BMO-TYPE FUNCTIONALS RELATED TO THE TOTAL VARIATION AND CONNECTION TO DENOISING MODELS

SERENA GUARINO LO BIANCO AND ROBERTA SCHIATTARELLA

Dedicated to G.Buttazzo on the occasion of his 70'th birthday[1]

ABSTRACT. The purpose of this paper is to analyze the asymptotic behaviour in the spirit of Γ-convergence of BMO-type functionals related to the total variation of a function u. Moreover, we deal with a minimization problem coming from applications in image processing.

Keywords: Total variation, bounded variation, BMO, Γ-convergence.
2020 Mathematics Subject Classification: 49Q20, 49J45, 26B30, 26D10

1. INTRODUCTION

Recently the study of characterizations of Sobolev and bounded variation functions by certain BMO-type seminorms has emerged as an intriguing research area. In [6] the authors, inspired by the celebrated space of John and Nirenberg of bounded mean oscillation (BMO), introduced the space B of functions from the unit cube $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ such that the following seminorm is finite

$$[u]_B := \sup_{0<\varepsilon<1} [u]_\varepsilon, \quad \text{where } [u]_\varepsilon = \varepsilon^{n-1} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| u(x) - \fint_{Q'} u \right| dx.$$

Here $\mathcal{G}_\varepsilon$ denotes a collection of mutually disjoint ε-cubes Q' of the type $Q' = x + \varepsilon Q$ whose cardinality does not exceed ε^{1-n}. The function space B contains in particular BMO, the space of bounded variation functions BV and the fractional space $W^{\frac{1}{p},p}$ for $1 \leq p < \infty$.

Many variants have been considered in the last years. In particular when the family involves ε-cubes of general orientation (this is the *isotropic case*), it is proved in [17] and [12] that given an open set $\Omega \subset \mathbb{R}^n$ and a function $u \in SBV(\Omega)$, the space of special BV functions, then

$$\lim_{\varepsilon \to 0} \varepsilon^{n-1} \sup_{\mathcal{J}_\varepsilon} \sum_{Q' \in \mathcal{J}_\varepsilon} \fint_{Q'} \left| u(x) - \fint_{Q'} u \right| dx = \frac{1}{4} \int_\Omega |\nabla u|\, dx + \frac{1}{2} |D^s u|(\Omega)\,. \tag{1.1}$$

Here $\mathcal{J}_\varepsilon$ is a family of pairwise disjoint cubes of side length ε contained in Ω. By considering in (1.1) the characteristic function of a measurable set $A \subset \mathbb{R}^n$ a characterization of finite perimeter $P(A)$ is obtained (this was originally proved in [1]).

The *anisotropic* version of (1.1) was considered in [2] and [15] (see also [14]). Given D a bounded and connected open set with Lipschitz boundary and a function $u \in L^1(\Omega)$, for any $\varepsilon > 0$ we consider the following functional

$$H_\varepsilon(u, \Omega) = \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| u(x) - \fint_{D'} u \right| dx \tag{1.2}$$

where $\mathcal{H}_\varepsilon$ is a family of pairwise disjoint translations D' of εD contained in Ω.
In [15] it is proved that if D is a bounded open set satisfying some mild regularity assumptions

[1]This paper is dedicated to Professor G. Buttazzo who inspired this reasearch.These results are an outgrowth of an informal discussion that took place during the workshop "Weekend di lavoro su Calcolo delle Variazioni" held in Montecatini Terme (PT) in November 2022.

and $u \in SBV(\Omega)$, then there exist two Lipschitz continuous 1-homogeneous functions φ, ψ : $\mathbb{R}^n \to (0, +\infty)$, $\psi \leq \varphi$, strictly positive on $\mathbb{R}^n \setminus \{0\}$, ψ convex, such that

$$\lim_{\varepsilon \to 0} H_\varepsilon(u, \Omega) = \int_\Omega \psi(\nabla u(x))\, dx + \int_{J_u} \big(u^+(x) - u^-(x)\big)\, \varphi(\nu_u(x))\, d\mathcal{H}^{n-1}(x)\,. \tag{1.3}$$

In (1.3) ∇u stands for the absolutely continuous part of the gradient measure Du, J_u is the jump set of u, $u^+ > u^-$ are the traces of u on both sides of J_u and ν_u is the generalized normal to J_u oriented in the direction going from u^- to u^+. The particular case $u = \chi_A$ where $A \subset \mathbb{R}^n$ is a set of finite perimeter, was studied in [2].
One may then be tempted to infer that the same conclusion holds for every $u \in BV(\Omega)$. In fact, the quantity $H_\varepsilon(u, \Omega)$ is strictly related to the total variation $|Du|(\Omega)$ of u (see [15]). Indeed, by using Hölder inequality and Poincaré-Wirtinger inequality, for any $u \in BV$, we get that there exists a constant $C > 0$ depending only on D such that if $D' = x_0 + \varepsilon D$, then

$$\varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| u(x) - \fint_{D'} u \right| dx \leq C|Du|(\Omega)$$

and thus,

$$H_\varepsilon(u, \Omega) \leq C|Du|(\Omega). \tag{1.4}$$

However, for general BV functions u, the point-wise limit of $H_\varepsilon(u, \Omega)$ is more difficult to grasp. In particular, for functions $u \in BV(\Omega) \setminus SBV(\Omega)$ having the so-called Cantor part of the derivative, it is possible that the limit of $H_\varepsilon(u, \Omega)$ as ε goes to 0 does not exist, as shown by a one dimensional example of [17]. Nevertheless, one can still characterize the functions in $BV(\Omega)$ as the functions $u \in L^1(\Omega)$ such that $\limsup_{\varepsilon \to 0^+} H_\varepsilon(u, \Omega) < +\infty$ (see (2.6) below).
The non existence of the point-wise limit for general BV functions suggests that the mode of convergence of H_ε to the total variation as ε goes to 0 is extremely delicate. It is natural to expect that the appropriate framework in this case to analyze the asymptotic behaviour of H_ε is the Γ- limit (in the sense of E. De Giorgi). Obviously, since we are considering an anisotropic variant of the BMO-type seminorm by using, instead of cubes, covering families made by translations of a given set D, as Γ-limit we expect an anisotropic version of the total variation. For this reason, by considering the 1- homogeneous, Lipschitz function $\psi : \Omega \to (0, +\infty)$ that appears in (1.3), we define

$$\Psi(Du)(\Omega) = \inf \left\{ \liminf_{h \to \infty} \int_\Omega \psi(\nabla u_h) \,:\, u_h \in C^\infty(\Omega), u_h \to u \text{ in } L^1(\Omega) \right\}.$$

This definition with $\psi = |\cdot|$ coincides with the usual BV total variation (see Theorem 3.9. in [3]). We observe that if $u \in BV(\Omega)$ then $\Psi(Du)(\Omega) < +\infty$ (see (2.9)).
The main result of this paper reads as follows.

Theorem 1.1. *The family of functionals (H_ε) defined in (1.2) for $\varepsilon > 0$, Γ-converges in L^1 to the functional H defined for any $u \in L^1(\Omega)$ by*

$$H(u, \Omega) = \begin{cases} \Psi(Du) & \text{if } u \in BV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Finally, the *higher order and isotropic* counterpart of H_ε is the following functional defined for functions u in the higher Sobolev space $W^{m-1,1}_{\text{loc}}(\Omega)$, $m \in \mathbb{N}$, for any $\varepsilon > 0$,

$$K_\varepsilon(u, m, \Omega) = \varepsilon^{n-m} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| u(x) - P^{m-1}_{Q'}[u](x) \right| dx, \tag{1.5}$$

where the families $\mathcal{G}_\varepsilon$ are made of disjoint cubes $Q' = x_0 + \varepsilon Q$ of side length ε, centered in x_0, with arbitrary orientation contained in Ω and $P^{m-1}_{Q'}[u]$ is the polynomial of degree $m - 1$

centered at x_0, given by

$$P_{Q'}^{m-1}[u](x) = \sum_{|\alpha|\leq m-1} (x-x_0)^\alpha \fint_{Q'} (D^\alpha u)(s)\, ds. \tag{1.6}$$

In [20] it was proved that if $u \in W^{m,1}_{\text{loc}}(\Omega)$ then

$$\lim_{\varepsilon\to 0} K_\varepsilon(u,m,\Omega) = \beta(n,m) \int_\Omega |\nabla^m u|\, dx, \tag{1.7}$$

where

$$\beta(n,m) := \max_{\nu\in\mathbb{S}^{N-1}} \frac{1}{m!} \int_Q \left| \nu\cdot x^m - \int_Q \nu\cdot y^m\, dy \right| dx. \tag{1.8}$$

We refer to Section 2 of [20] for the notation used in (1.8). Observe that this last result is the extension of Theorem 2.2 in [18].

Also, the point-wise limit of K_ε for functions in BV^m does not exists but it is possible to characterize the functions in $BV^m(\Omega)$, the space of functions of m-th order bounded variation (see [11]), as the functions such that $\limsup_{\varepsilon\to 0} K_\varepsilon(u,m,\Omega) < +\infty$ (see (2.14) below). Still using the Γ-convergence framework, we have the following result.

Theorem 1.2. *The family of functionals* (K_ε) *defined in* (1.5) *for* $\varepsilon > 0$, Γ*-converges in* L^1 *to the functional* K *defined for any* $u \in L^1(\Omega)$ *by*

$$\Gamma - \lim K_\varepsilon(u,m,\Omega) = \begin{cases} \beta(m,n)|D^m u|(\Omega) & \textit{if } u \in BV^m(\Omega) \\ +\infty & \textit{otherwise} \end{cases}$$

When $m = 1$ this result was established in [4]. The used techniques are different; their approach is based on piece-wise constant approximations rather than convolutions and our method makes the demonstration significantly shorter.

In the last section of the paper, motivated by applications in Image Processing, we deal with the minimization of functionals of the form

$$F_\varepsilon(u,\Omega) = \Lambda \int_\Omega |f-u|^q + K_\varepsilon(u,1,\Omega), \qquad q \geq 1$$

i.e. functionals that are the sum of a fidelity part and of a regularization term. Here for the sake of brevity, we denote $K_\varepsilon(u,\Omega) = K_\varepsilon(u,1,\Omega)$. We investigate the existence of a minimizer when ε is fixed and we analyze their behavior as $\varepsilon \to 0$. The proof of the existence of minimizer is based on the Γ−convergence result (Theorem 1.2). Precisely, we prove

Theorem 1.3. *Let* $q > 1$. *For every* $\varepsilon > 0$, *there exists a unique* $u_\varepsilon \in L^q(\Omega)$ *such that*

$$F_\varepsilon(u_\varepsilon,\Omega) = \min_{u\in L^q(\Omega)} F_\varepsilon(u,\Omega)$$

Let u_0 *the unique minimizer of* F *in* $L^q(\Omega)\cap BV(\Omega)$, *then as* $\varepsilon \to 0$ *we have that*

$$u_\varepsilon \to u_0, \qquad \textit{in } L^q(\Omega)$$

and

$$F_\varepsilon(u_\varepsilon,\Omega) \to F(u_0,\Omega),$$

where

$$F(u_0,\Omega) = \frac{1}{4}|Du_0|(\Omega) + \Lambda \int_\Omega |f-u_0|^q.$$

In the case $q = 1$ one can not apply the previous Theorem but one can always consider almost minimizers; a slight generalization is given in Theorem 4.1.

2. Preliminaries

We collect some preliminary results and properties of the functionals H_ε and K_ε useful in the next sections.

Here and in the rest of the paper Ω will be an open set in $\mathbb{R}^n$. We denote by Du the gradient measure of u and by ∇u its absolutely continuous part.

We recall the definition of BV functions. A function $u \in L^1(\Omega)$ is said to have bounded variation, $u \in BV(\Omega)$, if

$$|Du|(\Omega) := \sup\left\{\int_\Omega u \operatorname{div}\phi\, dx : \phi \in C_0^1(\Omega, \mathbb{R}^n), \|\phi\|_\infty \leq 1\right\} < +\infty.$$

BV is a Banach space endowed with the norm

$$\|u\|_{BV} := \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

Smooth functions are not dense in $BV(\Omega)$, but every function $u \in BV(\Omega)$ is approximable in a weak sense (see Theorem 3.9 in [3]); that is, there exists a sequence $(u_h)_h$ of $C^\infty(\Omega)$ functions such that

$$\begin{cases} \|u_h - u\|_{L^1(\Omega)} \to 0 \\ \\ \int_\Omega |\nabla u_h| \to |Du|(\Omega). \end{cases} \tag{2.1}$$

2.1. **Properties of the functional H_ε.** Given two functions $u, v \in L^1(\Omega)$, the following properties hold true:

- an estimate from above

$$|H_\varepsilon(u,\Omega) - H_\varepsilon(v,\Omega)| \leq H_\varepsilon(u - v, \Omega); \tag{2.2}$$

- convexity with respect to the function: for any $\lambda \in [0,1]$, we have

$$H_\varepsilon(\lambda u + (1-\lambda)v, \Omega) \leq \lambda H_\varepsilon(u,\Omega) + (1-\lambda) H_\varepsilon(v,\Omega).$$

Lemma 2.1. *For every open set $A \subset\subset \Omega$, let $0 < \sigma < dist(A, \partial\Omega)$, then*

$$H_\varepsilon(\rho_\sigma * u, A) \leq H_\varepsilon(u,\Omega) \qquad \forall u \in L^1(\Omega), \tag{2.3}$$

where $\rho_\sigma(x) = \sigma^{-n}\rho\left(\frac{x}{\sigma}\right)$ and ρ is a standard mollifier with compact support in the unit ball B.

Proof. By fixing an open set $A \subset\subset \Omega$ and $0 < \sigma < \operatorname{dist}(A, \partial\Omega)$, for all $x \in A$, we set $u_\sigma(x) := (\rho_\sigma * u)(x)$. Thus, given a family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets D' translations of εD contained in A, using the definition of u_σ, Jensen inequality and Fubini's theorem we have that, recalling that $\int_B \rho(y)\, dy = 1$,

$$\begin{aligned}
&\varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| u_\sigma(x) - \fint_{D'} u_\sigma \right| dx \\
&\qquad = \varepsilon^{n-1} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| \int_B \rho(y) u(x - \sigma y)\, dy - \fint_{D'} \int_B \rho(y) u(z - \sigma y)\, dy dz \right| dx \\
&\qquad \leq \varepsilon^{n-1} \int_B \rho(y) \left(\sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| u(x - \sigma y)\, dy - \fint_{D'} u(z - \sigma y) dz \right| dx \right) dy \\
&\qquad = \varepsilon^{n-1} \int_B \rho(y) \left(\sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D' - \sigma y} \left| u(x) - \fint_{D' - \sigma y} u \right| dx \right) dy \leq H_\varepsilon(u,\Omega).
\end{aligned}$$

Taking the supremum over all families $\mathcal{H}_\varepsilon$, we get (2.3).

□

In the particular case that u is the linear function, we set for $\nu \in \mathbb{S}^{n-1}$,

$$\psi(\nu) := H(x \cdot \nu, Q)$$

where $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ is the unit cube and $H(x \cdot \nu, Q) = \lim_{\varepsilon \to 0} H_\varepsilon(x \cdot \nu, Q)$. The function ψ is well defined as showed in Section 3.2. of [15]. Moreover, the function ψ is Lipschitz continuous, bounded away from zero and convex (see Propositions 3.4 and 3.6 of [15]). With a slightly abuse of notion, we shall denote by ψ the 1- homogeneous extension $\tilde{\psi}$ of ψ to $\mathbb{R}^n$, defined in the following way:

$$\tilde{\psi}(0) = 0$$

and

$$\tilde{\psi}(\tau) = |\tau| \psi\left(\frac{\tau}{|\tau|}\right) \qquad \forall \tau \in \mathbb{R}^n \setminus \{0\}.$$

Clearly, the 1-homogeneous extension of ψ is Lipschitz and there exists a positive constant c such that

$$\psi(\tau) \geq c|\tau| \qquad \forall \tau \in \mathbb{R}^n \setminus \{0\}.$$

Moreover, if $u \in C^\infty(\Omega)$ there exist two positive constants C_1, C_2 such that

$$C_1 \int_\Omega |\nabla u(x)|\, dx \leq \int_\Omega \psi(\nabla u(x))\, dx \leq C_2 \int_\Omega |\nabla u(x)|\, dx. \tag{2.4}$$

We summarize here some results of [15]:

Theorem 2.2. *Let D be a bounded connected open set with Lipschitz boundary. Then*

a) *if $u \in W^{1,1}(\Omega)$, then*

$$\lim_{\varepsilon \to 0} H_\varepsilon(u, \Omega) = \int_\Omega \psi(\nabla u)\, dx \tag{2.5}$$

b) *if $u \in L^1(\Omega)$, then $u \in BV(\Omega)$ if, and only if,*

$$\liminf_{\varepsilon \to 0} H_\varepsilon(u, \Omega) < +\infty \tag{2.6}$$

We introduce on BV an anisotropic norm equivalent to the total variation. We define the anisotropic variation $\Psi(Du)$ defined on open sets $A \subseteq \Omega$ by

$$\Psi(Du)(A) = \inf \left\{ \liminf_{h \to \infty} \int_A \psi(\nabla u_h) \,:\, u_h \in C^\infty(A), u_h \to u \text{ in } L^1(A) \right\}. \tag{2.7}$$

Moreover, Reshetnyak continuity theorem yields

$$\Psi(Du)(A) = \int_A \psi\left(\frac{dDu}{d|Du|}\right) d|Du|. \tag{2.8}$$

Clearly, this definition with $\psi = |\cdot|$ coincides with the usual BV total variation (see Theorem 3.9. in [3]).

For any $u \in BV$, from (2.8) and by (2.1), (2.4) we get that there exists two positive constants C_1, C_2 such that

$$C_1 |Du|(\Omega) \leq \Psi(Du)(\Omega) \leq C_2 |Du|(\Omega). \tag{2.9}$$

2.2. **Properties of the functional K_ε.** First we describe the properties of the polynomial that appears in (1.6).

For every $u \in L^1(\Omega)$, $\lambda, \mu \in \mathbb{R}$, and $Q' = x_0 + \varepsilon Q$ the following properties hold true:

- linearity:
$$\lambda P^{m-1}_{Q'}[u](x) + \eta P^{m-1}_{Q'}[v](x) = P^{m-1}_{Q'}[\lambda u + \eta v](x);$$
- scaling:
$$P^{m-1}_{\varepsilon Q'}[u](\varepsilon x) = P^{m-1}_{Q'}[u_\varepsilon](x),$$
where $u_\varepsilon(x) := u(\varepsilon x)$.

- convolution: for every open set U compactly contained in Ω such that $\operatorname{supp}_\Omega u \subseteq U \subset\subset \Omega$ and for every $\sigma > 0$, we consider $u_\sigma = \rho_\sigma * u$ with ρ a standard mollifier with compact support in the unit ball B and $\rho_\sigma(x) = \sigma^{-n}\rho(x/\sigma)$. We choose $\sigma < \operatorname{dist}(\operatorname{supp}_\Omega u, \partial U)$ so that $\operatorname{supp}_\Omega u_\sigma \subset U$. We have

$$P^{m-1}_{Q'}[u_\sigma](x) = P^{m-1}_{Q'}[\rho_\sigma * u](x) = \int_B \rho(y) P^{m-1}_{Q'-\sigma y}[u](x)\,dy. \tag{2.10}$$

We recall that

$$BV^m(\Omega) = \{u \in W^{m-1,1}(\Omega),\ D^{m-1}u \in BV(\Omega, S^{m-1}(\mathbb{R}^n))\}$$

is the space of (real valued) functions of m-th order bounded variation, i.e. the set of all functions, whose distributional gradients up to order $m-1$ are represented through 1-integrable tensor-valued functions and whose m-th distributional gradient is a tensor-valued Radon measure of finite total variation. Here $S^k(\mathbb{R}^n)$ denotes the set of all symmetric tensors of order k with real components, which is naturally isomorphic to the set of all k-linear symmetric maps $(\mathbb{R}^n)^k \to \mathbb{R}$ (see [13]).

The space $BV^m(\Omega)$ becomes a Banach space with the norm

$$\|u\|_{BV^m(\Omega)} = \|u\|_{W^{m-1,1}(\Omega)} + |D^m u|(\Omega).$$

Here the total variation of $D^{m-1}u$ is denoted by $|D^m u|(\Omega)$ and defined by

$$|D^m u|(\Omega) = \sup\left(\sum_{\alpha_1,\dots,\alpha_m=1}^{n} \int_\Omega D_{\alpha_1,\dots,\alpha_{m-1}} u \cdot \partial_{\alpha_m}\varphi_{\alpha_1,\dots,\alpha_m}\,dx\right),$$

where the supremum is taken over all $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$ with $\|\varphi\|_\infty = 1$. Obviously, $W^{m,1}(\Omega)$ is a subspace of $BV^m(\Omega)$.

The definition of BV^m generalizes that of the classical space of functions of bounded variation and many results about BV can be obtained in BV^m similarly (see [21]). We recall a higher-order variant of the famous Poincaré inequality, which will be useful throughout the sequel:

Theorem 2.3 (Poincarè inequality in BV^m [16, Lemma 2.2])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset with Lipschitz boundary, $m \in \mathbb{N}$, $1 \le p < \infty$. Then there exist a constant $C > 0$, depending only on Ω, m and n such that for all $u \in BV^m(\Omega)$*

$$\|u\|_{BV^m(\Omega)} \le C|D^m u|(\Omega).$$

In particular, the following version of Poincare's inequality holds. Let $u \in BV^m(Q')$ with Q' a translation of a cube of sidelenght ε, then there exists a constant $C = C(n,m)$ such that

$$\int_{Q'} |u - P^{m-1}_{Q'}[u]| \le C\varepsilon^m |D^m u|(Q'), \tag{2.11}$$

where $P^{m-1}_{Q'}[u]$ is defined in (1.6). This follows as in Lemma 2.1 of [22].

Finally, given two functions $u, v \in L^1(\Omega)$, the following properties hold true:

- an estimate from above:

$$|K_\varepsilon(u,m,\Omega) - K_\varepsilon(v,m,\Omega)| \le K_\varepsilon(u-v,m,\Omega) \tag{2.12}$$

- convexity: for any $\lambda \in [0,1]$, we have

$$K_\varepsilon(\lambda u + (1-\lambda)v, m, \Omega) \le \lambda K_\varepsilon(u,m,\Omega) + (1-\lambda)K_\varepsilon(v,m,\Omega) \tag{2.13}$$

We summarize here some results of [20]:

Theorem 2.4. *It holds true that*

- *if* $u \in W^{m,1}(\Omega)$, *then*

$$\lim_{\varepsilon\to 0} K_\varepsilon(u,m,\Omega) = \beta(m,n)\int_\Omega |\nabla^m u|\,dx\,,$$

where the constant $\beta(m,n)$ *is defined in* (1.8).

- *if* $u \in W^{m-1,1}_{loc}(\Omega)$, *then*

$$u \in BV^m(\Omega) \quad\Longleftrightarrow\quad \liminf_{\varepsilon\to 0} K_\varepsilon(u,m,\Omega) < +\infty. \tag{2.14}$$

Moreover, there are positive constants C_1 and C_2, independent of u, such that

$$C_1|D^m u|(\Omega) \le \liminf_{\varepsilon\to 0^+} K_\varepsilon(u,m,\Omega) \le \limsup_{\varepsilon\to 0^+} K_\varepsilon(u,m,\Omega) \le C_2|D^m u|(\Omega).$$

Remark 2.5. In the case $m = 1$ the constant $C_1 = \frac{1}{4}$ was obtained in [18]. We will find the constant C_1 for $m > 1$ in Proposition 3.1.

In the following statement, we prove that for $m = 1$ the truncation of function $u \in L^1$ does not increase the oscillation in every cube $Q' \subset \mathbb{R}^n$ and this will serve its purpose in the upcoming stage. Notice that for $m > 2$ the truncation could not preserve Sobolev space and the truncated function might not belong to $W^{m-1,1}$ anymore. We define the full truncation at level $k > 0$ as

$$T_k(u) = \begin{cases} u & \text{if } |u| \le k \\ k & \text{if } u > k \\ -k & \text{if } u < -k. \end{cases} \tag{2.15}$$

and prove the following.

Proposition 2.6. *Let* $m = 1$ *and* $u \in L^1_{loc}(\Omega)$, *then*

$$K_\varepsilon(T_k(u),\Omega) \le K_\varepsilon(u,\Omega). \tag{2.16}$$

Proof. Let $Q' \subset \Omega$ a cube centered in x_0. The polynomial $P^0_{Q'}[u]$ is described in (1.6) and precisely

$$P^0_{Q'}[u](x) = \fint_{Q'} u. \tag{2.17}$$

Firstly, we want to prove that

$$\fint_{Q'} \left| T_k u - \fint_{Q'} T_k u \right| \le \fint_{Q'} \left| u - \fint_{Q'} u \right|. \tag{2.18}$$

Clearly,

$$\fint_{Q'} \left| u - \fint_{Q'} u \right| = \frac{2}{|Q'|}\int_{\{u > \fint_{Q'} u\}} \left(u - \fint_{Q'} u \right) = \frac{2}{|Q'|}\int_{\{u < \fint_{Q'} u\}} \left(\fint_{Q'} u - u \right). \tag{2.19}$$

For a fixed $k \in \mathbb{R}$ let us denote $u_1(x) = u(x)$, $u_2(x) = k$ and define the truncation of u from above at level k by

$$g = \min(u_1,u_2) = \min(u,k).$$

Thus, let us consider

$$E_1 = \{x \in Q' \,:\, u_1(x) \le u_2(x)\} = \{x \in Q' \,:\, g(x) = u_1(x)\} \quad \text{and} \quad E_2 = Q' \setminus E_1.$$

We have, by (2.19)

$$\begin{aligned}\fint_{Q'} \left| g - \fint_{Q'} g \right| &= \frac{2}{|Q'|}\int_{\{g < \fint_{Q'} g\}} \left(\fint_{Q'} g - g \right) = \frac{2}{|Q|}\sum_{i=1}^{2}\int_{\{x\in E_i : u_i < \fint_{Q'} g\}} \left(\fint_{Q'} g - u_i \right) \\ &\le \frac{2}{|Q'|}\sum_{i=1}^{2}\int_{\{x\in Q' : u_i < \fint_{Q'} u_i\}} \left(\fint_{Q'} u_i - u_i \right) = \fint_{Q'} \left| u_1 - \fint_{Q'} u_1 \right|.\end{aligned}$$

Similarly, define the truncation from below at level k by

$$h = \max(u_1, u_2) = \max(u, k)$$

and we have

$$\fint_{Q'} \left| h - \fint_{Q'} h \right| \leq \fint_{Q'} \left| u_1 - \fint_{Q'} u_1 \right|.$$

Since

$$T_k u = \max(\min(u, k), -k),$$

(2.18) holds true and summing up over all cubes in the family $\mathcal{G}_\varepsilon$ we conclude. □

3. Γ−CONVERGENCE FOR H_ε AND K_ε

We recall (see for example [11]) that a family of functionals $\mathcal{F}_\varepsilon$ defined on $L^1(\Omega)$, Γ-converges in $L^1(\Omega)$ when ε goes to 0 to a functional $\mathcal{F}$ defined on $L^1(\Omega)$ if and only if, the following two conditions are satisfied:

i) (Γ−*liminf inequality*) $\forall u \in L^1(\Omega)$ and for every family $\{u_\varepsilon\}$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ as ε goes to 0, one has

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u);$$

ii) (Γ−*limsup inequality*) $\forall u \in L^1(\Omega)$, there exists a family $\{\tilde{u}_\varepsilon\}$ such that $\tilde{u}_\varepsilon$ converges to u in $L^1(\Omega)$ as ε goes to 0 and

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leq \mathcal{F}(u).$$

We give now the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. We assume without loss of generality that $\Psi(Du)(\Omega) < +\infty$ and then by (2.9) $f \in BV(\Omega)$. To prove the Γ−limsup inequality by (2.1) we consider a sequence $(v_h) \in C^\infty(\Omega)$ such that

$$v_h \to u \quad \text{in } L^1(\Omega) \quad \text{when } h \text{ goes to } 0$$

and

$$\int_\Omega |\nabla v_h| \to |Du|(\Omega).$$

By Reshetnyak's continuity theorem, it holds that

$$\int_\Omega \psi(\nabla v_h) \to \Psi(Du)(\Omega). \tag{3.1}$$

By (2.2), for each h let ε_h be such that

$$\left| H_\varepsilon(v_h, \Omega) - \int_\Omega \psi(\nabla v_h) \right| < h \qquad \forall \varepsilon < \varepsilon_h. \tag{3.2}$$

Without loss of generality we may assume that ε_h is an infinitesimal decreasing sequence with respect to h. We set

$$u_\varepsilon = v_h \qquad \text{if } \varepsilon_{h+1} < \varepsilon \leq \varepsilon_h.$$

Combining (3.1) and (3.2), it holds

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega)$$

and

$$\lim_{\varepsilon \to 0} H_\varepsilon(u_\varepsilon, \Omega) = \Psi(Du)$$

proving the Γ−limsup inequality.

To prove the Γ−liminf inequality we use the convolution strategy by considering $u_\sigma = u * \rho_\sigma$ and by fixing an open set A, $A \subset\subset \Omega$ as in Lemma 2.1. If $u \in BV$ then by a) of Theorem 2.2, we have

$$\lim_{\varepsilon \to 0} H_\varepsilon(u * \rho_\sigma, A) = \int_A \psi\big(\nabla(u * \rho_\sigma)\big)\, dx.$$

Moreover, by using (2.2) and the Poincarè inequality (1.4), we get

$$\begin{aligned}|H_\varepsilon(u_\varepsilon * \rho_\sigma, A) - &H_\varepsilon(u * \rho_\sigma, A)| \\ &\le C \int_A |\nabla\big((u_\varepsilon - u) * \rho_\sigma\big)| \\ &\le C\|\nabla \rho_\sigma\|_{L^1(A)} \|u_\varepsilon - u\|_{L^1(A)},\end{aligned} \tag{3.3}$$

where C depends only on A and the dimension n. Then applying (2.3) and (3.3), we have

$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon, A) \ge \lim_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon * \rho_\sigma, A) = \int_A \psi(\nabla u * \rho_\sigma)\, dx. \tag{3.4}$$

Letting $\sigma \to 0$, by Definition 2.7 we have

$$\lim_{\sigma\to 0} \int_A \psi\big(\nabla(u * \rho_\sigma)\big)\, dx \ge \Psi(Du)(A). \tag{3.5}$$

Thus, for every $u_\varepsilon \to u$, combining (3.5) and (3.4), letting $A \uparrow \Omega$, we get

$$\Psi(Du)(\Omega) \le \liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon, \Omega).$$

proving the $\Gamma-$liminf inequality. □

We prove now a lower bound for the functional $K_\varepsilon(u, m, \Omega)$ for $m \ge 2$. The case $m = 1$ has been proved in Proposition 3.4 in [18].

Proposition 3.1. *Let $m \ge 2$ and $u \in W^{m-1,1}_{loc}(\Omega)$. Then*

$$\liminf_{\varepsilon\to 0^+} K_\varepsilon(u, m, \Omega) \ge \beta(m, n)|D^m u|(\Omega)$$

Proof. Without loss of generality we can assume that

$$\liminf_{\varepsilon\to 0^+} K_\varepsilon(u, m, \Omega) < +\infty.$$

By Theorem 2.4 we have that $u \in BV^m$. We fix an open set $A \subset\subset \Omega$, $\sigma > 0$ and we set $u_\sigma = \rho_\sigma * u$, as in the proof of Lemma 2.1. Then, by using the definition of u_σ, Jensen's inequality and Fubini's Theorem, we have that

$$\begin{aligned}\varepsilon^{n-m} \sum_{Q'\in\mathcal{G}_\varepsilon} &\fint_{Q'} \Big|u_\sigma(x) - P^{m-1}_{Q'}[u_\sigma](x)\Big| dx \\ &= \varepsilon^{n-m} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left|\int_B \rho(y)\left(\tilde{u}(x) - P^{m-1}_{Q'}[\tilde{u}](x)\right) dy\right| dx \\ &\le \varepsilon^{n-m} \int_B \rho(y) \left(\sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \Big|\tilde{u}(x) - P^{m-1}_{Q'}[\tilde{u}](x)\Big| dx\right) dy,\end{aligned} \tag{3.6}$$

where $\tilde{u}(x) = u(x - \sigma y)$. Then, a change of variable shows that

$$P^{m-1}_{Q'}[\tilde{u}](x) = P^{m-1}_{Q'-\sigma y}[u](x - \sigma y)$$

and since $\int_B \rho(y)dy = 1$, we deduce

$$\varepsilon^{n-m} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \Big|u_\sigma(x) - P^{m-1}_{Q'}[u_\sigma](x)\Big| dx \le K_\varepsilon(u, m, \Omega). \tag{3.7}$$

Since $u_\sigma \in W^{m,1}(A)$ from Theorem 1.7 we have

$$\beta(m, n) \int_A |\nabla^m u_\sigma| = \liminf_{\varepsilon\to 0} K_\varepsilon(u_\sigma, m, A) \le \liminf_{\varepsilon\to 0} K_\varepsilon(u, m, \Omega).$$

We conclude letting $\sigma \to 0$ and then $A \uparrow \Omega$. □

Proof of Theorem 1.2. To prove the $\Gamma-$limsup inequality we proceed in the same way as in Theorem 1.1 by replacing (3.1) with the convergence of total variation in higher dimension and (3.2) with the analogous result holding by Theorem 1.7.

To prove the $\Gamma-$liminf inequality, from (3.7) in Proposition 3.1 we have

$$K_\varepsilon(u*\rho_\sigma, m, A) \le K_\varepsilon(u, m, \Omega)$$

and by Theorem 1.7

$$\lim_{\varepsilon\to 0} K_\varepsilon(u*\rho_\sigma, m, A) = \beta(m,n)\int_A |\nabla^m(u*\rho_\sigma)|\,dx.$$

Moreover, by using (2.12) and the Poincarè inequality in BV^m (equation (2.11)), we have

$$|K_\varepsilon(u_\varepsilon*\rho_\sigma, m, A) - K_\varepsilon(u*\rho_\sigma, m, A)| \le C\|\nabla^m\rho_\sigma\|_{L^1(A)}\|u_\varepsilon - u\|_{L^1(A)}.$$

Then combining these last three inequalities we have

$$\liminf_{\varepsilon\to 0} K_\varepsilon(u_\varepsilon, m, \Omega) \ge \lim_{\varepsilon\to 0} K_\varepsilon(u_\varepsilon*\rho_\sigma, m, A) = \beta(m,n)\int_A |\nabla^m(u*\rho_\sigma)|\,dx.$$

Then, letting $\sigma\to 0$ and then $A\uparrow\Omega$, we prove that for every $u_\varepsilon\to u$,

$$\liminf_{\varepsilon\to 0} K_\varepsilon(u_\varepsilon, m, \Omega) \ge \beta(m,n)|D^m u|(\Omega).$$

□

4. Application to image processing

Having an image of poor quality, a challenging problem is to find a better one not so far from the original. A popular strategy in image processing to improve an initial image f is to use a variational formulation by considering the problem

$$\inf\left\{F(u) + \Lambda\int_\Omega |f-u|^2 \,:\, u\in\mathcal{A}\right\}. \tag{4.1}$$

Here $\mathcal{A}$ is a suitable functional space, $\Lambda > 0$ is the fidelity parameter which fixed the grade of similarity with the original f and the functional F, called filter, has a regularization role. Minimizers of (4.1) are the better images.

Many kind of filters have been proposed starting from the most famous one by Rudin, Osher and Fatemi (see [23]) where

$$F(u) = |Du|(\Omega)$$

and the minimization problem is

$$\min\left\{|Du|(\Omega) + \int_\Omega |f-u|^2 \,:\, u\in L^2(\Omega)\right\}. \tag{ROF}$$

The advantages of the minimization model (ROF) are mainly the fact that the BV regularization term allows for discontinuities but disfavors large oscillations and moreover the strict convexity due to the L^2 approximation term uniquely determines the minimizers u in terms of the datum f and the chosen fidelity $\Lambda > 0$.

On the other hand, the ROF model is not contrast invariant, i.e. if u is the solution of (ROF) for the initial f, then cu may not be the solution for cf. This relies on the L^2 norm in the approximation term. A model that closes this lack of contrast invariance was proposed by Chan and Esedoglu (see [10]) by considering the model

$$\inf\left\{|Du|(\Omega) + \Lambda\int_\Omega |f-u| \,:\, u\in BV(\Omega)\right\} \tag{CE}$$

where the L^1 norm appears in the fidelity term. Nevertheless, this model is only convex thus the uniqueness of minimizers is not guaranteed.

Other variants have been proposed, by considering families of filters varying with respect to a small parameter ε. In [5] the authors proposed as filter a non local functional modeled on one considered by Bourgain, Brezis, Mironescu in [6]

$$AK_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon(|x - y|)\, dxdy.$$

In [19] the non local filter includes a given weight function:

$$GO(u) = \int_\Omega \left(\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^2} w(x, y) \right)^{1/2} dxdy,$$

and by choosing $w(x, y) = \rho_\varepsilon(|x - y|)$ one has the corresponding family GO_ε of filters.

Considering families of filters the aim is twofold: first, one investigates the existence of a minimizer for the approximated functional when ε is fixed and then one can study the behavior of these minimizers as $\varepsilon \to 0$. It is proved that the minimizers of AK_ε and GO_ε converge to the unique solution of the problem

$$(ROF_k) \qquad \inf \left\{ k|Du|(\Omega) + \Lambda \int_\Omega |f - u|^2 \,:\, u \in L^2(\Omega) \cap BV(\Omega) \right\}$$

with the constant $k = \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|\, d\sigma$ and $k = \left(\int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^2\, d\sigma \right)^{1/2}$ respectively. We remark that the functional in (ROF_k) is strictly convex and by standard functional analysis the solution is unique.

Here, in the spirit of functionals converging to the BV-norm considered in [7, 8, 9, 5], the following functional is studied

$$F_\varepsilon(u, \Omega) = K_\varepsilon(u, \Omega) + \Lambda \int_\Omega |f - u|^q, \quad q \geq 1 \tag{4.2}$$

where for brevity we denote $K_\varepsilon(u, \Omega) = K_\varepsilon(u, 1, \Omega)$. We deal with the corresponding minimization problem

$$\inf \{ F_\varepsilon(u, \Omega) \,:\, u \in L^q(\Omega) \}.$$

For $q > 1$, we claim that minimizers as $\varepsilon \to 0$ converge to the solution of the problem

$$(ROF^q) \qquad \inf \{ F(u, \Omega) \,:\, u \in L^q(\Omega) \cap BV(\Omega) \},$$

where

$$F(u, \Omega) = \frac{1}{4} |Du|(\Omega) + \Lambda \int_\Omega |f - u|^q.$$

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. For every $\varepsilon > 0$, the functional F_ε defined on $L^q(\Omega)$ in (4.2) is convex, since $K_\varepsilon(u, \Omega)$ is convex as showed in (2.13). Moreover, F_ε is lower semicontinuous for the strong L^q topology. Indeed, given $u_n \to u$ strongly in $L^q(\Omega)$, it converges also in $L^1(\Omega)$ and by Fatou's Lemma, we have

$$\liminf_{n \to +\infty} K_\varepsilon(u_n, \Omega) \leq K_\varepsilon(u, \Omega).$$

Therefore F_ε is also lower semicontinuous for the weak topology in $L^q(\Omega)$. When $q > 1$ the space $L^q(\Omega)$ is reflexive and then there exists a minimizer u_ε of F_ε in $L^q(\Omega)$. The uniqueness follows by the strict convexity.

In order to prove that $u_\varepsilon \to u_0$, since $q > 1$, we assume that there exists a subsequence u_{ε_k} such that $u_{\varepsilon_k} \rightharpoonup v$ weakly in $L^q(\Omega)$. We will prove that $v = u_0$ and we divide the proof in two steps.

Step 1. We first observe that, since the limsup inequality holds true in Theorem 1.2 for K_ε, there exists v_ε contained in $L^1(\Omega)$ such that $v_\varepsilon \to u_0$ in $L^1(\Omega)$ and

$$\limsup_{\varepsilon \to 0} K_\varepsilon(v_\varepsilon, \Omega) \leq \frac{1}{4} |Du_0|(\Omega).$$

Let us consider, for $\tau > 0$ the truncation function $T_\tau(\cdot)$ defined in (2.15). Since u_ε is a minimizer of F_ε we have

$$F_\varepsilon(u_\varepsilon,\Omega) \le F_\varepsilon(T_\tau v_\varepsilon,\Omega) \le K_\varepsilon(T_\tau v_\varepsilon,\Omega) + \Lambda \int_\Omega |f - T_\tau v_\varepsilon|^q. \tag{4.3}$$

Moreover, by Proposition 2.6, as the truncation may not increase the oscillation in every cube Q', for every w

$$\fint_{Q'} \left| T_\tau w - \fint_{Q'} T_\tau w \right| \, dx \le \fint_{Q'} \left| w - \fint_{Q'} w \right| \, dx. \tag{4.4}$$

Combining (4.3) and (4.4) we have

$$F_\varepsilon(u_\varepsilon,\Omega) \le K_\varepsilon(v_\varepsilon,\Omega) + \Lambda \int_\Omega |f - T_\tau v_\varepsilon|^q.$$

Letting $\varepsilon \to 0$ and using the limsup inequality, we have

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega) \le \frac{1}{4}|Du_0|(\Omega) + \Lambda \int_\Omega |f - T_\tau u_0|^q.$$

Moreover, letting $\tau \to +\infty$,

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega) \le \frac{1}{4}|Du_0|(\Omega) + \Lambda \int_\Omega |f - u_0|^q = F(u_0,\Omega). \tag{4.5}$$

Step 2. We claim that for every $w \in L^1(\Omega)$ and for every sequence $w_\varepsilon \in L^1(\Omega)$ such that $w_\varepsilon \rightharpoonup w$ weakly in $L^1(\Omega)$ we have

$$\liminf_{\varepsilon\to 0} K_\varepsilon(w_\varepsilon,\Omega) \ge \frac{1}{4}|Dw|(\Omega). \tag{4.6}$$

We fix fix an open set $A \subset\subset \Omega$, $\sigma > 0$ and we set $w_{\sigma,\varepsilon}(x) = (\varrho_\sigma * w_\varepsilon)(x)$, as in the proof of Lemma 2.1. Since $w_\varepsilon \rightharpoonup w$ weakly in $L^1(\Omega)$, for each fixed $\sigma > 0$,

$$w_{\sigma,\varepsilon} = \varrho_\sigma * w_\varepsilon \to \varrho_\sigma * w \ = w_\sigma \ \text{ strongly in } L^1(A) \text{ as } \varepsilon \to 0.$$

Moreover by (3.7) and the Γ-liminf inequality for K_ε, we have

$$\liminf_{\varepsilon\to 0} K_\varepsilon(w_\varepsilon,\Omega) \ge \liminf_{\varepsilon\to 0} K_\varepsilon(w_{\sigma,\varepsilon},A) \ge \frac{1}{4}|Dw_\sigma|(A).$$

Letting $\sigma \to 0$ and the $A \uparrow \Omega$, we conclude proving (4.6).

Now, applying this claim to the sequence $u_{\varepsilon_k} \rightharpoonup v$, we have

$$\liminf_{\varepsilon_k\to 0} K_{\varepsilon_k}(u_{\varepsilon_k},\Omega) \ge \frac{1}{4}|Dv|(\Omega)$$

and therefore

$$\liminf_{\varepsilon_k\to 0} F_{\varepsilon_k}(u_{\varepsilon_k},\Omega) \ge \frac{1}{4}|Dv|(\Omega) + \Lambda \int_\Omega |f - v|^q = F(v,\Omega). \tag{4.7}$$

Combining (4.7) and (4.5), by the uniqueness of minimizers, we have

$$u_0 = v.$$

It remains to prove that $F_\varepsilon(u_\varepsilon,\Omega) \to F(u_0,\Omega)$. We write

$$\Lambda \int_\Omega |f - u_\varepsilon|^q = F_\varepsilon(u_\varepsilon,\Omega) - K_\varepsilon(u_\varepsilon,\Omega)$$

and we use (4.7) and Γ-limsup inequality for K_ε to have

$$\limsup_{\varepsilon_k\to 0} \Lambda \int_\Omega |f - u_{\varepsilon_k}|^q \le F(u_0,\Omega) - \frac{1}{4}|Du_0|(\Omega) = \Lambda \int_\Omega |f - u_0|^q. \tag{4.8}$$

We know that u_{ε_k} such that $u_{\varepsilon_k} \rightharpoonup u_0$ weakly in $L^q(\Omega)$ so by (4.8) $u_{\varepsilon_k} \to u_0$ strongly in $L^q(\Omega)$. This implies $u_\varepsilon \to u_0$ strongly in $L^q(\Omega)$ and then

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega) \geq \frac{1}{4}|Du_0|(\Omega) + \Lambda \int_\Omega |f-u_0|^q = F(v,\Omega)$$

which combined with (4.5) conclude the proof. □

In Theorem 1.3, one cannot replace the condition $q > 1$ by $q = 1$. In this case we prove a slight generalization result by consider almost minimizers.

Theorem 4.1. *Let $q = 1$ and $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. Let $f \in L^1(\Omega)$ and let $\{\delta_\varepsilon\}$, $\{\tau_\varepsilon\}$ two positive sequences converging to 0 as $\varepsilon \to 0$. Let $\{u_\varepsilon\} \in L^1(\Omega)$ equi-bounded in $L^1(\Omega)$ such that*

$$F_{\delta_\varepsilon}(u_\varepsilon,\Omega) \leq \inf_{u\in L^1(\Omega)} F_{\delta_\varepsilon}(u,\Omega) + \tau_\varepsilon.$$

Then there exists a subsequence $\{u_{\varepsilon_k}\}$ of $\{u_\varepsilon\}$ such that u_{ε_k} converges to u_0 in $L^1(\Omega)$ where u_0 is a minimizer of the functional F defined on $L^1(\Omega) \cap BV(\Omega)$.

Proof. We immediately observe that $\liminf_{\varepsilon\to 0} K_\varepsilon(u_\varepsilon) < \infty$. Repeating the argument as in Theorem 2 of [4] we can consider an approximating sequence $u_\varepsilon^{\varepsilon/2}$ as defined in (19) of [4]. By Lemma 9, 10 and 11 in [4], $u_\varepsilon^{\varepsilon/2}$ has uniformly bounded total variation and $u_\varepsilon - u_\varepsilon^{\varepsilon/2} \to 0$ in $L^1(\Omega)$. Then, by BV compactness theorem [3, Thm. 3.23] there exists a subsequence ε_k such that $u_{\varepsilon_k}^{\varepsilon_k/2}$ converges to some u_0 and then $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$. By Fatou's Lemma and the Γ-liminf property of K_ε it follows

$$F(u_0,\Omega) \leq \liminf_{k\to\infty} F_{\delta_{\varepsilon_k}}(u_{\varepsilon_k},\Omega).$$

We prove now that u_0 is a minimizer of F in $L^1(\Omega) \cap BV(\Omega)$. Let $v \in L^1(\Omega) \cap BV(\Omega)$ be a minimizer of F. Applying Theorem 1.2 there exists $v_\varepsilon \in L^1(\Omega)$ such that $v_\varepsilon \to v$ in $L^1(\Omega)$ and by the Γ-limsup inequality,

$$\limsup_{\varepsilon\to 0} K_{\delta_\varepsilon}(v_\varepsilon,\Omega) \leq \frac{1}{4}|Dv|(\Omega).$$

For $A > 0$, let $T_A v$ the truncation of v at level A defined in (2.15). By Proposition 2.6 we have

$$K_{\delta_\varepsilon}(T_A v_\varepsilon,\Omega) \leq K_{\delta_\varepsilon}(v_\varepsilon,\Omega)$$

and by definition of u_ε, we get

$$\begin{aligned} F_{\delta_\varepsilon}(u_\varepsilon,\Omega) &\leq K_{\delta_\varepsilon}(T_A v_\varepsilon,\Omega) + \tau_\varepsilon + \Lambda \int_\Omega |T_A v_\varepsilon - f| \\ &\leq K_{\delta_\varepsilon}(v_\varepsilon,\Omega) + \tau_\varepsilon + \Lambda \int_\Omega |T_A v_\varepsilon - f|. \end{aligned} \tag{4.9}$$

Taking first the lim inf as $\varepsilon \to 0$ and then let $A \to +\infty$, it follows

$$F(u_0,\Omega) \leq \frac{1}{4}|Dv|(\Omega) + \Lambda \int_\Omega |v-f|$$

which implies that u_0 is a minimizer of F. □

Remark 4.2. Let us note that in the previous Theorem it is possible to choose a sequence of almost minimizers which is equibounded in $L^1(\Omega)$, at least for smooth datum f. Indeed, if $f \in L^\infty(\Omega)$ and $\{u_\varepsilon\}$ is a sequence of almost minimizers, then the sequence $\{T_{\|f\|_\infty}(u_\varepsilon)\}$ of truncation at level $\|f\|_\infty$ is equibounded in $L^1(\Omega)$ and a still almost minimizing.

Acknowledgments. The research of S.GLB. has been funded by PRIN Project 2022ZXZTN2. The research of R.S. has been partially funded by PRIN Project 2022XZSAFN and PRIN-PNRR P2022XSF5H. The authors are members of "GNAMPA" of Istituto Nazionale di Alta Matematica (INdAM), and they were partially supported by the project 2024 "Problemi variazionali associati alla variazione totale".

References

[1] L. Ambrosio, J. Bourgain, H. Brezis, A. Figalli, *BMO-type norms related to the perimeter of sets*, Comm. Pure Appl. Math., 69 (2016), 1062–1086. 1
[2] L. Ambrosio, G. Comi, *Anisotropic Surface Measures as Limits of Volume Fractions.* Current Research in Nonlinear Analysis 135 (2018), 1-32. 1, 2
[3] L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. 2, 4, 5, 13
[4] A. Arroyo-Rabasa, P. Bonicatto, G. Del Nin, *Representation of the total variation as a Γ-limit of BMO-type seminorms.*, available on arXiv:2112.03832. 3, 13
[5] G. Aubert, P. Kornprobst, *Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems?*, SIAM J. Numer. Anal. 47 (2009) 844–860. 11
[6] J. Bourgain, H. Brezis, P. Mironescu, *A new function space and applications*, Journal of the EMS, 17 (2015), 2083–2101. 1, 11
[7] J. Bourgain, H.-M. Nguyen, *A new characterization of Sobolev spaces*, C. R. Acad. Sci. Paris 343 (2006) 75–80. 11
[8] H. Brezis, HM. Nguyen, *Non-local Functionals Related to the Total Variation and Connections with Image Processing.* Ann. PDE 4(9) (2018). 11
[9] H. Brezis, HM. Nguyen, *Two subtle convex nonlocal approximations of the BV-norm*, Nonlinear Anal., 137, (2016), 222-245. 11
[10] T. F. Chan and S. Esedoglu, *Aspects of total variation regularized L^1 function approximation.* SIAM J. Appl. Math. , 65(5) (2005), 1817–1837. 10
[11] G. Dal Maso, *An Introduction to Γ-convergence.* Birkhauser, Boston, 1993. 3, 8
[12] G. De Philippis, N. Fusco, A. Pratelli: ,. .On the approximation of SBV functions. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 28 (2017), 369-413. 1
[13] F. Demengel, R. Temam, *Convex functions of a measure and applications*, Indiana University Mathematical Journal, 33 (5), (1984), 673–709. 6
[14] F. Farroni, S. Guarino Lo Bianco, R. Schiattarella, *BMO-type seminorms generating Sobolev functions*, J. Math. Anal. Appl. 491 (2020), 124298 1
[15] F. Farroni, N. Fusco, S. Guarino Lo Bianco, R. Schiattarella, *A formula for the anisotropic total variation of BV functions.* J. Funct. Anal. 278(9), 108–451 (2020) 1, 2, 5
[16] M. Fuchs, J. Muller, *A higher order TV-type variational problem related to the denoising and inpainting of images*, Nonlinear Anal. 154 (2017), 122-147. 6
[17] N. Fusco, G. Moscariello, C. Sbordone, *A formula for the total variation of SBV functions.* J. Funct. Anal. 270 (2016), no. 1, 419-446. 1, 2
[18] N. Fusco, G. Moscariello, C. Sbordone,*BMO-type seminorms and Sobolev functions.* ESAIM Control Optim. Calc. Var. 24 (2018), no. 2, 835-847. 3, 7, 9
[19] G. Gilboa, S. Osher, *Nonlocal operators with applications to image processing*, Multiscale Model. Simul. 7 (2008), 1005-1028. 11
[20] S. Guarino Lo Bianco, R. Schiattarella, *A BMO-Type Characterization of Higher Order Sobolev Spaces*, Potential Anal (2022). https://doi.org/10.1007/s11118-022-09987-8 3, 6
[21] R.L. Jerrard, H.M. Soner, *Functions of bounded higher variation*, Indiana Univ. Math. J., 51 (2002), 645–677. 6
[22] M. Marcus, *Exceptional sets with respect to Lebesgue differentiation of functions in Sobolev spaces*, Ann. Sc. Norm. Pisa, Serie VI, 1 (1974), 113-130. 6
[23] L. I. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D: Nonlinear Phenomena, 60 (1992), 259–268.10

Serena Guarino Lo Bianco:
Dipartimento di Scienze F.I.M., Università degli Studi di Modena e Reggio Emilia
Via Campi 213/A, 41125 Modena - ITALY
`serena.guarinolobianco@unimore.it`

Roberta Schiattarella:
Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II"
via Cintia, 80126 Napoli, ITALY
roberta.schiattarella@unina.it