closed curves, hence freely homotopic to simple closed geodesics c_1'' and c_2'', then c, c_1'', c_2'' are boundary components of a pair of pants $\Sigma_0 \subseteq \Sigma$,

since $c_1'' \cup \gamma$ or $c_2'' \cup \gamma$ could not create bigons, $c_1'' \cap \gamma = c_2'' \cap \gamma = \emptyset$, we have $\gamma \subseteq \Sigma_0$.

Next we prove $\ell(\gamma) \geqslant w_1(x)$. Σ_0 can be constructed by gluing 2 hexagons along 3 nonadjacent boundary segments. In the Poincaré disk model of $\mathbb{H}^2$, let $\mathbb{H}^1$ be the horizontal line. Let $A_1, A_2 \subseteq \mathbb{H}^1$, $A_1(-r,0), A_2(r,0)$ satisfying $\log \frac{1+r}{1-r} = \frac{\ell(c)}{4}$. ω_1, ω_2 are geodesics in $\mathbb{H}^2$ passes A_1, A_2 and perpendicular to $\mathbb{H}^1$. $P_1(-\frac{2r}{1+r^2}, \frac{1-r^2}{1+r^2})$, $P_2(\frac{2r}{1+r^2}, \frac{1-r^2}{1+r^2})$ are one of endpoints of ω_1, ω_2. ω_3' is the geodesic connecting P_1, P_2. Let ω_3 be the geodesic Q_1Q_2 where Q_1, Q_2 are both on the infinity boundary, $\omega_3 \cap \omega_1 = \emptyset$ and $\omega_3 \cap \omega_2 = \emptyset$. For $j = 1, 2$ assume ω_j' be the unique geodesic segment perpendicular to ω_j and ω_3. Then $A_1A_2, \omega_1, \omega_1', \omega_3, \omega_2', \omega_2$ compose the boundary of a hexagon, and suppose Σ_0 is constructed by gluing 2 copies of the hexagon along $\omega_1, \omega_2, \omega_3$.

Since $\gamma \subseteq \Sigma_0$, then

$$\begin{aligned}\ell(\gamma) &\geqslant 2d(\mathbb{H}^1, \omega_3) \geqslant 2d(\mathbb{H}^1, \omega_3') = 2\log\left(1 + \frac{1-r}{1+r}\right) - 2\log\left(1 - \frac{1-r}{1+r}\right) \\ &= 2\log\frac{e^{\ell(c)/4}+1}{e^{\ell(c)/4}-1} = w_1(\ell(c))\end{aligned}$$

□

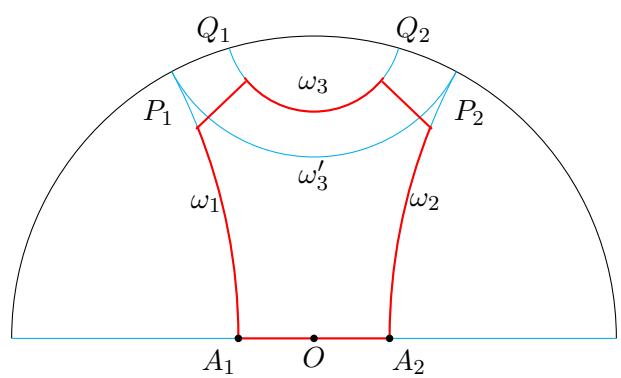

FIGURE 1. A hexagon of Σ_0

Since $w_1(x) > w(x)$ for $x > 0$, we have

$$N''(c'') = \left\{p \in \Sigma' : d(p, c'') < 2w(\ell(c)) - w_1(\ell(c))\right\} \subseteq \Sigma'$$

is either empty or an embedded annulus, and define $N''(c) = \phi'(N''(c'')) \subseteq \Sigma$, either empty or an embedded annulus also. Define

$$N_1(c) = N'(c) \cup N''(c)$$

Lemma 2.3. *$N_1(c)$ is an embedded annulus in Σ.*

Proof. If not, then there exists $y \in N'(c) \cap N''(c)$, connecting c and y by shortest geodesic ζ in $N'(c)$ and ζ' in $N''(c)$, and $\ell(\zeta) + \ell(\zeta') < 2w(\ell(c))$,

hence for all $z \in \zeta \cup \zeta'$, $d(z,c) \leqslant \frac{1}{2}(\ell(\zeta)+\ell(\zeta')) < w(\ell(c))$, hence $\zeta \cup \zeta' \subseteq N(c)$. $\zeta \cup \zeta'$ only intersects c at endpoints so it lies on one side of $N(c)$, this is impossible since $N'(c)$ and $N''(c)$ are on the different side of c. □

Similarly, for the ideal punctures of Σ we have the similar result. When Σ has punctures, we consider the universal covering $p : \mathbb{H}^2 \to \Sigma$, where $\mathbb{H}^2$ is the hyperbolic plane. Each puncture has a neighborhood whose boundary lifts to a union of horocycles that can intersect in at most points of tangency. Such a neighborhood is called a *cusp* of the surface.

In the upper half-plane model for $\mathbb{H}^2$. Let Γ be a cyclic group generated by a parabolic isometry of $\mathbb{H}^2$ fixing the point ∞. Let $H_c = \left\{(x,y) \in \mathbb{H}^2 \,\middle|\, y \geqslant c\right\}$ be a horoball. Each cusp can be modelled as H_c/Γ for some c up to isometry, and is diffeomorphic to $S^1 \times [c,\infty)$ so that each circle $S^1 \times \{t\}$ with $t \geqslant c$ is the image of a horocycle under p. Each circle is also called a *horocycle* by abuse of notation. The circle $S^1 \times \{t\}$ with $t \geqslant c$ is called an *Euclidean circle*. A cusp is *maximal* if it lifts to a union of horocycles with disjoint interiors such that there exists at least one point of tangency between different horocycles.

Lemma 2.4 (Adams, [1])**.** *For an orientable, metrically complete hyperbolic surface, the area with a maximal cusp is at least* 4*. The lower bound* 4 *is realized only in an ideal pair of pants.*

If c is a cusp, we define $N'(c)$ is a union of horocycles of c that length less than 4. It is an embedded cylinder.

3. GEODESICS ON PAIR OF PANTS

Definition 3.1. By *corkscrew geodesic* we mean a geodesic in the homotopy class as described in Figure 2. that is a curve consisting of the concatenation of a simple arc and another that winds k times around a boundary. The name "corkscrew" comes from [11].

There is a unique hyperbolic structure in the thrice punctured sphere $\Sigma_{0,3}$, the corkscrew geodesic consisting of the concatenation of a simple arc and another that winds 2 times around a cusp has 2 self-intersections, see Figure 2, which has length $2arccosh^{-1}(5) = 2\log(5+2\sqrt{6})$.

If the geodesic Γ is contained in a pair of pants, then we use the following results in [2]. Assume a pair of pants P with geodesic boundaries $\gamma_1, \gamma_2, \gamma_3$. For $i = 1,2,3$, define

$$c_i = \cosh\frac{\ell(\gamma_i)}{2} \quad c_{i,n} = \cosh\frac{n\ell(\gamma_i)}{2} \qquad s_i = \sinh\frac{\ell(\gamma_i)}{2} \quad s_{i,n} = \sinh\frac{n\ell(\gamma_i)}{2}$$

Let $\Gamma_{m,n}$ is the unique closed geodesic in P that has homotopy type of a curve winding around γ_1 m times and then winding around γ_2 n times. The length of $\Gamma_{m,n}$ can be computed as follows:

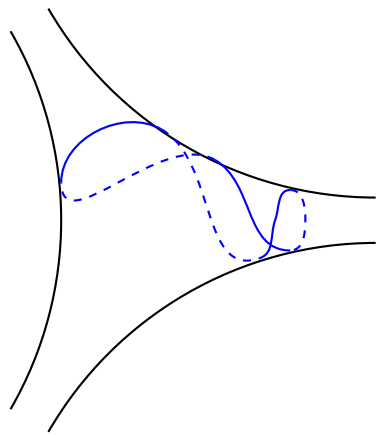

FIGURE 2. A corkscrew geodesic of $k = 2$

Theorem 3.2.

$$\cosh\frac{\ell(\Gamma_{m,n})}{2} = \frac{s_{1,m}}{s_1}\frac{s_{2,n}}{s_2}(c_3 + c_1c_2) + c_{1,m}c_{2,n} \tag{3}$$

Note that when P is an ideal pair of pants, i.e. some of $\gamma_1, \gamma_2, \gamma_3$ become cusps as their lengths tend to 0, then $c_i, c_{i,n}, s_i, s_{i,n}$ can be defined, and if γ_1 becomes cusp, $\frac{s_{1,m}}{s_1}$ can be altered by

$$\lim_{\ell(\gamma_1)\to 0+} \frac{\cosh\frac{m\ell(\gamma_1)}{2}}{\cosh\frac{\ell(\gamma_1)}{2}} = m$$

in (3). And if γ_2 becomes cusp, similarly $\frac{s_{2,n}}{s_2}$ can be altered by n in (3).

Corollary 3.3. *Suppose m, n are positive integers and $m + n \geqslant 3$. P is a pair of pants or an ideal pair of pants. $\Gamma_{m,n}$ is a closed geodesic in P as above. Then*

$$\ell(\Gamma_{m,n}) \geqslant 2\log(5 + 2\sqrt{6})$$

The equality holds if and only if P is a thrice punctured sphere and (m, n) is equal to $(1, 2)$ or $(2, 1)$.

Proof. Since $c_i = \cosh\frac{\ell(\gamma_i)}{2} \geqslant 1$ and similarly $c_{i,m} \geqslant 1$, and when $\ell(\gamma_1) > 0$

$$\frac{s_{1,m}}{s_1} = \frac{\sinh\frac{m\ell(\gamma_1)}{2}}{\sinh\frac{\ell(\gamma_1)}{2}} = \frac{\alpha^m - \alpha^{-m}}{\alpha - \alpha^{-1}} = \sum_{k=0}^{m-1}\alpha^{m-1-2k}$$
$$= \frac{1}{2}\sum_{k=0}^{m-1}(\alpha^{m-1-2k} + \alpha^{2k+1-m}) > m$$

Here $\alpha = e^{\ell(\gamma_1)/2}$. Similarly when $\ell(\gamma_2) > 0$ we have $\frac{s_{2,n}}{s_2} > n$. Hence we have

$$\cosh\frac{\ell(\Gamma_{m,n})}{2} = \frac{s_{1,m}}{s_1}\frac{s_{2,n}}{s_2}(c_3 + c_1c_2) + c_{1,m}c_{2,n} \geqslant 2mn + 1 \geqslant 5$$

Hence $\ell(\Gamma_{m,n}) \geqslant 2\log(5+2\sqrt{6})$. The equality holds if and only if $\frac{s_{1,m}}{s_1} = m$, $\frac{s_{2,n}}{s_2} = n$, and $c_3 = c_1 = c_2 = c_{1,m} = c_{2,n} = 1$, that is, $\ell(\gamma_1) = \ell(\gamma_2) = \ell(\gamma_3) = 0$, and $mn = 2$. □

4. Winding number of geodesic segments in a collar

Let $c \in \Sigma$ be a simple closed geodesic, and $w > 0$. This section we always assume the collar

$$N(c) = \{x \in \Sigma : d(x, c) < w\}$$

is an embedded annulus. And for cusps, assume c is a puncture and $N(c)$ is defined to be the cusp neighborhood with boundary horocycle of length 4. δ is a geodesic segment in $N(c)$ with endpoints x_1, x_2 on the same component of $\partial N(c)$.

Next we define the *winding number* $W(\delta)$ of the arc δ. The definitions are similar as [12].

(1) When c is a closed geodesic, every point of δ projects orthogonally to a well-defined point of c. The winding number of δ is given by the quotient of the length of the projection of δ divided by $\ell(c)$.
(2) Similarly when c is a cusp, every point of δ projects orthogonally to a well-defined point of the length h horocycle. The winding number of δ is given by the quotient of the length of the projection of δ divided by h.

Theorem 4.1. *If c is a closed geodesic, then*

$$\ell(\delta) = 2\sinh^{-1}\left(\sinh\frac{W(\delta)\ell(c)}{2} \cdot \cosh w\right)$$

Proof. The universal covering $p : \mathbb{H}^2 \to \Sigma$ from the Poincaré disk $\mathbb{H}^2$ to Σ is locally isometric. Let $\widetilde{\delta}$, $\widetilde{c}$ be a lift of δ and c. The connected component $\widetilde{N(c)}$ of $p^{-1}(N(c))$ containing $\widetilde{\delta}$ is a universal cover of the annulus $N(c)$. Let $\widetilde{x}_1$ and $\widetilde{x}_2$ be lifts of x_1,x_2 in $\widetilde{N(c)}$. Let $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ be the shortest geodesics from $\widetilde{x}_1$ and $\widetilde{x}_2$ to $\widetilde{c}$ respectively. Then $\eta_1 := p \circ \widetilde{\eta}_1$, $\eta_2 = p \circ \widetilde{\eta}_2$ is a geodesic connecting x and c and $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ are both perpendicular to c. Let $\widetilde{y}_1$ and $\widetilde{y}_2$ be the two feet. Without loss of generality, we may assume $\widetilde{c}$ is the horizontal diameter of $\mathbb{H}^2$ and the origin O is the middle point of $\widetilde{y}_1$ and $\widetilde{y}_2$, as illustrated in Figure 3.

The geodesic $\widetilde{\delta}$ between $\widetilde{x}_1$ and $\widetilde{x}_2$, $\widetilde{\eta}_1$, $\widetilde{\eta}_2$ and the geodesic δ_1 between $\widetilde{y}_1$ and $\widetilde{y}_2$ form a Saccheri quadrilateral, and half of it is a Lambert quadrilateral. Note that $d(\widetilde{y}_1, \widetilde{y}_2) = W(\delta)\ell(c)$. The property of the Lambert quadrilateral gives

$$\sinh\left(\frac{\ell(\delta)}{2}\right) = \sinh\left(\frac{W(\delta)\ell(c)}{2}\right)\cosh\left(\ell\left(\widetilde{\eta}_1\right)\right)$$

□

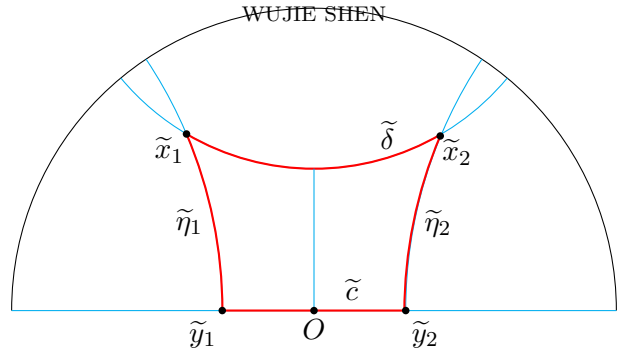

FIGURE 3. A covering of the annulus.

Theorem 4.2. *If c is a cusp with boundary length 4(it is embedded by Lemma 2.4), then*

$$\ell(\delta) = 2\log\left(2W(\delta) + \sqrt{4W^2(\delta)+1}\right)$$

Proof. When c is a cusp, consider the projection map p from the upper half plane model of $\mathbb{H}^2$ to Σ, where $p^{-1}(N(c)) = \{(x,y) \in \mathbb{H}^2 : y \geqslant 1\}$, and $A(-2,0)$ and $B(2,0)$ be two adjacent points of a same point $x \in \partial N(c)$, ℓ be the line $\{(x,y) \in \mathbb{H}^2 : y = 1\}$. Without loss of generality assume $\widetilde{\delta}$ is a lift of the arc δ and is an arc of a circle centered at the origin O with endpoints $P_1, P_2 \in \ell$. The hyperbolic length of the arc P_1P_2 is $d(P_1,P_2) = \ell(\delta)$, and $d_E(P_1,P_2) = 4W(\delta)$ where d_E means the Euclidean distance in $\mathbb{H}^2$, as in Figure 4 .Hence

$$\ell(\delta) = d(P_1,P_2) = 2\log\left(2W(\delta) + \sqrt{4W^2(\delta)+1}\right)$$

□

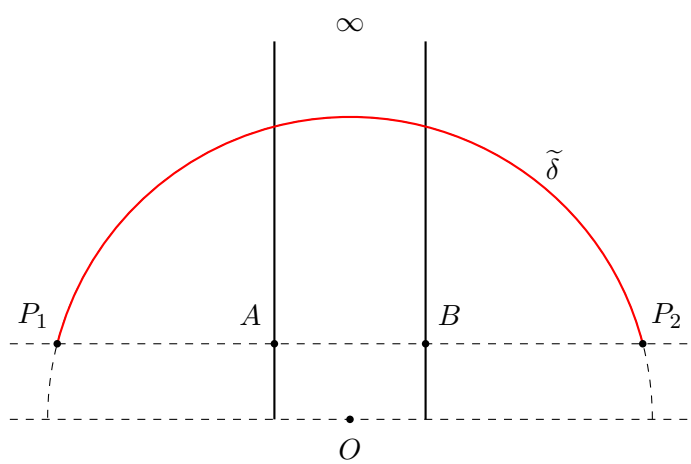

FIGURE 4. A covering of $N_0(c_i)$ when $c_i \in \mathcal{Y}$

5. Proof of theorem 1.3

The notations are as above, Γ is a closed geodesic in Σ with at least 2 self intersections, with length $\ell(\Gamma)$. Let $|\Gamma\cap\Gamma|$ be the self intersection number of Γ. Let γ be the shortest arc contained in Γ that is a closed curve in Σ, and $P\in\Sigma$ be its endpoint. Then γ is simple since the length of δ_0 is minimal. We finish the proof based on whether the length $\ell(\gamma)$ of γ satisfies

$$\ell(\gamma)\leqslant 2\log(5+2\sqrt{6})-4\log(1+\sqrt{2})<1.06$$

5.1. Case 1: $\ell(\gamma)\geqslant 1.06$.

Theorem 5.1. *If $\ell(\gamma)\geqslant 1.06$ and $\ell(\Gamma)<2\log(5+2\sqrt{6})$, then $|\Gamma\cap\Gamma|\leqslant 1$.*

Proof. Assume $\gamma'=\Gamma\setminus\gamma$, then γ' is a geodesic arc with same endpoints and $\ell(\gamma')<4\log(1+\sqrt{2})$. Using Yamada's result([9]) we have γ and γ' are freely homotopic to a multiple of simple closed geodesics(or cusp) β and β', and

$$\ell(\beta)\leqslant\ell(\gamma)\qquad \ell(\beta')\leqslant\ell(\gamma')<4\log(1+\sqrt{2})<3.5255$$

Using the collar lemma(2.1) we have

$$N(\beta)=\{p\in\Sigma\,:\,d(p,\beta)<w(\ell(\beta))\}$$

is an embedded annulus.

(1) If $\beta\cap\beta'=\emptyset$, then since γ' freely homotopic to a multiple of simple closed geodesic β' with multiplicity $k\in\mathbb{Z}_+$, say $k\beta'$. We have γ' homotopic to a curve with basepoint P with self-intersection number $k-1$, assume γ'' is the curve with infimum length satisfying this property. Then γ'' is a geodesic arc with same endpoint P, since it is freely homotopic to a multiple of β' and $\gamma''\cup\beta'$ could not create bigons, we have $\gamma''\cap\beta'=\emptyset$. Hence Γ is freely homotopic to a curve starting at P, winding around β one time, then goes to β' then winding around it k times, finally goes back to P. The curve has self intersection number k.

We can construct a pair of pants P with β and β' as two of three boundary components as follows.Connect P and β by shortest geodesic δ_1 in the annulus bounded by β and γ, and connect P and β' by shortest geodesic δ_2 in the annulus bounded by β' and a part of γ'. Consider a closed curve $\beta_1:=\delta_1\beta\delta_1^{-1}\delta_2\beta'\delta_2^{-1}$ (by connecting the endpoints) we can choose an orientation of β and β' such that β_1 is freely homotopic to a simple closed geodesic(or cusp) β'', which is the third boundary component of P. Since the homotopy type of Γ can be represented by a closed geodesic Γ' contained in P, hence $\Gamma=\Gamma'\subseteq P$ by the uniqueness of closed geodesic with given homotopy type. Using Corollary 3.3 we have $\ell(\Gamma)\geqslant 2\log(5+2\sqrt{6})$, a contradiction.

(2) If $\beta\cap\beta'\neq\emptyset$, then β is a simple closed geodesic, not a cusp. Assume $t=\frac{1}{2}\ell(\beta)>0$. Since γ and β freely homotopic, they bound an

annulus A, and assume $N'(\beta)$ and A are on the same side of $N(\beta)$. First note that if $\beta \cap \beta' \neq \emptyset$, since every arc β'' of $\beta' \cap N_1(\beta)$ satisfying $\beta'' \cap \beta = \emptyset$ is a geodesic segment get in and out $N_1(\beta)$ on different sides(in fact the function $d(x, \beta')$ first decreases to 0 and then increases when x goes along the arc), we have

$$\ell(\gamma') \geqslant \ell(\beta') \geqslant \ell(\beta'') \geqslant 2w(\ell(\beta)) = 2\log\frac{e^t+1}{e^t-1}$$

Let $\widetilde{\gamma} \subseteq \Gamma$ is the unique arc in $\Gamma \cap N'(\beta)$ containing or contained in γ'. The winding number $W(\widetilde{\gamma})$ can be defined above.

If $W(\widetilde{\gamma}) \geqslant 1$, then using Theorem 4.1 we have

$$\begin{aligned}\ell(\Gamma \setminus \beta'') \geqslant \ell(\widetilde{\gamma}) &= 2\sinh^{-1}\left(\sinh(tW(\widetilde{\gamma}))\cosh w_1(\ell(\beta))\right)\\ &\geqslant 2\sinh^{-1}\left(\sinh t\cosh\log\frac{e^{t/2}+1}{e^{t/2}-1}\right)\end{aligned}$$

If $w(\widetilde{\gamma}) < 1$, then in annulus A, $d(P, \beta) > w_1(\ell(\beta))$, hence

$$\begin{aligned}\ell(\Gamma \setminus \beta'') \geqslant \ell(\gamma) &= 2\sinh^{-1}\left(\sinh t\cosh d(P,\beta)\right)\\ &\geqslant 2\sinh^{-1}\left(\sinh t\cosh\log\frac{e^{t/2}+1}{e^{t/2}-1}\right)\\ &= 2\log\left(\frac{(e^t+1)^2}{2e^t} + \sqrt{\left(\frac{(e^t+1)^2}{2e^t}\right)^2+1}\right)\end{aligned}$$

In both two cases, let $T = \frac{(e^t+1)^2}{2e^t} > 2$ we have

$$\ell(\Gamma) \geqslant \ell(\beta'') + \ell(\Gamma \setminus \beta'') \geqslant 2w_1(\ell(\beta)) + \ell(\Gamma \setminus \beta'') \geqslant H$$

where

$$\begin{aligned}H =& 2\log\frac{e^t+1}{e^t-1} + 2\log\left(\frac{(e^t+1)^2}{2e^t} + \sqrt{\left(\frac{(e^t+1)^2}{2e^t}\right)^2+1}\right)\\ =& \log\frac{T}{T-2} + 2\log\left(T+\sqrt{T^2+1}\right)\end{aligned}$$

$$\frac{dH}{dT} = \frac{2}{\sqrt{T^2+1}} - \frac{2}{T(T-2)}$$

There exists $T_0 > 2$ such that when $2 \leqslant T \leqslant T_0$, $\frac{dH}{dT} \leqslant 0$, and when $T \geqslant T_0$, $\frac{dH}{dT} \geqslant 0$. When $T = 3$ $\frac{dH}{dT} < 0$, and when $T = \frac{25}{8}$ $\frac{dH}{dT} > 0$, hence $3 < T_0 < \frac{25}{8}$. Hence for $T > 2$ we have a contradiction by

$$H(T) \geqslant H(T_0) > \log\frac{25}{9} + 2\log\left(3+\sqrt{10}\right) > 2\log(5+2\sqrt{6})$$

□

5.2. **Case 2:** $\ell(\gamma) < 1.06$**.** The idea of the proof is to find the homotopy type of Γ and prove that Γ is in a pair of pants, then we use the conclusions in Section 3 to finish the proof.

Theorem 5.2. *If $\ell(\gamma) < 1.06$ and $\ell(\Gamma) < 2\log(5+2\sqrt{6})$, then we can find a pair of pants $P \subseteq \Sigma$, each boundary component is a simple closed geodesic or a cusp, and β is one of the boundary components.*

Proof. Notations as before, let β be the simple closed geodesic or cusp free homotopy to γ, choose the generalized collar $N_1(\beta)$ such that γ and the bigger halfcollar $N'(\beta)$ is on the same side of β, since $\ell(\gamma) < 1.06$, $\gamma \subseteq N'(\beta)$. Let $\widetilde{\gamma}$ be the arc in $\Gamma \cap N_1(\beta)$ containing γ. Assume $|\widetilde{\gamma} \cap \widetilde{\gamma}| = k$, and the endpoints of $\widetilde{\gamma}$ are $y_1, y_2 \in \partial N_1(\beta)$. Let $\alpha \in (k, k+1]$ be the winding number of $\widetilde{\gamma}$, and assume $\widetilde{\gamma} \cap \widetilde{\gamma} = \{P = P_1, P_2, ..., P_k\}$, and in $N_1(\beta)$, $d(P_i, \beta)$(or $d(P_i, h(\beta))$ when β is a cusp and $h(\beta)$ is a sufficiently small horocycle) is increasing on $i \in \{1, ..., k\}$. Let $\zeta \subseteq N_1(\beta)$ be the unique simple geodesic with endpoints y_1, y_2 and homotopy to the curve $\widetilde{\zeta}$ starting from y_1 and goes along $\widetilde{\gamma}$ to P_k, then goes along $\widetilde{\gamma}$ from P_k to y_2. ζ has winding number $\alpha - k \in (0, 1]$.

(1) If β is a closed geodesic, assume $\ell(\beta) = 2t > 0$, let

$$H_1(s,t) = \sinh^{-1}\left(\sinh(st)\cosh(w_1(2t))\right)$$

Let $u = 2(\cosh\frac{t}{2})^2$ we have

$$\begin{aligned}
&\ell(\widetilde{\gamma}) - \ell(\zeta) = 2H_1(k+\alpha, t) - 2H_1(\alpha, t)\\
\geqslant &2H_1(1+\alpha, t) - 2H_1(\alpha, t) \geqslant 2H_1(2, t) - 2H_1(1, t)\\
= &2\sinh^{-1}\left(\sinh 2t\cosh(w_1(2t))\right) - 2\sinh^{-1}\left(\sinh t\cosh(w_1(2t))\right)\\
= &2\sinh^{-1}(u(2u-2)) - 2\sinh^{-1}(u) \geqslant 2\sinh^{-1}(2u) - 2\sinh^{-1} u\\
\geqslant &2\sinh^{-1} 4 - 2\sinh^{-1} 2 > 1.06
\end{aligned}$$

The second inequality using the fact that $H_1(\alpha, t)$ is a concave function on α by taking second derivate of α. The third inequality holds since $2u - 2 > 2$ when $u > 2$. The fourth inequality using $2\sinh^{-1}(2u) - 2\sinh^{-1} u$ is an increasing function on $u > 2$. Hence we have

$$\ell(\Gamma \setminus \widetilde{\gamma}) + \ell(\zeta) < 4\log(1+\sqrt{2})$$

The closed curve $(\Gamma \setminus \widetilde{\gamma}) \cup \zeta$ is freely homotopic to a multiple of a simple closed geodesic, say β', and so does $(\Gamma \setminus \widetilde{\gamma}) \cup \widetilde{\zeta}$. Hence if $\beta \cap \beta' = \emptyset$, Γ is freely homotopic to a curve starting at P_k, winding around β finitely many times, then goes to β' then winding around finitely many times, finally goes back to P_k. Similar as case (1) in the proof of Theorem 5.1, the theorem holds. If $\beta \cap \beta' \neq \emptyset$ then same as case (2) in Theorem 5.1 to finish the proof.

(2) If β is a cusp, similar as the previous case, let $H_1(\alpha, t) = \log(2\alpha + \sqrt{4\alpha^2+1})$ and $\ell(\widetilde{\gamma}) - \ell(\zeta) = 2H_1(k+\alpha, t) - 2H_1(\alpha, t)$ also holds, similarly we can prove the case.

□

Proof of Theorem 1.3. If $\ell(\gamma) \geqslant 1.06$, then Theroem 5.1 implies that $\ell(\Gamma) \geqslant 2\log(5+2\sqrt{6})$. If $\ell(\gamma) < 1.06$, then Theroem 5.2 implies that Γ lies in a pair of pants and using Corollary 3 we get the conclusion. □

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