DIFFUSION LIMITS IN THE QUARTER PLANE AND NON-SEMIMARTINGALE REFLECTED BROWNIAN MOTION

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Abstract. We consider a continuous-time random walk in the quarter plane for which the transition intensities are constant on each of the four faces $(0,\infty)^2$, $F_1 = \{0\} \times (0,\infty)$, $F_2 = (0,\infty) \times \{0\}$ and $\{(0,0)\}$. We show that when rescaled diffusively it converges in law to a Brownian motion with oblique reflection direction $d^{(i)}$ on face F_i, $i = 1,2$, defined via the Varadhan-Williams submartingale problem [29]. A parameter denoted by α was introduced in [29], measuring the extent to which $d^{(i)}$ are inclined toward the origin. In the case of the quarter plane, α takes values in $(-2,2)$, and it is known that the reflected Brownian motion is a semimartingale if and only if $\alpha \in (-2,1)$. Convergence results via both the Skorohod map and the invariance principle for semimartingale reflected Brownian motion are known to hold in various settings in arbitrary dimension. In the case of the quarter plane, the invariance principle was proved for $\alpha \in (-2,1)$ whereas for tools based on the Skorohod map to be applicable it is necessary (but not sufficient) that $\alpha \in [-1,1)$. Another tool that has been used to prove convergence in general dimension is the extended Skorohod map, which in the case of the quarter plane provides convergence for $\alpha = 1$. This paper focuses on the range $\alpha \in (1,2)$, where the Skorohod problem and the extended Skorohod problem do not possess a unique solution, the limit process is not a semimartingale, and convergence to reflected Brownian motion has not been shown before. The result has implications on the asymptotic analysis of two Markovian queueing models: The *generalized processor sharing model with parallelization slowdown*, and the *coupled processor model*. In both cases, the diffusion limit in heavy traffic is characterized by the aforementioned reflected Brownian motion. The restriction of our treatment to dimension 2 is due to the fact that, for analogous models in higher dimension, the well posedness of the submartingale problem for the candidate limit process is an open problem.

1. Introduction

This paper studies a Markov process on $\mathcal{S}^1 \doteq \mathbb{Z}_+^2$ that, under diffusion scaling, converges to a reflected Brownian motion (RBM) in $\mathcal{S} \doteq [0,\infty)^2$ with oblique reflection on the boundary $\partial\mathcal{S}$. The focus is on the case where the directions of reflection, that are constant on each of the two faces $F_1 = \{0\} \times (0,\infty)$ and $F_2 = (0,\infty) \times \{0\}$, are in a range in which the RBM is not a semimartingale, and has not been constructed as a pathwise transformation of planar Brownian motion (BM) but rather via a submartingale problem introduced in [29] (where a general wedge was considered). The latter is defined along the lines of the Stroock-Varadhan submartingale problem [26] with an additional condition to account for the behavior of the process at the origin, where the reflection vector field is discontinuous. The existence and uniqueness of solutions was established in [29] in the case of zero drift, and extended to a constant drift in [17]. The Skorohod map [8] and the extended Skorohod map [21], which act in path space to transform a BM to an RBM, have been used in conjunction with the continuous mapping theorem to prove convergence to RBM in a wide variety of

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settings and in arbitrary dimension. In the setting under consideration, no pathwise transformation is known, and, in particular, these tools are not available. In [32], tools that go beyond continuous mapping techniques were introduced, providing invariance principles for processes that converge to a semimartingale RBM (SRBM) in the orthant in any dimension. These do not apply here either because the limit process is not a semimartingale. Whereas the question of convergence in absence of these tools provides the main motivation for this work, further interest stems from the fact that the setting also describes two Markovian queueing models. These are the generalized processor sharing (GPS) with parallelization slowdown, and the coupled processor model. The limit theorem we prove establishes the heavy traffic limit for both models, characterized via the submartingale problem.

To introduce our setting more precisely, let $\{e^{(1)}, e^{(2)}\}$ denote the standard basis in $\mathbb{R}^2$ (considered as column vectors) and assign an edge between $x, y \in \mathcal{S}^1$ whenever $x - y = \pm e^{(i)}$ for $i = 1$ or 2, a relation denoted as $x \sim y$. For $n \in \mathbb{N}$, let X^n be a nearest neighbor Markov process on the graph $(\mathcal{S}^1, \sim)$ with initial state $X^n(0) = x^n \in \mathcal{S}^1$ and transition intensities

$$r^n(x,y) = \begin{cases} \lambda_i^n & \text{if } y = x + e^{(i)},\ i = 1,2, \\ \mu_i^n & \text{if } x \in \mathcal{S}^o,\ y = x - e^{(i)},\ i = 1,2, \\ \mu_i^n + \nu_i^n & \text{if } x \in F_{i\#},\ y = x - e^{(i)},\ i = 1,2, \end{cases} \tag{1}$$

where throughout, $i^\# = 3 - i$ and $\mathcal{S}^o$ denotes the interior of $\mathcal{S}$. Here $\lambda_i^n, \mu_i^n, \nu_i^n$ are positive scalars for all $n \in \mathbb{N}$ and $i = 1, 2$. Thus, when one of the components is zero, the intensity of downward jumps of the other component increases. Assume that

$$\hat{\lambda}_i^n \doteq n^{-1/2}(\lambda_i^n - n\lambda_i) \to \hat{\lambda}_i, \tag{2}$$

$$\hat{\mu}_i^n \doteq n^{-1/2}(\mu_i^n - n\mu_i) \to \hat{\mu}_i, \tag{3}$$

$$\hat{\nu}_i^n \doteq n^{-1/2}(\nu_i^n - n\nu_i) \to \hat{\nu}_i, \tag{4}$$

for $i = 1, 2$, as $n \to \infty$, where $\lambda_i, \mu_i, \nu_i \in (0, \infty)$ and $\hat{\lambda}_i, \hat{\mu}_i, \hat{\nu}_i \in \mathbb{R}$. Assume moreover that $\lambda_i = \mu_i$, $i = 1, 2$; in the queueing literature this is referred to as the *heavy traffic* condition. Let $\hat{X}^n = n^{-1/2} X^n$ and assume that the rescaled initial conditions converge, namely $\hat{x}^n \doteq n^{-1/2} x^n \to \hat{x} \in \mathcal{S}$. As long as $\hat{X}^n$ visits only sites in $\mathcal{S}^o \cup F_i$, for either $i = 1$ or 2, it is well approximated by a BM with oblique reflection on this face. More precisely, let $\mathcal{O}$ be a smooth open planar domain with $\hat{x} \in \mathcal{O} \cap \mathcal{S}$. If, for either $i = 1$ or 2, $\bar{\mathcal{O}} \cap \{x \in S : x_{i\#} = 0\} = \emptyset$, then the sequence of processes $\hat{X}^n$ stopped on exiting $\mathcal{O}$, converges in law to an RBM on the half space $\{x \in \mathbb{R}^2 : x_i \geq 0\}$ stopped on exiting $\mathcal{O}$. This RBM has drift and diffusion coefficients

$$b = (b_1, b_2)', \qquad \Sigma = \operatorname{diag}(\sigma_1, \sigma_2), \qquad b_j = \hat{\lambda}_j - \hat{\mu}_j, \qquad \sigma_j^2 = \lambda_j + \mu_j = 2\mu_j, \qquad j = 1, 2. \tag{5}$$

Its reflection vector field on the boundary $\{x \in \mathbb{R}^2 : x_i = 0\}$ takes the constant value

$$d^{(i)} = e^{(i)} - \frac{\nu_{i\#}}{\mu_i} e^{(i\#)}. \tag{6}$$

It is therefore natural to guess that the sequence $\hat{X}^n$ itself converges in law to an RBM in $\mathcal{S}$, with drift and diffusion coefficients as above, and reflection vector field that, for each $i = 1, 2$, takes the value $d^{(i)}$ on F_i, which is a stochastic process rigorously defined through the submartingale problem [29, 17]. The goal of this work is to prove this convergence for a range of reflection directions $d^{(i)}$ not covered by existing results.

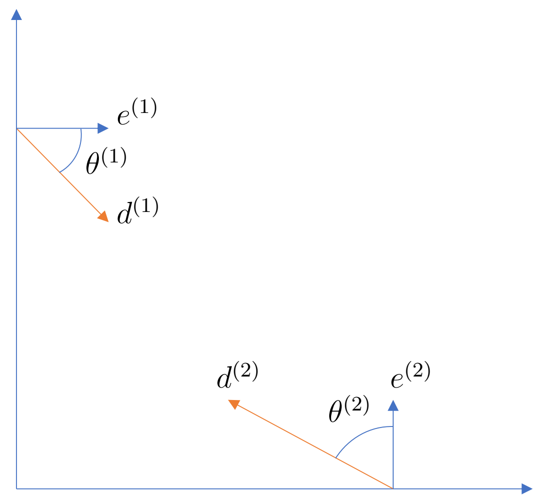

FIGURE 1. *For $i = 1, 2$, $d^{(i)}$ is the direction of reflection at face $x_i = 0$, and $\theta^{(i)}$ is the angle between $e^{(i)}$ and $d^{(i)}$, positive when $d^{(i)}$ is toward the origin.*

The geometry of the problem is captured in Fig. 1, where $\theta^{(i)}$ denotes the angle between the inward normal to face F_i, i.e., $e^{(i)}$, and $d^{(i)}$, considered positive when $d^{(i)}$ inclines toward the origin, namely

$$\theta^{(i)} \doteq \arcsin(-\|d^{(i)}\|^{-1} d^{(i)}_{i\#}), \qquad i = 1, 2. \tag{7}$$

To put the goal of this paper in context, consider, in the rest of this introduction, more general Σ and $d^{(i)}$ than the ones given by (5) and (6). Namely, let Σ be any member of $\mathfrak{S}$, the set of 2×2 diagonal matrices with strictly positive diagonal entries, and let $d^{(i)}$ merely satisfy the requirement $d^{(i)}_i > 0$, $i = 1, 2$. When $\Sigma = \mathrm{Id} \doteq \mathrm{diag}(1, 1)$, we will say that the RBM is a *unit variance* RBM. The original formulation from [29] considered a zero drift, unit variance RBM, with $d^{(i)}$ as above, and proved the existence and uniqueness in law of the process defined via the submartingale problem. These results were recently extended to the case of a general constant drift in [17], a result that is used in this paper. Although the results in [29, 17], were formulated for $\Sigma = \mathrm{Id}$, they cover the case of a general $\Sigma \in \mathfrak{S}$. That is, it is intuitively clear and will be made precise once the formal definition is introduced, that if X is an RBM with drift, diffusion coefficient, and reflection data $(b, \Sigma, \{d^{(i)}\})$, then $X^* \doteq \Sigma^{-1} X$ is an RBM with the data $(b^*, \mathrm{Id}, d^{*,(i)}\})$, where

$$b^* = \Sigma^{-1} b, \qquad d^{*,(i)} = \Sigma^{-1} d^{(i)}, \qquad i = 1, 2. \tag{8}$$

If, analogously to (7), we let

$$\theta^{(i)}_* \doteq \arcsin(-\|d^{*,(i)}\|^{-1} d^{*,(i)}_{i\#}), \qquad i = 1, 2, \tag{9}$$

then $\theta^{(i)}_*$ is the angle between $e^{(i)}$ and $d^{*,(i)}$, positive when $d^{*,(i)}$ inclines toward the origin. A key parameter introduced in [29] is

$$\alpha_* = \frac{\theta^{(1)}_* + \theta^{(2)}_*}{\pi/2}. \tag{10}$$

Denoted in [29] by the symbol α, this parameter measures the degree of "push" toward the origin exercised by the reflection at the boundary, and is known to determine various properties of the process. In particular, the process is a semimartingale if and only if $\alpha_* < 1$. Moreover, if $\alpha_* > 0$ then the origin is a.s. visited, and otherwise it is a.s. not visited. As it turns out, the tools available for proving convergence to this process also depend to a great extent on α_*. In this work we will be interested in the case $\alpha_* \in (1,2)$ which is the only range in which convergence has not been shown before. (In a manner analogous to (10), one could define a parameter in terms of the geometry of the original model, namely $\alpha \doteq (\pi/2)^{-1}(\theta^{(1)} + \theta^{(2)})$. It is easy to see that $\alpha_* \in R$ if and only if $\alpha \in R$ for each of the ranges $R = (-2,-1), \{-1\}, (-1,1), \{1\}$ and $(1,2)$. However, whereas α_* plays a major analytic role in our proofs, α is not as important and will not be used further in this paper.)

Let us briefly review the tools that have been used to prove convergence when $\alpha_* \in (-2,1]$. Results on the aforementioned Skorohod problem for polyhedral domains [13, 8, 9] give sufficient conditions for the transformation to be Lipschitz continuous in path space, and based on that, the convergence of diffusively scaled reflected random walks and related queueing models to an RBM has been shown for a variety of models. When specialized to the quarter plane, the setting of [13] covers the case where $\theta^{(i)} \geq 0$, $i = 1,2$ and $\alpha_* \in [0,1)$, and the broader setting of [8, 9] allows for $\alpha_* \in [-1,1)$, although the condition $\alpha_* \in [-1,1)$ is not sufficient for these results to hold. In the case $\alpha_* = 1$, the Skorohod map is not well-defined but the extended Skorohod map introduced in [21] gives a pathwise construction of an RBM and again provides convergence via continuity properties of this map.

For α_* in the range $(-2,-1) \cup (1,2)$, the Skorohod and extended Skorohod problem do not possess unique solutions, and only weak formulations are available. In [25], a weak formulation of an SRBM in the positive orthant $\mathbb{R}_+^N$ was given, akin to the notion of a weak solution of a stochastic differential equation. It was shown that a necessary condition for existence of this process is the so-called *completely-$\mathcal{S}$* condition, an algebraic condition on the matrix composed by the directions of reflection $\{d^{(i)}\}$. In [27], existence and uniqueness of this process were established under the completely-$\mathcal{S}$ condition, and in [32], under the same condition, an invariance principle was proved, yielding convergence to the SRBM. In the quarter plane, this condition holds if and only if $\alpha_* \in (-2,1)$, and in this range the process agrees with the one defined via the submartingale problem. The results on the submartingale problem cover the entire range $\alpha_* \in (-2,2)$, and thus this is the only formulation that handles the range $\alpha_* \in (1,2)$ that is of interest in this paper.

More broadly, reflected diffusions have been studied in a variety of settings. In [6], existence and uniqueness of an SRBM in convex polyhedral domains with constant direction of reflection on each face were proved. An extension of the results of [29] on the submartingale problem for a higher dimensional cone, with radially homogeneous direction of reflection, was studied in [16]. Related questions have been studied for domain with cusps. For such domains, determining whether a reflected diffusion is a semimartingale has been addressed in [7]. More recently, [4] gave existence and uniqueness results in a two-dimensional cusp with varying, oblique directions of reflection, and [5] provided such results for higher dimensional domains with a single singular point. Conditions under which an obliquely reflected diffusion process constitutes a Dirichlet process were studied in [14]. When specialized to the case of a wedge, these conditions correspond to $\alpha_* = 1$. Subsequently, the Dirichlet process property in the case of a wedge with $\alpha_* \in (1,2)$ was established in [18]. For reflected diffusions in piecewise smooth domains, [15] studied the equivalence between well-posedness of the submartingale problem and weak existence and uniqueness of solutions for stochastic differential equations with reflection, in settings that include non-semimartingale reflected diffusions.

There is a rich literature on convergence of discrete processes in domains with boundary to RBM in dimension 2 and higher. As far as queueing models are concerned, the first main result in this direction was the treatment of the generalized Jackson network in [24], where convergence of the diffusively scaled queue-length process to a multidimensional RBM was proved based on continuity of the underlying Skorohod map. We refer to [30, Ch. 14] for a survey on convergence results via continuity.

In settings where the Skorohod map does not exist but the RBM is a semimartingale, the aforementioned invariance principle tools of [32] have been used for proving convergence; see e.g. [31]. The paper [22] proves a heavy traffic limit theorem when the RBM is not a semimartingale but given through the extended Skorohod problem [21]. Using different techniques, invariance principles for random walks converging to RBM in highly non-smooth Euclidean domains were established in [2]. In this setting, the RBM is defined via the Dirichlet form, and loosely speaking, the reflection vector field is in the direction normal to the boundary.

A natural desired extension of the setting from [29] to higher dimension is to a polyhedral cone with reflection direction that is constant on each face F_i of codimension 1, satisfying merely $d^{(i)} \cdot n^{(i)} > 0$ for all i, where $n^{(i)}$ denotes the inward normal at face F_i. In particular, the case of an orthant in dimension ≥ 3 corresponds to higher dimensional versions of (1) and of the queueing models considered in this paper. However, the well-posedness of the corresponding submartingale problem in this generality remains an open problem.

Finally, we remark that the conjectured connection of diffusion limits of queueing models to the solution of the submartingale problem in the quarter plane when $\alpha_* \in (1, 2)$ has previously been proposed in [18] and [17].

1.1. **Paper organization.** The result and its applications for queueing are presented in Section 2, starting with Section 2.1 which states the definition of the submartingale problem and the main result. Implications to the heavy traffic limits of two queueing models are provided in Section 2.2. An outline of the proof is described in Section 2.3. Several preparatory lemmas are provided in Section 3. Finally the proof of the main result appears in Section 4. See Section 2.3 for a detailed description of the content of Sections 3–4.

1.2. **Notation.** Denote $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and $\mathbb{R}_+ = [0, \infty)$. For $x, y \in \mathbb{R}^2$, denote the standard inner product by $x \cdot y$ and the Euclidean norm by $\|x\|$. Denote the boundary and the interior of a set $S \in \mathbb{R}^2$ by ∂S and S^o, respectively. Let $\mathbb{B}_r = \{x \in \mathbb{R}^2 : \|x\| < r\}$, $\mathbb{S}_r = \mathcal{S} \cap \partial \mathbb{B}_r$, and

$$\mathcal{S}^n = (n^{-1/2}\mathbb{Z}_+)^2, \qquad n \in \mathbb{N}.$$

If $V \in \mathbb{R}^2$ then V_i, $i \in \{1, 2\}$ denote its components in the standard basis, and vice versa: Given V_i, $i \in \{1, 2\}$, V denotes the vector (V_1, V_2). Both these conventions hold also for random variables V_i and processes $V_i(\cdot)$. Members of $\mathbb{R}^2$ are considered as column vectors.

Denote by $C_b^2(\mathcal{S})$ the space of twice continuously differentiable functions defined on a neighborhood of $\mathcal{S}$ that are bounded and have bounded first two derivatives. For $f \in C_b^2(\mathcal{S})$, denote by

$\nabla f = (\nabla_i f)$ and $\nabla^2 f = (\nabla^2_{ij} f)$ the gradient and Hessian. For $\mathcal{S} \subset \mathcal{O} \subset \mathbb{R}^2$ and $f : \mathcal{O} \to \mathbb{R}$, denote

$$\partial^{n,i}_{\pm} f(x) = f(x \pm n^{-1/2} e^{(i)}) - f(x), \qquad x \in \mathcal{S}^n,\ i \in \{1,2\},\ n \in \mathbb{N},$$

well-defined provided $x - n^{-1/2} e^{(i)} \in \mathcal{O}$.

For $(\mathbf{X}, d_{\mathbf{X}})$ a Polish space, let $C(\mathbb{R}_+, \mathbf{X})$ and $D(\mathbb{R}_+, \mathbf{X})$ denote the space of continuous and, respectively, càdlàg paths, endowed with the topology of uniform convergence on compacts and, respectively, the J_1 topology.

For $\xi \in D(\mathbb{R}_+, \mathbb{R}^N)$, $N = 1, 2$, an interval $I \subset \mathbb{R}_+$, and $0 \le \delta \le S$, denote

$$\begin{aligned} \operatorname{osc}(\xi, I) &= \sup\{\|\xi(s) - \xi(t)\| : s, t \in I\}, \\ w_S(\xi, \delta) &= \sup\{\|\xi(t) - \xi(s)\| : s, t \in [0, S], |s - t| \le \delta\}, \\ \|\xi\|^*_S &= \sup\{\|\xi(t)\| : t \in [0, S]\}, \end{aligned}$$

and by $|\xi|(t)$ the total variation of ξ in $[0, t]$, taken with the Euclidean norm in $\mathbb{R}^N$.

$\iota : \mathbb{R}_+ \to \mathbb{R}_+$ is the identity map. For $k \in \mathbb{Z}$ and $x \in \mathbb{R}$ with $k \le x < k + 1$, denote by $\sum_{j=1}^{x}$ the sum over $j \in \mathbb{Z}$ such that $1 \le j \le k$.

For $x \in \mathbb{R}^2$, $b \in \mathbb{R}^2$ and $\Sigma \in \mathfrak{S}$, a (b, Σ)-BM is a two-dimensional BM with drift and diffusion coefficients b and Σ, respectively, starting at 0. The Borel σ-field on a Polish space $\mathbf{X}$ will be denoted as $\mathcal{B}(\mathbf{X})$. Convergence in distribution is denoted by $\Rightarrow$. A sequence $\{P_n\}$ of probability measures on $D(\mathbb{R}_+, \mathbf{X})$ is said to be C-tight if it is tight with the usual J_1 topology on $D(\mathbb{R}_+, \mathbf{X})$ and every weak limit point P is supported on $C(\mathbb{R}_+, \mathbf{X})$. With a slight abuse of standard terminology, a sequence of random elements (random variables or processes) is referred to as tight when their probability laws form a tight sequence of probability measures, and a similar use is made for the term C-tight. For a random variable X with values in some Polish space $\mathbf{X}$ and a probability measure P on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$, the notation $X \sim P$ denotes that the probability distribution of X equals P.

2. Main result and its applications

2.1. **Main result.** The processes X^n and their rescaled versions $\hat{X}^n$ introduced above are assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Henceforth, b, Σ and $d^{(i)}$ are given by (5) and (6), and $b^*, d^{*,(i)}, \theta^{(i)}_*$ and α_* by the corresponding quantities in (8), (9) and (10). As already mentioned, we shall focus on the range $\alpha_* \in (1, 2)$, a condition that can be rephrased as $\tan\theta^{(1)}_* > \cot\theta^{(2)}_*$, or, expressed directly in terms of the geometry of the model, $\tan\theta^{(1)} > \cot\theta^{(2)}$. In view of (6), the latter can also be written as

$$\nu_1 \nu_2 > \mu_1 \mu_2, \tag{11}$$

a condition assumed throughout what follows.

We recall the definition of the submartingale problem formulated in [29]. It originates from an approach to defining a measure on path space associated with a diffusion on a domain with boundary, introduced in [26], in which smooth domains were considered. In the case of a wedge, [29] supplemented the formulation of [26] by what is referred to below as the 'corner property'.

The coordinate process on $C(\mathbb{R}_+, \mathcal{S})$ will be denoted by Z throughout this paper and the filtration generated by this process will be referred to as the *canonical filtration*. Throughout, a process defined on $(C(\mathbb{R}_+,\mathcal{S}),\mathcal{B}(C(\mathbb{R}_+,\mathcal{S}))$ will be referred to as a martingale (resp., submartingale) if it is a martingale (resp., submartingale) with respect to the canonical filtration.

Definition 2.1. (The submartingale problem)
(i) A tuple $\tilde{\mathbf{D}}=(\tilde b,\tilde\Sigma,\{\tilde d^{(i)}\})$ *is an admissible data if* $\tilde b\in\mathbb{R}^2$, $\tilde\Sigma\in\mathfrak{S}$, *and* $\tilde d^{(i)}\in\mathbb{R}^2$, $\tilde d^{(i)}_i>0$ *for* $i=1,2$.
(ii) Let $z\in\mathcal{S}$. *A solution to the submartingale problem for the admissible data* $\tilde{\mathbf{D}}=(\tilde b,\tilde\Sigma,\{\tilde d^{(i)}\})$ *starting from* z *is a probability measure* P_z *on* $C(\mathbb{R}_+,\mathcal{S})$ *such that, with* E_z *denoting the corresponding expectation, the following statements hold.*
Initial condition. *One has*

$$P_z(Z(0)=z)=1.$$

The submartingale property. *Denoting*

$$\mathcal{L}f=\frac{1}{2}\mathrm{trace}(\tilde\Sigma\nabla^2 f\tilde\Sigma)+\tilde b\cdot\nabla f,\qquad\textit{and}\qquad M(t)=f(Z(t))-\int_0^t\mathcal{L}f(Z(s))ds,\tag{12}$$

$M(t)$ *is a* P_z*-submartingale for any function* $f\in C^2_b(\mathcal{S})$ *that is constant in a neighborhood of the origin and satisfies* $\tilde d^{(i)}\cdot\nabla f(x)\ge 0$ *for* $x\in F_i$, $i=1,2$.
The corner property. *One has*

$$E_z\Big[\int_0^\infty \mathbf{1}_{\{0\}}(Z(t))dt\Big]=0.$$

The following wellposedness result for the above submartingale problem is from [29, 17].

Theorem 2.2. [29, 17] *There exists a unique solution to the submartingale problem for* $\tilde{\mathbf{D}}$ *starting from* $z\in\mathcal{S}$ *whenever* $\tilde{\mathbf{D}}$ *is an admissible data.*

Proof. In [29], a more general wedge was considered, and it was shown, in the case $\tilde b=0$, $\tilde\Sigma=\mathrm{Id}$, that there exists a unique solution to the submartingale problem provided that the parameter denoted there by α (defined as α_* in (10)) is in the range $(-\infty,2)$. The same is true for general $\tilde b$ and $\tilde\Sigma=\mathrm{Id}$, studied in [17]. Now, in the case of the quarter plane, the parameter α of [29] is always in the range $(-2,2)$ (since $\tilde d^{(i)}_i>0$ for both i), and consequently, the assertion of the theorem holds in these cases, when $\tilde\Sigma=\mathrm{Id}$. In particular, for $\tilde b=0$, $\tilde\Sigma=\mathrm{Id}$, this is stated in [29, Theorems 2.5 and 3.10] and for general $\tilde b\in\mathbb{R}^2$ and $\tilde\Sigma=\mathrm{Id}$, this is [17, Theorem 2.3].

Next, for $\tilde\Sigma\in\mathfrak{S}$, note that if $\tilde X$ has the law P_x, a solution with data $(\tilde b,\tilde\Sigma,\{\tilde d^{(i)}\})$ starting from x, then $\tilde\Sigma^{-1}\tilde X$ has law P_y, a solution with data $(\tilde\Sigma^{-1}\tilde b,\mathrm{Id},\{\tilde\Sigma^{-1}\tilde d^{(i)}\})$, starting from $y=\tilde\Sigma^{-1}x$, and vice versa. Hence existence and uniqueness follow from the case $\tilde\Sigma=\mathrm{Id}$. □

Before stating the main result, we note that if P_x is the solution of the submartingale problem then $X\sim P_x$ can be regarded as a process with sample paths in $D(\mathbb{R}_+,\mathcal{S})$.

Theorem 2.3. *Let* $P_{\hat x}$ *be the unique solution to the submartingale problem with data* $\mathbf{D}=(b,\Sigma,\{d^{(i)}\})$ *starting from* $\hat x$ *and let* $X\sim P_{\hat x}$. *Then* $\hat X^n\Rightarrow X$ *in* $D(\mathbb{R}_+,\mathcal{S})$.

2.2. **Queueing models.**

2.2.1. *GPS with parallelization slowdown.* In the GPS model introduced in [19], a server divides its effort among several streams of jobs according to fixed proportions. Although it was introduced and analyzed with a general number of streams, we describe the model here for the case of two streams, which suffices for the purpose of relating it to our results. Given positive proportions ϕ_i, $i = 1, 2$ such that $\phi_1 + \phi_2 = 1$, and service rates μ_i^n, $i = 1, 2$, when jobs from both classes are present, the server's effort is split according to the proportions ϕ_i, and the job at the head of the line in queue i is served at rate $\phi_i \mu_i^n$, $i = 1, 2$. When only the i-th queue is non-empty, all effort is given to the job at the head of the line in this queue, and thus it is served at rate μ_i^n.

The GPS model with general service time distribution and general number of streams was considered in the heavy traffic limit in [22], where convergence was proved and the limit process was identified in terms of the extended Skorohod problem. To relate the model to the setting studied in this paper, we note that in the special case where the dimension is 2, the limit process can alternatively be identified as the unique solution to the submartingale problem of [29], with wedge angle $\pi/2$ and $\alpha = 1$, when the drift coefficient is 0; and as the process constructed in [17] when the drift is nonzero. An extension to a setting where some of the classes may be strictly subcritical was studied in [23]. Although the proof of convergence in [22] was not a mere application of the continuous mapping theorem, the continuity of the extended Skorohod map was instrumental to prove the convergence result. Such a pathwise mapping is not available for the RBM constructed in [29] when $\alpha > 1$.

In practice, the implementation of GPS is often performed by time slicing followed by a round robin scheduling. This design necessitates preempt and resume operations, and both may cause job switching overhead. The effect of switching overhead in processor sharing and round robin scheduling is widely acknowledged, and has been described and analyzed e.g. in [12, 28, 33]. (See also [20] for job switching overhead analysis for priority scheduling.) It is thus natural to posit that a fixed proportion of the processing capacity is lost at times when parallelization takes place.

With this in mind, we consider here a variation of the model that will be called *GPS with parallelization slowdown* (GPS–PS), which deviates from the GPS model in that $\phi_1 + \phi_2 < 1$ is assumed. In this paper we only aim at the Markovian setting of the model. Thus letting λ_i^n denote the arrival rates and $\mu_i^{\mathrm{PS},n}$ the service rates, the queue-length process X^n is a Markov process on $\mathcal{S}^1$ with transition intensities

$$r^{\mathrm{PS},n}(x, y) = \begin{cases} \lambda_i^n & \text{if } y = x + e^{(i)},\ i = 1, 2, \\ \phi_i \mu_i^{\mathrm{PS},n} & \text{if } x \in \mathcal{S}^o,\ y = x - e^{(i)},\ i = 1, 2, \\ \mu_i^{\mathrm{PS},n} & \text{if } x \in F_{i\#},\ y = x - e^{(i)},\ i = 1, 2. \end{cases}$$

As before, λ_i^n are assumed to satisfy (2), while for $\mu_i^{\mathrm{PS},n}$ we assume

$$\hat{\mu}_i^{\mathrm{PS},n} \doteq n^{-1/2}(\mu_i^{\mathrm{PS},n} - n\mu_i^{\mathrm{PS}}) \to \hat{\mu}_i^{\mathrm{PS}}.$$

The critical load (or heavy traffic) condition is expressed by

$$\lambda_i = \phi_i \mu_i^{\mathrm{PS}}, \qquad i = 1, 2.$$

To relate this setting to the main result, take $\mu_i^n = \phi_i \mu_i^{\mathrm{PS},n}$ and $\nu_i^n = (1-\phi_i)\mu_i^{\mathrm{PS},n}$. This gives $(\mu_i, \hat\mu_i) = \phi_i(\mu_i^{\mathrm{PS}}, \hat\mu_i^{\mathrm{PS}})$ and $\nu_i = (1-\phi_i)\mu_i^{\mathrm{PS}}$. Thus

$$d^{\mathrm{PS},(i)} = e^{(i)} - \frac{1-\phi_{i\#}}{\phi_i}\frac{\mu_{i\#}^{\mathrm{PS}}}{\mu_i^{\mathrm{PS}}} e^{(i\#)}.$$

Note that condition (11) takes here the form $(1-\phi_1)(1-\phi_2) > \phi_1\phi_2$, which is equivalent to our model assumption $\phi_1+\phi_2<1$. As before we assume that $X^n(0) = x^n \in \mathcal{S}^1$ and $\hat x^n = n^{-1/2}x^n \to \hat x \in \mathcal{S}$. As an application of Theorem 2.3 we obtain

Corollary 2.4. *For the above GPS–PS model, $\hat X^n \Rightarrow X \sim P_{\hat x}$, the unique solution to the submartingale problem with data $\mathbf{D}^{\mathrm{PS}} = (b^{\mathrm{PS}}, \Sigma^{\mathrm{PS}}, \{d^{\mathrm{PS},(i)}\})$, starting from $\hat x$, where*

$$b_i^{\mathrm{PS}} = \hat\lambda_i - \phi_i \hat\mu_i^{\mathrm{PS}}, \qquad (\sigma_i^{\mathrm{PS}})^2 = \lambda_i + \phi_i\mu_i^{\mathrm{PS}} = 2\phi_i\mu_i^{\mathrm{PS}}.$$

2.2.2. *The coupled processor model.* Here, two servers work in parallel, each one serving a queue. The rate of service offered to the job at the head of the line of each queue depends on whether the other queue is empty or not. A motivation for this model is a design in which a server facing an empty queue is available to help the other server; further motivation is given in [1]. In Markovian setting, this model was analyzed in [10]; see also [11, Section 9]. They characterized the steady state distribution via a Riemann–Hilbert boundary value problem. Later, [3] and [1] considered extensions to general service time distributions and, respectively, Levy-driven queues, and analyzed the stationary joint workload process.

To relate the model to our notation, let $\mu_i^{\mathrm{CP},n}$ and $\mu_i^{\mathrm{CP},n} + \nu_i^{\mathrm{CP},n}$, $i=1,2$ denote the service rates to class i when queue $i^\#$ is non-empty and, respectively, empty. Thus $\nu_i^{\mathrm{CP},n}$ can be regarded the service rate of the helping activity. The transition intensities of the queue-length proces X^n are then

$$r^{\mathrm{CP},n}(x,y) = \begin{cases} \lambda_i^n & \text{if } y = x + e^{(i)},\ i=1,2,\\ \mu_i^{\mathrm{CP},n} & \text{if } x \in \mathcal{S}^o,\ y = x - e^{(i)},\ i = 1,2,\\ \mu_i^{\mathrm{CP},n} + \nu_i^{\mathrm{CP},n} & \text{if } x \in F_{i\#},\ y = x - e^{(i)},\ i=1,2. \end{cases}$$

This is precisely (1) if one takes $\mu_i^n = \mu_i^{\mathrm{CP},n}$ and $\nu_i^n = \nu_i^{\mathrm{CP},n}$, and in full agreement with our main model once one imposes all the assumptions on λ_i^n, μ_i^n and ν_i^n from Section 2.1, namely (2)–(11), together with $n^{-1/2}x^n \to \hat x$. Under these conditions we have

Corollary 2.5. *For the above coupled processor model, $\hat X^n \Rightarrow X \sim P_{\hat x}$, the unique solution to the submartingale problem with data $\mathbf{D}^{\mathrm{CP}} = \mathbf{D}$, starting from $\hat x$, identical to that in Theorem 2.3.*

2.3. **Proof outline.** Owing to the uniqueness stated in Theorem 2.2, the main result can be established by showing that the sequence $\hat X^n$ is tight and that all its subsequential limits form solutions to the submartingale problem.

Since a continuous map or oscillation inequalities are not available in the setting under consideration, a different idea is required to obtain tightness. The argument is based on showing that the following property implies tightness: On sufficiently short intervals, a trajectory of $\hat X^n$ can either lie in a small neighborhood of the origin, or interact with at most one of the faces F_i, $i=1,2$.

The next main task is to prove that limits satisfy the corner property. Our approach is as follows. For a constant $c>1$ and arbitrarily small $\varepsilon>0$, we consider a sequence of upcrossings from below ε to above $c\varepsilon$, and downcrossings from above $c\varepsilon$ to below ε, of the process $\|\hat X^n(t)\|$. The goal is to show that the expected cumulative time spent in upcrossings, during any finite time interval $[0, S]$, converges to zero upon sending $n \to \infty$ and then $\varepsilon \to 0$. This can be achieved via upper estimates on a single upcrossing duration and lower estimates on a single downcrossing duration, which show that the former resides at a lower scale than the latter. During a downcrossing contained in $[0, S]$, the process is at least ε away from the origin. Hence the number of times it switches from visiting one of the faces F_i, $i = 1, 2$ to another, is tight, as a sequence indexed by n. The continuous mapping associated with reflection on each one of the faces can therefore be used to argue that $\hat{X}^n$, restricted to a single downcrossing, converges to such a restriction of an RBM. In particular, downcrossing durations of the former are well approximated by ones of the latter. Our RBM hitting time estimates are based on a test function that was constructed in [29] and played a central role there. The downcrossing duration estimate obtained this way scales like ε^{α_*}.

For upcrossings, the above approach is not applicable due to the lack of an analogous representation of the process via a continuous mapping when it is near the origin. Instead, the argument is based on the construction of another test function that is applied directly to the prelimit process. It gives an estimate on the expected hitting time from below ε to above $c\varepsilon$ that is uniform in n. This estimate scales like ε^2.

These are then combined to produce a bound on the expected cumulative time spent in $\mathbb{B}_\varepsilon$. The proof uses strong Markovity of both the prelimit and the candidate limit process. A crucial step is an argument showing that the time spent by the prelimit process in $F_1 \cup F_2$ is controlled by the time spent at the origin.

All the above steps are carried out in the special case where $\mu_i^n = n\mu_i = n\lambda_i = \lambda_i^n$, under which the transition intensities of $\hat{X}^n$ are symmetric in the interior, and as a result, the candidate limiting RBM has zero drift (but general diagonal diffusion matrix). A use of Girsanov's theorem to transform symmetric intensities to more general ones, followed by an estimate on the RN derivative, gives a bound on the time spent in $\mathbb{B}_\varepsilon$ for the general nonsymmetric case. The corner property then follows by Fatou's lemma.

To show the submartingale property, we start by deriving the semimartingale representation of $f(\hat{X}^n)$ for a test function f, followed by Taylor's expansion. 'Error' terms show up, involving the remainder terms in the expansion and the time spent by the process in $\partial\mathcal{S} = \{0\} \cup F_1 \cup F_2$. To show that these terms converge to zero in probability, the aforementioned argument that controls the time spent in $F_1 \cup F_2$ by that at the origin is used again. Combining it with the estimate on the time spent in $\mathbb{B}_\varepsilon$, this allows us to control the time spent in $\partial\mathcal{S}$ and show that the error terms vanish in the limit. The martingale terms in the prelimit representation, and the inequalities satisfied by the test function on the boundary, then give rise to the submartingale property in the limit as $n \to \infty$.

Sections 3–4 are structured as follows. Section 3 contains preliminary steps in the proof: In Section 3.1, certain elementary properties of Skorohod maps are recalled. Section 3.2 derives several equations satisfied by the processes $\hat{X}^n$ and Section 3.3 proves their C-tightness. Section 3.4 provides the argument for controlling the time in $F_1 \cup F_2$ in terms of the time at 0.

The proof is then provided in Section 4, starting with Section 4.1 where estimates on individual upcrossing and downcrossing durations are derived. In Section 4.2, the aforementioned sequence of upcrossings and downcrossings is constructed and a bound is obtained on the expected cumulative time spent in upcrossings. This bound is combined with a Girsanov transformation in Section 4.3 to deduce the corner property. Finally, the submartingale property is proved in Section 4.4.

3. Preliminaries

3.1. **Skorohod maps.** We recall the definition and some elementary properties of Skorohod maps on the half line and on the half plane. The map that constrains a 1-dimensional trajectory to $\mathbb{R}_+$ is defined as follows. Given $\psi \in D(\mathbb{R}_+, \mathbb{R})$ with $\psi(0) \geq 0$, there is a unique pair $(\phi, \eta) \in D(\mathbb{R}_+, \mathbb{R}_+)^2$ such that $\phi = \psi + \eta$, $\eta(0) = 0$, η is nondecreasing and $\int_{[0,\infty)} \phi(t) d\eta(t) = 0$. This pair is given by

$$\eta(t) = \sup_{s \in [0,t]} \psi^-(s), \qquad \phi(t) = \psi(t) + \eta(t), \qquad t \geq 0, \tag{13}$$

The map from $D(\mathbb{R}_+, \mathbb{R})$ to itself mapping $\psi \mapsto \phi$ is denoted by $\Gamma_{\mathbb{R}_+}$.

Given $h \in \mathbb{R}^2$ with $h_1 > 0$, the map that constrains a 2-dimensional trajectory to the half plane $\mathbb{R}_+ \times \mathbb{R}$ in the oblique direction h is defined as follows. Denoting $\bar{h} = h/\|h\|$, given $\psi \in D(\mathbb{R}_+, \mathbb{R}^2)$, $\psi(0) \in \mathbb{R}_+ \times \mathbb{R}$, there is a unique pair $(\phi, \eta) \in D(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})^2$ such that $\phi = \psi + \eta$, $\eta(0) = 0$, $\eta(t) = \bar{h}|\eta|(t)$, and $\int_{[0,\infty)} \phi_1(t) d|\eta|(t) = 0$. This pair is explicitly expressed as

$$\bar{h}_1 |\eta|(t) = \eta_1(t) = \sup_{s \in [0,t]} \psi_1^-(s), \qquad t \geq 0,$$

and $\phi = \psi + \eta = \psi + \bar{h}|\eta|$. The map from $D(\mathbb{R}_+, \mathbb{R}^2)$ to $D(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})$ mapping $\psi \mapsto \phi$ is denoted $\Gamma^h_{\mathbb{R}_+ \times \mathbb{R}}$. Analogously, for $h \in \mathbb{R}^2$ such that $h_2 > 0$, the map constraining a planar trajectory to $\mathbb{R} \times \mathbb{R}_+$, in the direction h, is denoted by $\Gamma^h_{\mathbb{R} \times \mathbb{R}_+}$.

With a slight abuse of notation, we will use the same notation $\Gamma_{\mathbb{R}_+}$, $\Gamma^h_{\mathbb{R}_+ \times \mathbb{R}}$ and $\Gamma^h_{\mathbb{R} \times \mathbb{R}_+}$ for the corresponding maps from $D([0, S], \mathbb{R})$ to itself or $D([0, S], \mathbb{R}^2)$ to itself for finite S.

As follows from the explicit construction above, for h in a compact subset H of $(0, \infty) \times \mathbb{R}$,

$$\operatorname{osc}(\phi, [s, t]) \leq \kappa \operatorname{osc}(\psi, [s, t]), \tag{14}$$

whenever $\phi = \Gamma^h_{\mathbb{R}_+ \times \mathbb{R}}(\psi)$, where $\kappa \in (0, \infty)$ is a constant that does not depend on ψ, s, t and $h \in H$.

3.2. **Equations for the rescaled process.** Let $\mathcal{A}_i$ and $\mathcal{D}_i$, $i = 1, 2$ be independent Poisson point processes on $\mathbb{R}^2_+$ with Lebesgue intensity measure, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathcal{A}^c_i(ds, dz) = \mathcal{A}^c_i(ds, dz) - ds \times dz$, $\mathcal{D}^c_i(ds, dz) = \mathcal{D}^c_i(ds, dz) - ds \times dz$, $i = 1, 2$ their compensated versions. The Markov process X^n introduced in Section 1 is equal in distribution to the first component (again denoted as X^n) of the tuple (X^n, A^n, D^n) that constitutes the unique solution of the system

$$\begin{aligned}
&X^n(t) = x^n + A^n(t) - D^n(t), \\
&A^n_i(t) = \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, \lambda^n_i]}(z) \mathcal{A}_i(ds, dz), \\
&D^n_i(t) = \int_{[0,t] \times \mathbb{R}_+} \Big[\mathbf{1}_{[0, \mu^n_i]}(z) \mathbf{1}_{\mathcal{S}^o}(X^n(s-)) + \mathbf{1}_{[0, \mu^n_i + \nu^n_i]}(z) \mathbf{1}_{F_{i\#}}(X^n(s-)) \Big] \mathcal{D}_i(ds, dz).
\end{aligned} \tag{15}$$

Define

$$\begin{aligned}
T^n_i(t) &\doteq \int_0^t \mathbf{1}_{F_i}(X^n(s)) ds, \qquad i = 1, 2, \qquad T^n_{12} \doteq T^n_1 + T^n_2, \\
T^n_0(t) &\doteq \int_0^t \mathbf{1}_{\{0\}}(X^n(s)) ds, \qquad T^n_\emptyset(t) \doteq \int_0^t \mathbf{1}_{\mathcal{S}^o}(X^n(s)) ds.
\end{aligned}$$

Note that D_i^n is a point process with compensator

$$\mu_i^n T_\emptyset^n(t) + (\mu_i^n + \nu_i^n) T_{i\#}^n(t).$$

In addition to the rescaled process $\hat X_i^n(t) = n^{-1/2} X_i^n(t)$ already defined, let

$$\hat A_i^n(t) \doteq n^{-1/2}[A_i^n(t) - \lambda_i^n t], \qquad \hat D_i^n(t) \doteq n^{-1/2}[D_i^n(t) - \mu_i^n T_\emptyset^n(t) - (\mu_i^n + \nu_i^n) T_{i\#}^n(t)]. \tag{16}$$

Then, for $t \ge 0$, we have

$$\hat X_i^n(t) = \hat x_i^n + \hat A_i^n(t) - \hat D_i^n(t) + \hat\lambda_i^n t + \lambda_i n^{1/2} t - n^{-1/2}\mu_i^n T_\emptyset^n(t) - n^{-1/2}(\mu_i^n + \nu_i^n) T_{i\#}^n(t). \tag{17}$$

Let $b_i^n = \hat\lambda_i^n - \hat\mu_i^n$ and note that $b^n \to b$. Also, let

$$d^{(i),n} = e^{(i)} - \frac{\nu_{i\#}^n}{\mu_i^n} e^{(i\#)},$$

and note that $d^{(i),n} \to d^{(i)}$. Then by (17) and the assumption $\lambda_i = \mu_i$,

$$\hat X^n(t) = \hat Y^n(t) + \hat R^n(t) \tag{18}$$

$$\hat Y^n(t) = \hat x^n + \hat A^n(t) - \hat D^n(t) + b^n t \tag{19}$$

$$\hat R^n(t) = \sum_i n^{-1/2}\mu_i^n d^{(i),n} T_i^n(t) + \sum_i n^{-1/2}\mu_i^n e^{(i)} T_0^n(t), \tag{20}$$

where, throughout, $\sum_{i=1}^2$ is abbreviated to $\sum_i$. In this representation, $\hat R^n$ is a boundary term that constrains in direction $d^{(i),n}$ when $\hat X^n$ is on face F_i, $i = 1, 2$, and in the direction $\sum_i \mu_i^n e^{(i)}$ when it is at the origin.

A variation of (18)–(20) is as follows. Let

$$\mathring D_i^n(t) = n^{-1/2} \int_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{[0,\mu_i^n]}(z) \mathcal{D}_i^c(ds, dz). \tag{21}$$

Letting $\mathring M^n = \mathring D^n - \hat D^n$, we have

$$\begin{aligned} \mathring M_i^n(t) &= n^{-1/2} \int_{[0,t]\times\mathbb{R}_+} \Big[\mathbf{1}_{[0,\mu_i^n]}(z) - \mathbf{1}_{[0,\mu_i^n]}(z)\mathbf{1}_{\mathcal{S}^o}(X^n(s-)) - \mathbf{1}_{[0,\mu_i^n+\nu_i^n]}(z)\mathbf{1}_{F_{i\#}}(X^n(s-))\Big] \mathcal{D}_i^c(ds, dz) \\ &= n^{-1/2} \int_{[0,t]\times\mathbb{R}_+} \Big[\mathbf{1}_{[0,\mu_i^n]}(z)\mathbf{1}_{\{0\}\cup F_i}(X^n(s-)) - \mathbf{1}_{(\mu_i^n,\mu_i^n+\nu_i^n]}(z)\mathbf{1}_{F_{i\#}}(X^n(s-))\Big] \mathcal{D}_i^c(ds, dz). \end{aligned} \tag{22}$$

Hence, with $\mathring Y^n = \hat Y^n + \hat D^n - \mathring D^n$, we obtain from (18)–(19),

$$\hat X^n(t) = \mathring Y^n(t) + \mathring M^n(t) + \hat R^n(t) \tag{23}$$

$$\mathring Y^n(t) = \hat x^n + \hat W^n(t) = \hat x^n + \hat A^n(t) - \mathring D^n(t) + b^n t. \tag{24}$$

Equations (23), (24), coupled with (20), give an alternative to (18)–(20), where now $\hat A^n$ and $\mathring D^n$ are mutually independent centered and scaled Poisson processes and it is obvious that $\hat W^n$ converges to a (b, Σ)-BM. Our proof uses both (18)–(20) and (23)–(24).

3.3. C-tightness.

Proposition 3.1. *The sequence of processes* $\{\hat{X}^n, n \in \mathbb{N}\}$ *is* C*-tight in* $D(\mathbb{R}_+, \mathbb{R}^2_+)$.

Proof. We will use (18)–(20). We first argue that $\{\hat{Y}^n\}$ is C-tight. Note that the sequence $\hat{A}^n$ is C-tight as it converges to a BM. The sequence $\hat{D}^n$ is C-tight as its i-th component is a martingale with quadratic variation $[\hat{D}^n_i](t) = n^{-1}D^n_i(t)$, which itself is a tight sequence of processes, all limits of which are a.s. c-Lipschitz for a suitable constant c. Furthermore $b^n \to b$. Combining these facts, $\{\hat{Y}^n\}$ is a C-tight sequence.

Now we turn to proving C-tightness of $\{\hat{X}^n\}$. Fix $S > 0$. Because the initial conditions $\hat{x}^n$ converge, it suffices to show that for any $\varepsilon > 0$ and $\eta > 0$ there is $\delta > 0$ such that

$$\limsup_n \mathbb{P}(w_S(\hat{X}^n, \delta) > \varepsilon) < \eta.$$

In the rest of this proof, δ is always of the form S/N, some $N \in \mathbb{N}$. For $\delta > 0$ and $k \in K_0 \doteq \{1, 2, \dots, S/\delta\}$, denote $I_k = I^\delta_k = [(k-1)\delta, k\delta]$. Then

$$w_S(\hat{X}^n, \delta) \le 2 \max_{k \in K_0} \operatorname{osc}(\hat{X}^n, I_k). \tag{25}$$

Consider the following partitioning of K_0:

$$\begin{aligned} K^n_1 = \{k \in K_0 :\ &\text{for some } s, t \in I_k,\ s < t,\ \text{ there exist } i \in \{1, 2\},\ \text{such that} \\ &\hat{X}^n(s-) \in F_i, \hat{X}^n(t) \in F_{i\#}, \hat{X}^n(u) \in \mathcal{S}^o \cap \mathbb{B}^c_\varepsilon,\ \text{for all } u \in [s, t)\}, \\ K^n_2 = \{k \in K_0 \setminus K^n_1 &: \min_{t \in I_k} \|\hat{X}^n(t)\| \le \varepsilon\}, \\ K^n_3 = \{k \in K_0 \setminus K^n_1 &: \min_{t \in I_k} \|\hat{X}^n(t)\| > \varepsilon\}. \end{aligned}$$

Suppose that given $\varepsilon > 0$ and $\eta > 0$ there is $\delta > 0$ such that

$$\limsup_n \mathbb{P}(K^n_1 \ne \emptyset) < \eta, \tag{26}$$

$$\limsup_n \mathbb{P}(K^n_2 \ne \emptyset \text{ and } \max_{k \in K^n_2} \max_{t \in I_k} \|\hat{X}^n(t)\| > 2\varepsilon) < \eta, \tag{27}$$

$$\limsup_n \mathbb{P}(K^n_3 \ne \emptyset \text{ and } \max_{k \in K^n_3} \operatorname{osc}(\hat{X}^n, I_k) > \varepsilon) < \eta. \tag{28}$$

Let us argue that, in this case,

$$\limsup_n \mathbb{P}(w_S(\hat{X}^n, \delta) > 6\varepsilon) < 3\eta. \tag{29}$$

By (25), on the event $w_S(\hat{X}^n, \delta) > 6\varepsilon$, there exists $k \in K_0$ for which $\operatorname{osc}(\hat{X}^n, I_k) > 3\varepsilon$. If there exists such $k \in K^n_1$ then $K^n_1 \ne \emptyset$, an event having probability $< \eta$ from (26). If there exists such $k \in K^n_3$ then in particular $\operatorname{osc}(\hat{X}^n, I_k) > \varepsilon$, an event having probability $< \eta$ from (28). If there exists such $k \in K^n_2$ then it is impossible that

$$\max_{t \in I_k} \|\hat{X}^n(t)\| \le 2\varepsilon$$

because the diameter of the set $\mathcal{S} \cap \mathbb{B}_{2\varepsilon}$ is $2\sqrt{2}\varepsilon < 3\varepsilon$. Hence, from (27), this event also has probability $< \eta$, showing (29). Note that (29) gives C-tightness. It thus remains to prove (26)–(28).

To show (26), note that if K_1^n is nonempty then for s and t as in the definition of K_1^n, one has (for n sufficiently large) $\|\hat{X}^n(t)-\hat{X}^n(s)\|>\varepsilon$. Moreover, during the interval $[s,t)$, $\hat{X}^n$ takes values in $\mathcal{S}^o$, hence, by (20), and recalling the definition of T^n, we have that $\hat{R}^n$ is flat over the interval. Hence by (18),

$$\mathbb{P}(K_1^n\neq\emptyset)\leq\mathbb{P}(w_S(\hat{Y}^n,\delta)>\varepsilon),$$

and the claim follows by C-tightness of $\hat{Y}^n$.

Next, under the event indicated in (27), there exist $k\in K_0$ and $s,t\in I_k$ satisfying $s<t$, such that either $\|\hat{X}^n(s)\|\leq\varepsilon$, $\|\hat{X}^n(t)\|>2\varepsilon$ or $\|\hat{X}^n(s)\|>2\varepsilon$, $\|\hat{X}^n(t)\|\leq\varepsilon$. Because the size of jumps is $n^{-1/2}$, it follows that there exist two members of I_k, that are again denoted by $s<t$, such that $\|\hat{X}^n(t)-\hat{X}^n(s)\|\geq\varepsilon-n^{-1/2}$, and during $[s,t]$, $\hat{X}^n$ does not visit $\mathbb{B}_\varepsilon$. In addition, by the definition of K_2^n, $\hat{X}^n$ visits at most one of the faces F_1 or F_2 (in addition to $\mathcal{S}^o$) during $[s,t]$.

Consider now the event in (28). Under this event, there exists a $k\in K_0$, and $[s,t]\subset I_k$ with $\operatorname{osc}(\hat{X}^n,[s,t])>\varepsilon$, during which at most one of F_1 and F_2 is visited.

Hence, in both cases, we can find a $k\in K_0$ and $s,t\in I_k$ with $s<t$, such that $\operatorname{osc}(\hat{X}^n,[s,t])\geq\varepsilon-n^{-1/2}$, and for all $u\in[s,t]$, $\hat{X}^n(u)\in S^o\cup F_i$ for either $i=1$ or 2. Suppose the former holds. Then by (18)–(20), for $u\in[s,t]$,

$$\begin{aligned}\hat{X}^n(u)-\hat{X}^n(s)&=\hat{Y}^n(u)-\hat{Y}^n(s)+\hat{R}^n(u)-\hat{R}^n(s),\\ \hat{R}^n(u)-\hat{R}^n(s)&=n^{-1/2}\mu_1^n d^{(1),n}(T_1^n(u)-T_1^n(s)).\end{aligned}$$

In particular, $\int_{[s,t)}\hat{X}_1^n(u)d|\hat{R}^n|(u)=0$. If we let

$$\check{X}^n(u)=\hat{X}^n(s+u),\qquad \check{Y}^n(u)=\hat{X}^n(s)+\hat{Y}^n(s+u)-\hat{Y}^n(s),\qquad 0\leq u\leq t-s,$$

we see that $\check{X}^n(u)=\Gamma^{d^{(1),n}}_{\mathbb{R}_+\times\mathbb{R}}(\check{Y}^n)(u)$, $u\in[0,t-s]$. Recalling that $d^{(1),n}$ converge, it follows by (14) that, for some positive constant c,

$$\varepsilon-n^{-1/2}\leq\operatorname{osc}(\hat{X}^n,[s,t])\leq c\operatorname{osc}(\hat{Y}^n,[s,t]).$$

Clearly this conclusion also holds in the case where F_2 is visited (and F_1 is not). This gives

$$\begin{aligned}&\mathbb{P}(K_2^n\neq\emptyset\text{ and }\max_{k\in K_2^n}\max_{t\in I_k}\|\hat{X}^n(t)\|>2\varepsilon)\vee\mathbb{P}(K_3^n\neq\emptyset\text{ and }\max_{k\in K_3^n}\operatorname{osc}(\hat{X}^n,I_k)>\varepsilon)\\ &\quad\leq\mathbb{P}(c\,w_S(\hat{Y}^n,\delta)\geq\varepsilon-n^{-1/2}).\end{aligned}$$

Now (27) and (28) follow on using once again the fact that $\{\hat{Y}^n\}$ are C-tight. □

3.4. **A relation between the boundary processes.** Let

$$\hat{\mathcal{F}}_t=\sigma\{\mathcal{A}_i([0,s]\times[0,z]),\mathcal{D}_i([0,s]\times[0,z]),i=1,2,\ s\in[0,t],\ z\in\mathbb{R}_+\},\qquad t\in\mathbb{R}_+.$$

Then, for all n, all the processes that were constructed in Section 3.2 are adapted to this filtration.

Lemma 3.2. *Fix $S>0$. Let s^n be a sequence of $\hat{\mathcal{F}}_t$-stopping times, and t^n a sequence of random variables satisfying $s^n\leq t^n\leq s^n+S$. Then there exist a constant $\kappa_1\in(0,\infty)$ and a sequence of random variables $\xi^n\to0$ in probability such that*

$$T_{12}^n(t^n)-T_{12}^n(s^n)\leq\kappa_1(T_0^n(t^n)-T_0^n(s^n))+\kappa_1 n^{-1/2}\|\hat{X}^n(s^n)\|+\xi^n.$$

Note that the result does not require tightness of s^n.

Proof. Denote

$$\zeta_i^n = n^{-1}\mu_i^n + n^{-1}\nu_i^n, \qquad \beta_i^n = \mu_i^n(\mu_i^n + \nu_i^n)^{-1}. \tag{30}$$

By (3)–(4) one has, as $n \to \infty$,

$$\beta_i^n = \frac{\mu_i^n}{\mu_i^n + \nu_i^n} \to \beta_i \doteq \frac{\mu_i}{\mu_i + \nu_i}.$$

Also, by (11), $\beta_1 + \beta_2 < 1$. Hence there are $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\beta_1^n + \beta_2^n < 1 - \varepsilon$. In what follows, $n \geq n_0$.

Denote

$$\bar{H}^n = n^{-1/2}\hat{H}^n,$$

where $\hat{H}^n$ is each of the processes $\hat{A}^n, \hat{D}^n, \hat{X}^n, \hat{Y}^n, \hat{R}^n$. Denote $h^n = (1/\zeta_1^n, 1/\zeta_2^n)$. Then by (18), $h^n \cdot \bar{X}^n = h^n \cdot \bar{Y}^n + h^n \cdot \bar{R}^n$. Recall that $d_i^{(i),n} = 1$ and $d_{i\#}^{(i),n} = -\nu_{i\#}^n/\mu_i^n$. Hence by (20), with $\bar{\mu}^n = n^{-1}\mu^n$ and $\bar{\nu}^n = n^{-1}\nu^n$,

$$\bar{R}_i^n = \bar{\mu}_i^n T_i^n - \bar{\nu}_i^n T_{i\#}^n + \bar{\mu}_i^n T_0^n,$$

and

$$\begin{aligned} h^n \cdot \bar{R}^n &= \frac{\bar{\mu}_1^n T_1^n - \bar{\nu}_1^n T_2^n + \bar{\mu}_1^n T_0^n}{\bar{\mu}_1^n + \bar{\nu}_1^n} + \frac{\bar{\mu}_2^n T_2^n - \bar{\nu}_2^n T_1^n + \bar{\mu}_2^n T_0^n}{\bar{\mu}_2^n + \bar{\nu}_2^n} \\ &= (\beta_1^n + \beta_2^n - 1)T_{12}^n + (\beta_1^n + \beta_2^n)T_0^n. \end{aligned}$$

This gives

$$(1 - \beta_1^n - \beta_2^n)T_{12}^n(t) = (\beta_1^n + \beta_2^n)T_0^n(t) - h^n \cdot \bar{X}^n(t) + h^n \cdot \bar{Y}^n(t).$$

In view of the bound $(1 - \beta_1^n - \beta_2^n) > \varepsilon$ and the nonnegativity of $h^n \cdot \bar{X}^n$,

$$\varepsilon(T_{12}^n(t^n) - T_{12}^n(s^n)) \leq (\beta_1^n + \beta_2^n)(T_0^n(t^n) - T_0^n(s^n)) + h^n \cdot \bar{X}^n(s^n) + \xi_0^n,$$

where

$$\xi_0^n = \|h^n \cdot \bar{Y}^n(s^n + \cdot) - h^n \cdot \bar{Y}^n(s^n)\|_S^*.$$

By the boundedness of β_i^n and h_i^n, the result will follow once it is shown that $\xi_0^n \to 0$ in probability.

To this end, note that $\bar{D}_i^n$ is an $\hat{\mathcal{F}}_t$-martingale with quadratic variation $[\bar{D}_i^n] = n^{-2}D_i^n$. Moreover, by (15), one has for some $C \in (0, \infty)$,

$$D_i^n(s^n + S) - D_i^n(s^n) \leq \mathcal{D}_i([s^n, s^n + S] \times [0, Cn]).$$

As a result, $\mathbb{E}\{[\bar{D}_i^n](s^n + S) - [\bar{D}_i^n](s^n)\} \leq CSn^{-1}$, and by the Burkholder-Davis-Gundy inequality, $\|\bar{D}_i^n(s^n + \cdot) - \bar{D}_i^n(s^n)\|_S^* \to 0$ in probability. A similar estimate holds for $\bar{A}_i^n$. Using this along with the boundedness of h^n, $\hat{x}^n$ and b^n in (19) shows that $\xi_0^n \to 0$ in probability. $\square$

4. Proof of main result

Here we provide the proof of Theorem 2.3, where the main two steps are stated as parts (i) and (ii) of the following result. We will assume, without loss of generality, that weak limit points X of $\hat{X}^n$ are defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 4.1. *Let X be a weak limit point of the sequence $\hat{X}^n$.*
(i) One has

$$\mathbb{E}\Big[\int_0^\infty \mathbf{1}_{\{0\}}(X(t))dt\Big] = 0.$$

(ii) The process

$$f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds, \qquad \mathcal{L}f \doteq \frac{1}{2}\text{trace}(\Sigma\nabla^2 f\Sigma) + b\cdot\nabla f,$$

with $f \in C_b^2(\mathcal{S})$ that is constant in a neighborhood of the origin and satisfies $d^{(i)}\cdot\nabla f(x) \geq 0$ for $x \in F_i$, $i = 1, 2$, is a $\mathbb{P}$-submartingale with respect to the filtration generated by X. Equivalently, if $P_{\hat{x}}$ is the law induced by X on $C(\mathbb{R}_+, \mathcal{S})$ and M is given by (12) *with data* $\mathbf{D}$ *(recalling Z is the coordinate process) then M is a $P_{\hat{x}}$-submartingale.*

Parts (i) and (ii) of the above result are proved in Section 4.3 and Section 4.4, respectively.

Proof of Theorem 2.3. In view of the tightness of the sequence $\hat{X}^n$ stated in Proposition 3.1 and the uniqueness of solutions to the submartingale problem stated in Theorem 2.2, it suffices to show that the law $P_{\hat{x}}$ of any weak limit point X is a solution of the submartingale problem with data $\mathbf{D}$ starting from $\hat{x}$. To this end, note that the initial condition follows from the assumed convergence $\hat{x}^n \to \hat{x}$, whereas the corner property and the submartingale property follow from parts (i) and (ii) of Proposition 4.1, respectively. □

In the next two subsections we establish some estimates in preparation for the proof of Proposition 4.1.

4.1. **Estimates on upcrossing and downcrossing durations.** This subsection is concerned with upcrossings of the modulus $\|\hat{X}^n(t)\|$ of the rescaled process and downcrossings of the modulus $\|X(t)\|$ of the candidate limit process, and develops bounds on their durations. These bounds are proved under a special choice of the rate parameters,

$$\lambda_i^n = n\lambda_i, \qquad \mu_i^n = n\mu_i, \qquad \nu_i^n = n\nu_i, \qquad n \in \mathbb{N},\ i = 1, 2. \tag{31}$$

For $n \in \mathbb{N}$ and $x \in \mathcal{S}^n$, let $\mathbb{P}_x = \mathbb{P}_x^n$ and $\mathbb{E}_x = \mathbb{E}_x^n$ denote the probability measure on $(\Omega, \mathcal{F})$ for which $\hat{X}^n(0) = x$ a.s. and the corresponding expectation.

Lemma 4.2. *Assume* (31) *holds. For $\varepsilon > 0$ let $\eta_\varepsilon^n \doteq \inf\{t \geq 0 : \|\hat{X}^n(t)\| \geq \varepsilon\}$. Then there is a constant $\kappa \in (0, \infty)$, and for each $\varepsilon > 0$, an $n_0 \in \mathbb{N}$, such that, for all $n \geq n_0$,*

$$\sup_{x\in\mathcal{S}^n} \mathbb{E}_x[\eta_{\kappa\varepsilon}^n] \leq \varepsilon^2.$$

Proof. Under assumption (31), the generator of the process $\hat{X}^n$ is given by

$$\mathcal{L}^n f(x) = \begin{cases} \sum_i n\lambda_i(\partial_+^{n,i} f(x) + \partial_-^{n,i} f(x)) & x \in \mathcal{S}^n \cap \mathcal{S}^o, \\ \sum_i n\lambda_i\partial_+^{n,i} f(x) + n(\lambda_i + \nu_i)\partial_-^{n,i} f(x) & x \in \mathcal{S}^n \cap F_{i\#},\ i = 1, 2, \\ \sum_i n\lambda_i\partial_+^{n,i} f(x) & x = 0, \end{cases}$$

for $f : \mathcal{S}^n \to \mathbb{R}$, where we recall the notation $\sum_i = \sum_{i=1}^2$.

Fix $\varepsilon > 0$. Let $a_0 \doteq \varepsilon(\lambda_1 \wedge \lambda_2)^{-1/2}$ and $A \doteq (\nu_1 \vee \nu_2)(\lambda_1\lambda_2)^{-1/2}$. Define $\Psi : \mathbb{R}_+^2 \to \mathbb{R}$ as

$$\Psi(x) \doteq a_0^2 - (x_1^2\lambda_1^{-1} + x_2^2\lambda_2^{-1} + 2Ax_1x_2(\lambda_1\lambda_2)^{-1/2}), \qquad x \in \mathbb{R}_+^2.$$

For $x \in \mathbb{R}^2_+$, let $\|x\|_\lambda \doteq (x_1^2\lambda_1^{-1} + x_2^2\lambda_2^{-1})^{1/2}$. Using the inequality $2x_1x_2(\lambda_1\lambda_2)^{-1/2} \le x_1^2\lambda_1^{-1} + x_2^2\lambda_2^{-1}$ we see that

$$\Psi(x) > 0 \quad \text{when} \quad \|x\|_\lambda < a_0(1+A)^{-1/2}.$$

Now, for $x \in \mathcal{S}^n \cap \mathcal{S}^o$, we have

$$\mathcal{L}^n\Psi(x) = \sum_i n\lambda_i(\partial_+^{n,i}\Psi(x) + \partial_-^{n,i}\Psi(x)) = -4.$$

For $x \in \mathcal{S}^n \cap F_1$,

$$\begin{aligned}
\mathcal{L}^n\Psi(x) &= \sum_i n\lambda_i\partial_+^{n,i}\Psi(x) + n(\lambda_2+\nu_2)\partial_-^{n,2}\Psi(x) \\
&= -n\Big(3n^{-1} + 2Ax_2n^{-1/2}\frac{\lambda_1^{1/2}}{\lambda_2^{1/2}}\Big) - n\frac{\nu_2}{\lambda_2}(n^{-1} - 2n^{-1/2}x_2) \\
&\le -3,
\end{aligned}$$

where the last inequality follows on recalling that $A \ge \nu_2(\lambda_1\lambda_2)^{-1/2}$. Similarly, for $x \in \mathcal{S}^n \cap F_2$, $\mathcal{L}^n\Psi(x) \le -3$. Finally,

$$\mathcal{L}^n\Psi(0) = \sum_i n\lambda_i\partial_+^{n,i}\Psi(0) = -2.$$

Fix $A_0 < (1+A)^{-1/2}$. Let $\gamma^n_{A_0a_0} \doteq \inf\{t \ge 0 : \|\hat{X}^n(t)\|_\lambda \ge A_0a_0\}$. Then for $x \in \mathcal{S}^n$ with $\|x\|_\lambda \le A_0a_0$,

$$\Psi(\hat{X}^n(t\wedge\gamma^n_{A_0a_0})) = \Psi(x) + \int_0^{t\wedge\gamma^n_{A_0a_0}} \mathcal{L}^n\Psi(\hat{X}^n(s))ds + M^n_t \le \Psi(x) - 2(t\wedge\gamma^n_{A_0a_0}) + M^n_t,$$

where, under $\mathbb{P}_x$, M^n is a martingale starting at 0. We can find $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, and $x \in \mathcal{S}^n$ with $\|x\|_\lambda \le A_0a_0$, we have that $\|x + n^{-1/2}e^{(i)}\|_\lambda < (1+A)^{-1/2}a_0$. Taking expectations, and noting that $\Psi(\hat{X}^n(t\wedge\gamma^n_{A_0a_0})) > 0$ when $n \ge n_0$, we see that, for $n \ge n_0$,

$$\mathbb{E}_x(t\wedge\gamma^n_{A_0a_0}) \le \frac{1}{2}\Psi(x) \le \frac{1}{2}a_0^2.$$

Sending $t \to \infty$ and noting that $\gamma^n_{A_0a_0} \ge \eta^n_{A_0\varepsilon}$, we now see that, for $n \ge n_0$, $\mathbb{E}_x(\eta^n_{A_0\varepsilon}) \le \frac{1}{2}a_0^2 = \frac{1}{2}\varepsilon^2(\lambda_1\wedge\lambda_2)^{-1}$. The result follows on taking $\kappa = A_0(2(\lambda_1\wedge\lambda_2))^{1/2}$. $\square$

Under assumption (31), the candidate limiting RBM has zero drift. The next lemma is concerned with such a process. The proof uses hitting time estimates from [29], that were developed for a zero drift, unit variance RBM. To this end, it uses the diagonal transformation Σ^{-1} to transform an RBM with data $(0, \Sigma, \{d^{(i)}\})$ to one with data $(0, \mathrm{Id}, d^{*,(i)}\})$. We recall from (9)–(10) that the parameter $\alpha^* \in (1,2)$ was defined in terms of the latter.

Lemma 4.3. *For $x \in \mathcal{S}$, let P_x denote the solution to the submartingale problem for the admissible data $\mathbf{D} = (0, \Sigma, \{d^{(i)}\})$ starting from x, E_x the corresponding expectation. Then there exist constants $\varepsilon_0 > 0$, $a > 0$ and $c > 1$, such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$\sup_{x\in\mathbb{S}_{c\varepsilon}} E_x[e^{-\tau^\varepsilon}] \le 1 - a\varepsilon^{\alpha_*},$$

where

$$\tau^\varepsilon = \inf\{t \ge 0 : \|Z(t)\| \le \varepsilon\}.$$

Proof. We note that in [29, Sec. 3.3] it is shown that, in the unit variance, zero drift case, the solution to the submartingale problem is a strong Markov process and it has the Feller property. Since one can obtain a solution to the submartingale problem with data $(0, \Sigma, \{d^{(i)}\})$ from one with data $(0, \mathrm{Id}, d^{*,(i)}\})$ via a diagonal transformation, it follows that the former is a Feller strong Markov process as well.

Consider $c > 1$ (the value of which is to be determined later in the proof), let $\varepsilon_0 > 0$ be such that $c\varepsilon_0 < 1$, and $\varepsilon \in (0, \varepsilon_0)$. Fix $x \in \mathbb{R}^2_+$ with $\|x\| = c\varepsilon$. Denote $\eta = \inf\{t : \|Z(t)\| \geq 1\}$. In this proof we suppress ε in the notation for τ^ε. One has $\eta \wedge \tau < \infty$ P_x-a.s., as follows from [29, eq. (2.21)] in the unit variance case, and consequently in our case as well. Let, for $t \geq 0$, $\mathcal{F}_t \doteq \sigma\{Z(s) : s \leq t\}$. Then, using strong Markovity,

$$\begin{aligned}
E_x[e^{-\tau}] &\leq E_x[\mathbf{1}_{\{\tau<\eta\}} + e^{-\tau}\mathbf{1}_{\{\eta<\tau\}}] \\
&\leq P_x(\tau < \eta) + E_x[E_x[e^{-(\tau-\eta)}\mathbf{1}_{\{\eta<\tau\}}|\mathcal{F}_\eta]] \\
&= P_x(\tau < \eta) + E_x[\mathbf{1}_{\{\eta<\tau\}}E_{Z(\eta)}[e^{-\tau}]] \\
&\leq P_x(\tau < \eta) + P_x(\eta < \tau)\sup_{\|y\|=1} E_y[e^{-\tau}] \\
&= 1 - P_x(\eta < \tau) + P_x(\eta < \tau)\kappa_1 \\
&= 1 - aP_x(\eta < \tau),
\end{aligned}$$

where $\kappa_1 = \sup_{\|y\|=1} E_y[e^{-\tau}]$ and $a = 1 - \kappa_1$. To prove that $\kappa_1 < 1$, argue by contradiction and assume $\kappa_1 = 1$. Then by the Feller property, there exists y, $\|y\| = 1$ such that $\tau = 0$ P_y-a.s., contradicting sample path continuity of Z. It follows that $\kappa_1 < 1$ and $a > 0$.

Next, let $Y \doteq \Sigma^{-1}Z$ and note that its law is a solution to the submartingale problem with data $(0, \mathrm{Id}, \{d^{*,(i)}\})$ starting from $\Sigma^{-1}x$. Let $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ be the test function from [29] defined as

$$\Phi(y) = \|y\|^{\alpha_*}\cos(\alpha_*\theta - \theta_*^{(1)}),$$

where $y = (\|y\|\cos\theta, \|y\|\sin\theta)$. Then the proof given in [29] to equation [29, (2.13)], which regards a different pair of stopping times, applies to (η, τ) thanks to the fact that these stopping times on the filtration generated by Z are also stopping times on the filtration generated by Y, as the two filtrations are equal. It gives

$$E_x[\Phi(Y(\eta))\mathbf{1}_{\{\eta<\tau\}} + \Phi(Y(\tau))\mathbf{1}_{\{\tau<\eta\}}] = \Phi(\Sigma^{-1}x).$$

Note that, for $\theta \in [0, \pi/2]$,

$$\cos(\alpha_*\theta - \theta_*^{(1)}) \geq \kappa_2 \doteq \cos(|\theta_*^{(1)}| \vee |\theta_*^{(2)}|) > 0.$$

Therefore, with $\kappa_3 \doteq \kappa_2(\sigma_1 \vee \sigma_2)^{-\alpha_*}$

$$\kappa_3\|x\|^{\alpha_*} \leq \kappa_2\|\Sigma^{-1}x\|^{\alpha_*} \leq \Phi(\Sigma^{-1}x) \leq (\sigma_1 \wedge \sigma_2)^{-\alpha_*}(P_x(\eta < \tau) + \varepsilon^{\alpha_*}).$$

Recalling that $\|x\| = c\varepsilon$, we have

$$P_x(\eta < \tau) \geq \kappa_3 c^{\alpha_*}\varepsilon^{\alpha_*}(\sigma_1 \wedge \sigma_2)^{\alpha_*} - \varepsilon^{\alpha_*}.$$

Choose $c > 1$ such that $\kappa_3 c^{\alpha_*}(\sigma_1 \wedge \sigma_2)^{\alpha_*} \geq 2$ to obtain $P_x(\eta < \tau) \geq \varepsilon^{\alpha_*}$. The result follows. $\square$

4.2. **A sequence of upcrossings and downcrossings.** The goal of this subsection is to prove Proposition 4.4 below, which gives a bound on the expected cumulative time that $\{\hat{X}^n(t), t \in [0,S]\}$ spends in $\mathbb{B}_\varepsilon$ in the special case when (31) holds. This is done by constructing successive upcrossings and downcrossings of $\|\hat{X}^n\|$ and using the estimates from Section 4.1. Lemma 4.2 is directly applicable to individual upcrossings, whereas Lemma 4.3 regards an RBM, and transforming it to an estimate on downcrossing durations of $\|\hat{X}^n\|$, as in Lemma 4.5 below, involves a convergence argument.

Proposition 4.4. *Assume* (31) *holds. Then for all $S < \infty$,*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mathbb{E}_{\hat{x}^n}\int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(\hat{X}^n(t))dt = 0.$$

Throughout what follows, fix $\varepsilon_0 > 0$, $a > 0$ and $c > 1$ as in Lemma 4.3. For $\varepsilon \in (0, \varepsilon_0)$, consider the following sequence of stopping times for each $n \geq n_0$. Namely, $\gamma^n_{-1} = 0$, and for $k \in \mathbb{Z}_+$,

$$\begin{aligned} \gamma^n_{2k} &= \inf\{t \geq \gamma^n_{2k-1} : \|\hat{X}^n(t)\| \leq \varepsilon\},\\ \gamma^n_{2k+1} &= \inf\{t \geq \gamma^n_{2k} : \|\hat{X}^n(t)\| \geq c\varepsilon\}, \end{aligned} \tag{32}$$

suppressing here, and in the entire construction below, the dependence on ε.

Lemma 4.5. *Assume* (31) *holds. Let $N \in \mathbb{N}$ and $S \in (0,\infty)$. Then*

$$\limsup_{n\to\infty}\mathbb{P}\Big(\sum_{k=1}^N(\gamma^n_{2k} - \gamma^n_{2k-1}) \leq S\Big) \leq \exp\{S - aN\varepsilon^{\alpha_*}\}.$$

In preparation to proving Lemma 4.5 we state and prove Lemma 4.6 below. The proof of Lemma 4.5 and Proposition 4.4 appear afterwards.

Fix $0 < c_0 < 1$ throughout what follows. For any given $f \in D(\mathbb{R}_+, \mathbb{R}^2)$ with finitely many jumps in any compact time interval, satisfying

$$\sup_{t\in(0,\infty)}\|f(t) - f(t-)\| \leq c_0\varepsilon/4 \quad \text{ and } \quad \|f(0)\| > c_0\varepsilon,$$

we can construct recursively a unique path $g \in D(\mathbb{R}_+, \mathbb{R}^2_+)$ reflected obliquely on $\partial\mathcal{S}$, along the direction $d^{(i)}$ on face F_i, $i = 1, 2$, and absorbed when first visiting $\mathcal{S}\cap\mathbb{B}_{c_0\varepsilon}$. Although quite standard, we provide a construction of such a path in Appendix A. We will denote this reflected/absorbed path g obtained from f as $\Lambda^\varepsilon(f)$. The following result gives a continuity property and a relation to the submartingale problem.

Lemma 4.6. *(i) Let $\{Y^n\}$ be a sequence of processes with sample paths in $D(\mathbb{R}_+, \mathbb{R}^2)$ such that for each n, a.s., Y^n has finitely many jumps in any compact interval, $\sup_{t\in(0,\infty)}\|Y^n(t) - Y^n(t-)\| \to 0$ as $n \to \infty$, and $Y^n(0) \in \mathbb{S}_{c\varepsilon}$; in particular, $\Lambda^\varepsilon(Y^n)$ is well defined for all large n. Suppose moreover that $Y^n \to \xi + W$, a.s., uniformly on compacts, where ξ is an $\mathbb{S}_{c\varepsilon}$-valued random variable and W is a $(0,\Sigma)$-BM independent of ξ. Then $U^n \doteq \Lambda^\varepsilon(Y^n)$ converges a.s., uniformly on compacts, to $U \doteq \Lambda^\varepsilon(\xi + W)$.*
(ii) For any $x \in \mathcal{S}$, $\|x\| > c_0\varepsilon$, the process $U = \Lambda^\varepsilon(x + W)$ is equal in distribution to the process $Z(\cdot \wedge \tau^\varepsilon)$ under the solution to the submartingale problem with data $(0, \Sigma, \{d^{(i)}\})$ starting at x, where $\tau^\varepsilon = \inf\{t : \|Z(t)\| \leq c_0\varepsilon\}$.

Proof. i. Assume without loss that the above processes are defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\Omega_0 \in \mathcal{F}$ be the full measure set on which all the a.s. properties in the statement of the lemma hold (for every n). Then there exist $n_0 \in \mathbb{N}$, and for each $\omega \in \Omega_0$ and $n \geq n_0$, $k_n \in \mathbb{Z}_+ \cup \{\infty\}$, $0 \leq \sigma_1^n < \sigma_2^n < \cdots < \sigma_{k_n-1}^n < \sigma_{k_n}^n$ (precisely defined in Appendix A) such that σ_1^n is the first time U^n hits one of the faces F_i, $i = 1, 2$, and for $2 \leq N \leq k_n - 1$, σ_N^n is the first time after σ_{N-1}^n when U^n hits a face distinct from the one it hit at σ_{N-1}^n, and finally at $\sigma_{k_n}^n$, it first hits $\mathbb{B}_{c_0\varepsilon}$, after which it is absorbed ($k_n = \infty$ corresponds to never visiting $\mathbb{B}_{c_0\varepsilon}$). Denote the analogous sequence for U by $0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k$. All these quantities depend on ω which is suppressed in the notation. We will use the standard fact that the hitting time of a closed set is a continuous function of the trajectory at $\mathbb{P}$-a.e. path of a nondegenerate two-dimensional RBM in the half space. Fix $S_0 \in (0, \infty)$. Then using this result, by a recursive argument, it follows that, for ω in a full measure set $\Omega_1 \subset \Omega_0$, there is an $n_1 \geq n_0$, such that for all $n \geq n_1$, $\sigma_j^n \wedge S_0 \to \sigma_j \wedge S_0$, and $\sup_{t \in [0, \sigma_j \wedge S_0]} \|U^n(t) - U^n(t)\| \to 0$ for all $j \leq k$. The result follows.

ii. This is immediate from the fact that U is a $(0, \Sigma)$-BM that reflects on $\partial\mathcal{S}$ in the direction $d^{(i)}$ on face F_i, $i = 1, 2$, and gets absorbed in $\mathbb{B}_{c_0\varepsilon}$. □

Proof of Lemma 4.5. In this proof, a continuous time Markov process on $(n^{-1/2}\mathbb{Z})^2$ with generator $\sum n\lambda_i(\partial_+^{n,i} + \partial_-^{n,i})$, starting at 0, will be called an *n-simple random walk* (n-SRW). Due to the special choice of parameters in (31), note that $\lambda_i^n = \mu_i^n = \mu_i n$ and $b^n = 0$. Hence by the definitions (16), (21) and (24) of $\hat{A}^n$, $\mathring{D}^n$ and $\hat{W}^n$, it is seen that $\hat{W}^n$ is an n-SRW.

The statement of the lemma is concerned with the lengths of the time intervals $[\gamma_{2k+1}^n, \gamma_{2k+2}^n]$, which are determined by the paths of $\hat{X}^n$. However, the proof will use a construction that, for each k, extends the paths $\hat{X}^n|_{[\gamma_{2k+1}^n, \gamma_{2k+2}^n]}$ to $[\gamma_{2k+1}^n, \infty)$ in a way that depends on k and is distinct from $\hat{X}^n$. For this extension we introduce $\{\hat{W}_k^{*,n}, k \in \mathbb{Z}_+\}$, which is a collection of mutually independent n-SRWs, independent of the processes $\mathcal{A}_i, \mathcal{D}_i$ that were used for constructing $\hat{X}^n$ in (15).

To construct the extension we go back to equations (23)–(24). Under (31), we also have $d^{(i),n} = d^{(i)}$. Moreover, by (32), for $t \in [\gamma_{2k+1}^n, \gamma_{2k+2}^n]$, $T_0^n(t) = T_0^n(\gamma_{2k+1}^n)$. Let $\Theta_k = \Theta_{\varepsilon,k}^n$ denote the shift operator acting on $D(\mathbb{R}_+, \mathbb{R}^2)$ as

$$\Theta_k\zeta(t) = \zeta(\gamma_{2k+1}^n + t) - \zeta(\gamma_{2k+1}^n), \qquad t \geq 0.$$

Then, denoting $\hat{\xi}_k^n = \hat{X}^n(\gamma_{2k+1}^n)$, we obtain from (21)–(24),

$$\hat{X}^n(\gamma_{2k+1}^n + t) = \hat{\xi}_k^n + \Theta_k\Big\{\hat{W}^n + \mathring{M}^n + \sum_i n^{1/2}\mu_i d^{(i)} T_i^n\Big\}(t), \qquad t \in [0, \gamma_{2k+2}^n - \gamma_{2k+1}^n].$$

Since $c_0\varepsilon < \varepsilon$, this gives for all large n,

$$\hat{X}^n(\gamma_{2k+1}^n + t) = \Lambda^\varepsilon(\hat{\xi}_k^n + \Theta_k\hat{W}^n + \Theta_k\mathring{M}^n)(t), \qquad t \in [0, \gamma_{2k+2}^n - \gamma_{2k+1}^n]. \tag{33}$$

Moreover, since γ_{2k+1}^n are stopping times on the filtration generated by $(\hat{X}^n, \hat{W}^n)$, and the latter forms a strong Markov process, it follows that $\Theta_k\hat{W}^n$ are also n-SRW for each k. We now define, for each k, new processes that agree with $\hat{X}^n(\gamma_{2k+1}^n + \cdot)$, $\Theta_k\hat{W}^n$ and $\Theta_k\mathring{M}^n$ on $[0, \gamma_{2k+2}^n - \gamma_{2k+1}^n]$, thus, in particular, respect relation (33), but may differ from them on $(\gamma_{2k+2}^n - \gamma_{2k+1}^n, \infty)$. Namely, for $k \in \mathbb{Z}_+$, define

$$\hat{W}_k^n(t) \doteq \begin{cases} \Theta_k\hat{W}^n(t), & \text{for } t \in [0, \gamma_{2k+2}^n - \gamma_{2k+1}^n] \\ \hat{W}^n(\gamma_{2k+2}^n) - \hat{W}^n(\gamma_{2k+1}^n) + \hat{W}_k^{*,n}(t - \gamma_{2k+2}^n + \gamma_{2k+1}^n) & \text{for } t \in (\gamma_{2k+2}^n - \gamma_{2k+1}^n, \infty). \end{cases}$$

Noting that both $\Theta_k \hat{W}^n$ and $\hat{W}_k^{n,*}$ are n-SRW and $(\hat{W}^n, \gamma_{2k+2}^n)$ are independent of $\hat{W}_k^{n,*}$, it follows that $\hat{W}_k^n$ is an n-SRW for each k. It is also clear from the construction that, for $k \in \mathbb{Z}_+$,

$$\hat{W}_k^n \text{ is independent of } (\{\hat{W}_j^n, j < k\}, \{\hat{\xi}_j^n, j \le k\}). \tag{34}$$

If we now let

$$\hat{M}_k^n(t) \doteq \mathring{M}^n((\gamma_{2k+1}^n + t) \wedge \gamma_{2k+2}^n) - \mathring{M}^n(\gamma_{2k+1}^n), \qquad t \ge 0, \tag{35}$$

then by (33), we have

$$\hat{X}^n(\gamma_{2k+1}^n + t) = \hat{U}_k^n(t), \qquad t \in [0, \gamma_{2k+2}^n - \gamma_{2k+1}^n],$$

where

$$\hat{U}_k^n(t) = \Lambda^\varepsilon(\hat{\xi}_k^n + \hat{W}_k^n + \hat{M}_k^n)(t), \qquad t \in \mathbb{R}_+. \tag{36}$$

Denoting $\tau_k^n \doteq \inf\{t \ge 0 : \|\hat{U}_k^n(t)\| \le \varepsilon\}$, we see that $\tau_k^n = \gamma_{2k+2}^n - \gamma_{2k+1}^n$. Whereas, for $t \in [0, \gamma_{2k+2}^n - \gamma_{2k+1}^n]$, (36) expresses the same relation as (33), the construction achieves in addition an independence structure for $\hat{W}_k^n$, namely (34), which $\Theta_k \hat{W}^n$ do not possess.

Toward taking the n limit, note that $\{\hat{\xi}_k^n, k \in \mathbb{Z}_+\}$ take values in a compact set and so this sequence is automatically tight. As n-SRW, $\hat{W}_k^n$ converge in distribution, as $n \to \infty$, to W_k which is a $(0, \Sigma)$-BM. Furthermore, if along some convergent subsequence $\{(\xi_k, W_k), k \in \mathbb{Z}_+\}$ denotes the weak limit of $\{(\hat{\xi}_k^n, \hat{W}_k^n), k \in \mathbb{Z}_+\}$, then $\|\xi_k\| = c\varepsilon$ a.s., for every k and moreover, the dependence structure (34) transfers to the limit, namely, for $k \in \mathbb{Z}_+$,

$$W_k \text{ is a } (0, \Sigma)\text{-BM independent of } (\{W_j, j < k\}, \{\xi_j, j \le k\}). \tag{37}$$

Next, using the fact that $T_0^n(\gamma_{2k+2}^n) = T_0^n(\gamma_{2k+1}^n)$ we apply Lemma 3.2 with $s^n = s_k^n = \gamma_{2k+1}^n$ and $t^n = t_k^n = (s_k^n + S_0) \wedge \gamma_{2k+2}^n$, for a fixed finite S_0. It is clear from (32) that $n^{-1/2}\|\hat{X}^n(s_k^n)\| \to 0$ in probability. Hence Lemma 3.2 shows that $T_{12}^n(t_k^n) - T_{12}^n(s_k^n) \to 0$ in probability. By (35) and (22), $\mathbb{E}\{[\hat{M}_k^n](S_0)\} \le C\mathbb{E}\{T_{12}^n(t_k^n) - T_{12}^n(s_k^n)\}$ for some constant C. Noting that $T_{12}^n(t_k^n) - T_{12}^n(s_k^n) \le S_0$ shows that $\mathbb{E}\{[\hat{M}_k^n](S_0)\} \to 0$, hence by the Burkholder-Davis-Gundy inequality and the fact that S_0 is arbitrary, one has for $k \in \mathbb{Z}_+$,

$$\hat{M}_k^n \to 0 \text{ in probability, as } n \to \infty.$$

Combining these observations, we have, along the above subsequence,

$$(\hat{\xi}_k^n, \hat{W}_k^n, k \in \mathbb{Z}_+) \Rightarrow (\xi_k, W_k, k \in \mathbb{Z}_+) \qquad \text{in} \quad (\mathcal{S} \times D(\mathbb{R}_+, \mathbb{R}^2))^{\mathbb{Z}_+}.$$

By appealing to Skorohod's representation theorem we assume without loss of generality that the convergence is a.s. Then, letting $U_k \doteq \Lambda^\varepsilon(\xi_k + W_k)$ and using Lemma 4.6.i we now see that, along the subsequence,

$$(\hat{\xi}_k^n, \hat{W}_k^n, \hat{U}_k^n, k \in \mathbb{Z}_+) \to (\xi_k, W_k, U_k, k \in \mathbb{Z}_+), \quad \text{a.s.} \tag{38}$$

Let, for $k \in \mathbb{Z}_+$,

$$\tau_k \doteq \inf\{t \ge 0 : \|U_k(t)\| \le \varepsilon\}.$$

Then, along the subsequence,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\sum_{k=1}^{N}(\gamma^n_{2k}-\gamma^n_{2k-1})\le S\Big) = \limsup_{n\to\infty} \mathbb{P}\Big(\sum_{k=1}^{N}\tau^n_{k-1}\le S\Big)$$
$$\le \mathbb{P}\Big(\sum_{k=1}^{N}\tau_{k-1}\le S\Big),$$

where the inequality follows from the convergence in (38), the continuity of U_k, and the lower semicontinuity, at any continuous trajectory in $D(\mathbb{R}_+,\mathbb{R}^2)$, of the hitting time of a closed set, regarded as a function on $D(\mathbb{R}_+,\mathbb{R}^2)$. (In fact, as in the proof of Lemma 4.6, this hitting time is continuous at U_k a.s., however that is not needed at this part of the proof.)

It remains to show that $\mathbb{P}\big(\sum_{k=1}^{N}\tau_{k-1}\le S\big)\le e^{S-aN\varepsilon^{\alpha_*}}$. For $x\in\mathcal{S}$, $\|x\|>c_0\varepsilon$, denote

$$g(x)=\mathbb{E}[e^{-\tau(x,W)}] \qquad \text{where} \qquad \tau(x,W)=\inf\{t\ge 0:\|\Lambda^\varepsilon(x+W)(t)\|\le\varepsilon\},$$

and W is a $(0,\Sigma)$-BM. Let $\mathcal{G}_k \doteq \sigma\{\xi_j, j\le k\}\vee\sigma\{W_j, j<k\}$. Then, using (37), the fact that $\|\xi_k\|=c\varepsilon$ a.s., and then Lemma 4.6.ii and Lemma 4.3, one has for all $\varepsilon\in(0,\varepsilon_0)$

$$\mathbb{E}[e^{-\tau_k}\mid\mathcal{G}_k]=g(\xi_k)\le\sup_{x\in\mathbb{S}_{c\varepsilon}}g(x)\le 1-a\varepsilon^{\alpha_*} \qquad \text{a.s.} \tag{39}$$

Thus by successive conditioning,

$$\mathbb{P}\Big(\sum_{k=1}^{N}\tau_{k-1}\le S\Big)\le e^S\mathbb{E}[e^{-\sum_{k=1}^{N}\tau_{k-1}}]$$
$$\le e^S(1-a\varepsilon^{\alpha_*})^N$$
$$\le e^{S-aN\varepsilon^{\alpha_*}},$$

where the next to last inequality uses (39). This completes the proof of Lemma 4.5. □

Proof of Proposition 4.4. The dependence of the stopping times $\{\gamma^n_j\}$ on ε will now be emphasized by writing $\gamma^{n,\varepsilon}_j$. Let

$$N^\varepsilon_n=\max\{k\in\mathbb{Z}_+:\gamma^{n,\varepsilon}_{2k}\le S\}.$$

Let $\beta\in(\alpha_*,2)$. Then for all $\varepsilon\in(0,\varepsilon_0)$ and all large n,

$$\mathbb{E}\int_0^S \mathbf{1}_{\{\|\hat X^n(t)\|\le\varepsilon\}}dt\le\mathbb{E}\Big[\mathbf{1}_{\{N^\varepsilon_n\le\varepsilon^{-\beta}\}}\sum_{k=0}^{N^\varepsilon_n}(\gamma^{n,\varepsilon}_{2k+1}-\gamma^{n,\varepsilon}_{2k})\Big]+S\mathbb{P}(N^\varepsilon_n>\varepsilon^{-\beta})$$
$$\le\mathbb{E}\Big[\sum_{k=0}^{\varepsilon^{-\beta}}(\gamma^{n,\varepsilon}_{2k+1}-\gamma^{n,\varepsilon}_{2k})\Big]+S\mathbb{P}\Big(\sum_{k=1}^{\varepsilon^{-\beta}}(\gamma^{n,\varepsilon}_{2k}-\gamma^{n,\varepsilon}_{2k-1})\le S\Big).$$

Let κ be as in Lemma 4.2. Applying this lemma with ε replaced by $c\kappa^{-1}\varepsilon$ gives

$$\sup_{x\in\mathcal{S}^n}\mathbb{E}_x[\eta^n_{c\varepsilon}]\le c^2\kappa^{-2}\varepsilon^2.$$

Hence

$$\mathbb{E}\int_0^S \mathbf{1}_{\{\|\hat X^n(t)\|\le\varepsilon\}}dt\le c^2\kappa^{-2}\varepsilon^{2-\beta}+S\mathbb{P}\Big(\sum_{k=1}^{\varepsilon^{-\beta}}(\gamma^{n,\varepsilon}_{2k}-\gamma^{n,\varepsilon}_{2k-1})\le S\Big).$$

Since the weakly convergent subsequence considered above (37) was arbitrary, we have from Lemma 4.5 that

$$\limsup_{n\to\infty} \mathbb{E} \int_0^S \mathbf{1}_{\{\|\hat{X}^n(t)\| \le \varepsilon\}} dt \le c^2 \kappa^{-2} \varepsilon^{2-\beta} + S \exp\{S - a\varepsilon^{-(\beta-\alpha_*)}\}.$$

The result follows on sending $\varepsilon \to 0$. ☐

4.3. **Proof of the corner property.**

Proof of Proposition 4.1.i. The first step is to extend Proposition 4.4 from the special case (31) to λ^n, μ^n, ν^n as in the main result. Fix $S \in (0, \infty)$. Let $\mathbb{P}^n$ denote the probability law on $D([0,S], \mathcal{S}^n)$ induced by the Markov process $\{\hat{X}^n(t) : 0 \le t \le S\}$ starting at $\hat{x}^n \in \mathcal{S}^n$, under $\mathbb{P}$. To distinguish the special case, let $\tilde{\mathbb{P}}^n$ denote the probability law on $D([0,S], \mathcal{S}^n)$ induced by $\{\hat{X}^n(s) : 0 \le t \le S\}$ starting at $\hat{x}^n \in \mathcal{S}^n$, in the special case (31). Denote by $\mathbb{E}^n$ and $\tilde{\mathbb{E}}^n$ the corresponding expectations. Denote the coordinate process on $D([0,S], \mathcal{S}^n)$ by Z^n. Let $L_S^n \doteq \frac{d\mathbb{P}^n}{d\tilde{\mathbb{P}}^n}$. Then there exists $K \in (0, \infty)$, such that for all n and $x \in \mathcal{S}^n$,

$$\tilde{\mathbb{E}}^n [L_S^n]^2 \le K. \tag{40}$$

This fact is standard but for completeness we give a proof in Appendix B. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E} \int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(\hat{X}^n(t)) dt &= \mathbb{E}^n \int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(Z^n(t)) dt \\
&\le (\tilde{\mathbb{E}}^n [L_S^n]^2)^{1/2} \Big(\tilde{\mathbb{E}}^n \Big(\int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(Z^n(t)) dt \Big)^2 \Big)^{1/2} \\
&\le K^{1/2} \Big(\tilde{\mathbb{E}}^n \int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(Z^n(t)) dt \Big)^{1/2}.
\end{aligned}$$

Hence by Proposition 4.4, $\lim_{\varepsilon \to 0} \kappa(\varepsilon) = 0$, where

$$\kappa(\varepsilon) = \limsup_{n\to\infty} \mathbb{E} \int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(\hat{X}^n(t)) dt.$$

Recall that X denotes a weak limit point of $\hat{X}^n$. By appealing to the Skorohod representation theorem, we can assume that $\|\hat{X}^n - X\|_S^* \to 0$ a.s. as $n \to \infty$. This convergence and Fatou's lemma imply

$$\begin{aligned}
\mathbb{E} \int_0^S \mathbf{1}_{\{0\}}(X(t)) dt &\le \mathbb{E} \int_0^S \liminf_{n\to\infty} \mathbf{1}_{\mathbb{B}_\varepsilon}(\hat{X}^n(t)) dt \\
&\le \liminf_{n\to\infty} \mathbb{E} \int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(\hat{X}^n(t)) dt \le \kappa(\varepsilon).
\end{aligned}$$

Sending $\varepsilon \to 0$ shows that the left-hand side is 0. Sending $S \to \infty$ and applying the monotone convergence theorem proves the result. ☐

We record the following consequence of the above proof and Lemma 3.2.

Corollary 4.7. *For all $S < \infty$, $T_0^n(S) + T_1^n(S) + T_2^n(S) \to 0$ in probability as $n \to \infty$.*

Proof. For $S < \infty$ and $\varepsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{E}_{\hat{x}^n} T_0^n(S) = \limsup_{n\to\infty} \mathbb{E}_{\hat{x}^n} \int_0^S \mathbf{1}_{\{0\}}(\hat{X}^n(t)) dt \le \limsup_{n\to\infty} \mathbb{E}_{\hat{x}^n} \int_0^S \mathbf{1}_{\mathbb{B}_\varepsilon}(\hat{X}^n(t)) dt.$$

The right-hand side was denoted in the proof of Proposition 4.1.i by $\kappa(\varepsilon)$, and it was shown that $\kappa(0+) = 0$. It follows that $T_0^n(S) \to 0$ in probability. The result now follows upon applying Lemma 3.2 with $0 = s^n < t^n = S$. □

4.4. **Proof of the submartingale property.** The last step of the proof is to show that limit points satisfy the submartingale property.

Proof of Proposition 4.1.ii. Let $f \in C_b^2(\mathcal{S})$ be constant in a neighborhood of origin and $d^{(i)} \cdot \nabla f(x) \geq 0$ for all $x \in F_i$, $i = 1, 2$. Let, for $t \geq 0$, $\mathcal{F}_t \doteq \sigma\{X(s) : 0 \leq s \leq t\}$. We need to show that $\{M(t)\}_{t \geq 0}$ defined as

$$M(t) \doteq f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds, \ t \geq 0,$$

is an $\mathcal{F}_t$-submartingale.

Fix a subsequence, denoted again as $\{n\}$, along which $\hat{X}^n \Rightarrow X$. From (15), it follows that

$$\begin{aligned} f(\hat{X}^n(t)) = f(\hat{x}^n) &+ \sum_i \int_{[0,t]\times\mathbb{R}_+} \partial_+^{n,i} f(\hat{X}^n(s-))\mathbf{1}_{[0,\lambda_i^n]}(z)\mathcal{A}_i(ds, dz) \\ &+ \sum_i \int_{[0,t]\times\mathbb{R}_+} \partial_-^{n,i} f(\hat{X}^n(s-))\Big[\mathbf{1}_{[0,\mu_i^n]}(z)\mathbf{1}_{\mathcal{S}^o}(\hat{X}^n(s-)) \\ &\qquad + \mathbf{1}_{[0,\mu_i^n+\nu_i^n]}(z)\mathbf{1}_{F_{i\#}}(\hat{X}^n(s-))\Big]\mathcal{D}_i(ds, dz). \end{aligned} \tag{41}$$

Fix $0 \leq t_1 \leq t_2 < \infty$ and let $\psi : C(\mathbb{R}_+, \mathcal{S}) \to \mathbb{R}$ be a nonnegative bounded continuous function. To prove the submartingale property, it suffices to show that

$$E\left[\psi(X(\cdot \wedge t_1))(M(t_2) - M(t_1))\right] \geq 0. \tag{42}$$

Letting, for $i = 1, 2$,

$$\begin{aligned} M_i^{n,A}(t) &\doteq \int_{[0,t]\times\mathbb{R}_+} \partial_+^{n,i} f(\hat{X}^n(s-))\mathbf{1}_{[0,\lambda_i^n]}(z)\mathcal{A}_i^c(ds, dz) \\ M_i^{n,D}(t) &\doteq \int_{[0,t]\times\mathbb{R}_+} \partial_-^{n,i} f(\hat{X}^n(s-))\Big[\mathbf{1}_{[0,\mu_i^n]}(z)\mathbf{1}_{\mathcal{S}^o}(\hat{X}^n(s-) \\ &\qquad + \mathbf{1}_{[0,\mu_i^n+\nu_i^n]}(z)\mathbf{1}_{F_{i\#}}(\hat{X}^n(s-))\Big]\mathcal{D}_i^c(ds, dz), \end{aligned}$$

we can write, for n large enough,

$$\begin{aligned} &f(\hat{X}^n(t_2)) - f(\hat{X}^n(t_1)) - \sum_i [(M_i^{n,A}(t_2) - M_i^{n,A}(t_1)) + (M_i^{n,D}(t_2) - M_i^{n,D}(t_1))] \\ &= \sum_i \int_{t_1}^{t_2} \lambda_i^n \partial_+^{n,i} f(\hat{X}^n(s))ds + \sum_i \int_{t_1}^{t_2} \partial_-^{n,i} f(\hat{X}^n(s))\Big[\mu_i^n \mathbf{1}_{\mathcal{S}^o}(\hat{X}^n(s)) + (\mu_i^n + \nu_i^n)\mathbf{1}_{F_{i\#}}(\hat{X}^n(s))\Big]ds \\ &= \sum_i \int_{t_1}^{t_2} \Big[\lambda_i^n \partial_+^{n,i} f(\hat{X}^n(s)) + \mu_i^n \partial_-^{n,i} f(\hat{X}^n(s))\Big]ds - \sum_i \int_{t_1}^{t_2} \partial_-^{n,i} f(\hat{X}^n(s))\mu_i^n \mathbf{1}_{F_i}(\hat{X}^n(s))ds \\ &\quad + \sum_i \int_{t_1}^{t_2} \partial_-^{n,i} f(\hat{X}^n(s))\nu_i^n \mathbf{1}_{F_{i\#}}(\hat{X}^n(s))ds, \end{aligned}$$

where in proving the second equality, we have used the fact that f is constant in a neighborhood of the origin and so $\partial_-^{n,i} f(\hat{X}^n(s))\mathbf{1}_{\{0\}}(\hat{X}^n(s)) = 0$ for n large enough.

Next, using Taylor's approximation, write for $i = 1, 2$,

$$\partial_+^{n,i} f(\hat{X}^n(s)) = n^{-1/2} \nabla_i f(\hat{X}^n(s)) + n^{-1} \frac{1}{2} \nabla_{ii} f(\hat{X}^n(s)) + G_i^{n,A}(s),$$
$$\partial_-^{n,i} f(\hat{X}^n(s)) = -n^{-1/2} \nabla_i f(\hat{X}^n(s)) + n^{-1} \frac{1}{2} \nabla_{ii} f(\hat{X}^n(s)) + G_i^{n,D}(s).$$

Here $G_i^{n,A}(s) = (2n)^{-1}(\nabla_{ii} f(\hat{Y}^{(i),n}(s)) - \nabla_{ii} f(\hat{X}^n(s))$ where $\hat{Y}^{(i),n}(s)$ is a point on the line segment joining $\hat{X}^n(s)$ with $\hat{X}^n(s) + n^{-1/2} e^{(i)}$. In particular, $\|\hat{Y}^{(i),n}(s) - \hat{X}^n(s)\| \le n^{-1/2}$. Similar relations holds for $G_i^{n,D}$. By the tightness of $\hat{X}^n$ and the boundedness and continuity of $\nabla_{ii} f$,

$$\sup_{t_1 \le s \le t_2} \sum_i n[|G_i^{n,A}(s)| + |G_i^{n,D}(s)|] \to 0 \tag{43}$$

in L^1 as $n \to \infty$. Now note that, for $i = 1, 2$, and $s \in [t_1, t_2]$,

$$\begin{aligned} &[-\partial_-^{n,i} f(\hat{X}^n(s)) \mu_i^n + \partial_-^{n,i^\#} f(\hat{X}^n(s)) \nu_{i^\#}^n] \\ &= [\hat{\mu}_i^n \nabla_i f(\hat{X}^n(s)) - \hat{\nu}_{i^\#}^n \nabla_{i^\#} f(\hat{X}^n(s))] + n^{1/2}[\mu_i \nabla_i f(\hat{X}^n(s)) - \nu_{i^\#} \nabla_{i^\#} f(\hat{X}^n(s))] + \tilde{G}_i^n(s), \end{aligned} \tag{44}$$

where

$$\tilde{G}_i^n(s) = \bar{\nu}_{i^\#}^n \Big(\frac{1}{2} \nabla_{i^\# i^\#} f(\hat{X}^n(s)) + n G_{i^\#}^{n,D}(s)\Big) - \bar{\mu}_i^n \Big(\frac{1}{2} \nabla_{ii} f(\hat{X}^n(s)) + n G_i^{n,D}(s)\Big).$$

Since, $\bar{\mu}_i^n$, $\bar{\nu}_{i^\#}^n$ are bounded, $f \in C_b^2$, and (43) holds, we have from Corollary 4.7 that

$$\int_{t_1}^{t_2} |\tilde{G}_i^n(s)| \mathbf{1}_{F_i}(\hat{X}^n(s)) ds \to 0 \text{ in } L^1, \text{ as } n \to \infty.$$

Similarly, since $\hat{\mu}_i^n$ and $\hat{\nu}_{i^\#}^n$ are bounded

$$\int_{t_1}^{t_2} |\hat{\mu}_i^n \nabla_i f(\hat{X}^n(s)) - \hat{\nu}_{i^\#}^n \nabla_{i^\#} f(\hat{X}^n(s))| \mathbf{1}_{F_i}(\hat{X}^n(s)) ds \to 0 \text{ in } L^1, \text{ as } n \to \infty.$$

Also, by the property $d^{(i)} \cdot \nabla f(x) \ge 0$ for all $x \in F_i$, we have that

$$[\mu_i \nabla_i f(\hat{X}^n(s)) - \nu_{i^\#} \nabla_{i^\#} f(\hat{X}^n(s))] \mathbf{1}_{F_i}(\hat{X}^n(s)) \ge 0.$$

Using the above observations in (44), we see that

$$\sum_i \int_{t_1}^{t_2} \Big[-\partial_-^{n,i} f(\hat{X}^n(s)) \mu_i^n + \partial_-^{n,i^\#} f(\hat{X}^n(s)) \nu_{i^\#}^n \Big] \mathbf{1}_{F_i}(\hat{X}^n(s)) ds \ge H_1^n,$$

where $H_1^n \to 0$ in L^1. Next, using (43),

$$\begin{aligned} &\sum_i [\lambda_i^n \partial_+^{n,i} f(\hat{X}^n(s)) + \mu_i^n \partial_-^{n,i} f(\hat{X}^n(s))] \\ &= \sum_i [\hat{\lambda}_i^n - \hat{\mu}_i^n] \nabla_i f(\hat{X}^n(s)) + \frac{1}{2} \sum_i n^{-1} [\lambda_i^n + \mu_i^n] \nabla_{ii} f(\hat{X}^n(s)) + \check{G}^n(s), \end{aligned}$$

where $\sup_{t_1 \le s \le t_2} |\check{G}^n(s)| \to 0$ in L^1. Since $\hat{\lambda}_i^n - \hat{\mu}_i^n \to b_i$ and $n^{-1}[\lambda_i^n + \mu_i^n] \to \lambda_i + \mu_i = 2\lambda_i$, it then follows that

$$\sum_i \int_{t_1}^{t_2} [\lambda_i^n \partial_+^{n,i} f(\hat{X}^n(s)) + \mu_i^n \partial_-^{n,i} f(\hat{X}^n(s))] ds = \int_{t_1}^{t_2} \mathcal{L} f(\hat{X}^n(s)) ds + H_2^n,$$

where $H_2^n \to 0$ in L^1. Combining the above, and letting $\bar{M}^n(t) \doteq \sum_i (M_i^{n,A}(t) + M_i^{n,D}(t))$, we now see that

$$f(\hat{X}^n(t_2)) - f(\hat{X}^n(t_1)) - \int_{t_1}^{t_2} \mathcal{L}f(\hat{X}^n(s))ds \geq \bar{M}^n(t_2) - \bar{M}^n(t_1) + H^n,$$

where $H^n \to 0$ in L^1. Thus, since ψ is nonnegative,

$$\begin{aligned}
&\mathbb{E}\left[\psi(\hat{X}^n(\cdot \wedge t_1))\left(f(\hat{X}^n(t_2)) - f(\hat{X}^n(t_1)) - \int_{t_1}^{t_2} \mathcal{L}f(\hat{X}^n(s))ds\right)\right] \\
&\geq \mathbb{E}\left[\psi(\hat{X}^n(\cdot \wedge t_1))\left(\bar{M}^n(t_2) - \bar{M}^n(t_1) + H^n\right)\right] = \mathbb{E}\left[\psi(\hat{X}^n(\cdot \wedge t_1))H^n\right],
\end{aligned}$$

where in the last equality we have used the martingale property of $\bar{M}^n$. Now sending $n \to \infty$ and recalling the weak convergence of $\hat{X}^n$ to X we have

$$\begin{aligned}
&\mathbb{E}\left[\psi(X(\cdot \wedge t_1))\left(f(X(t_2)) - f(X(t_1)) - \int_{t_1}^{t_2} \mathcal{L}f(X(s))ds\right)\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\psi(\hat{X}^n(\cdot \wedge t_1))\left(f(\hat{X}^n(t_2)) - f(\hat{X}^n(t_1)) - \int_{t_1}^{t_2} \mathcal{L}f(\hat{X}^n(s))ds\right)\right] \geq 0.
\end{aligned}$$

This proves (42) and completes the proof of the proposition. □

Appendix A. Construction of Λ^ε

Fix $f \in D(\mathbb{R}_+, \mathbb{R}^2)$ with $f(0) \in \mathcal{S}$ and $|f(0)| > c_0\varepsilon$, satisfying, for all $t > 0$, $\|f(t) - f(t-)\| \leq \tilde{\varepsilon} \doteq c_0\varepsilon/4$ and having locally finitely many jumps. For notational simplicity, denote the map $\Gamma^{d^{(1)}}_{\mathbb{R}_+\times\mathbb{R}}$ (resp. $\Gamma^{d^{(2)}}_{\mathbb{R}\times\mathbb{R}_+}$) as Γ_1 (resp. Γ_2). Also let $\tilde{F}_i \doteq \{x \in \mathcal{S}^c : \inf_{y\in F_i} \|x - y\| \leq \tilde{\varepsilon}\}$. Define a sequence $g_N \in D(\mathbb{R}_+, \mathbb{R}^2)$, $N \in \mathbb{Z}_+$ recursively as follows. Let $\sigma_0 = 0$ and

$$g_0(t) \doteq f(t),\ t \geq 0.$$

For $N \in \mathbb{N}$, if $\sigma_{N-1} < \infty$, let

$$\gamma_N \doteq \inf\{t \geq \sigma_{N-1} : g_{N-1}(t) \in \mathcal{S}^c\},\ \eta_N \doteq \inf\{t \geq \sigma_{N-1} : |g_{N-1}(t)| \leq c_0\varepsilon\},\ \sigma_N \doteq \gamma_N \wedge \eta_N.$$

If $\gamma_N < \infty$ and $\gamma_N < \eta_N$, let i_N be the index $i \in \{1, 2\}$ for which $g_{N-1}(\gamma_N) = g_{N-1}(\sigma_N) \in \tilde{F}_i$. If $\sigma_{N-1} = \infty$ let $\sigma_N = \infty$. Let $g_N(t) = g_{N-1}(t)$ for $t \in [0, \sigma_N)$. If $\sigma_N < \infty$, for $t \in [0, \infty)$, let

$$g_N(t) = \begin{cases} g_{N-1}(\sigma_N) & \text{if } \eta_N \leq \gamma_N \\ \Gamma_{i_N}(g_{N-1})(t) & \text{if } \gamma_N < \eta_N. \end{cases}$$

Note that one always has $i_N \neq i_{N-1}$, and so the construction alternates between Γ_1 (reflection at F_1) and Γ_2 (reflection at F_2).

This construction produces a sequence g_N, $N \in \mathbb{Z}_+$ satisfying $g_N(t) = g_{N-1}(t)$ for $t \in [0, \sigma_N)$. If $\sigma_N \to \infty$ then the pointwise limit $g(t) \doteq \lim_N g_N(t)$ exists for every t, and we set $\Lambda^\varepsilon(f) = g$. In this case the trajectory never gets absorbed in $\mathbb{B}_{c_0\varepsilon}$. If, on the other hand, σ_N remain bounded then absorption must occur, namely there must exist N for which $\eta_N < \gamma_N$. Let K be the first such N and set $\Lambda^\varepsilon(f) = g_K$. This completes the construction.

Appendix B. Proof of estimate (40).

We begin by observing that one can give the following distributionally equivalent construction of X^n. Let $\{\mathcal{A}_i^n, \mathcal{D}_i^n, \mathcal{B}_i^n, i = 1, 2\}$ be mutually independent Poisson processes with intensities λ_i^n, μ_i^n and $\mu_i^n + \nu_i^n$, $i = 1, 2$, respectively. Define

$$\begin{aligned} X^n(t) &= x^n + A^n(t) - D^n(t), \\ A_i^n(t) &= \mathcal{A}_i^n(t), \\ D_i^n(t) &= \int_{[0,t]} \mathbf{1}_{\mathcal{S}^o}(X^n(s-)) d\mathcal{D}_i^n(s) + \int_{[0,t]} \mathbf{1}_{F_{i\#}}(X^n(s-)) \mathcal{B}_i^n(s). \end{aligned} \tag{45}$$

Then the process X^n has the same law as the process X^n introduced in (15). When $\lambda_i^n, \mu_i^n, \nu_i^n$ are replaced by $n\lambda_i, n\mu_i, n\nu_i$, the corresponding processes above are denoted as $\tilde{\mathcal{A}}_i^n, \tilde{\mathcal{D}}_i^n, \tilde{\mathcal{B}}_i^n, \tilde{X}^n$. We can find $C \in (0, \infty)$ such that, for all $n \in \mathbb{N}$,

$$\sum_i \left(\left| \frac{\lambda_i^n}{n\lambda_i} - 1 \right| + \left| \frac{\mu_i^n}{n\mu_i} - 1 \right| + \left| \frac{\nu_i^n + \mu_i^n}{n\nu_i + n\mu_i} - 1 \right| \right) \leq \frac{C}{n^{1/2}}. \tag{46}$$

Fix $S < \infty$ and let $\Sigma = D([0, S], \mathbb{R})$. Let $P_a^{n,i}$ denote the law of $\mathcal{A}_i^n|_{[0,S]}$, regarded as a probability measure on $(\Sigma, \mathcal{B}(\Sigma))$. Similarly, denote the law of $\tilde{\mathcal{A}}_i^n|_{[0,S]}$ as $\tilde{P}_a^{n,i}$. Then by Girsanov's theorem for point processes,

$$\frac{dP_a^{n,i}}{d\tilde{P}_a^{n,i}} = \exp\left\{ \log\left(\frac{\lambda_i^n}{n\lambda_i}\right) \mathcal{A}_i^n(S) + n\lambda_i S \left(1 - \frac{\lambda_i^n}{n\lambda_i}\right) \right\}.$$

Denoting the expectation under $\tilde{P}_a^{n,i}$ as $\tilde{E}_a^{n,i}$, it then follows from (46) that

$$\tilde{E}_a^{n,i} \left(\frac{dP_a^{n,i}}{d\tilde{P}_a^{n,i}}\right)^2 \leq \exp\left\{ n\lambda_i S \left(1 - \frac{\lambda_i^n}{n\lambda_i}\right)^2 \right\} \leq e^{C^2 \lambda_i S}.$$

Similar estimates hold when $(\mathcal{A}_i^n, \tilde{\mathcal{A}}_i^n)$ is replaced by $(\mathcal{D}_i^n, \tilde{\mathcal{D}}_i^n)$ and $(\mathcal{B}_i^n, \tilde{\mathcal{B}}_i^n)$. From the mutual independence of the Poisson processes it now follows that L_S^n satisfies

$$\tilde{\mathbb{E}}^n (L_S^n)^2 \leq e^{C^2 \sum_i (\lambda_i + 2\mu_i + \nu_i) S}.$$

This proves (40). ∎

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