On the sharp constants in the regional fractional Sobolev inequalities

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Abstract

In this paper, we study the sharp constants in fractional Sobolev inequalities associated with the regional fractional Laplacian in domains.

1 Introduction

Let $n \geq 1$, $\sigma \in (0, 1)$ (with the additional assumption that $\sigma < 1/2$ if $n = 1$), and $\Omega \subset \mathbb{R}^n$ be an open set. Consider the sharp constant of the fractional Sobolev inequality

$$Y_{n,\sigma}(\Omega) := \inf \left\{ I_{n,\sigma,\mathbb{R}^n}[u] : u \in C_c^\infty(\Omega), \int_\Omega |u|^{\frac{2n}{n-2\sigma}} \,\mathrm{d}x = 1 \right\},$$

where

$$I_{n,\sigma,\mathbb{R}^n}[u] := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y \tag{1}$$

is the fractional Sobolev semi-norm of u. Using the dilation and translation invariance of $Y_{n,\sigma}(\mathbb{R}^n)$, it is not difficult to see that $Y_{n,\sigma}(\Omega) = Y_{n,\sigma}(\mathbb{R}^n)$. Moreover, Lieb [13] classifies all the minimizers for $Y_{n,\sigma}(\mathbb{R}^n)$ and shows that they do not vanish anywhere on $\mathbb{R}^n$. Therefore, the infimum $Y_{n,\sigma}(\Omega)$ is not attained unless $\Omega = \mathbb{R}^n$.

Together with Xiong, the first two authors in [9] considered the sharp constant of the fractional Sobolev inequality on the domain Ω:

$$S_{n,\sigma}(\Omega) := \inf \left\{ I_{n,\sigma,\Omega}[u] : u \in C_c^\infty(\Omega), \int_\Omega |u|^{\frac{2n}{n-2\sigma}} \,\mathrm{d}x = 1 \right\}, \tag{2}$$

where

$$I_{n,\sigma,\Omega}[u] := \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y \tag{3}$$

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is another fractional Sobolev semi-norm for u. In probability, $I_{n,\sigma,\Omega}$ defined in (3) is called the Dirichlet form of the censored 2σ-stable process [2] in Ω. Its generator

$$(-\Delta)^{\sigma}_{\Omega}u := 2\lim_{\varepsilon\to 0}\int_{\{y\in\Omega:\ |y-x|\geq\varepsilon\}}\frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \tag{4}$$

is usually called the regional fractional Laplacian [11, 12]. Therefore, in this paper, we call (3) as the regional fractional Sobolev semi-norm of u. When $S_{n,\sigma}(\Omega)>0$, we call (2) the sharp constant of the regional fractional Sobolev inequality in Ω. It follows from [6] that if $n\geq 2$ and $\sigma>1/2$, then $S_{n,\sigma}(\Omega)>0$. If $\sigma<1/2$ and Ω is of finite measure with sufficiently regular boundary, then Lemma 16 in [9] shows that $S_{n,\sigma}(\Omega)=0$. If $\sigma<1/2$ and Ω is the complement of the closure of a bounded Lipschitz domain or a domain above the graph of a Lipschitz function, then it follows from the fractional Sobolev inequality on $\mathbb{R}^n$ and the Hardy inequality in [5] that $S_{n,\sigma}(\Omega)>0$.

It was discovered in [9] that the minimization problem for $S_{n,\sigma}(\Omega)$ behaves differently from that for $Y_{n,\sigma}(\Omega)$. Unlike $Y_{n,\sigma}(\Omega)$, which always equals to $Y_{n,\sigma}(\mathbb{R}^n)$ and is never achieved unless $\Omega=\mathbb{R}^n$, the constant $S_{n,\sigma}(\Omega)$ depends on the domain Ω, and can be achieved in many cases assuming that $n\geq 4\sigma$:

- If the complement Ω^c has an interior point, then $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n)$.
- If $\sigma\neq 1/2$, then $S_{n,\sigma}(\mathbb{R}^n_+)$ is achieved (see also Musina-Nazarov[15]).
- If $\sigma>1/2$, Ω is a bounded domain such that $B^+_\varepsilon\subset\Omega\subset\mathbb{R}^n_+$ for some $\varepsilon>0$, then $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n_+)$. Moreover, if $\partial\Omega$ is smooth then $S_{n,\sigma}(\Omega)$ is achieved.

Here, we used the notations that $\mathbb{R}^n_+=\{x=(x',x_n)\in\mathbb{R}^n : x_n>0\}$, $B_r=\{x\in\mathbb{R}^n : |x|<r\}$ and $B^+_r=B_r\cap\mathbb{R}^n_+$.

Recently, Fall-Temgoua [8] proved that if Ω is a bounded C^1 domain and $\sigma>1/2$ is very close to $1/2$, then $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n_+)$, and consequently, $S_{n,\sigma}(\Omega)$ is achieved, by showing the upper semicontinuity of $S_{n,\sigma}(\Omega)$ on $\sigma\in[1/2,1)$ and using the fact that $S_{n,1/2}(\Omega)=0$.

As explained in [9], the discrepancy between the $S_{n,\sigma}(\Omega)$ and $Y_{n,\sigma}(\Omega)$ problems can be explained as a Brézis-Nirenberg [3] effect :

$$\begin{aligned} I_{n,\sigma,\Omega}[u] &= I_{n,\sigma,\mathbb{R}^n}[u]-2\int_\Omega u^2(x)\,\mathrm{d}x\int_{\mathbb{R}^n\setminus\Omega}\frac{1}{|x-y|^{n+2\sigma}}\,\mathrm{d}y\\ &\approx I_{n,\sigma,\mathbb{R}^n}[u]-c_{n,\sigma}\int_\Omega\frac{u^2(x)}{\mathrm{dist}(x,\partial\Omega)^{2\sigma}}\,\mathrm{d}x\quad\forall\,u\in C^\infty_c(\Omega).\end{aligned}$$

Therefore, the $S_{n,\sigma}(\Omega)$ problem is the $Y_{n,\sigma}(\Omega)$ problem with an additional negative term, and it is this term that for $n\geq 4\sigma$ lowers the value of the infimum and produces a minimizer. This fact was first observed by Brézis-Nirenberg [3]. The difference between the $S_{n,\sigma}(\Omega)$ and $Y_{n,\sigma}(\Omega)$ problems is also related to the difference between the regional fractional Laplacian and the "full" fractional Laplacian on $\mathbb{R}^n$, and in turn by the nonlocal Hardy-type term's dependence on Ω.

As mentioned earlier, it was proved in [9] that if $n\geq 4\sigma$, $1/2<\sigma<1$, and Ω is a smooth bounded domain such that $B^+_\varepsilon\subset\Omega\subset\mathbb{R}^n_+$ for some $\varepsilon>0$, then $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n_+)$, and consequently, $S_{n,\sigma}(\Omega)$ is achieved. The assumption that $B^+_\varepsilon\subset\Omega$ means that the boundary of Ω near the origin is flat. In this paper, we would like explore the strict inequality $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n_+)$ for non-flat domains.

Theorem 1.1. *Let $n \geq 4$, $\frac{1}{2} < \sigma < 1$ and $\Omega \subset \mathbb{R}^n$ be an open set. Suppose there exists a point $a \in \partial\Omega$ such that $\partial\Omega$ is C^3 near the point a. Then there exist two positive constants c and C, both of which depend only on n, σ and Ω, such that*

$$S_{n,\sigma}(\Omega) \leq S_{n,\sigma}(\mathbb{R}^n_+) - \frac{c\Gamma_0 H(a)}{\lambda} + C\lambda^{-2\sigma}$$

for all large λ, where $H(a)$ is the mean curvature of $\partial\Omega$ at a, and

$$\Gamma_0 := \iint_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \frac{(\xi_n - \zeta_n)(|\xi'|^2 - |\zeta'|^2)|\Theta(\xi) - \Theta(\zeta)|^2}{|\xi - \zeta|^{n+2\sigma+2}} \mathrm{d}\xi \mathrm{d}\zeta < +\infty \tag{5}$$

with Θ being a minimizer of $S_{n,\sigma}(\mathbb{R}^n_+)$ that is radially symmetric in the first $n-1$ variables. In particular, $S_{n,\sigma}(\Omega) \leq S_{n,\sigma}(\mathbb{R}^n_+)$.

We do not know what the sign of Γ_0 is or whether it is zero, and we leave it as an open question. We do not have an explicit form of Θ. Some of its estimates are given in Proposition 2.2.

Since $S_{n,\sigma}(\Omega)$ is preserved under reflections, rotations, translations and dilations, we can assume that a is the origin 0. The smoothness condition assumed in Theorem 1.1 indicates that if the principal curvatures of $\partial\Omega$ at 0 are denoted as α_i ($i = 1, 2, ..., n-1$), then the boundary $\partial\Omega$ near the origin can be represented (up to rotating coordinates if necessary) by

$$x_n = h(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2 + g(x')|x'|^2, \tag{6}$$

where g is a bounded Lipschitz continuous function of the x' variables defined in a small ball in $\mathbb{R}^{n-1}$ such that $g(0) = 0$. To prove Theorem 1.1, we first flatten the boundary near the point a, and then we use a cut-off of a rescaled minimizer of $S_{n,\sigma}(\mathbb{R}^n_+)$ as a test function.

To prove the strict inequality $S_{n,\sigma}(\Omega) < S_{n,\sigma}(\mathbb{R}^n_+)$ without knowing the sign of Γ_0, we need a global smallness condition, that is, we need to assume that part of the boundary Ω is represented by the function in (6) with small α_i, and Ω is above its graph.

Theorem 1.2. *Let $n \geq 4$, $\frac{1}{2} < \sigma < 1$, $\alpha_1, \cdots, \alpha_{n-1}$ be real numbers, g be a locally Lipschitz continuous function on $\mathbb{R}^{n-1}$ such that $g(0) = 0$, h be defined as in (6), and*

$$\mathscr{R} := \{x = (x', x_n) \in \mathbb{R}^n : x_n > h(x')\}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set such that for some $\delta_0 > 0$ and $R_0 > 0$,

$$(B_{\delta_0} \cap \mathscr{R}) \subset \Omega \subset (\{x \in \mathbb{R}^n : |x'| < R_0\} \cap \mathscr{R}).$$

Then there exists a positive constant ε_0 depending only on n, σ, R_0 and δ_0 such that if

$$|\nabla_{x'} g(x')| \leq \varepsilon_0 \text{ for every } |x'| < R_0, \text{ and } |\alpha_i| \leq \varepsilon_0 \text{ for every } i = 1, \cdots, n-1,$$

then

$$S_{n,\sigma}(\Omega) < S_{n,\sigma}(\mathbb{R}^n_+).$$

An important intermediate step in the proof of Theorem 1.1 is that the minimizers of $S_{n,\sigma}(\mathbb{R}^n_+)$ are radially symmetric in the first $n-1$ variables.

Theorem 1.3. *Assume that $n \geq 2$, $1/2 < \sigma < 1$ and $u \in \mathring{H}^\sigma(\mathbb{R}^n_+)$ is a minimizer of $S_{n,\sigma}(\mathbb{R}^n_+)$. Then u must be radially symmetric about some point in $\mathbb{R}^{n-1}$ for the first $n-1$ variables.*

In fact, this symmetry holds not only for the minimizers of $S_{n,\sigma}(\mathbb{R}^n_+)$, but also for the solutions of its Euler-Lagrange equation.

Theorem 1.4. *Assume that $n \geq 2$, $1/2 < \sigma < 1$ and $u \in \mathring{H}^\sigma(\mathbb{R}^n_+)$ is a nonnegative solution of*

$$(-\Delta)^\sigma_{\mathbb{R}^n_+} u = u^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in } \mathbb{R}^n_+, \tag{7}$$

then u must be radially symmetric about some point in $\mathbb{R}^{n-1}$ for the first $n-1$ variables.

To prove Theorem 1.4, we use the method of moving planes for the regional fractional Laplacian. In this step, we adapt ideas in [4] for the full fractional Laplacian $(-\Delta)^\sigma$ to the regional fractional Laplacian $(-\Delta)^\sigma_{\mathbb{R}^n_+}$ in our case. Although Theorem 1.3 follows from Theorem 1.4, we also provide a proof using the rearrangement arguments, which are of independent interests.

This paper is organized as follows. In Section 2, we prove the radial symmetry in Theorem 1.3 and Theorem 1.4. In Section 3, we show the properties of the sharp constant $S_{n,\sigma}(\Omega)$ stated in Theorems 1.1 and 1.2. In the Appendix A, we include the technical calculations for some quantitative integrals of the minimizers Θ of $S_{n,\sigma}(\mathbb{R}^n_+)$.

2 Radial symmetry

Let $n \geq 2$. If u is a function on $\mathbb{R}^n_+$ and such that for a.e. $x_n \in \mathbb{R}_+$ and every $\lambda > 0$ one has $|\{x' \in \mathbb{R}^{n-1} : |u(x', x_n)| > \lambda\}| < \infty$, where $|\cdot|$ denotes the Lebesgue measure, then we define its rearrangement

$$u^\sharp(x', x_n) := u(\cdot, x_n)^*(x') \,.$$

Here $*$ denotes symmetric decreasing rearrangement in $\mathbb{R}^{n-1}$.

Proposition 2.1. *Let $n \geq 2$ and $\sigma \in (0,1)$. Then for any $u \in \mathring{H}^\sigma(\mathbb{R}^n_+)$, one has*

$$I_{n,\sigma,\mathbb{R}^n_+}[u] \geq I_{n,\sigma,\mathbb{R}^n_+}[u^\sharp] \,.$$

If the equality holds, then there is an $a' \in \mathbb{R}^{n-1}$ such that either

$$u(x', x_n) = u^\sharp(x' - a', x_n) \qquad \text{for a.e. } (x', x_n) \in \mathbb{R}^n_+$$

or

$$u(x', x_n) = -u^\sharp(x' - a', x_n) \qquad \text{for a.e. } (x', x_n) \in \mathbb{R}^n_+ \,.$$

Proof of Theorem 1.3. By the equimeasurability property of symmetric decreasing rearrangment in $\mathbb{R}^{n-1}$ we have $\int_{\mathbb{R}^{n-1}} (u^\sharp(x', x_n))^p \, dx' = \int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p \, dx'$ for any $p > 0$. Thus, as a consequence of Proposition 2.1, we infer that any minimizer u of $S_{n,\sigma}(\mathbb{R}^n_+)$ satisfies either $u(x', x_n) = u^*(x' - a', x_n)$ for a.e. $(x', x_n) \in \mathbb{R}^n_+$ or $u(x', x_n) = -u^*(x' - a', x_n)$ for a.e. $(x', x_n) \in \mathbb{R}^n_+$, for some $a' \in \mathbb{R}^{n-1}$. $\square$

For the proof of the inequality in Proposition 2.1, we use an argument due to Almgren-Lieb [1]. To characterize the cases of equality, we use a strengthening of this argument due to Frank-Seiringer [10].

Proof of Proposition 2.1. We write

$$I_{n,\sigma,\mathbb{R}^n_+}[u] = \iint_{\mathbb{R}_+\times\mathbb{R}_+} J_{|x_n-y_n|}[u(\cdot,x_n),u(\cdot,y_n)]\,\mathrm{d}x_n\,\mathrm{d}y_n$$

with

$$J_r[f,g] := \iint_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} \frac{(f(x')-g(y'))^2}{(|x'-y'|^2+r^2)^{\frac{n+2\sigma}{2}}}\,\mathrm{d}x'\,\mathrm{d}y'\,.$$

Note that when $r>0$, the kernel $(|z'|^2+r^2)^{-\frac{n+2\sigma}{2}}$ is integrable. Therefore, we can expand the square $(f(x')-f(y'))^2$ and, in the "diagonal terms" perform one of the integrals, which leads to the square of the L^2-norms of f and g. Since these norms coincide with those of f^* and g^*, we obtain

$$J_r[f,g]-J_r[f^*,g^*] = 2\iint_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} \frac{f^*(x')\,g^*(y')-f(x')g(y')}{(|x'-y'|^2+r^2)^{\frac{n+2\sigma}{2}}}\,\mathrm{d}x'\,\mathrm{d}y'\,.$$

By the Riesz rearrangement inequality (see, e.g., Theorem 3.7 in Lieb–Loss [14]),

$$J_r[f,g]-J_r[f^*,g^*]\geq 0\,.$$

Inserting this with $f=u(\cdot,x_n)$ and $g=u(\cdot,y_n)$ into the above formula we obtain $I_{n,\sigma,\mathbb{R}^n_+}[u]\geq I_{n,\sigma,\mathbb{R}^n_+}[u^{\#}]$, as claimed.

In the above argument, we use the square integrability of $u(\cdot,x_n)$ for a.e. x_n, which is not a priori clear. We can argue more carefully as follows. We first observe that $(u(x)-u(y))^2\geq(|u(x)|-u(y)|)^2$, so $I_{n,\sigma,\mathbb{R}^n_+}[u]\geq I_{n,\sigma,\mathbb{R}^n_+}[|u|]$. Now we apply the above argument to $\min\{(|u|-\epsilon)_+,M\}$ with two positive constants ϵ and M, which belongs to L^∞ and has support on a set of finite measure, so is in L^2. So for this cut off function we have the claimed inequality and then we can remove the cut-offs by applying the monotone convergence theorem.

Now assume that we have the equality $I_{n,\sigma,\mathbb{R}^n_+}[u]=I_{n,\sigma,\mathbb{R}^n_+}[u^{\#}]$. Then we also must have the equality $I_{n,\sigma,\mathbb{R}^n_+}[u]=I_{n,\sigma,\mathbb{R}^n_+}[|u|]$ and, by the above argument we easily see that either $u(x)=|u(x)|$ for a.e. $x\in\mathbb{R}^n_+$ or $u(x)=-|u(x)|$ for a.e. $x\in\mathbb{R}^n_+$. Next, the equality $I_{n,\sigma,\mathbb{R}^n_+}[|u|]=I_{n,\sigma,\mathbb{R}^n_+}[u^{\#}]$ implies that for a.e. $(x_n,y_n)\in\mathbb{R}_+\times\mathbb{R}_+$,

$$J_{|x_n-y_n|}[|u(\cdot,x_n)|,|u(\cdot,y_n)|] = J_{|x_n-y_n|}[u^\sharp(\cdot,x_n),u^\sharp(\cdot,y_n)]\,.$$

Thus, by Lieb's theorem (see, e.g., Theorem 3.9 in Lieb–Loss [14]) for a.e. $(x_n,y_n)\in\mathbb{R}_+\times\mathbb{R}_+$ there is an $a'(x_n,y_n)\in\mathbb{R}^{n-1}$ such that

$$|u(x',x_n)| = u^\sharp(x'-a'(x_n,y_n),x_n) \qquad\text{and}\qquad |u(y',y_n)| = u^\sharp(y'-a'(x_n,y_n),y_n)$$

for a.e. $x',y'\in\mathbb{R}^{n-1}$. Since the left hand side in the first equation is independent of y_n and in the second one of x_n, we deduce that $a'(x_n,y_n)$ is independent of x_n and y_n, that is, it is a constant $a'\in\mathbb{R}^{n-1}$. This implies the assertion of the proposition. $\square$

Next, we will prove Theorem 1.4 using the method of moving planes.

Proposition 2.2. *Let $n\geq 2$ and $1/2<\sigma<1$. Let $0\not\equiv u\in\mathring{H}^\sigma(\mathbb{R}^n_+)$ be non-negative and satisfy* (7). *Then $u\in C^{2\sigma-1}_{loc}(\overline{\mathbb{R}}^n_+)\cap C^\infty(\mathbb{R}^n_+)$, and there are constants $0<c\leq C<+\infty$ (depending on u) such that*

$$c\frac{x_n^{2\sigma-1}}{(1+|x|)^{n+2\sigma-2}}\leq u(x)\leq C\frac{x_n^{2\sigma-1}}{(1+|x|)^{n+2\sigma-2}}. \tag{8}$$

Furthermore, $x_n^{1-2\sigma}u(x) \in C^1(\overline{\mathbb{R}}^n_+)$ and there is a constant $\widetilde{C} > 0$ (depending on u) such that

$$|\nabla(x_n^{1-2\sigma}u(x))| \le \frac{\widetilde{C}}{(1+|x|)^{n+2\sigma-1}}. \tag{9}$$

for $x \in \mathbb{R}^n_+$.

Proof. The estimate (8) was proved in Proposition 1.5 in [9]. The estimate (9) for $|x| \le 1$ follows from (8) and the regularity that $x_n^{1-2\sigma}u(x) \in C^1(\overline{\mathbb{R}}^n_+)$ proved in Fall-Ros-Oton [7]. Let

$$\tilde{u}(x) = |x|^{2\sigma-n}u\left(\frac{x}{|x|^2}\right).$$

Then $\tilde{u}$ satisfies (7) as well. Thus, $\tilde{u}$ satisfies (8) and $|\nabla(x_n^{1-2\sigma}\tilde{u}(x))| \le C$ in $\overline{B}_1^+$ for some $C > 0$ (depending on u). Since

$$u(x) = |x|^{2\sigma-n}\tilde{u}\left(\frac{x}{|x|^2}\right)$$

as well, we have

$$x_n^{1-2\sigma}u(x) = |x|^{2-2\sigma-n}y_n^{1-2\sigma}\tilde{u}\,(y)\,, \quad \text{where} \quad y = \frac{x}{|x|^2}.$$

The estimate (9) for $|x| \ge 1$ follows from that $|y_n^{1-2\sigma}\tilde{u}(y)| + |\nabla(y_n^{1-2\sigma}\tilde{u}(y))| \le C$ in $\overline{B}_1^+$. □

Proof of Theorem 1.4. For $\lambda \in \mathbb{R}$ we define

$$T_\lambda = \left\{x \in \mathbb{R}^n_+ : x_1 = \lambda\right\}, \quad x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$$
$$u_\lambda(x) = u(x^\lambda), \quad w_\lambda(x) = u_\lambda(x) - u(x)$$

and

$$\Sigma_\lambda = \left\{x \in \mathbb{R}^n_+ : x_1 < \lambda\right\}, \quad \widetilde{\Sigma}_\lambda = \left\{x^\lambda : x \in \Sigma_\lambda\right\}.$$

By Proposition 2.2, we have $\lim\limits_{|x|\to\infty,\, x\in\mathbb{R}^n_+} \omega_\lambda(x) = 0$ for any fixed λ. Hence, if ω_λ is negative somewhere in Σ_λ, then the minimum of ω_λ in $\overline{\Sigma}_\lambda$ would be attained in Σ_λ. Let

$$\Sigma_\lambda^- = \left\{x \in \Sigma_\lambda : \omega_\lambda(x) < 0\right\}.$$

Then for $x \in \Sigma_\lambda^-$, we have

$$\begin{aligned}(-\Delta)^\sigma_{\mathbb{R}^n_+} w_\lambda(x) &= u_\lambda(x)^{\frac{n+2\sigma}{n-2\sigma}} - u(x)^{\frac{n+2\sigma}{n-2\sigma}} \\ &= \frac{n+2\sigma}{n-2\sigma}\left(\int_0^1 (tu_\lambda(x) + (1-t)u(x))^{\frac{4\sigma}{n-2\sigma}}\ \mathrm{d}t\right) w_\lambda(x) \\ &\ge \frac{n+2\sigma}{n-2\sigma} u(x)^{\frac{4\sigma}{n-2\sigma}} w_\lambda(x).\end{aligned}$$

That is

$$(-\Delta)^\sigma_{\mathbb{R}^n_+} w_\lambda(x) + c(x)\omega_\lambda(x) \ge 0 \quad \text{in } \Sigma_\lambda^-, \tag{10}$$

where

$$c(x) := -\frac{n+2\sigma}{n-2\sigma} u(x)^{\frac{4\sigma}{n-2\sigma}}.$$

Also, by Proposition 2.2, we have

$$|x|^{2\sigma}|c(x)| \le \frac{C}{|x|^{\frac{4\sigma^2+2n\sigma-4\sigma}{n-2\sigma}}},$$

and thus,

$$\liminf_{|x|\to\infty,\, x\in\mathbb{R}^n_+} |x|^{2\sigma}c(x) = 0.$$

Let $x^0 \in \Sigma_\lambda$ be such that $w(x^0) = \min_{\overline{\Sigma}_\lambda} w < 0$. Then

$$\begin{aligned}
(-\Delta)^\sigma_{\mathbb{R}^n_+} w_\lambda(x^0) &= 2P.V. \int_{\mathbb{R}^n_+} \frac{w_\lambda(x^0) - w_\lambda(y)}{|x^0 - y|^{n+2\sigma}} dy \\
&= 2P.V. \left\{ \int_{\Sigma_\lambda} \frac{w_\lambda(x^0) - w_\lambda(y)}{|x^0 - y|^{n+2\sigma}} dy + \int_{\widetilde{\Sigma_\lambda}} \frac{w_\lambda(x^0) - w_\lambda(y)}{|x^0 - y|^{n+2\sigma}} dy \right\} \\
&= 2P.V. \left\{ \int_{\Sigma_\lambda} \frac{w_\lambda(x^0) - w_\lambda(y)}{|x^0 - y|^{n+2\sigma}} dy + \int_{\Sigma_\lambda} \frac{w_\lambda(x^0) - w_\lambda(y^\lambda)}{|x^0 - y^\lambda|^{n+2\sigma}} dy \right\} \\
&\le 2P.V. \left\{ \int_{\Sigma_\lambda} \frac{w_\lambda(x^0) - w_\lambda(y)}{|x^0 - y^\lambda|^{n+2\sigma}} dy + \int_{\Sigma_\lambda} \frac{w_\lambda(x^0) + w_\lambda(y)}{|x^0 - y^\lambda|^{n+2\sigma}} dy \right\} \\
&= 4 \int_{\Sigma_\lambda} \frac{w_\lambda(x^0)}{|x^0 - y^\lambda|^{n+2\sigma}} dy. \qquad (11)
\end{aligned}$$

Moreover, if $|x_0| > |\lambda|$ is sufficiently large, then

$$\begin{aligned}
\int_{\Sigma_\lambda} \frac{1}{|x^0 - y^\lambda|^{n+2\sigma}} dy &\ge \int_{\{y\in\widetilde{\Sigma}_\lambda : 2|x_0| \le |y-x^0| \le 3|x_0|} \frac{1}{|x^0 - y|^{n+2\sigma}} dy \\
&\ge \frac{m}{|x^0|^{2\sigma}},
\end{aligned}$$

where $m > 0$ is a constant. Together with (10), we obtain

$$0 \le (-\Delta)^\sigma_{\mathbb{R}^n_+} u(x^0) + c(x^0)u(x^0) \le \left[\frac{m}{|x^0|^{2\sigma}} + c(x^0) \right] u(x^0) < 0, \qquad (12)$$

which is a contradiction.

This proves that if λ is sufficiently negative, then

$$w_\lambda \ge 0 \quad \text{in } \Sigma_\lambda.$$

Therefore, we can define

$$\bar{\lambda} = \sup \{ \lambda \in \mathbb{R} : w_\mu(x) \ge 0, \quad \forall x \in \Sigma_\lambda, \mu \le \lambda \}.$$

If $\bar{\lambda} = +\infty$, then since $u(x) \to 0$ as $|x| \to \infty$, we have that $u \equiv 0$, which is a contradiction. Hence, $\bar{\lambda} < \infty$. We will prove in the below that $w_{\bar{\lambda}} \equiv 0$ in $\Sigma_{\bar{\lambda}}$.

We argue by contradiction that we suppose $w_{\bar{\lambda}} > 0$ at some point, and thus in some open subset of $\Sigma_{\bar{\lambda}}$.

Then $w_{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}}$, since otherwise, if there exists $z \in \Sigma_{\bar{\lambda}}$ such that $w_{\bar{\lambda}}(z) = 0$, then by the equation of $w_{\bar{\lambda}}$, it follows that

$$\begin{aligned} 0 = (-\Delta)^{\sigma}_{\mathbb{R}^n_+} w_{\bar{\lambda}}(z) &= 2P.V. \int_{\mathbb{R}^n_+} \frac{w_{\bar{\lambda}}(z) - w_{\bar{\lambda}}(y)}{|z-y|^{n+2\sigma}} dy \\ &= 2P.V. \left\{ \int_{\Sigma_{\bar{\lambda}}} \frac{0 - w_{\bar{\lambda}}(y)}{|z-y|^{n+2\sigma}} dy + \int_{\widetilde{\Sigma_{\bar{\lambda}}}} \frac{0 - w_{\bar{\lambda}}(y)}{|z-y|^{n+2\sigma}} dy \right\} \\ &= 2P.V. \int_{\Sigma_{\bar{\lambda}}} w_{\lambda}(y) \left(\frac{1}{|z-y^{\bar{\lambda}}|^{n+2\sigma}} - \frac{1}{|z-y|^{n+2\sigma}} \right) dy \\ &< 0, \end{aligned}$$

which is a contradiction.

Now, from (12), we have that there exists $R_0 > 0$ such that for every $\lambda \in [\bar{\lambda}, \bar{\lambda}+1]$,

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \setminus B_{R_0}.$$

Since we just proved that $w_{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}}$, by continuity, we have that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$w_\lambda > 0 \quad \text{in } \overline{B_{R_0} \cap \Sigma_{\bar{\lambda}-\varepsilon} \cap \{x_n > \varepsilon\}} \text{ for all } \lambda \in [\bar{\lambda}, \bar{\lambda}+\delta].$$

We are going to show that

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \text{ for all } \lambda \in [\bar{\lambda}, \bar{\lambda}+\delta] \tag{13}$$

if we choose ε and δ to be small enough. Suppose there exists $\bar{x}$ satisfying $\bar{x}_n \in (0, \varepsilon)$ or $\bar{x}_1 \in (\bar{\lambda}-\varepsilon, \lambda)$ such that

$$w_\lambda(\bar{x}) = \min_{\overline{\Sigma}_\lambda} w_\lambda < 0.$$

Then from (10) and (11), we have

$$\frac{n+2\sigma}{n-2\sigma} u(\bar{x})^{\frac{4\sigma}{n-2\sigma}} w_\lambda(\bar{x}) \leq (-\Delta)^{\sigma}_{\mathbb{R}^n_+} w_\lambda(\bar{x}) \leq 4 \int_{\Sigma_\lambda} \frac{2 w_\lambda(\bar{x})}{|\bar{x} - y^\lambda|^{n+2\sigma}} dy.$$

That is,

$$\int_{\Sigma_\lambda} \frac{1}{|\bar{x} - y^\lambda|^{n+2\sigma}} dy \leq C u(\bar{x})^{\frac{4\sigma}{n-2\sigma}} \leq C \bar{x}_n^{\frac{4\sigma(2\sigma-1)}{n-2\sigma}}. \tag{14}$$

If $\bar{x}_1 \in (\bar{\lambda}-\varepsilon, \lambda)$, then

$$\int_{\Sigma_\lambda} \frac{1}{|\bar{x} - y^\lambda|^{n+2\sigma}} dy \geq \frac{C}{(\varepsilon+\delta)^{2\sigma}} \to \infty \quad \text{as } \varepsilon + \delta \to 0,$$

contradicting to (14) since $\bar{x} \in B_{R_0}$.

If $\bar{x}_n \in (0, \varepsilon)$, then since $\bar{x} \in B_{R_0}$, we have

$$\int_{\Sigma_\lambda} \frac{1}{|\bar{x} - y^\lambda|^{n+2\sigma}} dy \geq C,$$

contradicting to (14) if ε is small.

This proves (13), which contradicts with the definition of $\bar{\lambda}$. Hence, we have proved that $w_{\bar{\lambda}} \equiv 0$ in $\Sigma_{\bar{\lambda}}$, that is, u is symmetric about the plane $T_{\bar{\lambda}}$ in $\mathbb{R}^n_+$. Since the x_1 direction can be chosen arbitrarily for the first $n-1$ variables, we have actually shown that u is radially symmetric with respect to some point in $\partial\mathbb{R}^n_+$ in the first $n-1$ variables. $\square$

Proposition 2.3. *Assume that $n \geq 4$, $1/2 < \sigma < 1$, $\lambda > 0$, and $\gamma > 0$ that $\gamma \neq 2\sigma$. Let Θ be a minimizer of $S_{n,\sigma}(\mathbb{R}^n_+)$ that is radially symmetric in the first $n-1$ variables. Then*

$$\iint_{B^+_\lambda \times B^+_\lambda} \frac{(|\xi'|^2 + |\zeta'|^2)^{\gamma/2}|\Theta(\xi) - \Theta(\zeta)|^2}{|\xi - \zeta|^{n+2\sigma}} d\xi d\zeta < C(n,\sigma,\gamma)(1 + \lambda^{\gamma-2\sigma}). \tag{15}$$

If $\gamma < 2\sigma$, then

$$\iint_{(\mathbb{R}^n_+ \times \mathbb{R}^n_+)\setminus(B^+_\lambda \times B^+_\lambda)} \frac{(|\xi'|^2 + |\zeta'|^2)^{\gamma/2}|\Theta(\xi) - \Theta(\zeta)|^2}{|\xi - \zeta|^{n+2\sigma}} \,\mathrm{d}\xi\mathrm{d}\zeta \leq C(n,\sigma)\lambda^{\gamma-2\sigma}. \tag{16}$$

The proof of this proposition is given in the Appendix A.

3 Sharp constants

Proof of Theorem 1.2. First of all, we observe that $S_{n,\sigma}(\Omega)$ is preserved under reflections, rotations, translations and dilations. Hence, we can assume $\delta_0 = 4$.

Let $\Phi : \Omega \to \mathbb{R}^n$ defined as

$$\xi = \Phi(x) = (x_1, \cdots, x_{n-1}, x_n - h(x')).$$

Let Θ be a minimizer of $S_{n,\sigma}(\mathbb{R}^n_+)$ that is radially symmetric in the first $n-1$ variables. For every $\lambda > 0$, we let

$$\Theta_\lambda(x) = \lambda^{\frac{n-2\sigma}{2}}\Theta(\lambda x).$$

Let η be a cut off function such that $\eta \in C^1(\mathbb{R}^n)$, $\eta \equiv 1$ in B_2, $0 \leq \eta \leq 1$ in B_3 and $\eta \equiv 0$ in B^c_3. Let $\theta_\lambda(x) = (\eta\Theta_\lambda)(x)$ and $v_\lambda = \theta_\lambda \circ \Phi(x)$. Then we have

$$\begin{aligned} I_{n,\sigma,\Omega}[v_\lambda] &= \iint_{\Omega\times\Omega} \frac{|v_\lambda(x) - v_\lambda(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\mathrm{d}y \\ &= \iint_{U\times U} \frac{|\theta_\lambda(\xi) - \theta_\lambda(\zeta)|^2}{\Big(|\xi' - \zeta'|^2 + [\xi_n + h(\xi') - \zeta_n - h(\zeta')]^2\Big)^{\frac{n+2\sigma}{2}}} d\xi d\zeta, \end{aligned} \tag{17}$$

where $U = \Phi(\Omega)$. We would like to analyze the denominator

$$\begin{aligned} A(\xi,\zeta) &= \Big(|\xi' - \zeta'|^2 + \big[\xi_n + h_(\xi') - \zeta_n - h(\zeta')\big]^2\Big)^{-\frac{n+2\sigma}{2}} \\ &= |\xi - \zeta|^{-(n+2\sigma)}[1 + B(\xi,\zeta) + C(\xi,\zeta) + D(\xi,\zeta)]^{-\frac{n+2\sigma}{2}}, \end{aligned} \tag{18}$$

where

$$
\begin{aligned}
B(\xi,\zeta) &= \frac{1}{|\xi-\zeta|^2}(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right),\\
C(\xi,\zeta) &= \frac{2(\xi_n-\zeta_n)}{|\xi-\zeta|^2}\left(g\left(\xi'\right)|\xi'|^2-g\left(\zeta'\right)|\zeta'|^2\right),\\
D(\xi,\zeta) &= \frac{(h(\xi')-h(\zeta'))^2}{|\xi-\zeta|^2}.
\end{aligned}
$$

We will show that each term in the above is sufficiently small so that we can have a Taylor expansion for $A(\xi,\zeta)$.

For $B(\xi,\zeta)$, since

$$
\begin{aligned}
\left|\sum_{i=1}^{n-1}\alpha_i(\xi_i^2-\zeta_i^2)\right| &\le \left(\sum_{i=1}^{n-1}\alpha_i^2(\xi_i+\zeta_i)^2\right)^{1/2}\left(\sum_{i=1}^{n-1}(\xi_i-\zeta_i)^2\right)^{1/2}\\
&\le 2\varepsilon_0(|\xi'|^2+|\zeta'|^2)^{1/2}|\xi'-\zeta'|,
\end{aligned}
$$

where we used $|\alpha_i|\le\varepsilon_0$ for every $i=1,\cdots,n-1$, then we have

$$
|B(\xi,\zeta)|\le\varepsilon_0(|\xi'|^2+|\zeta'|^2)^{1/2}. \tag{19}
$$

For $C(\xi,\zeta)$ and $D(\xi,\zeta)$, since

$$
\begin{aligned}
&\left|g\left(\xi'\right)|\xi'|^2-g\left(\zeta'\right)|\zeta'|^2\right|\\
&\le\left|g\left(\xi'\right)|\xi'|^2-g\left(\zeta'\right)|\xi'|^2\right|+\left|g\left(\zeta'\right)|\xi'|^2-g\left(\zeta'\right)|\zeta'|^2\right|\\
&\le\varepsilon_0|\xi'-\zeta'||\xi'|^2+\varepsilon_0|\zeta'|\left(|\xi'|+|\zeta'|\right)\cdot\left(|\xi'|-|\zeta'|\right)\\
&\le\left[\varepsilon_0|\xi'|^2+\varepsilon_0|\zeta'|(|\xi'|+|\zeta'|)\right]|\xi'-\zeta'|\\
&\le\frac{3\varepsilon_0(|\xi'|^2+|\zeta'|^2)}{2}\cdot|\xi'-\zeta'|,
\end{aligned}
$$

where we used $g(0)=0$ and $|\nabla_{x'}g(x')|\le\varepsilon_0$, then we have

$$
|C(\xi,\zeta)|\le\frac{3\varepsilon_0(|\xi'|^2+|\zeta'|^2)}{2} \tag{20}
$$

and

$$
\begin{aligned}
|D(\xi,\zeta)| &= \frac{(h(\xi')-h(\zeta'))^2}{|\xi-\zeta|^2}\\
&\le\frac{2\left(\frac{1}{2}\sum_{i=1}^{n-1}\left(\alpha_i\xi_i^2-\alpha_i\zeta_i^2\right)\right)^2}{|\xi-\zeta|^2}+\frac{2\left(g\left(\xi'\right)|\xi'|^2-g\left(\zeta'\right)|\zeta'|^2\right)^2}{|\xi-\zeta|^2}\\
&\le\frac{(|\xi'|^2+|\zeta'|^2)\left(\sum_{i=1}^{n-1}\alpha_i^2\right)|\xi'-\zeta'|^2}{|\xi-\zeta|^2}+\frac{2\left(\frac{3\varepsilon_0(|\xi'|^2+|\zeta'|^2)}{2}|\xi'-\zeta'|\right)^2}{|\xi-\zeta|^2}\\
&\le(n-1)\varepsilon_0^2(|\xi'|^2+|\zeta'|^2)+\frac{9\varepsilon_0^2}{2}(|\xi'|^2+|\zeta'|^2)^2.
\end{aligned} \tag{21}
$$

Hence, for every $(\xi,\zeta)\in U\times U$, which satisfies $|\xi'|<R_0$ and $|\zeta'|<R_0$, there holds

$$B(\xi,\zeta)+C(\xi,\zeta)+D(\xi,\zeta)\le 2R_0\varepsilon_0+3R_0^2\varepsilon_0+2(n-1)R_0^2\varepsilon_0^2+18R_0^4\varepsilon_0^2.$$

Thus, if we choose ε_0 to be sufficiently small, each of $B(\xi,\zeta), C(\xi,\zeta)$ and $D(\xi,\zeta)$ is small, so that we can have the Taylor expansion of $A(\xi,\zeta)$. To be more explicitly, first we can choose a proper ε_0 such that $|B(\xi,\zeta)|+|C(\xi,\zeta)|+|D(\xi,\zeta)|<\frac{1}{2}$ for all $(\xi,\zeta)\in U\times U$. Then we can choose a constant $A_1>0$ such that when $|a|<1/2$,

$$(1+a)^{-\frac{n+2\sigma}{2}}\le 1-\frac{n+2\sigma}{2}a+A_1a^2.$$

Therefore, if we denote $E(\xi,\zeta):=B(\xi,\zeta)+C(\xi,\zeta)+D(\xi,\zeta)$, then we have

$$\begin{aligned}
&A(\xi,\zeta)|\xi-\zeta|^{n+2\sigma}\\
&\le 1-\frac{(n+2\sigma)}{2}B(\xi,\zeta)-\frac{n+2\sigma}{2}C(\xi,\zeta)-\frac{n+2\sigma}{2}D(\xi,\zeta)+A_1E(\xi,\zeta)^2\\
&\le 1-\frac{(n+2\sigma)}{|\xi-\zeta|^2}(\xi_n-\zeta_n)\left(\frac{1}{2}\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\frac{1}{2}\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)+F(\xi,\zeta),
\end{aligned}\tag{22}$$

where

$$F(\xi,\zeta)=\frac{n+2\sigma}{2}|C(\xi,\zeta)|+\frac{n+2\sigma}{2}D(\xi,\zeta)+A_1E(\xi,\zeta)^2.$$

Therefore, it follows from (17), (18) and (22) that

$$\begin{aligned}
&I_{n,\sigma,\Omega_\mu}[v_\lambda]\\
&=\iint_{U\times U}\frac{|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{\left(|\xi'-\zeta'|^2+(\xi_n+h_\mu(\xi')-\zeta_n-h_\mu(\zeta'))^2\right)^{\frac{n+2\sigma}{2}}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&\le\iint_{U\times U}\frac{|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&\quad-\frac{(n+2\sigma)}{2}\iint_{U\times U}\frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&\quad+\iint_{U\times U}\frac{F(\xi,\zeta)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta.
\end{aligned}\tag{23}$$

We are going to estimate each term in the right hand side of (23).

We start with estimating the third term there. By using (19), (20) and (21), there exists a positive constant C which depends only on n, σ and R_0 such that

$$F(\xi,\zeta)\le C\varepsilon_0(|\xi'|^2+|\zeta'|^2)$$

for all $(\xi,\zeta)\in U\times U\subset B_{R_0}^+\times B_{R_0}^+$. Therefore,

$$\iint_{U\times U}\frac{F(\xi,\zeta)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta$$

$$
\begin{aligned}
&\leq \frac{C\varepsilon_0}{\lambda^2}\iint_{B^+_{\lambda R_0}\times B^+_{\lambda R_0}} \frac{(|\xi'|^2+|\zeta'|^2)|\eta(\lambda^{-1}\xi)\Theta(\xi)-\eta(\lambda^{-1}\zeta)\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{2C\varepsilon_0}{\lambda^2}\iint_{B^+_{\lambda R_0}\times B^+_{\lambda R_0}} \frac{|\zeta'|^2|\eta(\lambda^{-1}\xi)\Theta(\xi)-\eta(\lambda^{-1}\zeta)\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&|\eta(\lambda^{-1}\xi)\Theta(\xi)-\eta(\lambda^{-1}\zeta)\Theta(\zeta)|^2\\
&\leq 2|\eta(\lambda^{-1}\xi)|^2|\Theta(\xi)-\Theta(\zeta)|^2+2|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2|\Theta(\zeta)|^2\\
&\leq 2|\Theta(\xi)-\Theta(\zeta)|^2+2|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2|\Theta(\zeta)|^2.
\end{aligned}\tag{24}
$$

Since for every $\zeta\in\mathbb{R}^n_+$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}\frac{|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\\
&\leq \int_{\{|\xi-\zeta|<\lambda\}}\frac{C}{\lambda^2|\xi-\zeta|^{n+2\sigma-2}}\,\mathrm{d}\xi+\int_{\{|\xi-\zeta|\geq\lambda\}}\frac{4}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\\
&\leq \frac{C}{\lambda^{2\sigma}},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\iint_{B^+_{\lambda R_0}\times B^+_{\lambda R_0}}\frac{|\zeta'|^2|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2|\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma-2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq C\int_{B^+_{\lambda R_0}}|\zeta'|^2|\Theta(\zeta)|^2\int_{\mathbb{R}^n_+}\frac{|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\,\mathrm{d}\zeta\\
&\leq C\int_{B^+_{\lambda R_0}}|\zeta'|^2|\Theta(\zeta)|^2\,\mathrm{d}\zeta\\
&\leq C(1+\lambda^{4-n})\lambda^{-2\sigma}\\
&\leq C,
\end{aligned}
$$

where we used Proposition 2.2 and $n\geq 3$. Then by using (15), we obtain that

$$
\iint_{U\times U}\frac{F(\xi,\zeta)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}d\xi d\zeta\leq\frac{C\varepsilon_0}{\lambda^{2\sigma}}.\tag{25}
$$

Next, we estimate the second term in the right hand side of (23). For every $i=1,\cdots,n-1$, we have

$$
\begin{aligned}
&\iint_{U\times U}\frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&=\iint_{B_1^+\times B_1^+}\frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\Theta_\lambda(\xi)-\Theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\iint_{(U\times U)\setminus(B_1^+\times B_1^+)} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
=&\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\Theta_\lambda(\xi)-\Theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&-\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B_1^+\times B_1^+)} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\Theta_\lambda(\xi)-\Theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&+\iint_{(U\times U)\setminus(B_1^+\times B_1^+)} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta. \qquad (26)
\end{aligned}
$$

Using Theorem 1.4, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\Theta_\lambda(\xi)-\Theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&=\frac{\sum_{i=1}^{n-1}\alpha_i}{n-1}\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{(\xi_n-\zeta_n)\left(|\xi'|^2-|\zeta'|^2\right)|\Theta_\lambda(\xi)-\Theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&=\frac{H\Gamma_0}{\lambda}, \qquad (27)
\end{aligned}
$$

where Γ_0 is given in (5), and

$$
H=\frac{1}{n-1}\sum_{i=1}^{n-1}\alpha_i \quad \text{is the mean curvature.}
$$

Since

$$
\frac{|\xi_n-\zeta_n|\left|\xi_i^2-\zeta_i^2\right|}{|\xi-\zeta|^{n+2\sigma+2}}\le\frac{(|\xi_i|+|\zeta_i|)}{2|\xi-\zeta|^{n+2\sigma}},
$$

we have

$$
\begin{aligned}
&\left|\lambda\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B_1^+\times B_1^+)} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\Theta_\lambda(\xi)-\Theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\right|\\
&=\left|\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B_\lambda^+\times B_\lambda^+)} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\right|\\
&\le\frac{\varepsilon_0}{2}\sum_{i=1}^{n-1}\left|\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B_\lambda^+\times B_\lambda^+)} \frac{(|\xi_i|+|\zeta_i|)|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta\right|\\
&\le C\varepsilon_0\lambda^{1-2\sigma}, \qquad (28)
\end{aligned}
$$

where we used (16) in the last inequality, and

$$
\left|\lambda\iint_{(U\times U)\setminus(B_1^+\times B_1^+)} \frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\right|
$$

$$
\begin{aligned}
&\le \frac{\varepsilon_0}{2}\sum_{i=1}^{n-1}\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B^+_\lambda\times B^+_\lambda)} \frac{(|\xi_i|+|\zeta_i|)|\eta(\lambda^{-1}\xi)\Theta(\xi)-\eta(\lambda^{-1}\zeta)\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&= \frac{\varepsilon_0}{2}\sum_{i=1}^{n-1}\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B^+_\lambda\times B^+_\lambda)} \frac{|\zeta_i||\eta(\lambda^{-1}\xi)\Theta(\xi)-\eta(\lambda^{-1}\zeta)\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&\le \frac{\varepsilon_0}{2}\sum_{i=1}^{n-1}\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B^+_\lambda\times B^+_\lambda)} \frac{2|\zeta_i||\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta\\
&\quad+\frac{\varepsilon_0}{2}\sum_{i=1}^{n-1}\iint_{(\mathbb{R}^n_+\times\mathbb{R}^n_+)\setminus(B^+_\lambda\times B^+_\lambda)} \frac{2|\zeta_i||\Theta(\zeta)|^2|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta, \qquad (29)
\end{aligned}
$$

where we used (24) in the last inequality. Since

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+\setminus B^+_\lambda}|\zeta_i||\Theta(\zeta)|^2\,\mathrm{d}\zeta\int_{\mathbb{R}^n}\frac{|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\\
&=\int_{B^+_{4\lambda}\setminus B^+_\lambda}|\zeta_i||\Theta(\zeta)|^2\,\mathrm{d}\zeta\int_{\mathbb{R}^n}\frac{|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\\
&\quad+\int_{\mathbb{R}^n_+\setminus B^+_{4\lambda}}|\zeta_i||\Theta(\zeta)|^2\,\mathrm{d}\zeta\int_{B^+_{3\lambda}}\frac{1}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\\
&\le C\lambda^{3-n-2\sigma}+C\lambda^n\int_{\mathbb{R}^n_+\setminus B^+_{4\lambda}}|\zeta|^{1-n-2\sigma}|\Theta(\zeta)|^2\,\mathrm{d}\zeta\\
&\le C\lambda^{3-n-2\sigma}\\
&\le C\lambda^{1-2\sigma},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{B^+_\lambda}|\zeta_i||\Theta(\zeta)|^2\,\mathrm{d}\zeta\int_{\mathbb{R}^n_+\setminus B^+_\lambda}\frac{|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\\
&=\int_{B^+_\lambda}|\zeta_i||\Theta(\zeta)|^2\,\mathrm{d}\zeta\int_{\mathbb{R}^n_+\setminus B^+_{2\lambda}}\frac{|\eta(\lambda^{-1}\xi)-\eta(\lambda^{-1}\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\\
&\le C\lambda^{-2\sigma}\int_{B^+_\lambda}|\zeta_i||\Theta(\zeta)|^2\,\mathrm{d}\zeta\\
&\le C\lambda^{-2\sigma}(1+\lambda^{3-n})\\
&\le C\lambda^{1-2\sigma},
\end{aligned}
$$

we obtain from (29) that

$$
\begin{aligned}
&\left|\lambda\iint_{(U\times U)\setminus(B^+_1\times B^+_1)}\frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta\right|\\
&\le C\varepsilon_0\lambda^{1-2\sigma}. \qquad (30)
\end{aligned}
$$

Therefore, it follows from (26), (27), (28) and (30) that we obtain the estimate for the second term in the right hand side of (23):

$$\left|\iint_{U\times U}\frac{(\xi_n-\zeta_n)\left(\sum_{i=1}^{n-1}\alpha_i\xi_i^2-\sum_{i=1}^{n-1}\alpha_i\zeta_i^2\right)|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma+2}}\,\mathrm{d}\xi\mathrm{d}\zeta-\frac{H\Gamma_0}{\lambda}\right|$$
$$\le C\varepsilon_0\lambda^{-2\sigma}. \tag{31}$$

Combining (23), (25) and (31), we have

$$I_{n,\sigma,\Omega}[v_\lambda]\le\iint_{U\times U}\frac{|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\mathrm{d}\xi\mathrm{d}\zeta-\frac{(n+2\sigma)H\Gamma_0}{2\lambda}+\frac{C\varepsilon_0}{\lambda^{2\sigma}}. \tag{32}$$

From the proof of Theorem 1.3 in [9], we have

$$\iint_{U\times U}\frac{|\theta_\lambda(\xi)-\theta_\lambda(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\xi\mathrm{d}\zeta\le S_{n,\sigma}(\mathbb{R}^n_+)-c\lambda^{-2\sigma}+C\lambda^{-n-2\sigma+2} \tag{33}$$

$$\int_\Omega|v_\lambda|^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\ge\int_{B_4}|\theta_\lambda|^{\frac{2n}{n-2\sigma}}\,\mathrm{d}\xi\ge 1-c\lambda^{-\frac{n(n+2\sigma-2)}{n-2\sigma}}, \tag{34}$$

where c and C are positive constants depending only on n and σ. Hence, we have from (32), (33) and (34) that

$$\frac{I_{n,\sigma,\Omega}[v_\lambda]}{\left(\int_\Omega|v_\lambda|^{\frac{2n}{n-2\sigma}}dx\right)^{\frac{n-2\sigma}{n}}}$$
$$\le\left(1+C\lambda^{-\frac{n(n+2\sigma-2)}{n-2\sigma}}\right)\cdot\left(S_{n,\sigma}(\mathbb{R}^n_+)-\frac{(n+2\sigma)H\Gamma_0}{2\lambda}-(c-C\varepsilon_0)\lambda^{-2\sigma}+C\lambda^{-n-2\sigma+2}\right)$$
$$\le S_{n,\sigma}(\mathbb{R}^n_+)-\frac{(n+2\sigma)H\Gamma_0}{2\lambda}-(c-C\varepsilon_0)\lambda^{-2\sigma}+C\lambda^{-n+1}. \tag{35}$$

Without knowing the sign of Γ_0, we use the crude estimate that $|H|\le\varepsilon_0$. Therefore,

$$\frac{I_{n,\sigma,\Omega}[v_\lambda]}{\left(\int_\Omega|v_\lambda|^{\frac{2n}{n-2\sigma}}dx\right)^{\frac{n-2\sigma}{n}}}\le S_{n,\sigma}(\mathbb{R}^n_+)-\frac{(n+2\sigma)\Gamma_0\varepsilon_0}{2\lambda}-(c-C\varepsilon_0)\lambda^{-2\sigma}+C\lambda^{-n+1}$$

By choosing λ large and then choosing ε_0 small, we obtain $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n_+)$. □

Proof of Theorem 1.1. As mentioned earlier, we can assume $a=0$ and there exists $\delta_0>0$ such that $\partial\Omega\cap B_{\delta_0}$ can be represented by (6) after a necessary coordinate rotation. Since the sharp constant $S_{n,\sigma}(\Omega)$ does not change under dilations, we can have a dilation of Ω with a sufficiently large number μ. The domain after dilation is denoted as

$$\Omega_\mu:=\{\mu x:x\in\Omega\}.$$

Then the boundary $\partial\Omega_\mu\cap B_{\mu\delta_0}$ is presented by

$$x_n=h_\mu(x'):=\frac{1}{2\mu}\sum_{i=1}^{n-1}\alpha_i x_i^2+\frac{1}{\mu}g\left(\frac{1}{\mu}x'\right)|x'|^2.$$

Choose μ large so that $(B_8 \cap \mathscr{R}_\mu) \subset \Omega_\mu$, where $\mathscr{R}_\mu := \{x \in \mathbb{R}^n : x_n > h_\mu(x')\}$.

Let $\Phi_\mu : \Omega_\mu \to \mathbb{R}^n$ defined as $\xi = \Phi_\mu(x) = (x_1, \cdots, x_{n-1}, x_n - h_\mu(x'))$. Let $\Theta(x)$ be a minimizer of $S_{n,\sigma}(\mathbb{R}^n_+)$ that is radially symmetric in the first $n-1$ variables as before, and $\Theta_\lambda(x) = \lambda^{\frac{n-2\sigma}{2}}\Theta(\lambda x)$ for $\lambda > 0$. Let η be a cut off function such that $\eta \in C^1(\mathbb{R}^n)$, $\eta \equiv 1$ in B_2, $0 \le \eta \le 1$ in B_3 and $\eta \equiv 0$ in B_3^c. Let $\theta_\lambda(x) = (\eta\Theta_\lambda)(x)$ and $v_\lambda = \theta_\lambda \circ \Phi_\mu(x)$. Then we have

$$\begin{aligned} I_{n,\sigma,\Omega_\mu}[v_\lambda] &= \iint_{\Omega_\mu \times \Omega_\mu} \frac{|v_\lambda(x) - v_\lambda(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\mathrm{d}y \\ &= \iint_{(\Omega_\mu \cap B_8)\times(\Omega_\mu \cap B_8)} \frac{|v_\lambda(x) - v_\lambda(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\mathrm{d}y \\ &\quad + 2\int_{\Omega_\mu \cap B_8} |v_\lambda(x)|^2 \int_{\Omega_\mu \setminus B_8} \frac{\mathrm{d}y}{|x-y|^{n+2\sigma}} \,\mathrm{d}x. \end{aligned}$$

Since

$$\begin{aligned} &\int_{\Omega_\mu \cap B_8} |v_\lambda(x)|^2 \int_{\Omega_\mu \setminus B_8} \frac{\mathrm{d}y}{|x-y|^{n+2\sigma}} \,\mathrm{d}x \\ &= 2\int_{\Omega_\mu \cap B_4} |v_\lambda(x)|^2 \int_{\Omega_\mu \setminus B_8} \frac{\mathrm{d}y}{|x-y|^{n+2\sigma}} \,\mathrm{d}x \\ &\le \frac{C(n,\sigma)}{\lambda^{2\sigma}} \int_{\mathbb{R}^n_+} |\Theta(x)|^2 dx \\ &\le \frac{C(n,\sigma)}{\lambda^{2\sigma}}, \end{aligned}$$

where we used Proposition 2.2 in the last inequality, it follows from (32) and (33) that

$$I_{n,\sigma,\Omega_\mu}[v_\lambda] \le S_{n,\sigma}(\mathbb{R}^n_+) - \frac{(n+2\sigma)\Gamma_0 H}{2\mu\lambda} + C\lambda^{-2\sigma}.$$

This together with (34) shows that

$$S_{n,\sigma}(\Omega) \le S_{n,\sigma}(\mathbb{R}^n_+) - \frac{(n+2\sigma)\Gamma_0 H}{2\mu\lambda} + C\lambda^{-2\sigma}.$$

□

A Proof of Proposition 2.3

We first prove (15).

Proof of (15). We suppose $\lambda > 100$ is very large. Note that

$$\iint_{B^+_\lambda \times B^+_\lambda} \frac{(|\xi'|^\gamma + |\zeta'|^\gamma)|\Theta(\xi) - \Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}} \mathrm{d}\xi\mathrm{d}\zeta = 2\iint_{B^+_\lambda \times B^+_\lambda} \frac{|\xi'|^\gamma|\Theta(\xi) - \Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}} \mathrm{d}\xi\mathrm{d}\zeta.$$

First, it is clear that

$$\int_{B^+_{10}} |\xi'|^\gamma \,\mathrm{d}\xi \int_{B^+_\lambda} \frac{|\Theta(\xi) - \Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le 10^\gamma \iint_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \frac{|\Theta(\xi) - \Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}} \mathrm{d}\xi\mathrm{d}\zeta < \infty. \tag{36}$$

Secondly, using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\geq 1\}}\frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\\
&\leq C\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\Theta(\xi)^2\,\mathrm{d}\xi+\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\geq 1\}}\frac{2\Theta(\zeta)^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta. \qquad (37)
\end{aligned}
$$

We have

$$
\begin{aligned}
\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\geq 1,\ |\zeta|\leq|\xi|/2\}}\frac{2\Theta(\zeta)^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta &\leq C\int_{B_\lambda^+\setminus B_{10}^+}\frac{|\xi'|^\gamma}{|\xi|^{n+2\sigma}}\,\mathrm{d}\xi\\
&\leq C(1+\lambda^{\gamma-2\sigma}),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\geq 1,\ |\zeta|\geq|\xi|/2\}}\frac{2\Theta(\zeta)^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta &\leq C\int_{B_\lambda^+\setminus B_{10}^+}\frac{|\xi'|^\gamma}{|\xi|^{2n-2}}\,\mathrm{d}\xi\\
&< C(1+\lambda^{\gamma-2\sigma}),
\end{aligned}
$$

where we used (8) and $n\geq 4>2+2\sigma$. Hence, it follows from (37) that

$$
\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\geq 1\}}\frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\leq C(1+\lambda^{\gamma-2\sigma}). \qquad (38)
$$

Finally, if we denote $\widetilde{\Theta}(\xi)=\Theta(\xi)\xi_n^{1-2\sigma}$, then

$$
\begin{aligned}
&\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\leq 1\}}\frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\\
&=\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\leq 1\}}\frac{|\widetilde{\Theta}(\xi)\xi_n^{2\sigma-1}-\widetilde{\Theta}(\zeta)\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\\
&\leq\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\widetilde{\Theta}(\xi)^2\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\leq 1\}}\frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\\
&\quad+\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\leq 1\}}\frac{|\widetilde{\Theta}(\xi)-\widetilde{\Theta}(\zeta)|^2\zeta_n^{4\sigma-2}}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta. \qquad (39)
\end{aligned}
$$

Note that if $\xi_n\geq 3/2$, then

$$
\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\leq 1\}}\frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\leq C\xi_n^{2\sigma-2}<C.
$$

Now let us consider $\xi_n<3/2$. Then

$$
\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\leq\frac{\xi_n}{2}\}}\frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\leq C\xi_n^{2\sigma-2},
$$

$$
\int_{\{\zeta\in B_\lambda^+:\,\frac{\xi_n}{2}<|\zeta-\xi|\leq 1,\ \zeta_n<2\xi_n\}}\frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\leq C\xi_n^{4\sigma-2}\cdot\xi_n^{-2\sigma}=C\xi_n^{2\sigma-2},
$$

$$
\int_{\{\zeta\in B_\lambda^+:\,\frac{\xi_n}{2}<|\zeta-\xi|\leq 1,\ \zeta_n\geq 2\xi_n\}}\frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\leq C\int_{\{\zeta\in B_\lambda^+:\,|\zeta|\leq 1,\ \zeta_n\geq\xi_n\}}\frac{\zeta_n^{4\sigma-2}}{|\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\leq C\xi_n^{2\sigma-2}.
$$

Therefore,

$$\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\widetilde{\Theta}(\xi)^2\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\le 1\}}\frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\le C\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\xi_n^{2\sigma-2}\widetilde{\Theta}(\xi)^2\,\mathrm{d}\xi$$
$$\le C(1+\lambda^{\gamma-2\sigma}),$$

where we used (8) and $n\ge 4$ in the last inequality. Furthermore, using (9), we have

$$\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\le 1\}}\frac{|\widetilde{\Theta}(\xi)-\widetilde{\Theta}(\zeta)|^2\zeta_n^{4\sigma-2}}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\le C\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\frac{1}{|\xi|^{2n}}\,\mathrm{d}\xi$$
$$\le C(1+\lambda^{\gamma-2\sigma}).$$

Hence, it follows from (39) that

$$\int_{B_\lambda^+\setminus B_{10}^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in B_\lambda^+:\,|\zeta-\xi|\le 1\}}\frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\le C(1+\lambda^{\gamma-2\sigma}).$$

This, together with (36) and (38), proves (15). □

Next, we prove (16).

Proof of (16). We suppose $\lambda>100$ is very large. Note that

$$\iint_{(\mathbb{R}_+^n\times\mathbb{R}_+^n)\setminus(B_\lambda^+\times B_\lambda^+)}\frac{(|\xi'|^\gamma+|\zeta'|^\gamma)|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\mathrm{d}\xi\mathrm{d}\zeta$$
$$\le 4\iint_{(\mathbb{R}_+^n\setminus B_\lambda^+)\times\mathbb{R}_+^n}\frac{|\xi'|^\gamma|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\mathrm{d}\xi\mathrm{d}\zeta. \tag{40}$$

Again, using the Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}_+^n\setminus B_\lambda^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in\mathbb{R}_+^n:\,|\zeta-\xi|\ge 1\}}\frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta$$
$$\le C\int_{\mathbb{R}_+^n\setminus B_\lambda^+}|\xi'|^\gamma\Theta(\xi)^2\,\mathrm{d}\xi+\int_{\mathbb{R}^n\setminus B_\lambda^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in\mathbb{R}_+^n:\,|\zeta-\xi|\ge 1\}}\frac{2\Theta(\zeta)^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta. \tag{41}$$

Since $\gamma<2\sigma$, we have

$$\int_{\mathbb{R}_+^n\setminus B_\lambda^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in\mathbb{R}_+^n:\,|\zeta-\xi|\ge 1,\ |\zeta|\le|\xi|/2\}}\frac{2\Theta(\zeta)^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\le C\int_{\mathbb{R}_+^n\setminus B_\lambda^+}\frac{|\xi'|^\gamma}{|\xi|^{n+2\sigma}}\,\mathrm{d}\xi\le C\lambda^{\gamma-2\sigma},$$

and

$$\int_{\mathbb{R}_+^n\setminus B_\lambda^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in\mathbb{R}_+^n:\,|\zeta-\xi|\ge 1,\ |\zeta|\ge|\xi|/2\}}\frac{2\Theta(\zeta)^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\le C\int_{\mathbb{R}_+^n\setminus B_\lambda^+}\frac{|\xi'|^\gamma}{|\xi|^{2n-2}}\,\mathrm{d}\xi\le C\lambda^{\gamma-2\sigma},$$

where we used (8) and $n\ge 4>2+2\sigma$. Hence, it follows from (41) that

$$\int_{\mathbb{R}_+^n\setminus B_\lambda^+}|\xi'|^\gamma\,\mathrm{d}\xi\int_{\{\zeta\in\mathbb{R}_+^n:\,|\zeta-\xi|\ge 1\}}\frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}}\,\mathrm{d}\zeta\le C\lambda^{\gamma-2\sigma}. \tag{42}$$

Finally, if we denote $\widetilde{\Theta}(\xi) = \Theta(\xi)\xi_n^{1-2\sigma}$, then

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \,\mathrm{d}\xi \int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \\
&= \int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \,\mathrm{d}\xi \int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\widetilde{\Theta}(\xi)\xi_n^{2\sigma-1}-\widetilde{\Theta}(\zeta)\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \\
&\le \int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \widetilde{\Theta}(\xi)^2 \,\mathrm{d}\xi \int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \\
&\quad + \int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \,\mathrm{d}\xi \int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\widetilde{\Theta}(\xi)-\widetilde{\Theta}(\zeta)|^2\zeta_n^{4\sigma-2}}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta.
\end{aligned}
\tag{43}
$$

Note that if $\xi_n \ge 3/2$, then

$$
\int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le C\xi_n^{2\sigma-2} < C.
$$

Now let us consider $\xi_n < 3/2$. Then

$$
\begin{aligned}
&\int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le \frac{\xi_n}{2}\}} \frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le C\xi_n^{2\sigma-2}, \\
&\int_{\{\zeta\in\mathbb{R}^n_+ :\, \frac{\xi_n}{2}<|\zeta-\xi|\le 1,\, \zeta_n<2\xi_n\}} \frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le C\xi_n^{4\sigma-2}\cdot\xi_n^{-2\sigma} = C\xi_n^{2\sigma-2}, \\
&\int_{\{\zeta\in\mathbb{R}^n_+ :\, \frac{\xi_n}{2}<|\zeta-\xi|\le 1,\, \zeta_n\ge 2\xi_n\}} \frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le C\int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta|\le 1,\, \zeta_n\ge \xi_n\}} \frac{\zeta_n^{4\sigma-2}}{|\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le C\xi_n^{2\sigma-2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \widetilde{\Theta}(\xi)^2 \,\mathrm{d}\xi \int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\xi_n^{2\sigma-1}-\zeta_n^{2\sigma-1}|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta &\le C\int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \xi_n^{2\sigma-2}\widetilde{\Theta}(\xi)^2 \,\mathrm{d}\xi \\
&\le C\lambda^{\gamma-2\sigma},
\end{aligned}
$$

where we used (8) and $n \ge 4 > 4\sigma$ in the last inequality. Furthermore, using (9), we have

$$
\int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \,\mathrm{d}\xi \int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\widetilde{\Theta}(\xi)-\widetilde{\Theta}(\zeta)|^2\zeta_n^{4\sigma-2}}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le C\int_{\mathbb{R}^n_+ \setminus B^+_\lambda} |\xi'|^\gamma \frac{1}{|\xi|^{2n}} \,\mathrm{d}\xi \le C\lambda^{\gamma-2\sigma}.
$$

Hence, it follows from (43) that

$$
\int_{\mathbb{R}^n_+ \setminus B^+_{10}} |\xi'|^\gamma \,\mathrm{d}\xi \int_{\{\zeta\in\mathbb{R}^n_+ :\, |\zeta-\xi|\le 1\}} \frac{|\Theta(\xi)-\Theta(\zeta)|^2}{|\xi-\zeta|^{n+2\sigma}} \,\mathrm{d}\zeta \le C\lambda^{\gamma-2\sigma}.
$$

This, together with (40) and (42), proves (15). □

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