

THE TOTAL ABSOLUTE CURVATURE OF CLOSED CURVES WITH SINGULARITIES

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ABSTRACT. In this paper, we give a generalization of Fenchel's theorem for closed curves as frontals in Euclidean space $\mathbb{R}^n$. We prove that, for a non-co-orientable closed frontal in $\mathbb{R}^n$, its total absolute curvature is greater than or equal to π . It is equal to π if and only if the curve is a planar locally L -convex closed frontal whose rotation index is $1/2$ or $-1/2$. Furthermore, if the equality holds and if every singular point is a cusp, then the number N of cusps is an odd integer greater than or equal to 3, and $N = 3$ holds if and only if the curve is simple.

1. INTRODUCTION

We fix an integer n greater than 1. Let $\gamma(s)$ ($s \in [0, L]$) be an arclength parametrization of a closed regular curve in Euclidean n -space $\mathbb{R}^n$, where L is the length of $\gamma(s)$. Denote by $k(s)$ (≥ 0) the curvature function of $\gamma(s)$. Then,

$$K(\gamma) = \int_0^L k(s) ds \quad \left(k = \left\| \frac{d^2\gamma}{ds^2} \right\| \right)$$

is called the *total absolute curvature*. The following holds:

Fenchel's theorem ([11, 12, 4, 17]). *The total absolute curvature of a closed regular curve in $\mathbb{R}^n$ is greater than or equal to 2π . It is equal to 2π if and only if the curve is an oval.*

Here, an oval is a locally convex simple closed plane curve. We remark that a locally convex closed plane curve is simple if and only if its rotation index is equal to 1 or -1 . So far, several generalizations of Fenchel's theorem have been obtained: knots in $\mathbb{R}^3$ [10, 16]; curves in a non-positively curved Riemannian manifolds [22, 5]; curves in a sphere [20, 21]; open curves in $\mathbb{R}^n$ [9]; curves in the Lorentz-Minkowski space [3]; curves in $\text{CAT}(\kappa)$ space [18]. See also [2]. Furthermore, Chern-Lashof [7, 8] generalized Fenchel's theorem to closed submanifolds in $\mathbb{R}^n$. The lower bound of the total absolute curvature is related to topological invariant of the submanifold. Therefore, Fenchel's theorem is one of the main results in global differential geometry. In this paper, we extend Fenchel's theorem for curves with *singular points*.

1.1. Statement of results. For a smooth map $\gamma : I \rightarrow \mathbb{R}^n$ defined on a non-empty interval I , a point $c \in I$ is called a *regular point* (resp. a *singular point*) of γ if the derivative $\gamma'(c)$ does not vanish (resp. does vanish), where the prime $'$ means d/dt . In this paper, a *curve* is a smooth map

$$\gamma : I \longrightarrow \mathbb{R}^n$$

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whose regular set $\text{Reg}(\gamma)$ is dense in I . A curve $\gamma : I \rightarrow \mathbb{R}^n$ is called a *frontal* if there exists a smooth map $\mathbf{e} : I \rightarrow S^{n-1}$ such that $\gamma'(t)$ and $\mathbf{e}(t)$ are linearly dependent for each $t \in I$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Such an $\mathbf{e}(t)$ is said to be a *unit tangent vector field* along γ . Moreover, a frontal γ is said to be a *wave front* (or a *front*, for short), if $L = (\gamma, \mathbf{e})$ is a regular curve in $\mathbb{R}^n \times S^{n-1}$.

On the regular set $\text{Reg}(\gamma)$ of a frontal $\gamma : I \rightarrow \mathbb{R}^n$, the *curvature function* is defined as

$$k(t) = \frac{\sqrt{\|\gamma'\|^2 \|\gamma''\|^2 - (\gamma' \cdot \gamma'')^2}}{\|\gamma'\|^3}.$$

Here, we denote by $\|\mathbf{a}\|$ the norm $\|\mathbf{a}\| := \sqrt{\mathbf{a} \cdot \mathbf{a}}$ of a vector $\mathbf{a} \in \mathbb{R}^n$, and $\mathbf{a} \cdot \mathbf{b}$ is the canonical Euclidean inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Since $\mathbf{e}(t) = \pm \gamma'(t) / \|\gamma'(t)\|$ holds on $\text{Reg}(\gamma)$, the curvature function $k(t)$ is written as

$$k(t) = \|\mathbf{e}'(t)\| / \|\gamma'(t)\|.$$

In general, the curvature function k of a frontal may diverge at a singular point. However,

$$(1.1) \quad k ds = \|\mathbf{e}'(t)\| dt$$

is a bounded continuous 1-form even at a singular point, where $ds = \|\gamma'(t)\| dt$ is the arclength measure. We call $k ds$ the *curvature measure*.

If a frontal $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ is a periodic map, γ is called a *closed frontal*. By multiplying the parameter by a constant, we may assume that γ is 2π -periodic. Then, the domain of definition of γ is regarded as $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For each $a \in \mathbb{R}$, we represent the 2π -periodic closed curve $\gamma(t)$ by the restriction $\gamma|_{[a, a+2\pi]}$.

If $\mathbf{e}$ is also a 2π -periodic map, that is, if

$$\mathbf{e}(t + 2\pi) = \mathbf{e}(t) \quad (t \in \mathbb{R}),$$

then γ is said to be *co-orientable*. Then, the domain of definition of $\mathbf{e}$ is regarded as S^1 . On the other hand, if

$$\mathbf{e}(t + 2\pi) = -\mathbf{e}(t) \quad (t \in \mathbb{R}),$$

then γ is called *non-co-orientable*. In this case, $\mathbf{e}$ is a 4π -periodic map, and hence, the domain of definition of $\mathbf{e}$ is regarded as $\tilde{S}^1 = \mathbb{R}/4\pi\mathbb{Z}$.

For a closed frontal $\gamma : S^1 \rightarrow \mathbb{R}^n$, as the curvature measure is a continuous 1-form on S^1 ,

$$(1.2) \quad K(\gamma) = \int_{S^1} k ds$$

is a bounded non-negative number. We call $K(\gamma)$ the *total absolute curvature*.

We remark that, in general, the total absolute curvature $K(\gamma)$ does not possess a non-trivial lower bound. In fact, the line segment $\gamma(t) = (\cos t, 0)$ is a closed frontal with vanishing total absolute curvature $K(\gamma) = 0$. Similar examples exist even when restricted to wave fronts, see Example 4.1.

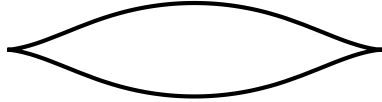


FIGURE 1. A co-orientable closed front having arbitrary small total absolute curvature (Example 4.1).

Hence, in this paper, we deal with non-co-orientable closed frontals in $\mathbb{R}^n$. More precisely, we prove the following Fenchel-type theorem.

Theorem A. *The total absolute curvature of a non-co-orientable closed frontal in $\mathbb{R}^n$ is greater than or equal to π . It is equal to π if and only if the curve is a planar locally L -convex frontal whose rotation index is equal to $1/2$ or $-1/2$.*

Here, the definitions of the local L -convexity and the rotation index are given in Section 2. In the case of regular curves, Fenchel's theorem says that every closed regular curve having minimum total absolute curvature 2π must be a simple closed curve. However, in the case of frontals, there exist non-simple examples having minimum total absolute curvature $K(\gamma) = \pi$, see Figure 2.

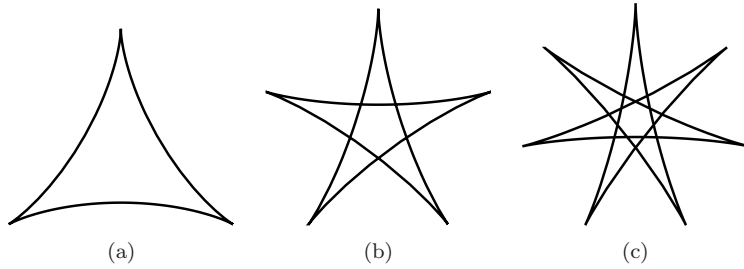


FIGURE 2. Closed non-co-orientable fronts with total absolute curvature $K(\gamma) = \pi$ in $\mathbb{R}^2$. All curves (a), (b) and (c) are given by hypocycloids (Example 4.2). Although the leftmost (a) is a simple closed curve, the others, (b) and (c), are not simple. All the singular points of these curves are cusp.

Thus, a natural question to consider is when a non-co-orientable closed frontal with minimum total absolute curvature becomes a simple closed curve. In this paper, we solve this question when the singularities are all *cusps*. Here, for a smooth map $\gamma : I \rightarrow \mathbb{R}^2$, a singular point $c \in I$ of γ is called *cusp* if the map-germ $\gamma : (I, c) \rightarrow (\mathbb{R}^2, \gamma(c))$ is right-left equivalent to the map-germ $t \mapsto (t^2, t^3)$. It is known that the cusp singularity is stable among the singularities of plane curves. We have the following.

Theorem B. *Let $\gamma : S^1 \rightarrow \mathbb{R}^2$ be a non-co-orientable closed front with minimum total absolute curvature $K(\gamma) = \pi$. Suppose that every singular point is a cusp. Then the number N of cusps is an odd integer satisfying $N \geq 3$. Moreover, $N = 3$ if and only if γ is simple.*

On the other hand, for a given convex closed curve, its focal set (i.e., the image of the caustic) admits singular points. Gounai-Umehara [14] classified the diffeomorphic type of focal sets of convex curves which admit at most four cusps under a generic assumption. Such the caustics are wave fronts having no inflection points. Theorem B can be regarded as a generalization of Gounai-Umehara's classification theorem in the case that the focal sets have exactly three singular points.

1.2. Organization of the paper. This paper is organized as follows. We prove Theorem A in Section 2, and Theorem B in Section 3. In Section 4, we exhibit some examples. In particular, we show an example of a co-orientable closed wave front with arbitrarily small total absolute curvature (Example 4.1), and non-co-orientable closed hypocycloids with total absolute curvature π (Example 4.2), as shown in Figure 2.

2. PROOF OF THEOREM A

In this section, we give a proof of Theorem A. First, we prove that for a non-co-orientable closed frontal $\gamma : S^1 \rightarrow \mathbb{R}^n$, the total absolute curvature $K(\gamma)$ satisfies

$K(\gamma) \geq \pi$, and that $K(\gamma) = \pi$ implies $\gamma(t)$ must be planar (Proposition 2.4). Next, after introducing the definitions of the local L -convexity and the rotation index, we prove that a non-co-orientable closed frontal $\gamma : S^1 \rightarrow \mathbb{R}^2$ with total absolute curvature $K(\gamma) = \pi$ must be locally L -convex and the rotation index is $\pm 1/2$, and vice versa (Proposition 2.7).

2.1. Length of tangent indicatrix. By (1.1) and (1.2), we have the following.

Lemma 2.1. *For a closed frontal $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^n$, the length*

$$\mathcal{L}(\mathbf{e}) = \int_0^{2\pi} \|\mathbf{e}'(t)\| dt$$

of $\mathbf{e} : [0, 2\pi] \rightarrow S^{n-1}$ as a spherical curve in S^{n-1} coincides with the total absolute curvature $K(\gamma)$ of the frontal γ .

The non-co-orientability yields the following.

Lemma 2.2. *Let $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^n$ be a non-co-orientable closed frontal with a unit tangent vector field $\mathbf{e} : [0, 2\pi] \rightarrow S^{n-1}$. Then, every great hypersphere of S^{n-1} intersects the image $\mathbf{e}([0, 2\pi])$.*

Proof. Let G be a great hypersphere of S^{n-1} . Then there exists a vector $\xi \in S^{n-1}$ such that $G = \{\mathbf{x} \in S^{n-1}; \mathbf{x} \cdot \xi = 0\}$. Hence, it suffices to prove that there exists $t_0 \in [0, 2\pi]$ such that $\mathbf{e}(t_0) \cdot \xi = 0$. We set $f : [0, 2\pi] \rightarrow \mathbb{R}$ by $f(t) = \mathbf{e}(t) \cdot \xi$. Since γ is non-co-orientable, $\mathbf{e}(0) = -\mathbf{e}(2\pi)$ holds and hence, we have $f(0) = -f(2\pi)$. If $f(0) = 0$, then the claim is true, so let us assume that $f(0) \neq 0$. Since $f(0)$ and $f(2\pi)$ have the opposite sign, the intermediate value theorem yields that there exists $t_0 \in [0, 2\pi]$ such that $f(t_0) = 0$, which proves the assertion. $\square$

We use the following fact called the *Rutishauser-Samelson lemma* [17]. This lemma is also introduced in an expository article written by S.S. Chern [6].

Fact 2.3 ([17], see also [6]). *Let $\mathbf{c} : S^1 \rightarrow S^{n-1}$ be a C^1 closed curve of length $\mathcal{L}(\mathbf{c})$. We denote by Γ the image $\mathbf{c}(S^1)$. Suppose that, for any great hypersphere G of S^{n-1} , the intersection $\Gamma \cap G$ is non-empty. Then $\mathcal{L}(\mathbf{c}) \geq 2\pi$ holds. Moreover, if $\mathcal{L}(\mathbf{c}) = 2\pi$, then Γ consists of two half-arcs of great circles.*

Now we have the following.

Proposition 2.4. *Let $\gamma : S^1 \rightarrow \mathbb{R}^n$ be a non-co-orientable closed frontal. Then, $K(\gamma) \geq \pi$ holds. Moreover, if the equality holds, then γ is planar.*

Proof. Let $\tilde{\gamma}$ and $\tilde{\mathbf{e}}$ be the extensions of γ and $\mathbf{e}$ so that their domains of definition are $[0, 4\pi]$. Then, the domain of definition of $\tilde{\mathbf{e}}$ is regarded as $\tilde{S}^1 = \mathbb{R}/4\pi\mathbb{Z}$, and the image $\tilde{\Gamma} = \tilde{\mathbf{e}}(\tilde{S}^1)$ defines a closed curve in S^{n-1} . From Lemma 2.2, $\tilde{\Gamma}$ intersects with any great hypersphere of S^{n-1} . Therefore, Fact 2.3 yields $K(\tilde{\gamma}) \geq 2\pi$. Since $K(\tilde{\gamma}) = 2K(\gamma)$, we have $K(\gamma) \geq \pi$. The equality holds if and only if $\tilde{\Gamma}$ consists of two half-arcs of great circles, that is, $\Gamma = \mathbf{e}([0, 2\pi])$ is a half-arc of a great circle. Therefore, the image of $\gamma(t)$ is a subset of a 2-dimensional subspace of $\mathbb{R}^n$, which proves the assertion. $\square$

2.2. Local L -convexity of planar frontals. On the regular set $\text{Reg}(\gamma)$ of a frontal $\gamma : I \rightarrow \mathbb{R}^2$, we set

$$\kappa(t) = \frac{1}{\|\gamma'\|^3} \det(\gamma', \gamma'').$$

To distinguish between $\kappa(t)$ and $k(t)$, we call $\kappa(t)$ the *oriented curvature function* of $\gamma(t)$. Since $\mathbf{e}(t) = \pm \gamma'(t)/\|\gamma'(t)\|$ holds on $\text{Reg}(\gamma)$, the oriented curvature function $\kappa(t)$ is written as $\kappa(t) = \tilde{\kappa}(t)/\|\gamma'(t)\|$, where we set

$$(2.1) \quad \tilde{\kappa}(t) = \det(\mathbf{e}(t), \mathbf{e}'(t)) \quad (\mathbf{e}' = d\mathbf{e}/dt).$$

Letting $ds = \|\gamma'(t)\| dt$ be the arclength measure of γ , we have that

$$(2.2) \quad \kappa ds = \tilde{\kappa} dt$$

is a bounded smooth 1-form even at a singular point. We call κds the *oriented curvature measure*.

Definition 2.5. A frontal $\gamma : I \rightarrow \mathbb{R}^2$ is called *locally L-convex*, if either $\tilde{\kappa} \geq 0$ or $\tilde{\kappa} \leq 0$ holds on I , where $\tilde{\kappa}$ is the function defined by (2.1).

Since the oriented curvature function κ is written as $\kappa = \tilde{\kappa}/\|\gamma'\|$ on the regular set $\text{Reg}(\gamma)$, the local L -convexity condition is a generalization of the local convexity of regular curves. We remark that Li-Wang [15] defined *ℓ -convex curves*, cf. [13]. An ℓ -convex curve is a closed wave front in $\mathbb{R}^2$ such that $\tilde{\kappa}$ has no zeros, and hence, ℓ -convex curves are locally L -convex.

Let $\gamma : S^1 \rightarrow \mathbb{R}^2$ be a non-co-orientable closed frontal with a unit tangent vector field $\mathbf{e} : [0, 2\pi] \rightarrow S^1$. We set a smooth function $\theta(t)$ as

$$(2.3) \quad \mathbf{e}(t) = (\cos \theta(t), \sin \theta(t)).$$

We call $\theta(t)$ the *angle function*. Since $\mathbf{e} : [0, 2\pi] \rightarrow S^1$ satisfies $\mathbf{e}(2\pi) = -\mathbf{e}(0)$, there exists an integer $m \in \mathbb{Z}$ such that $\theta(2\pi) - \theta(0) = (2m + 1)\pi$ holds. We call the half-integer

$$\text{ind}_\gamma = m + \frac{1}{2}$$

the *rotation index* of $\gamma(t)$. We remark that $2\pi \text{ind}_\gamma = \theta(2\pi) - \theta(0)$.

Lemma 2.6. For a frontal $\gamma : I \rightarrow \mathbb{R}^2$, we set a smooth function $\theta(t)$ as (2.3). Then $\theta'(t) = \tilde{\kappa}(t)$ holds.

Proof. Substituting (2.3) into (2.1), we obtain the desired result. $\square$

Proposition 2.7. Let $\gamma : S^1 \rightarrow \mathbb{R}^2$ be a non-co-orientable closed frontal. Then, the total absolute curvature $K(\gamma)$ is greater than or equal to π . It is equal to π if and only if γ is an L -convex frontal whose rotation index is equal to $1/2$ or $-1/2$.

Proof. Let $\mathbf{e} : [0, 2\pi] \rightarrow S^1$ be a unit tangent vector field along $\gamma(t)$. We set $\theta(t)$ the angle function of $\mathbf{e}(t)$ as in (2.3). The rotation index is written as $\text{ind}_\gamma = m + 1/2$, where $m \in \mathbb{Z}$ is an integer. By the triangle inequality, we have

$$K(\gamma) = \int_0^{2\pi} |\theta'(t)| dt \geq \left| \int_0^{2\pi} \theta'(t) dt \right| = |\theta(2\pi) - \theta(0)| \geq |2m + 1|\pi \geq \pi.$$

The equality holds if and only if $\theta'(t)$ has constant sign and $m = 0, -1$. Lemma 2.6 yields that $K(\gamma) = \pi$ holds if and only if $\gamma(t)$ is locally L -convex and $\text{ind}_\gamma = \pm 1/2$. $\square$

Proof of Theorem A. By Propositions 2.4 and 2.7, we have the assertion. $\square$

3. PROOF OF THEOREM B

In this section, we prove Theorem B. First, in subsection 3.1, we discuss the sign function of frontals having cusp singularities. Next, in subsection 3.2, several estimates of the total absolute curvature are given. Finally, in subsection 3.3, we give a proof of Theorem B.

3.1. Sign function. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a smooth curve. Then a singular point $t = c$ is called *cusp*, if there exist a coordinate change $t = t(s)$ and a local diffeomorphism Φ of $\mathbb{R}^2$ such that $t(0) = c$, $\Phi(\gamma(c)) = (0, 0)$ and $\Phi \circ \gamma(t(s)) = (s^2, s^3)$ hold.

It is well-known that a plane curve $\gamma(t)$ has cusp at $t = c$ if and only if

$$(3.1) \quad \gamma'(c) = \mathbf{0} \quad \text{and} \quad \det(\gamma''(c), \gamma'''(c)) \neq 0$$

hold (cf. [23, Theorem 1.3.2]). It is known that the curvature function κ diverges at cusp [23, Corollary 1.3.11].

Let $\gamma : I \rightarrow \mathbb{R}^2$ be a frontal equipped with a unit tangent vector field $\mathbf{e}$. On the regular set $\text{Reg}(\gamma)$, we set $T = \gamma' / \|\gamma'\|$. Then, we define a continuous function $\epsilon : \text{Reg}(\gamma) \rightarrow \{1, -1\}$ as

$$\epsilon(t) = T(t) \cdot \mathbf{e}(t).$$

We call $\epsilon(t)$ the *sign function*.

Lemma 3.1. *The sign function $\epsilon(t)$ changes its sign across cusp singularities.*

Proof. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a frontal having a cusp at $c \in I$. Without loss of generality, we may assume that $c = 0$. The criterion for the cusp (3.1) yields that $\gamma'(0) = \mathbf{0}$ and $\gamma''(0) \neq \mathbf{0}$ hold. Therefore, there exists a smooth map $\mathbf{x} : I \rightarrow \mathbb{R}^2$ such that $\gamma'(t) = t\mathbf{x}(t)$. Since $\gamma''(0) \neq \mathbf{0}$, we have $\mathbf{x} \neq \mathbf{0}$ on a neighborhood of $t = 0$. We set $E = \mathbf{x} / \|\mathbf{x}\|$. Then $T = \gamma' / \|\gamma'\|$ is written as $T(t) = \text{sgn}(t) E(t)$. On the other hand, either

$$(i) \quad \mathbf{e}(t) = E(t) \quad \text{or} \quad (ii) \quad \mathbf{e}(t) = -E(t)$$

holds. In the case (i), we have $\epsilon(t) = \text{sgn}(t)$. In the case (ii), we have $\epsilon(t) = -\text{sgn}(t)$. Therefore, the assertion holds. $\square$

Corollary 3.2. *Let $\gamma(t)$ be a front in $\mathbb{R}^2$ defined on a bounded closed interval $[a, b]$ ($a < b$), and let j be a positive integer. Suppose that every singular point is cusp, and the restriction $\gamma|_{(a,b)}$ has exactly j singular points. Then the sign function $\epsilon(t)$ satisfies $\epsilon(a) = (-1)^j \epsilon(b)$, where we set $\epsilon(a) = \lim_{t \rightarrow a+0} \epsilon(t)$ and $\epsilon(b) = \lim_{t \rightarrow b-0} \epsilon(t)$.*

Proof. By Lemma 3.1, we have that $\epsilon(t)$ change its sign at cusps. Hence, $\epsilon(a) = \epsilon(b)$ (resp. $\epsilon(a) = -\epsilon(b)$), if j is even (resp. odd). $\square$

Lemma 3.3. *Let $\gamma(t)$ be a frontal in $\mathbb{R}^2$ defined on a bounded closed interval $[a, b]$ ($a < b$). Then it holds that $K(\gamma) \geq \arccos(\mathbf{e}(a) \cdot \mathbf{e}(b))$.*

Proof. By a proof similar to Proposition 2.7, We have $K(\gamma) \geq |\theta(a) - \theta(b)|$. Since $|\theta(a) - \theta(b)| \geq \arccos(\mathbf{e}(a) \cdot \mathbf{e}(b))$, we have the desired result. $\square$

3.2. Angle at the endpoints of frontals. The following fact is known (see [1, Proposition 3.21], for example), cf. Figure 3.

Fact 3.4. *Let $\gamma(t)$ be a frontal in $\mathbb{R}^2$ defined on a bounded closed interval $[a, b]$ ($a < b$). If $\gamma(a) = \gamma(b)$, and the restriction $\gamma|_{(a,b)}$ is a regular curve, then $K(\gamma) > \pi$ holds.*

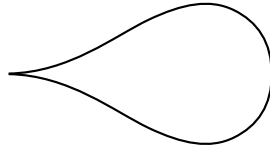


FIGURE 3. A frontal having the same endpoints but with no singular point in the interior.

We generalize Fact 3.4 to frontals (Propositions 3.6 and 3.7).

Definition 3.5. Let $\gamma(t)$ be a frontal in $\mathbb{R}^2$ defined on a bounded closed interval $[a, b]$ ($a < b$). Suppose that $\gamma(a) = \gamma(b)$, and the restriction $\gamma|_{(a,b)}$ has finite singular points. Then the number $\varphi \in [0, \pi]$ defined by

$$\varphi = \arccos(-T(a) \cdot T(b))$$

is called the *angle at the endpoints* of γ , where $T = \gamma'/\|\gamma'\|$ and

$$T(a) = \lim_{t \rightarrow a+0} T(t), \quad T(b) = \lim_{t \rightarrow b-0} T(t).$$

Proposition 3.6. Let $\gamma(t)$ be a frontal in $\mathbb{R}^2$ defined on a bounded closed interval $[a, b]$ ($a < b$). Suppose that every singular point is cusp, $\gamma(a) = \gamma(b)$, and the restriction $\gamma|_{(a,b)}$ has exactly two singular points. Then $K(\gamma) \geq \pi - \varphi$ holds, where $\varphi \in [0, \pi]$ is the angle at endpoints of γ .

Proof. By Corollary 3.2, we have $\epsilon(a) = \epsilon(b)$. Since $T(t) = \epsilon(t)\mathbf{e}(t)$, it holds that $T(a) \cdot T(b) = (\epsilon(a)\mathbf{e}(a)) \cdot (\epsilon(b)\mathbf{e}(b)) = \mathbf{e}(a) \cdot \mathbf{e}(b)$. Hence we have

$$\varphi = \arccos(-T(a) \cdot T(b)) = \arccos(-\mathbf{e}(a) \cdot \mathbf{e}(b)) = \pi - \arccos(\mathbf{e}(a) \cdot \mathbf{e}(b)).$$

Since the length $\mathcal{L}(\mathbf{e})$ of $\mathbf{e}$ satisfies $\mathcal{L}(\mathbf{e}) \geq \arccos(\mathbf{e}(a) \cdot \mathbf{e}(b))$, Lemma 2.1 yields that $K(\gamma) = \mathcal{L}(\mathbf{e}) \geq \pi - \varphi$. $\square$

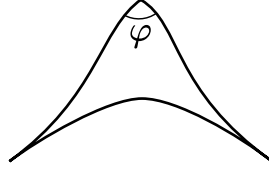


FIGURE 4. A frontal having the same endpoints but with two singular points in the interior.

Proposition 3.7. Let $\gamma(t)$ be a frontal in $\mathbb{R}^2$ defined on a bounded closed interval $[a, b]$ ($a < b$). Suppose that every singular point is cusp, $\gamma(a) = \gamma(b)$, and the restriction $\gamma|_{(a,b)}$ has exactly one singular point. Then $K(\gamma) > \varphi$ holds, where $\varphi \in [0, \pi]$ is the angle at endpoints of γ .

Proof. Let $c \in (a, b)$ be the singular point of $\gamma(t)$. By Corollary 3.2, we have $\epsilon(a) = -\epsilon(b)$. Since $T(t) = \epsilon(t)\mathbf{e}(t)$, it holds that

$$\varphi = \arccos(-T(a) \cdot T(b)) = \arccos(\mathbf{e}(a) \cdot \mathbf{e}(b)).$$

If necessary, by a rotation and translation of $\mathbb{R}^2$, we may assume that $\gamma(c) = (0, 0)$ and $\mathbf{e}(c) = (0, 1)$ hold. We denote by $\gamma(t) = (x(t), y(t))$. Since $x(t)$ is a continuous function on a closed interval $[a, b]$, there exist $t_1, t_2 \in [a, b]$ such that $\max_{t \in [a, b]} x(t) = x(t_1)$ and $\min_{t \in [a, b]} x(t) = x(t_2)$ hold. If $t_1 = t_2$, then $\max_{t \in [a, b]} x(t) = \min_{t \in [a, b]} x(t)$ holds. Hence we have $x(t)$ is identically zero, and the image of $\gamma(t)$ is a subset of the y -axis. Then the curvature function is identically zero, which contradicts the fact that the curvature diverges at cusps ([23, Corollary 1.3.11]). Hence $t_1 \neq t_2$ holds. If necessary, by changing the orientation of t , we may assume that $t_1 < t_2$.

Also we have that $t_1, t_2 \neq c$. In fact, if $t_1 = c$, then $x(t) \leq 0$ holds. The tangent line¹ of $\gamma(t)$ at the cusp $t = c$ divides the curve $\gamma(t)$ into two parts, one on each

¹The tangent line of a front at a cusp is also called the *centerline* [23, Corollary 1.3.11].

side ([23, Corollary 1.3.11]). Since the centerline of $\gamma(t)$ at $t = c$ is the y -axis, [23, Corollary 1.3.11] yields a contradiction, and hence $t_1 \neq c$. Similarly, we have $t_2 \neq c$.

We set

$$\arccos(\mathbf{e}(a) \cdot \mathbf{e}(c)) = \varphi_a, \quad \arccos(\mathbf{e}(b) \cdot \mathbf{e}(c)) = \varphi_b.$$

Then, $\varphi_a + \varphi_b \geq \varphi$ holds.

(I) The case of $a < t_1 < c < t_2 < b$. The total absolute curvature $K(\gamma)$ is divided into

$$(3.2) \quad K(\gamma) = K(\gamma|_{[a,t_1]}) + K(\gamma|_{[t_1,c]}) + K(\gamma|_{[c,t_2]}) + K(\gamma|_{[t_2,b]}).$$

Then it holds that

$$(3.3) \quad K(\gamma|_{[t_1,c]}) > 0 \quad \text{and} \quad K(\gamma|_{[c,t_2]}) > 0.$$

In fact, if we suppose $K(\gamma|_{[t_1,c]}) = 0$, then $\kappa(t)$ is identically zero on $[t_1, c]$, which contradicts the fact that the curvature diverges at cusps ([23, Corollary 1.3.11]). Hence, we have $K(\gamma|_{[t_1,c]}) > 0$. Similarly, $K(\gamma|_{[c,t_2]}) > 0$ holds.

Since $x'(t_1) = x'(t_2) = 0$, we have $\mathbf{e} = (0, \pm 1)$ at t_1, t_2 , namely, $\mathbf{e}(t_i) = \pm \mathbf{e}(c)$ holds ($i = 1, 2$). Thus, either (I-i), (I-ii), (I-iii) or (I-iv) occurs, cf. Figure 5.

(I-i) If $\mathbf{e}(t_1) = \mathbf{e}(t_2) = \mathbf{e}(c)$, $K(\gamma|_{[a,t_1]}) \geq \varphi_a$ and $K(\gamma|_{[t_2,b]}) \geq \varphi_b$ hold. By (3.2) and (3.3), we have $K(\gamma) > K(\gamma|_{[a,t_1]}) + K(\gamma|_{[t_2,b]}) \geq \varphi_a + \varphi_b \geq \varphi$.

(I-ii) If $\mathbf{e}(t_1) = -\mathbf{e}(t_2) = \mathbf{e}(c)$, Lemma 3.3 yields $K(\gamma|_{[c,t_2]}) \geq \arccos(\mathbf{e}(c) \cdot \mathbf{e}(t_2)) = \pi$. By (3.2) and (3.3), it holds that $K(\gamma) > \pi \geq \varphi$.

(I-iii) If $-\mathbf{e}(t_1) = \mathbf{e}(t_2) = \mathbf{e}(c)$, as in the case (I-ii), we have $K(\gamma|_{[t_1,c]}) \geq \pi$. By (3.2) and (3.3), it holds that $K(\gamma) > \pi \geq \varphi$.

(I-iv) If $\mathbf{e}(t_1) = \mathbf{e}(t_2) = -\mathbf{e}(c)$, as in the cases (I-ii) and (I-iii), we have $K(\gamma|_{[t_1,c]}) \geq \pi$. By (3.2) and (3.3), it holds that $K(\gamma) > \pi \geq \varphi$.

Therefore, in all cases (I-i), (I-ii), (I-iii), and (I-iv), we have $K(\gamma) > \varphi$.

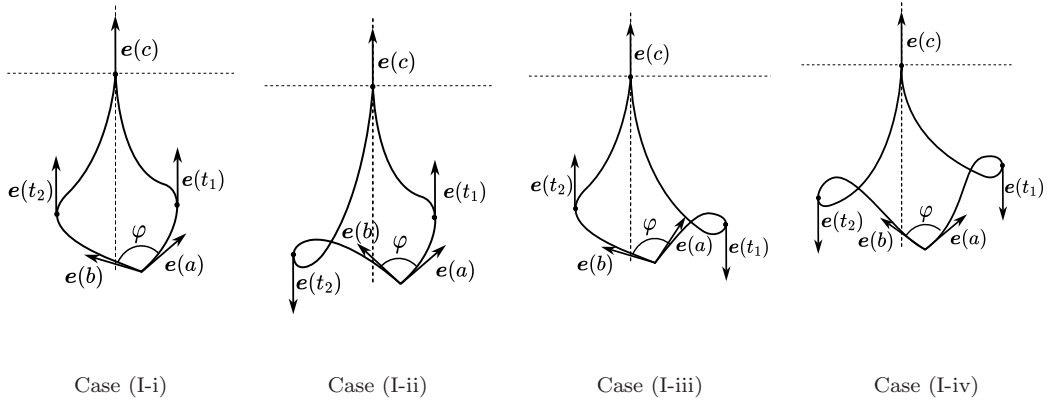


FIGURE 5. Frontals having the same endpoints but with one singular point in the interior, the case (I).

(II) The case of $a < t_1 < t_2 < c < b$. The total absolute curvature $K(\gamma)$ is written as $K(\gamma) = K(\gamma|_{[a,t_1]}) + K(\gamma|_{[t_1,t_2]}) + K(\gamma|_{[t_2,c]}) + K(\gamma|_{[c,b]})$. As in (3.3), we have

$$(3.4) \quad K(\gamma|_{[t_2,c]}) > 0.$$

Similarly,

$$(3.5) \quad K(\gamma|_{[t_1,t_2]}) > 0$$

holds. Then either (II-i), (II-ii), (II-iii) or (II-iv) occurs, cf. Figure 6.

- (II-i) If $\mathbf{e}(t_1) = \mathbf{e}(t_2) = \mathbf{e}(c)$, $K(\gamma|_{[a,t_1]}) \geq \varphi_a$ and $K(\gamma|_{[c,b]}) \geq \varphi_b$ hold. By (3.4), we have $K(\gamma) > K(\gamma|_{[a,t_1]}) + K(\gamma|_{[c,b]}) \geq \varphi_a + \varphi_b \geq \varphi$.
- (II-ii) If $\mathbf{e}(t_1) = -\mathbf{e}(t_2) = \mathbf{e}(c)$, Lemma 3.3 yields $K(\gamma|_{[t_1,t_2]}) \geq \arccos(\mathbf{e}(t_1) \cdot \mathbf{e}(t_2)) = \pi$ and $K(\gamma|_{[c,t_2]}) \geq \arccos(\mathbf{e}(c) \cdot \mathbf{e}(t_2)) = \pi$. Hence, it holds that $K(\gamma) \geq 2\pi > \varphi$.
- (II-iii) If $-\mathbf{e}(t_1) = \mathbf{e}(t_2) = \mathbf{e}(c)$, as in the case (II-ii), we have $K(\gamma|_{[t_1,t_2]}) \geq \pi$. By (3.4), it holds that $K(\gamma) > \pi \geq \varphi$.
- (II-iv) If $\mathbf{e}(t_1) = \mathbf{e}(t_2) = -\mathbf{e}(c)$, as in the cases (II-ii) and (II-iii), we have $K(\gamma|_{[t_2,c]}) \geq \pi$. By (3.5), it holds that $K(\gamma) > \pi \geq \varphi$.

Therefore, in all cases (II-i), (II-ii), (II-iii), and (II-iv), we have $K(\gamma) > \varphi$.

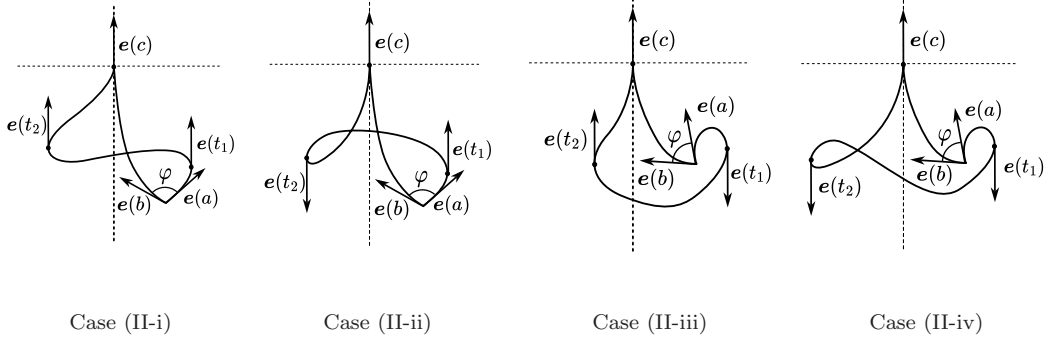


FIGURE 6. Frontals having the same endpoints but with one singular point in the interior, the case (II).

(III) The case of $a = t_1 < c < t_2 < b$. The total absolute curvature $K(\gamma)$ is written as $K(\gamma) = K(\gamma|_{[a,c]}) + K(\gamma|_{[c,t_2]}) + K(\gamma|_{[t_2,b]})$. As in (3.3), we have $K(\gamma|_{[c,t_2]}) > 0$ and $K(\gamma|_{[a,c]}) > 0$. As in the cases (I), (II), since $x'(t_2) = 0$, we have $\mathbf{e}(t_2) = \pm \mathbf{e}(c)$. Thus, either (III-i) or (III-ii) occurs, cf. Figure 7.

- (III-i) If $\mathbf{e}(t_2) = \mathbf{e}(c)$, $K(\gamma|_{[a,c]}) \geq \varphi_a$ and $K(\gamma|_{[t_2,b]}) \geq \varphi_b$ hold. By $K(\gamma|_{[c,t_2]}) > 0$, we have $K(\gamma) > K(\gamma|_{[a,c]}) + K(\gamma|_{[t_2,b]}) \geq \varphi_a + \varphi_b \geq \varphi$.
- (III-ii) If $\mathbf{e}(t_2) = -\mathbf{e}(c)$, we have $K(\gamma|_{[c,t_2]}) \geq \arccos(\mathbf{e}(c) \cdot \mathbf{e}(t_2)) = \pi$. Hence, by $K(\gamma|_{[a,c]}) > 0$, we have $K(\gamma) > \pi \geq \varphi$.

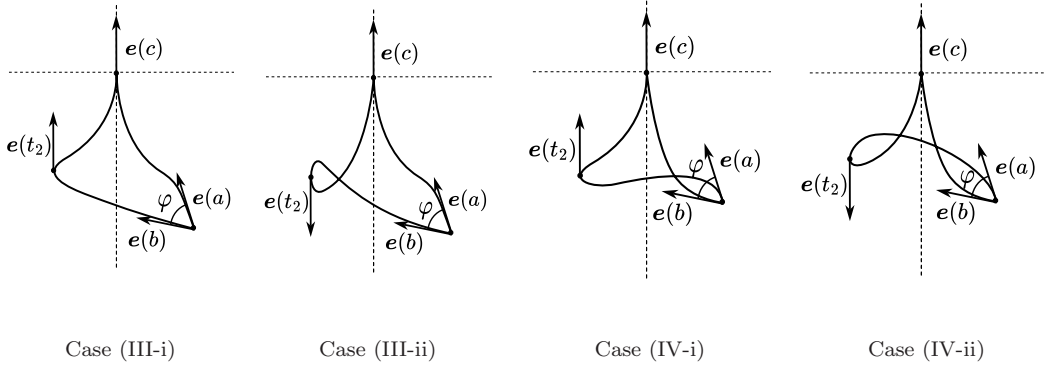


FIGURE 7. Frontals having the same endpoints but with one singular point in the interior, the cases (III) and (IV).

(IV) The case of $a = t_1 < t_2 < c < b$. The total absolute curvature $K(\gamma)$ is written as $K(\gamma) = K(\gamma|_{[a,t_2]}) + K(\gamma|_{[t_2,c]}) + K(\gamma|_{[c,b]})$. As in (3.3), we have $K(\gamma|_{[t_2,c]}) > 0$ and $K(\gamma|_{[c,b]}) > 0$.

As in the cases (I), (II), (III), since $x'(t_2) = 0$, we have $e(t_2) = \pm e(c)$. Thus, either (IV-i) or (IV-ii) occurs, cf. Figure 7.

(IV-i) If $e(t_2) = e(c)$, $K(\gamma|_{[a,t_2]}) \geq \varphi_a$ and $K(\gamma|_{[c,b]}) \geq \varphi_b$ hold. By $K(\gamma|_{[t_2,c]}) > 0$, we have $K(\gamma) > K(\gamma|_{[a,t_2]}) + K(\gamma|_{[c,b]}) \geq \varphi_a + \varphi_b \geq \varphi$.

(IV-ii) If $e(t_2) = -e(c)$, Lemma 3.3 yields $K(\gamma|_{[t_2,c]}) \geq \arccos(e(t_2) \cdot e(c)) = \pi$. By $K(\gamma|_{[c,b]}) > 0$, we have $K(\gamma) > \pi \geq \varphi$.

(V) The case of $a < t_1 < c < t_2 = b$. By an argument similar to the case (III), we have $K(\gamma) > \varphi$, cf. Figure 8.

(VI) The case of $a < c < t_1 < t_2 = b$. By an argument similar to the case (IV), we have $K(\gamma) > \varphi$, cf. Figure 8.

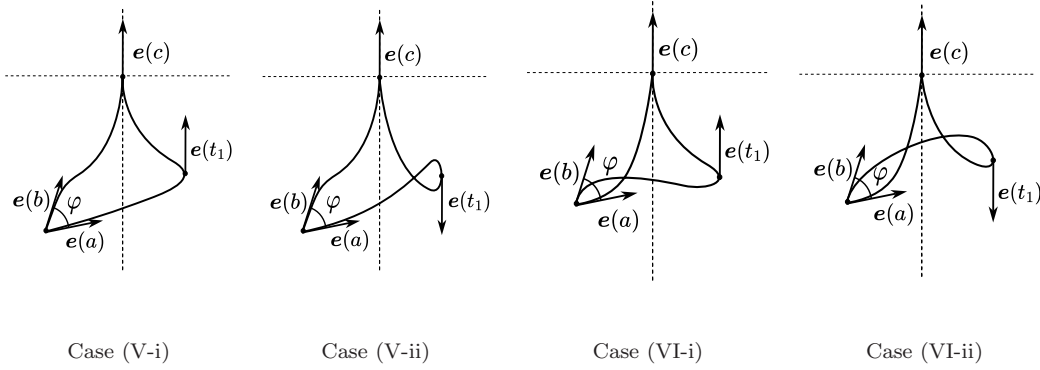


FIGURE 8. Frontals having the same endpoints but with one singular point in the interior, the cases (V) and (VI).

By the cases (I)–(VI), we have $K(\gamma) > \varphi$. □

3.3. Simplicity and the number of singular points. Using the estimates of the total absolute curvature in Fact 3.4 and Propositions 3.6, 3.7, we prove the following.

Theorem 3.8. *Let $\gamma : S^1 \rightarrow \mathbb{R}^2$ be a non-co-orientable closed front satisfying $K(\gamma) = \pi$. If every singular point of γ is cusp, and the number of the singular points of γ is 3, then γ is a simple closed curve.*

Proof. Suppose $t = 0, c_1, c_2$ ($0 < c_1 < c_2 < 2\pi$) are cusps. We prove that $\gamma : [0, 2\pi) \rightarrow \mathbb{R}^2$ is injective by contradiction. So, we assume that there exists $a, b \in [0, 2\pi)$ ($a < b$) satisfies $\gamma(a) = \gamma(b)$. By a translation of t , if necessary, we may suppose that $0 \leq a < c_1$. Either one of the cases (1), (2) or (3) occurs, cf. Figure 9.

- (1) In the case of $a < b \leq c_1$, since the restriction $\gamma|_{[a,b]}$ satisfies the assumption of Fact 3.4, we have $K(\gamma) \geq \int_a^b |\kappa| ds > \pi$. Similarly, in the cases of $a = 0$, $c_2 < b \leq 2\pi$, or $0 < a < c_1$, $c_2 < b \leq 2\pi$, the restriction $\gamma|_{[b, 2\pi+a]}$ satisfies the assumption of Fact 3.4, and hence, we have $K(\gamma) > \pi$. See Figure 9 (1).
- (2) In the case of $0 < a < c_1$ and $c_1 < b < c_2$, We set $\gamma_1 = \gamma|_{[a,b]}$ and $\gamma_2 = \gamma|_{[b, 2\pi+a]}$. We let $\varphi \in [0, \pi]$ be the angle at the endpoints of γ_1 . By Proposition 3.7, we have $K(\gamma_1) > \varphi$. On the other hand, the angle at the endpoints of γ_2 is written as φ . Thus, by Proposition 3.6, we have $K(\gamma_2) \geq \pi - \varphi$. Hence, we

obtain $K(\gamma) = K(\gamma_1) + K(\gamma_2) > \pi$. The case of $0 < a < c_1$ and $c_2 < b < 2\pi$ is proved in the same way. See Figure 9 (2).

- (3) In the case of $a = 0$ and $c_1 < b < c_2$, we set $\gamma_1 = \gamma|_{[a,b]}$ and $\gamma_2 = \gamma|_{[b,2\pi]}$. We let $\varphi \in [0, \pi]$ be the angle at the endpoints of γ_1 . Then $\varphi = \arccos(-T(a) \cdot T(b))$. By Proposition 3.7, we have $K(\gamma_1) > \varphi$. On the other hand, the angle at the endpoints of γ_2 is written as

$$\arccos(-T(b) \cdot T(2\pi)) = \arccos(T(a) \cdot T(b)) = \pi - \varphi.$$

Thus, by Proposition 3.7, we have $K(\gamma_2) > \pi - \varphi$. Hence, we obtain $K(\gamma) = K(\gamma_1) + K(\gamma_2) > \pi$. The case of $0 < a < c_1$ and $b = c_2$ is proved in the same way, by setting $\gamma_1 = \gamma|_{[a,b]}$, $\gamma_2 = \gamma|_{[b,2\pi+a]}$. See Figure 9 (3).

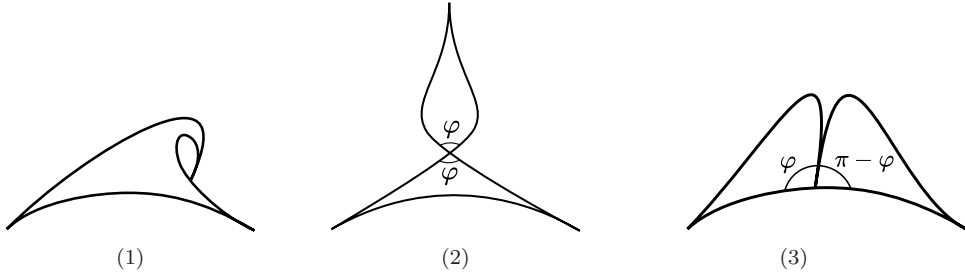


FIGURE 9. Possible curve shapes when having self-intersections.

Therefore, by the cases (1), (2) and (3), we have $K(\gamma) > \pi$, a contradiction. $\square$

Proof of Theorem B. Let $\gamma : S^1 \rightarrow \mathbb{R}^2$ be a non-co-orientable closed front with $K(\gamma) = \pi$. Suppose that every singular point is a cusp. By the non-co-orientability of γ , the number N of cusps is an odd integer (for example, see [23]). If $N = 1$, then Fact 3.4 yields that $K(\gamma) > \pi$, a contradiction. Hence, we have $N \geq 3$.

By Theorem 3.8, it suffices to prove that if $\gamma : S^1 \rightarrow \mathbb{R}^2$ is simple then $N = 3$. Denote by Ω the interior domain of γ . Changing the orientation of $\gamma(t)$, if necessary, we may assume that the left-hand side of $\gamma(t)$ is the interior domain Ω . Let $c_j \in [0, 2\pi)$ ($j = 1, \dots, N$) be the singular points of $\gamma(t)$, and $\angle C_j$ be the interior angle of $\gamma(t)$ at $C_j = \gamma(c_j)$ for $j = 1, \dots, N$. Then, by the Gauss-Bonnet formula, we have

$$(3.6) \quad \sum_{j=1}^N \angle C_j = (N-2)\pi + \int_{\partial\Omega} \kappa ds.$$

Since every singular point of $\gamma(t)$ is cusp, we have $\angle C_j = 0$ for $j = 1, \dots, N$. Moreover, as $\gamma(t)$ is locally L -convex, we have $\tilde{\kappa}(t) \geq 0$ or $\tilde{\kappa}(t) \leq 0$. Thus, we may set $\tilde{\kappa}(t) = \sigma |\tilde{\kappa}(t)|$ where $\sigma \in \{+, -\}$. Then

$$\int_{\partial\Omega} \kappa ds = \int_{S^1} \tilde{\kappa} dt = \sigma \int_{S^1} |\tilde{\kappa}| dt = \sigma \pi$$

holds. Then, (3.6) yields that $0 = (N-2)\pi + \sigma\pi$. If $\sigma = +$, we have $N = 1$. By Fact 3.4, $K(\gamma) > \pi$, a contradiction. Hence, we have $\sigma = -$, and $N = 3$ holds. $\square$

4. EXAMPLES

If a frontal γ admits a cusp singularity, then κ is unbounded, and hence, $K(\gamma) > 0$ holds. However, there are examples of a closed front having cusps whose total absolute curvature $K(\gamma)$ is less than a given positive number.

Example 4.1 (Eye-shaped closed frontal). For a positive number a , we set

$$(4.1) \quad \gamma_a(t) = \frac{1}{5 - 3 \cos 2t} (3 \cos t - \cos 3t, 4a \sin^3 t).$$

Then, $\gamma_a(t)$ is a 2π -periodic front with unit tangent vector field

$$\mathbf{e}(t) = \frac{1}{\sqrt{G(t, a)}} \left(-(2 \sin^4 t + \sin^2 t + 1), a \sin 2t (\sin^2 t + 1) \right),$$

where we set $H(t) = 4 \sin^8 t + 4 \sin^6 t + 5 \sin^4 t + 2 \sin^2 t + 1$ and $G(t, a) = H(t) + 4a^2 \sin^2 t (-\sin^6 t - \sin^4 t + \sin^2 t + 1)$. We remark that $G(t, a) > 0$ for each $a > 0$ and $t \in \mathbb{R}$. Since $\mathbf{e}(t + 2\pi) = \mathbf{e}(t)$ holds for each t , the front γ_a is co-orientable. Every singular point is given by $t = m\pi$ for an integer m . Since $\gamma'_a(m\pi) = 0$ and $\det(\gamma''_a(m\pi), \gamma'''_a(m\pi)) = -36a$, every singular point is cusp.

For a given positive number ε , we set $a = \varepsilon/M$, where $M = \int_0^{2\pi} \frac{F(t)}{H(t)} dt (> 0)$. We also set $F(t) = |3 \cos 2t - 1|(5 - 3 \cos 2t)$. Since $G(t, a) \geq H(t)$, the total absolute curvature $K(\gamma_a)$ satisfies

$$K(\gamma_a) = \frac{a}{2} \int_0^{2\pi} \frac{F(t)}{G(t, a)} dt < \frac{a}{2} \int_0^{2\pi} \frac{F(t)}{H(t)} dt < \varepsilon.$$

Therefore, γ_a is a closed co-orientable front whose singular set consists of cusps, and its total absolute curvature $K(\gamma_a)$ is less than a given positive number.

Example 4.2 (Hypocycloid). Let m be a positive integer. We set $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ as

$$\gamma(t) = (m \cos(m+1)t + (m+1) \cos mt, m \sin(m+1)t - (m+1) \sin mt).$$

For each singular point $t = c$, it holds that $\gamma'(c) = 0$ and $\det(\gamma''(c), \gamma'''(c)) \neq 0$. Every singular point is a cusp. Hence, γ is a closed front with period 2π . Since $\mathbf{e}(t) = (\cos(t/2), \sin(t/2))$ gives a unit tangent vector field satisfying $\mathbf{e}(t + 2\pi) = -\mathbf{e}(t)$, γ is non-co-orientable. Then the total absolute curvature $K(\gamma)$ is given by

$$K(\gamma) = \int_0^{2\pi} |\kappa(t)| \|\gamma'(t)\| dt = \pi.$$

For the figure of these hypocycloids, see Figure 2. Figure 2 (a), (b) and (c) are the figures of the hypocycloids of $m = 1, 2$ and 3 , respectively. In these cases, the total absolute curvature $K(\gamma)$ is π .

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