

SHARP SPECTRAL GAP OF ADAPTIVE LANGEVIN DYNAMICS

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ABSTRACT. We consider a degenerated Fokker-Planck type differential operator associated to an adaptive Langevin dynamic. We prove Eyring–Kramers formulas for the bottom of the spectrum of this operator in the low temperature regime. The main ingredients are resolvent estimates obtained via hypocoercive techniques and the construction of sharp Gaussian quasimodes through an adaptation of the WKB method.

1. INTRODUCTION

1.1. **Motivations.** In order to describe the dynamic that rules the evolution of a molecular system at temperature of order $h > 0$, the following homogeneous Langevin process is widely used

$$(1.1) \quad dX_t = \xi(X_t)dt + \sqrt{2h}\sigma(X_t)dB_t$$

where $(X_t)_{t \geq 0}$ gives the positions of the particles, the vector field ξ is the drift coefficient, the matrix field σ is the diffusion coefficient and $(B_t)_{t \geq 0}$ denotes a d -dimensional Brownian motion. In the low temperature regime, i.e. $h \rightarrow 0$, we observe metastable behaviour of the solution of (1.1), this can be obtained via the study of the exit problem for this SDE. Considering an open set Ω and a point $x \in \Omega$ the question is to know where and when does the process exit Ω having set $X_0 = x$. This problem has been intensively studied in [12] or [7] for some pioneer work, we also refer to [8], [30] and [25] for recent progress concerning this question.

Another approach is to look at the Fokker-Planck associated to (1.1). Considering

$$\mathcal{L} = h \sum_{i,j} a_{i,j} \partial_i \partial_j + \sum_k \xi_k \partial_k,$$

where $(a_{i,j})_{i,j} = \sigma \sigma^T$, the infinitesimal generator associated to the semigroup solving the following PDE

$$(1.2) \quad \begin{cases} \partial_t u - \mathcal{L}u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

whose solution is $u(t, x) = \mathbb{E}(u_0(X_t) | X_0 = x)$. Its adjoint problem is

$$(1.3) \quad \begin{cases} \partial_t \psi - \mathcal{L}^* \psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

where A^* denotes the formal adjoint of any differential operator A . It is this last equation which is mainly called the Fokker-Planck equation whose solution is given by

the density $\psi(t, X)$ that the random variable X_t follows when it makes sense. Therefore studying \mathcal{L} or \mathcal{L}^* is a good way to obtain results concerning X_t .

A main question about this PDE is its resolution, implied by the maximal accretivity of $-\mathcal{L}$ which is a real problem when it is not self-adjoint. In order to study the long time behaviour of (1.2) an efficient strategy is to study the spectral properties of \mathcal{L} and particularly its smallest eigenvalues when they have non-negative real part. Determining its spectral gap informs us on equilibrium states and the metastability of (1.1). When considering self-adjoint operators, the spectral Theorem directly implies the decreasing in time of the solution of (1.2), but it needs some more results otherwise. Although we will not go that far in this paper, let us mention that the non-self-adjoint setting yields consequent additional difficulties which are solved using the Gearhart-Prüss Theorem. In [17] and [18], the authors have established a quantitative version of this Theorem uniform in h which is the main argument to prove for example [1, Corollary 1.5 and 1.6] or likewise [31, Corollary 1.8 and 1.9].

The SDE (1.1) and its generator have been largely studied in the past decades, in particular when $h \rightarrow 0$ and under more assumptions : for example taking $\xi = -\nabla V$ for $V : \mathbb{R}^d \rightarrow \mathbb{R}$ a potential and $\sigma = \text{Id}$ we recover the overdamped Langevin process

$$dX_t = -\nabla V(X_t) + \sqrt{2h}dB_t,$$

whose generator is the Kramers-Smoluchowski operator

$$(1.4) \quad \mathcal{L} = h\Delta - \nabla V \cdot \nabla$$

which is conjugated to the Witten Laplacian :

$$-e^{-V/2h}h\mathcal{L}e^{V/2h} = \Delta_{\frac{V}{2}} := -h^2\Delta + \frac{1}{4}|\nabla V|^2 - \frac{h}{2}\Delta V.$$

Introduced in [35] in order to prove the Morse inequalities, this Laplacian is a positive self-adjoint operator that arises in many different domains such as control theory [24] or dynamical system [6]. As discussed previously, one main goal is to determine its spectrum (which we already know is included in \mathbb{R}_+) and especially the bottom of its spectrum for metastability questions and exit time estimates. At first, only estimates about the order of the bottom of the spectrum were proved in [16]. They obtained that Δ_V has as much low lying eigenvalues as V has minima and these eigenvalues satisfy the bound :

$$\lambda = O(e^{-c/h}),$$

for a certain $c > 0$ uniform in h . No more could have been proven on the small eigenvalues for 20 years because of the topological restrains, but in [2] and [15] this barrier has been crossed and sharp estimates have been proven, obtaining the right order and describing precisely the prefactor

$$\lambda_j = z_j e^{-2S_j/h} (1 + o(1))$$

with explicit $z_j > 0$ and $S_j > 0$ and $1 \leq j \leq n_0$ where n_0 denotes the number of minima of V . These kind of formulae go back to pioneer work [23], [11] and are called Eyring-Kramers laws. Through functional analysis for self-adjoint operators, such

estimates induce results about convergence rates of the semigroup associated to (1.4), the return to equilibrium being in $1/\lambda_2$ (because $\lambda_1 = 0$). More precisely, due to the different S_j , we seem to have a sort of stability of the system during exponentially large time intervals corresponding to the inverse of the λ_j , each one around the eigenfunction associated to λ_j , it is this phenomenon which is called metastability.

It has also been proven (see [7], [12], [30], [25]) that for this special operator, its first positive eigenvalue is the inverse of the mean exit time of the processus solving (1.1) in the boundary case.

These works and approaches which were doing just fine with this operator (1.4) does not directly apply to non-self-adjoint operators which arise naturally from the homogeneous Langevin process. For example in \mathbb{R}^{2d} taking $V : \mathbb{R}^d \rightarrow \mathbb{R}$ a potential, $\xi(x, v) = (v, -\nabla V(x) - v)$, $\gamma > 0$ a friction coefficient and $\sigma = 0_x \oplus \text{Id}_v$ we obtain

$$(1.5) \quad \begin{cases} dx_t = v_t dt, \\ dv_t = (-\nabla V(x_t) - \gamma v_t) dt + \sqrt{2\gamma h} dB_t, \end{cases}$$

whose generator is the Kramers-Fokker-Planck operator

$$\mathcal{L} = v \cdot \partial_x - \partial_x V \cdot \partial_v + \gamma(h\Delta_v - v \cdot \partial_v)$$

where Δ_v denotes the Laplacian acting only on the v coordinates.

Due to the lack of self-adjointness because of the hamiltonian part $v \cdot \partial_x - \partial_x V \cdot \partial_v$, the previous method had to be adapted, the main issue was resolvent estimates which were not free anymore. Through microlocal analysis, this problem was first solved in the non-semiclassical framework [21] and then in the semiclassical one [20].

Equation (1.1) is used not only for modelling particle system, but its efficiency has been spread to other domains such as molecular dynamics [34] or high dimensional data analysis [4]. The main advantage using the Langevin dynamic for numerical simulations instead of a usual Monte Carlo random walk is the use of the gradient of the potential V , which result in less wasted computation [32], [3]. Despite this benefit, it requires some precise information about that gradient which is a very challenging task. Thus the Adaptive Langevin dynamic was introduced in [22], [27] from a fusion of a deterministic Nosé-Hoover scheme and a more usual overdamped Langevin process in order to reduce the needed knowledge about the gradient of V . By that time they mainly show their results through numerical simulations (we refer to [28] for further details about the numerical study and modelisation of these processes). The adaptive Langevin dynamic was next studied in [26] at fixed temperature where the authors determines some of its properties, namely spectral gap (see [26, Theorem 2.1]) using hypocoercive estimates. The main question we want to address in this paper is to study how these properties depend on the semiclassical parameter h . In short, we model the gradient noise by another stochastic process which results in adding another unknown Brownian motion (that we can combine with the already existing one). But in order to retrieve the standard Gibbs state, we need to consider the friction coefficient to be

a new variable, and all this leads to a slight modification in the SDE (1.5) :

$$(1.6) \quad \begin{cases} dx_t = v_t dt, \\ dv_t = (-\nabla V(x_t) - \nu y_t v_t - \gamma v_t) dt + \sqrt{2\gamma h} dB_t, \\ dy_t = \nu(|v_t|^2 - dh) dt. \end{cases}$$

where $\gamma, \nu > 0$ denotes positive parameters and the variable $X_t = (x_t, v_t, y_t)$ lives in $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. This SDE is [26, (2.4)] with several names changed. Its associated generator called \mathcal{L}_{AdL} in [26] is

$$(1.7) \quad \mathcal{L} = v \cdot \partial_x - \partial_x V \cdot \partial_v + \nu(|v|^2 - dh) \partial_y - yv \cdot \partial_v + \gamma(h\Delta_v - v \cdot \partial_v).$$

Even if it shares some similarities with the overdamped Langevin dynamics, (1.6) and its generator (1.7) do not satisfy some of the crucial hypotheses made by [1]. Mainly, they require that the operator is at most quadratic microlocally. In (1.6) the terms that do not respect this assumption are $-\nu y_t v_t dt$ and $\nu|v_t|^2 dt$ which will result in the cubic terms $\nu(|v|^2 \partial_y - yv \cdot \partial_v)$ in (1.7). That hypothesis inherited from [20] was crucial for their microlocal estimates. In our work we manage to avoid that necessity by using the separated variable property of our particular operator which justifies the use of hypocoercivity methods in the spirit of [10]. In that sense, [1] is more general because it applies to a wide class of operator, but our work is not contained in their because of the operator's cubic term.

Therefore, this article is at the edge between [26] and [1], trying to use the arguments of the second reference in order to generalize the results of the first one in a semiclassical way and describe the low lying eigenvalues of their degenerate operator. Here we will obtain hypocoercivity, resolvent estimates and rough description of the eigenvalues uniform in the parameters γ and ν (depending on h) but for sharp estimates, we had to fix γ and ν independant of h .

1.2. Statements. Considering \mathcal{L} from (1.7), we clearly have that $\mathcal{L}1 = 0$ and taking

$$f(x, v, y) = \frac{V(x)}{2} + \frac{|v|^2 + y^2}{4}$$

one can show that $\mathcal{L}^* e^{-2f/h} = 0$. For this paper, we consider the conjugate operator $P = -e^{f/h} h \mathcal{L}^* e^{-f/h}$. In our context, \mathcal{L} has only real coefficient, thus $\sigma(\mathcal{L}^*) = \sigma(\mathcal{L})$, and we obtain

$$P = H_0 + \nu Y + \gamma \mathcal{O}$$

where

$$\begin{cases} H_0 = v \cdot h \partial_x - \partial_x V \cdot h \partial_v, \\ Y = (v h \partial_y - y h \partial_v) \circ v - h d(h \partial_y - \frac{y}{2}), \\ \mathcal{O} = -h^2 \Delta_v + \frac{|v|^2}{4} - h \frac{d}{2}. \end{cases}$$

We observe that we have the algebraic relations:

$$(1.8) \quad H_0^* = -H_0, \quad Y^* = -Y, \quad \mathcal{O}^* = \mathcal{O}$$

and inherited from the properties of \mathcal{L} ,

$$(1.9) \quad P(e^{-f/h}) = P^*(e^{-f/h}) = 0$$

Proposition 1.1. *The operator P initially defined on $C_c^\infty(\mathbb{R}^{2d+1})$ admits a unique maximally accretive extension that we still denote by $(P, D(P))$.*

We postpone the proof of this Proposition to the Appendix.

Assumption 1. *There exist $C > 0$ and a compact set $K \subset \mathbb{R}^d$ such that*

$$V(x) \geq -C, \quad |\nabla V(x)| \geq \frac{1}{C} \quad \text{and} \quad |\text{Hess } V(x)| \leq C.$$

for all $x \in \mathbb{R}^d \setminus K$.

Under this assumption, it is known (see for example [29, Lemma 3.14]) that there exists $b > 0$, such that $V(x) \geq -b + b|x|$.

Lemma 1.2. *Suppose that Assumption 1 holds true. One has $e^{-f/h} \in D(P)$ and*

$$(1.10) \quad H_0(e^{-f/h}) = Y(e^{-f/h}) = \mathcal{O}(e^{-f/h}) = 0$$

Proof. The proof of (1.10) is a simple computation, and therefore, we retrieve (1.9), thus $e^{-f/h} \in D(P)$ thanks to the maximal accretivity of P and $e^{-f/h} \in L^2(\mathbb{R}^{2d})$ thanks to Assumption 1. □

Assumption 2. *The function V is a Morse function.*

Under Assumptions 1 and 2, the set \mathcal{U} of critical points of V is finite. We denote by $\mathcal{U}^{(0)}$ the set of minima of V and $\mathcal{U}^{(1)}$ the set of critical points of index 1. We shall also denote $n_0 = \#\mathcal{U}^{(0)}$. As the critical points of f are the $(x^*, 0, 0)$ for $x^* \in \mathcal{U}$, with the same index, we will identify those two and use x^* instead of $(x^*, 0, 0)$ where it is clear which one we are really talking about (x^* will mostly be denoted either \mathbf{m} if of index 0 or \mathbf{s} if of index 1).

Throughout the paper, we suppose that V satisfies Assumptions 1 and 2.

Theorem 1. *There exists $h_0 > 0$, $c_0, c_1, c, c' > 0$, such that for all $h \in]0, h_0]$, there exists G_h subspace of $L^2(\mathbb{R}^{2d})$ of finite dimension such that for all $\gamma, \nu > 0$, and all $u \in D(P) \cap G_h^\perp$, one has*

$$\|(P - z)u\|_{L^2} \geq c_1 g(h) \|u\|_{L^2}$$

uniformly with respect to $z \in \mathbb{C}$ such that $\text{Re}(z) \leq c_0 g(h)$, where

$$(1.11) \quad g(h) = h \min \left(\nu^2 h \gamma, \frac{1}{\gamma}, \frac{\gamma}{\nu^2 h}, \frac{\nu^2 h}{\gamma} \right).$$

If $g(h)$ satisfies

$$(1.12) \quad g(h) \geq e^{-\frac{\tilde{c}}{2h}} \text{ for any } \tilde{c} < c_f \text{ where } c_f = \inf_{\mathbf{m} \in \mathcal{U}^{(0)}} \inf_{\text{supp } \nabla \chi_{\mathbf{m}}} f - f(\mathbf{m}) > 0,$$

then there exists $\lambda_{\mathbf{m}}(h) \in \mathbb{C}$ for all $\mathbf{m} \in \mathcal{U}^{(0)}$ such that

$$\begin{aligned} \sigma(P) \cap \{\operatorname{Re} z \leq c_0 g(h)\} &= \{\lambda_{\mathbf{m}}(h), \mathbf{m} \in \mathcal{U}^{(0)}\}, \text{ and} \\ \forall \mathbf{m} \in \mathcal{U}^{(0)}, |\lambda_{\mathbf{m}}(h)| &\leq c' e^{-c_f/h}. \end{aligned}$$

Moreover, for all $0 < c'_0 < c_1$

$$\forall |z| > c'_0 g(h), \text{ such that } \operatorname{Re} z \leq c_0 g(h), \|(P - z)^{-1}\|_{L^2} \leq \frac{2}{c'_0 g(h)}.$$

Remark 1.3. *Formally, when taking $h = 1$, we recognize the conclusion of [26, Corollary 1] noticing ν in our paper is ε^{-1} in theirs. Through similar hypocoercive methods we achieve to generalize their result to the semiclassical regime.*

This Theorem, true in its general form will allow us to prove the following one, which describes a much more restrain case for the purpose of this paper : the double well. We will only consider this case because of its simplicity compared to the general one, the aim of this paper is to show that the sharp quasimodes and the methods developed in [1] can be adapted to our operator although it does not satisfy some key assumption they made. To detail a bit more the technicality avoided here, in order to deal with a more generic case, one need to introduce several topological definitions regarding the minima of V and sets around theses minima that will be essential to defined sharp quasimodes and have the most precise estimates. Moreover, at the end we obtain a matrix whose eigenvalues are the eigenvalues we are looking for, but in our case it is a mere 2×2 matrix with three zeros. In the general case, the matrix is not even diagonal and it needs a non trivial study to extract its eigenvalues. We will get into the general case in a forecasting paper.

Theorem 2. *Let us suppose $\mathcal{U}^{(0)} = \{\underline{\mathbf{m}}, \widehat{\mathbf{m}}\}$ where $\underline{\mathbf{m}}$ is the unique global minimum of V , $\mathcal{U}^{(1)} = \{\mathbf{s}\}$ and $\gamma, \nu > 0$ are fixed. There exists $c_0, h_0 > 0$, such that for all $h \in]0, h_0]$, one has*

$$\sigma(P) \cap \{\operatorname{Re} z \leq c_0 g(h)\} = \{0, \lambda\},$$

where $g(h) \propto h^2$ is as in (1.11) and with

$$\lambda = \frac{\mu(\mathbf{s})(\det \operatorname{Hess} V(\widehat{\mathbf{m}}))^{\frac{1}{2}}}{2\pi |\det \operatorname{Hess} V(\mathbf{s})|^{\frac{1}{2}}} h e^{-S(\widehat{\mathbf{m}})/h} (1 + O(\sqrt{h})),$$

where $\mu(\mathbf{s}) = \frac{1}{2}(-\gamma + \sqrt{\gamma^2 + 4\eta}) > 0$ with η the sole negative eigenvalue of $\operatorname{Hess}_{\mathbf{s}} V$ and

$$S(\widehat{\mathbf{m}}) = V(\mathbf{s}) - V(\widehat{\mathbf{m}}).$$

Remark 1.4. *If familiar with this sort of results, one should expect to have an Arrhenius law of the form $\lambda = O(e^{-2S(\widehat{\mathbf{m}})/h})$ but here we do not have that 2, this is the result of having a Gibbs state $e^{-f/h} \propto e^{-\frac{V}{2h}}$.*

Remark 1.5. *One can prove a similar Theorem without the double well assumption, it requires much more geometric constructions. Going further in the development of w*

(defined in (3.3)) in order to obtain higher principal orders in (3.10), one can transform the $1+O(\sqrt{h})$ into $1+O(h)$ and even obtain a full semiclassical asymptotic development $1+h\sum_{j\geq 0} a_j h^j$ with explicit a_j . Consult [1] for more details and an explicit way to obtain that generalisation.

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2. HYPOCOERCIVE ESTIMATES

We introduce the function $\rho(v) = (2\pi h)^{-\frac{d}{4}} e^{-\frac{|v|^2}{4h}}$ and the projector onto the kernel of \mathcal{O} defined on $L^2(\mathbb{R}^{2d+1})$ by

$$\Pi_\rho u(x, v, y) = \int_{\mathbb{R}^d} u(x, v', y) \rho(v') dv' \rho(v) = u_\rho(x, y) \rho(v),$$

where we denoted

$$u_\rho = \langle u, \rho \rangle_{L^2_v(\mathbb{R}^d)}.$$

Let us denote $Z = H_0 + \nu Y$ the skew-adjoint part of P and notice that we have

$$\Pi_\rho Z \Pi_\rho = \mathcal{O} \Pi_\rho = 0.$$

Indeed using (2.3):

$$\begin{aligned} \Pi_\rho Z \Pi_\rho &= \Pi_\rho (v \cdot \delta_x + \nu(|v|^2 - dh) \delta_y) \Pi_\rho \\ \Pi_\rho v_j \Pi_\rho u &= c u_\rho \int v_j e^{-\frac{|v|^2}{2h}} dv = 0 \\ \Pi_\rho (|v|^2 - dh) \Pi_\rho u &= c h u_\rho \int (|v|^2 - d) e^{-\frac{|v|^2}{2}} dv = 0 \end{aligned}$$

whence $\Pi_\rho Z \Pi_\rho = 0$ (with c a constant that changed from line two to three). Moreover we have the following Lemma that will be useful many times in the following.

Lemma 2.1. *For any $j = 1, \dots, d$, the operator $v_j \Pi_\rho$ is bounded on L^2 and*

$$\forall k \in \mathbb{N}, \quad \|v_j^k \Pi_\rho\|_{L^2 \rightarrow L^2} = O(h^{k/2})$$

Proof. We notice that for $u \in L^2(\mathbb{R}^{2d+1})$, $v_j \Pi_\rho u = 2h \partial_{v_j} \Pi_\rho u$ hence the result. □

We define for $\alpha > 0$

$$(2.1) \quad A = (h\alpha + h^{-1}(Z\Pi_\rho)^*(Z\Pi_\rho))^{-1}(Z\Pi_\rho)^*.$$

This auxiliary operator is introduced in [10] and used in [26] in order to ease the calculus in the proof of Theorem 1. This kind of method to compute hypocoercivity was mainly introduced and used at first in [33], [21] and [19].

Lemma 2.2. *The operator A is bounded on $L^2(\mathbb{R}^{2d+1})$, it satisfies*

$$A = \Pi_\rho A = A(1 - \Pi_\rho)$$

and one has the estimate

$$\|A\|_{L^2} \leq \frac{1}{\sqrt{\alpha}}$$

Proof. See [10] for the bound, and $A = A(1 - \Pi_\rho)$ comes from $\Pi_\rho Z \Pi_\rho = 0$. □

We introduce also the semiclassical Witten Laplacian associated to the function $\frac{V}{2}$, acting on $L^2(\mathbb{R}_x^d)$

$$\Delta_{\frac{V}{2}} = -h^2 \Delta_x + \frac{1}{4} |\nabla V|^2 - \frac{h}{2} \Delta V$$

and the semiclassical Witten Laplacian associated to the function $y \mapsto \frac{y^2}{4}$, acting on $L^2(\mathbb{R}_y)$

$$N_y = -h^2 \partial_y^2 + \frac{y^2}{4} - \frac{h}{2}.$$

Throughout the paper, we denote $\delta_{x_i} = h \partial_{x_i} + \frac{\partial_{x_i} V}{2}$ and $\delta_y = h \partial_y + \frac{y}{2}$ the associated twisted derivatives. One has the identities

$$\Delta_{\frac{V}{2}} = \sum_{i=1}^d \delta_{x_i}^* \delta_{x_i} \quad \text{and} \quad N_y = \delta_y^* \delta_y.$$

Along with $\delta_v = h \partial_v + \frac{v}{2}$ which then gives $\mathcal{O} = \delta_v^* \delta_v$, these twisted derivatives allow us to rewrite H_0 and Y in a more fancy way through direct computations

$$(2.2) \quad \begin{aligned} H_0 &= v \cdot \delta_x - \partial_x V \cdot \delta_v, \\ Y &= (|v|^2 - dh) \delta_y - yv \cdot \delta_v. \end{aligned}$$

And because Π_ρ is a projector on the kernel of δ_v , we thus have

$$(2.3) \quad H_j \Pi_\rho = v_j \delta_{x_j} \Pi_\rho \quad \text{and} \quad Y \Pi_\rho = (|v|^2 - dh) \delta_y \Pi_\rho$$

where $H_0 = \sum_{j=1}^d H_j$ and thus $H_j = v_j h \partial_{x_j} - \partial_{x_j} V h \partial_{v_j}$. We also recall the commutation rules

$$(2.4) \quad [\delta_{x_i}, \delta_{x_j}] = 0, \quad [\delta_{x_i}, \delta_{x_j}^*] = h \partial_{ij}^2 V, \quad [\delta_{x_i}, \delta_{x_j}^* \delta_{x_j}] = h \partial_{ij}^2 V \delta_{x_j}$$

and

$$(2.5) \quad [\delta_y, N_y] = h \delta_y.$$

Lemma 2.3. *One has*

$$(2.6) \quad (Z\Pi_\rho)^*(Z\Pi_\rho) = dhB\Pi_\rho$$

where

$$B = \Delta_{\frac{v}{2}} + 2\nu^2 h N_y.$$

Proof. We have $Z\Pi_\rho = (H_0 + \nu Y)\Pi_\rho = (v \cdot \delta_x + \nu(|v|^2 - dh)\delta_y)\Pi_\rho$ but, because of the parity of ρ :

$$\begin{aligned} \langle Z\Pi_\rho u, Z\Pi_\rho w \rangle &= \langle v \cdot \delta_x \Pi_\rho u, v \cdot \delta_x \Pi_\rho w \rangle + \nu^2 \langle (|v|^2 - dh)\delta_y \Pi_\rho u, (|v|^2 - dh)\delta_y \Pi_\rho w \rangle \\ &\quad + 2\nu \operatorname{Re} \underbrace{\langle v \cdot \delta_x \Pi_\rho u, (|v|^2 - dh)\delta_y \Pi_\rho w \rangle}_{=0}, \end{aligned}$$

the last scalar product involves an integral over \mathbb{R}^d of an odd function of v it is therefore null. With the same argument, in the double sum, we only have the diagonal terms :

$$\begin{aligned} \langle v \cdot \delta_x \Pi_\rho u, v \cdot \delta_x \Pi_\rho w \rangle &= \sum_{i=1}^d \langle v_i \delta_{x_i} \Pi_\rho u, v_i \delta_{x_i} \Pi_\rho w \rangle = \sum_{i=1}^d \langle v_i^2 \delta_{x_i}^* \delta_{x_i} \Pi_\rho u, \Pi_\rho w \rangle \\ &= \sum_{i=1}^d \langle v_i^2 \delta_{x_i}^* \delta_{x_i} u_\rho \rho, w_\rho \rho \rangle = \sum_{i=1}^d \langle \delta_{x_i}^* \delta_{x_i} u_\rho, w_\rho \rangle_{L^2_{x,y}} \langle v_i^2 \rho, \rho \rangle_{L^2_v} \\ &= \sum_{i=1}^d \langle \delta_{x_i}^* \delta_{x_i} u_\rho \rho, w \rangle_{L^2_{x,y}} \langle v_i^2 \rho, \rho \rangle_{L^2_v} \end{aligned}$$

By integration by parts, we note

$$\langle v_i^2 \rho, \rho \rangle = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} v_i^2 e^{-\frac{|v|^2}{2h}} \frac{dv}{h^{\frac{d}{2}}} = h\alpha_d$$

and so :

$$\langle v \cdot \delta_x \Pi_\rho u, v \cdot \delta_x \Pi_\rho w \rangle = \langle h\alpha_d \Delta_{\frac{v}{2}} \Pi_\rho u, w \rangle.$$

And with very similar computations, we obtain :

$$\begin{aligned} \langle (|v|^2 - dh)\delta_y \Pi_\rho u, (|v|^2 - dh)\delta_y \Pi_\rho w \rangle &= \langle \delta_y^* \delta_y u_\rho \rho, w \rangle \langle (|v|^2 - dh)^2 \rho, \rho \rangle \\ \langle (|v|^2 - dh)^2 \rho, \rho \rangle &= h^2 \beta_d \\ \langle (|v|^2 - dh)\delta_y \Pi_\rho u, (|v|^2 - dh)\delta_y \Pi_\rho w \rangle &= \langle h^2 \beta_d N_y \Pi_\rho u, w \rangle. \end{aligned}$$

By naming $I_n = (2\pi)^{-d/2} \int |v|^n e^{-\frac{|v|^2}{2}} dv$ one has

$$\alpha_d = I_2 = dI_0 = d$$

and

$$\beta_d = I_4 - 2dI_2 + d^2 I_0 = (d(d+2) - 2d^2 + d^2)I_0 = 2d$$

Finally, we observe that we indeed have proved 2.6 :

$$(Z\Pi_\rho)^*(Z\Pi_\rho) = dhB\Pi_\rho$$

□

One direct consequence of Lemma 2.3 and (2.1) is that we have

$$A = (h\alpha + dB)^{-1}(Z\Pi_\rho)^*$$

Let $\chi_{\mathbf{m}}$, $\mathbf{m} \in \mathcal{U}^{(0)}$ be some cutoffs in $\mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\chi_{\mathbf{m}}$ is supported in $B(\mathbf{m}, r)$ for some $r > 0$ to be chosen small enough and $\chi_{\mathbf{m}}$ is constant near \mathbf{m} . We then introduce the quasimodes

$$f_{\mathbf{m}}(x, v, y) = \chi_{\mathbf{m}}(x)e^{-(f(x, v, y) - f(\mathbf{m}))/h},$$

and we set the constant $\chi_{\mathbf{m}}(\mathbf{m})$ such that $f_{\mathbf{m}}$ is of norm one in $L^2(\mathbb{R}^{2d+1})$. For $r > 0$ small enough, these functions have disjoint support and hence the vector space

$$F_h = \text{span}\{f_{\mathbf{m}}, \mathbf{m} \in \mathcal{U}^{(0)}\}$$

has dimension n_0 .

We in fact have that G_h in Theorem 1 is F_h we just defined. It is a natural space to consider noticing that $e^{-f/h}\mathbb{R}$ is the kernel of $B\Pi_\rho$ which should not be surprising since $B\Pi_\rho$ is a self-adjoint operator built to behave like P .

Lemma 2.4. *There exists $c_0, h_0 > 0$ such that for all $h \in]0, h_0]$, $\nu > 0$ and $u \in F_h^\perp$, one has*

$$\langle B\Pi_\rho u, u \rangle \geq c_0 h \min(1, \nu^2 h) \|\Pi_\rho u\|^2$$

Proof. We set $W(x, y) = \frac{V(x)}{2} + \frac{y^2}{4}$, hence $\Delta_W = \Delta_{\frac{V}{2}} + N_y$, and we see that W has the same property as V : if $V \geq -C$ then so is W , $|\nabla W|^2 = \frac{1}{4}(|\nabla V|^2 + y^2)$ and $\text{Hess}_W(x, y) = \frac{1}{2} \begin{pmatrix} \text{Hess}_V(x) & 0 \\ 0 & 1 \end{pmatrix}$. Therefore W satisfies Assumptions 1 and 2 as much as V , and the minima of W are the $(\mathbf{m}, 0)$ where $\mathbf{m} \in \mathcal{U}^{(0)}$. In order to lighten the notations we will identify \mathbf{m} and $(\mathbf{m}, 0)$, likewise we will identify $\mathcal{U}^{(0)}$ with $\mathcal{U}^{(0)} \times \{0\}$. We also denote $\delta_W = h\nabla + \nabla W$, and $X = (x, y) \in \mathbb{R}^{d+1}$.

Using known facts about the Witten laplacian (see for example [5, Theorem 11.1] or [16] for the exponential bound) we have that

$$(2.7) \quad \exists c, \varepsilon, h_0 > 0, \forall h \in]0, h_0] \sigma(\Delta_W) \cap]ce^{-c/h}, \varepsilon h[= \emptyset,$$

and Δ_W has exactly n_0 eigenvalues in $[0, ce^{-c/h}]$ that we denote $E_n(\Delta_W)$.

We also denote $\tilde{F}_h = \{u_\rho, u \in F_h\} = \text{span}(\tilde{f}_{\mathbf{m}})_{\mathbf{m} \in \mathcal{U}^{(0)}}$ where $\tilde{f}_{\mathbf{m}} = f_{\mathbf{m}}(\cdot, 0, \cdot)$ up to a normalization factor, then we will admit for now that because of what we stated,

$$\exists \varepsilon' > 0, \forall u \in \tilde{F}_h^\perp, \langle \Delta_W u, u \rangle \geq \varepsilon' h \|u\|^2.$$

Therefore, for $u \in F_h^\perp$

$$\begin{aligned} \langle B\Pi_\rho u, u \rangle &= \langle B\Pi_\rho^2 u, u \rangle = \langle \Pi_\rho B \Pi_\rho u, u \rangle = \langle B \Pi_\rho u, \Pi_\rho u \rangle \\ &= \langle \Delta_{\frac{\nu}{2}} \Pi_\rho u, \Pi_\rho u \rangle + 2\nu^2 h \langle N_y \Pi_\rho u, \Pi_\rho u \rangle \\ &\geq \min(1, \nu^2 h) \langle \Delta_W \Pi_\rho u, \Pi_\rho u \rangle \\ &\geq \varepsilon' h \min(1, \nu^2 h) \|\Pi_\rho u\|^2 \end{aligned}$$

□

And so we proved the lemma, let's now show what we have admitted :

Lemma 2.5. *There exists $\varepsilon' > 0$, such that for all $u \in \tilde{F}_h^\perp$, $\langle \Delta_W u, u \rangle \geq \varepsilon' h \|u\|^2$.*

Proof. We first have the spectral projector on the eigenvectors associated to the small eigenvalues : $\Pi_W = \frac{1}{2i\pi} \int_C (z - \Delta_W)^{-1} dz$ where we denote $C = C(0, \frac{\varepsilon}{2}h)$ the circle centered in 0 of radius $\frac{\varepsilon}{2}h$ positively oriented, where ε is defined in (2.7). Thus,

$$\Pi_W - \text{Id} = \frac{1}{2i\pi} \int_C ((z - \Delta_W)^{-1} - z^{-1}) dz = \frac{1}{2i\pi} \int_C (z - \Delta_W)^{-1} \Delta_W z^{-1} dz,$$

applied to the \tilde{f}_m , we get

$$(2.8) \quad \Pi_W \tilde{f}_m - \tilde{f}_m = \frac{1}{2i\pi} \int_C \underbrace{(z - \Delta_W)^{-1}}_{=O(h^{-1})} \underbrace{\Delta_W(\tilde{f}_m)}_{=O(e^{-c/h})} \frac{dz}{z} = O(e^{-c/h})$$

Then by the spectral theorem (which we can use because Δ_W is self-adjoint), noting φ_n eigenvectors of Δ_W associated to $E_n(\Delta_W)$:

$$\begin{aligned} \langle \Delta_W u, u \rangle &= \sum_{n \leq n_0} E_n(\Delta_W) |\langle u, \varphi_n \rangle|^2 + \int_{\varepsilon h}^{\infty} \lambda d\langle E_\lambda u, u \rangle \\ &\geq \int_{\varepsilon h}^{\infty} \lambda d\langle E_\lambda u, u \rangle \geq \varepsilon h \left(\|u\|^2 - \sum_{n \leq n_0} |\langle u, \varphi_n \rangle|^2 \right) \end{aligned}$$

We now want to show that $\exists c > 0, \forall u \in \tilde{F}_h^\perp, \|u\|^2 - \sum_{n \leq n_0} |\langle u, \varphi_n \rangle|^2 \geq c \|u\|^2$, or in an

equivalent way $\sum_{n \leq n_0} |\langle u, \varphi_n \rangle|^2 \leq c' \|u\|^2$ with $c' < 1$. But as $\forall m \in \mathcal{U}^{(0)}, \langle u, \tilde{f}_m \rangle = 0$ be-

cause $u \in \tilde{F}_h^\perp$, and since $\langle \tilde{f}_m, \tilde{f}_{m'} \rangle = \delta_{m,m'} + O(e^{-C/h})$, noticing that $\text{span}(\Pi_W \tilde{f}_m)_{\mathcal{U}^{(0)}} = \text{span}(\varphi_n)_{n \leq n_0}$, hence using (2.8) we obtain

$$\sum_{n \leq n_0} \langle \cdot, \varphi_n \rangle \varphi_n = \sum_{m \in \mathcal{U}^{(0)}} a_m \langle \cdot, \Pi_W \tilde{f}_m \rangle \Pi_W \tilde{f}_m = \sum_{m \in \mathcal{U}^{(0)}} a_m \langle \cdot, \tilde{f}_m \rangle \tilde{f}_m + O(e^{-c/h})$$

and thus

$$\sum_{n \leq n_0} |\langle u, \varphi_n \rangle|^2 \leq C e^{-C/h} \|u\|^2.$$

□

Lemma 2.6. *One has the following identities*

$$(2.9) \quad H_i^* H_j \Pi_\rho = (-v_i v_j \delta_{x_i} \delta_{x_j} + \delta_{i,j} h \partial_{x_i} V \delta_{x_j}) \Pi_\rho$$

$$(2.10) \quad Y^* Y \Pi_\rho = ((|v|^2 - dh)^2 \delta_y^* \delta_y - ((|v|^2 - dh)^2 - 2h|v|^2) y \delta_y) \Pi_\rho$$

$$(2.11) \quad H_i^* Y \Pi_\rho = (-v_i (|v|^2 - dh) \delta_{x_i} \delta_y + 2v_i h \partial_{x_i} V \delta_y) \Pi_\rho$$

$$(2.12) \quad Y^* H_i \Pi_\rho = (-v_i (|v|^2 - dh) \delta_{x_i} \delta_y + v_i h y \delta_{x_i}) \Pi_\rho$$

These identities will be useful for the following lemma but since its proof is mere calculus we postpone it to the Appendix.

Lemma 2.7. *There exists $C, h_0 > 0$ such that for all $h \in]0, h_0]$, $\nu > 0$ and for all $u \in F_h^\perp$, one has*

$$(2.13) \quad |\langle AZ(1 - \Pi_\rho)u, u \rangle| \leq C(1 + \nu \sqrt{h} \alpha^{-\frac{1}{2}} + \alpha^{-\frac{1}{2}}) h \|\Pi_\rho u\| \|(1 - \Pi_\rho)u\|$$

$$(2.14) \quad |\langle A\mathcal{O}u, u \rangle| \leq C \alpha^{-\frac{1}{2}} h \|\Pi_\rho u\| \|(1 - \Pi_\rho)u\|$$

$$(2.15) \quad |\langle Zu, Au \rangle| \leq Ch \|(1 - \Pi_\rho)u\|^2$$

Proof. Within this proof, C will denote a positive constant that may only depends on the dimension d and may change from line to line.

Let us start with the proof of (2.13). Since $A = \Pi_\rho A$, by the Cauchy-Schwarz inequality it is sufficient to show that the operator AZ (or equivalently its adjoint) is bounded on L^2 . One has

$$\begin{aligned} Z^* A^* &= Z^* Z \Pi_\rho (h\alpha + dB)^{-1} \Pi_\rho \\ &= \left(\sum_{i,j} H_i^* H_j + \nu^2 Y^* Y + \nu \sum_i (H_i^* Y + Y H_i^*) \right) \Pi_\rho (h\alpha + dB)^{-1} \Pi_\rho \end{aligned}$$

and we estimate each term separately. We start with the term involving $H_i^* H_j$. From (2.9), we deduce that

$$(2.16) \quad H_i^* H_j \Pi_\rho (h\alpha + dB)^{-1} = (-v_i v_j \delta_{x_i} \delta_{x_j} + \delta_{i,j} h \partial_{x_i} V \delta_{x_j}) (h\alpha + dB)^{-1} \Pi_\rho$$

First we observe that

$$\begin{aligned}
\|\delta_{x_i}\delta_{x_j}(h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\|^2 &\leq \sum_{k,l} \|\delta_{x_k}\delta_{x_l}(h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\|^2 \\
&\leq \sum_{k,l} \langle \delta_{x_k}^* \delta_{x_k} \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\leq \sum_l \langle \Delta_{\frac{V}{2}} \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\leq \sum_l \langle \Delta_{\frac{V}{2}} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l}^* \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\quad + \sum_l \langle [\Delta_{\frac{V}{2}}, \delta_{x_l}] (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\leq \|\Delta_{\frac{V}{2}} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\|^2 \\
&\quad - \sum_{k,l} \langle h\partial_{kl}^2 V \delta_{x_k} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\leq C(1 + h \max_k \|\delta_{x_k} (h\alpha + d\Delta_{\frac{V}{2}})^{-\frac{1}{2}}\|^2) \|(h\alpha + d\Delta_{\frac{V}{2}})^{-\frac{1}{2}}\|^2 \|u\|^2
\end{aligned}$$

where we used (2.4) to compute the commutator, Assumption 1 and Lemma A.2 to get the last estimates. Thanks to Lemma A.3 and Lemma A.2, this implies

$$(2.17) \quad \|\delta_{x_i}\delta_{x_j}(h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\| \leq C(1 + h^{\frac{1}{2}}h^{-\frac{1}{2}}\alpha^{-\frac{1}{2}})\|u\| \leq C(1 + \alpha^{-\frac{1}{2}})\|u\|$$

Using Lemma 2.1 and Lemma A.1 this implies

$$(2.18) \quad v_i v_j \delta_{x_i} \delta_{x_j} (h\alpha + dB)^{-1} \Pi_\rho = (1 + \alpha^{-\frac{1}{2}}) O(h)$$

Similarly, since $|\nabla V|^2 \leq 4(\Delta_{\frac{V}{2}} + \frac{h}{2}\Delta V)$ in the sense of operators, we have

$$\begin{aligned}
\|\partial_{x_i} V \delta_{x_j} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\|^2 &\leq \sum_{k,l} \langle |\partial_{x_k} V|^2 \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\leq C \sum_l \langle \Delta_{\frac{V}{2}} \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\quad + Ch \max_k \||\Delta V|^{\frac{1}{2}} \delta_{x_k} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\|^2 \\
&\leq C \sum_l \langle \Delta_{\frac{V}{2}} \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u, \delta_{x_l} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u \rangle \\
&\quad + Ch \max_k \|\delta_{x_k} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\|^2
\end{aligned}$$

using Assumption 1, which implies by the same arguments as above that

$$\|\partial_{x_i} V \delta_{x_j} (h\alpha + d\Delta_{\frac{V}{2}})^{-1}u\| \leq C(1 + \alpha^{-\frac{1}{2}})\|u\|.$$

Using again Lemma A.1, it follows that

$$(2.19) \quad h\partial_{x_i} V \delta_{x_j} (h\alpha + dB)^{-1} \Pi_\rho = (1 + \alpha^{-\frac{1}{2}}) O(h) \Pi_\rho$$

Combining (2.16), (2.18) and (2.19), we finally get

$$(2.20) \quad H_i^* H_j \Pi_\rho (h\alpha + dB)^{-1} = (1 + \alpha^{-\frac{1}{2}}) O(h) \Pi_\rho.$$

Now, from (2.10), we have :

$$\begin{aligned} \nu^2 Y^* Y \Pi_\rho (h\alpha + dB)^{-1} &= \nu^2 ((|v|^2 - dh)^2 \delta_y^* \delta_y - (|v|^2 - dh)^2 - 2h|v|^2) y \delta_y \\ &\quad \times (h\alpha + dB)^{-1} \Pi_\rho \end{aligned}$$

First we notice that due to Lemma A.2,

$$\|\delta_y^* \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1}\| = \|N_y (h\alpha + 2d\nu^2 h N_y)^{-1}\| \leq C\nu^{-2} h^{-1}$$

And using Lemma 2.1 and A.1, we then get :

$$\begin{aligned} \|(|v|^2 - dh)^2 \delta_y^* \delta_y (h\alpha + dB)^{-1} \Pi_\rho\| &\leq \|(|v|^2 - dh)^2 \Pi_\rho\| \|\delta_y^* \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1}\| \\ &\quad \times \|(h\alpha + 2d\nu^2 h N_y)(h\alpha + dB)^{-1}\| \\ &\leq C\nu^{-2} h \end{aligned}$$

Hence

$$(2.21) \quad (|v|^2 - dh)^2 \delta_y^* \delta_y (h\alpha + dB)^{-1} \Pi_\rho = \nu^{-2} O(h) \Pi_\rho.$$

Now since $y^2 \leq 4N_y + 2h$ in the sense of operators, we have :

$$\begin{aligned} (*_1) &:= \|y \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u\|^2 \\ &= \langle y^2 \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u, \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u \rangle \\ &\leq 4 \langle N_y \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u, \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u \rangle \\ &\quad + 2h \|\delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u\|^2 \\ &= 4 \|N_y (h\alpha + 2d\nu^2 h N_y)^{-1} u\|^2 + 2h \|\delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u\|^2 \\ &\quad + 4 \langle [N_y, \delta_y] (h\alpha + 2d\nu^2 h N_y)^{-1} u, \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u \rangle \\ &= 4 \|N_y (h\alpha + 2d\nu^2 h N_y)^{-1} u\|^2 - 2h \|\delta_y (h\alpha + 2d\nu^2 h N_y)^{-1} u\|^2 \\ &\leq 4\nu^{-4} h^{-2} \|u\|^2, \end{aligned}$$

where we used (2.5) to compute the commutator and Lemma A.3 for the last estimate.

Thus, thanks to Lemma 2.1 and A.1, it implies :

$$\begin{aligned} (*_2) &:= \|((|v|^2 - dh)^2 - 2h|v|^2) y \delta_y (h\alpha + dB)^{-1} \Pi_\rho\| \\ &\leq \|y \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1}\| \|(h\alpha + 2d\nu^2 h N_y)(h\alpha + dB)^{-1}\| \\ &\quad \times \|((|v|^2 - dh)^2 - 2h|v|^2) \Pi_\rho\| \\ &\leq C\nu^{-2} h^{-1} h^2 \\ &\leq C\nu^{-2} h \end{aligned}$$

which proves

$$(2.22) \quad (|v|^2 - dh)^2 - 2h|v|^2 y \delta_y (h\alpha + dB)^{-1} \Pi_\rho = \nu^{-2} O(h) \Pi_\rho$$

Combining (2.21) and (2.22) we have :

$$(2.23) \quad \nu^2 Y^* Y \Pi_\rho (h\alpha + dB)^{-1} = O(h) \Pi_\rho.$$

Let us now study simultaneously the last two terms, using lemma 2.6 we have

$$\begin{aligned} Y^* H_i \Pi_\rho (h\alpha + dB)^{-1} &= (-v_i(|v|^2 - dh)\delta_{x_i}\delta_y + v_i h y \delta_{x_i})(h\alpha + dB)^{-1} \Pi_\rho \\ H_i^* Y \Pi_\rho (h\alpha + dB)^{-1} &= (-v_i(|v|^2 - dh)\delta_{x_i}\delta_y + 2v_i h \partial_{x_i} V \delta_y)(h\alpha + dB)^{-1} \Pi_\rho \end{aligned}$$

Thanks to Lemma A.3 and since δ_x and δ_y commute

$$\begin{aligned} (*_3) &:= \left\| \delta_{x_i} \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1/2} (h\alpha + d\Delta_{\frac{V}{2}})^{-1/2} \right\| \\ &\leq \left\| \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1/2} \right\| \left\| \delta_{x_i} (h\alpha + d\Delta_{\frac{V}{2}})^{-1/2} \right\| \\ &\leq C\nu^{-1} h^{-\frac{1}{2}} \end{aligned}$$

Therefore, since B commute with $\Delta_{\frac{V}{2}}$ and N_y , we get with Lemma 2.1 and Lemma A.1

$$\begin{aligned} (*_4) &:= \left\| v_i(|v|^2 - dh)\delta_{x_i}\delta_y (h\alpha + dB)^{-1} \Pi_\rho \right\| \\ &\leq \left\| \delta_{x_i}\delta_y (h\alpha + 2d\nu^2 h N_y)^{-1/2} (h\alpha + d\Delta_{\frac{V}{2}})^{-1/2} \right\| \left\| v_i(|v|^2 - dh)\Pi_\rho \right\| \\ &\quad \times \left\| (h\alpha + 2d\nu^2 h N_y)^{1/2} (h\alpha + dB)^{-1/2} \right\| \left\| (h\alpha + d\Delta_{\frac{V}{2}})^{1/2} (h\alpha + dB)^{-1/2} \right\| \\ &\leq C\nu^{-1} h^{-\frac{1}{2}} h^{\frac{3}{2}} \end{aligned}$$

Consequently,

$$v_i(|v|^2 - dh)\delta_y \delta_{x_i} (h\alpha + dB)^{-1} \Pi_\rho = \nu^{-1} O(h) \Pi_\rho$$

Similarly, thanks to Lemma A.3 and then Lemma A.2 :

$$\begin{aligned} (*_5) &:= \left\| y \delta_{x_i} (h\alpha + 2d\nu^2 h N_y)^{-1/2} (h\alpha + d\Delta_{\frac{V}{2}})^{-1/2} \right\|^2 \\ &\leq \left\| \delta_{x_i} (h\alpha + d\Delta_{\frac{V}{2}})^{-1/2} \right\|^2 \left\| y (h\alpha + 2d\nu^2 h N_y)^{-1/2} \right\|^2 \\ &\leq C \left\| y (h\alpha + 2d\nu^2 h N_y)^{-1/2} \right\|^2 \\ &\leq C \langle N_y (h\alpha + 2d\nu^2 h N_y)^{-1/2} u, (h\alpha + 2d\nu^2 h N_y)^{-1/2} u \rangle \\ &\quad + Ch \left\| (h\alpha + 2d\nu^2 h N_y)^{-1/2} u \right\|^2 \\ &\leq C \left(\left\| N_y (h\alpha + 2d\nu^2 h N_y)^{-1} \right\| + h \left\| (h\alpha + 2d\nu^2 h N_y)^{-1/2} \right\|^2 \right) \|u\|^2 \\ &\leq C(\nu^{-2} h^{-1} + h h^{-1} \alpha^{-1}) \|u\|^2 \\ &\leq C\nu^{-2} h^{-1} (1 + \nu^2 h \alpha^{-1}) \|u\|^2 \end{aligned}$$

Which leads to

$$\left\| y \delta_{x_i} (h\alpha + 2d\nu^2 h N_y)^{-1/2} (h\alpha + d\Delta_{\frac{V}{2}})^{-1/2} \right\| \leq C\nu^{-1} h^{-\frac{1}{2}} (1 + \nu\sqrt{h}\alpha^{-\frac{1}{2}})$$

And with very identical arguments, we get :

$$\left\| \partial_{x_i} V \delta_y (h\alpha + 2d\nu^2 h N_y)^{-1/2} (h\alpha + d\Delta_{\frac{V}{2}})^{-1/2} \right\| \leq C\nu^{-1} h^{-\frac{1}{2}} (1 + \alpha^{-\frac{1}{2}})$$

Therefore, with Lemma 2.1 and Lemma A.1:

$$\|v_i h y \delta_{x_i} (h\alpha + dB)^{-1} \Pi_\rho\| + \|2v_i h \partial_{x_i} V \delta_y (h\alpha + dB)^{-1} \Pi_\rho\| \leq C\nu^{-1} h (1 + \alpha^{-\frac{1}{2}} + \nu \sqrt{h} \alpha^{-\frac{1}{2}})$$

which leads to

$$(2.24) \quad \nu(Y^* H_i + H_i^* Y)(h\alpha + dB)^{-1} \Pi_\rho = (1 + \alpha^{-\frac{1}{2}} + \nu \sqrt{h} \alpha^{-\frac{1}{2}}) O(h) \Pi_\rho.$$

Combining (2.20), (2.23) and (2.24) we have finally completely proved (2.13).

To prove (2.14), we first show that $\mathcal{O}A^* \Pi_\rho$ is bounded :

$$\begin{aligned} \mathcal{O}A^* \Pi_\rho &= \mathcal{O}Z \Pi_\rho (h\alpha + dB)^{-1} \Pi_\rho \\ \mathcal{O}Z \Pi_\rho u &= (\mathcal{O}(v \cdot \delta_x + \nu(|v|^2 - dh)\delta_y) \Pi_\rho) u \\ &= [\mathcal{O}, v \cdot \delta_x + \nu(|v|^2 - dh)\delta_y] \Pi_\rho u \\ &= -2dh^2 \nu \delta_y \Pi_\rho u - 2h^2 (\delta_x + 2\nu \delta_y v) \cdot \left(-\frac{v}{2h}\right) \Pi_\rho u \\ &= h (v \cdot \delta_x + 2\nu(|v|^2 - dh)\delta_y) \Pi_\rho u. \end{aligned}$$

Using Lemmas 2.1, A.1, A.2, A.3, it proves

$$\mathcal{O}A^* \Pi_\rho = \alpha^{-\frac{1}{2}} O(h) \Pi_\rho$$

Which leads to

$$\begin{aligned} \langle A\mathcal{O}u, u \rangle &= \langle \Pi_\rho A \mathcal{O}(1 - \Pi_\rho)u, u \rangle = \langle (1 - \Pi_\rho)u, \mathcal{O}A^* \Pi_\rho u \rangle \\ |\langle A\mathcal{O}u, u \rangle| &\leq \|\mathcal{O}A^* \Pi_\rho\| \|\Pi_\rho u\| \|(1 - \Pi_\rho)u\| \leq C\alpha^{-\frac{1}{2}} h \|\Pi_\rho u\| \|(1 - \Pi_\rho)u\| \end{aligned}$$

which proves (2.14).

Then, for (2.15) we show that $Z \Pi_\rho (h\alpha + dB)^{-1/2}$ is bounded :

$$\begin{aligned} \|Z \Pi_\rho (h\alpha + dB)^{-1/2} u\|^2 &= \langle (Z \Pi_\rho)^* (Z \Pi_\rho) (h\alpha + dB)^{-1/2} u, (h\alpha + dB)^{-1/2} u \rangle \\ &= h \|u\|^2 - h^2 \alpha \|(h\alpha + dB)^{-1/2} u\|^2 \end{aligned}$$

which gives us

$$Z \Pi_\rho (h\alpha + dB)^{-1/2} = O(\sqrt{h}).$$

Thanks to that result,

$$\begin{aligned} \langle Zu, Au \rangle &= \langle (1 - \Pi_\rho)u, (\Pi_\rho Z)^* A (1 - \Pi_\rho)u \rangle \\ |\langle Zu, Au \rangle| &\leq \|(\Pi_\rho Z)^* A\| \|(1 - \Pi_\rho)u\|^2 = \|-Z \Pi_\rho A\| \|(1 - \Pi_\rho)u\|^2 \\ &\leq \|Z \Pi_\rho (h\alpha + dB)^{-1} (Z \Pi_\rho)^*\| \|(1 - \Pi_\rho)u\|^2 \\ &\leq \|Z \Pi_\rho (h\alpha + dB)^{-1/2}\| \|(h\alpha + dB)^{-1/2} (Z \Pi_\rho)^*\| \|(1 - \Pi_\rho)u\|^2 \\ &\leq \|Z \Pi_\rho (h\alpha + dB)^{-1/2}\|^2 \|(1 - \Pi_\rho)u\|^2 \leq Ch \|(1 - \Pi_\rho)u\|^2 \end{aligned}$$

which completes the Lemma 2.7.

□

Proposition 2.8. *There exists $C, \delta_0, h_0 > 0$ such that for all $h \in]0, h_0]$, $\gamma, \nu > 0$ and for all $u \in D(P) \cap F_h^\perp$, one has*

$$\operatorname{Re} \langle Pu, (1 + \delta(h)(A + A^*))u \rangle_{L^2} \geq Cg(h)\|u\|_{L^2}^2$$

choosing $\alpha = \min(1, \nu^2 h)$, where $\delta(h) = \delta_0 \frac{g(h)}{h}$ and with $g(h)$ defined in (1.11).

Proof. For all $\delta > 0$, let us define

$$I_\delta = \operatorname{Re} \langle Pu, (1 + \delta(A + A^*))u \rangle_{L^2}$$

Using the decomposition $P = Z + \gamma\mathcal{O}$, and the skew-adjointness of Z coming from (1.8), one gets

$$I_\delta = \gamma \langle \mathcal{O}u, u \rangle + \delta \operatorname{Re} \langle Pu, (A + A^*)u \rangle$$

From the spectral properties of \mathcal{O} , it follows that

$$(2.25) \quad I_\delta \geq \gamma h \|(1 - \Pi_\rho)u\|^2 + \delta \operatorname{Re} \langle Pu, (A + A^*)u \rangle.$$

Denoting $J = \langle Pu, (A + A^*)u \rangle$, one has

$$J = \langle AZu, u \rangle + \gamma \langle A\mathcal{O}u, u \rangle + \langle Zu, Au \rangle + \gamma \langle \mathcal{O}u, Au \rangle$$

and since $A = \Pi_\rho A$ and $\Pi_\rho \mathcal{O} = 0$ it follows that

$$J = \langle AZ\Pi_\rho u, u \rangle + J'$$

with

$$J' = \langle AZ(1 - \Pi_\rho)u, u \rangle + \gamma \langle A\mathcal{O}u, u \rangle + \langle Zu, Au \rangle.$$

Moreover, by definition, one has $AZ\Pi_\rho = (h\alpha + dB\Pi_\rho)^{-1}dhB\Pi_\rho$. Combined with Lemma 2.4 and taking $\alpha = \min(1, \nu^2 h)$, this shows that

$$(2.26) \quad AZ\Pi_\rho \geq dh \frac{c_0 h \min(1, \nu^2 h)}{h \min(1, \nu^2 h) + c_0 dh \min(1, \nu^2 h)} \Pi_\rho = \frac{c_0 dh}{1 + c_0 d} \Pi_\rho = c'_0 h \Pi_\rho$$

Consequently, there exists $c'_0 > 0$ such that

$$\operatorname{Re} J \geq c'_0 h \|\Pi_\rho u\|^2 + \operatorname{Re} J'$$

Plugging this estimate into (2.25) we get

$$(2.27) \quad I_\delta \geq \gamma h \|(1 - \Pi_\rho)u\|^2 + \delta c'_0 h \|\Pi_\rho u\|^2 + \delta \operatorname{Re} J'.$$

Recall that

$$(2.28) \quad J' = \langle AZ(1 - \Pi_\rho)u, u \rangle + \gamma \langle A\mathcal{O}u, u \rangle + \langle Zu, Au \rangle.$$

Combining (2.27), (2.28) and Lemma 2.7, we get

$$\begin{aligned} I_\delta &\geq -Ch\delta(1 + (1 + \nu\sqrt{h} + \gamma)\alpha^{-\frac{1}{2}})\|\Pi_\rho u\| \|(1 - \Pi_\rho)u\| \\ &\quad + (\gamma h - Ch\delta)\|(1 - \Pi_\rho)u\|^2 + \delta c'_0 h \|\Pi_\rho u\|^2 \\ &\geq h \left(\gamma - C\delta - \frac{C^2\delta(1 + (1 + \nu\sqrt{h} + \gamma)\alpha^{-\frac{1}{2}})^2}{2c'_0} \right) \|(1 - \Pi_\rho)u\|^2 + h \frac{\delta c'_0}{2} \|\Pi_\rho u\|^2 \end{aligned}$$

Optimizing the right hand side by taking

$$\delta = \frac{2\gamma c'_0}{c_0'^2 + 2C c'_0 + C^2(1 + (1 + \nu\sqrt{h} + \gamma)\alpha^{-\frac{1}{2}})^2},$$

we get

$$I_\delta \geq \frac{\delta h c'_0}{2} \|u\|^2.$$

We shall say that $\delta(h) \asymp \tilde{\delta}(h)$ if there exists $C_1, C_2 > 0$ such that for h small enough, $C_1 \tilde{\delta}(h) \leq \delta(h) \leq C_2 \tilde{\delta}(h)$. Therefore we have that

$$\delta \asymp \frac{\gamma}{1 + (1 + \nu^2 h + \gamma^2)\alpha^{-1}}$$

Recalling we took $\alpha = \min(1, \nu^2 h)$, hence $\alpha^{-1} = \max(1, (\nu^2 h)^{-1})$, one has

Case 1 : $\nu^2 h \leq 1$

$$\delta \asymp \frac{\gamma}{1 + (1 + \nu^2 h + \gamma^2)(\nu^2 h)^{-1}} \asymp \frac{\nu^2 h \gamma}{1 + \gamma^2} \asymp \min(\nu^2 h \gamma, \frac{\nu^2 h}{\gamma}).$$

Case 2 : $1 \leq \nu^2 h$

$$\delta \asymp \frac{\gamma}{1 + \nu^2 h + \gamma^2} \asymp \frac{\gamma}{\nu^2 h + \gamma^2} \asymp \min\left(\frac{\gamma}{\nu^2 h}, \frac{1}{\gamma}\right).$$

This yields $\delta(h) \asymp \frac{g(h)}{h}$ and therefore

$$I_\delta \geq C g(h) \|u\|^2$$

for some new constant $C > 0$ independent of h, γ and ν . This proves the proposition. \square

We finally have the tools to prove Theorem 1. On one hand, with a mere Cauchy-Schwartz, we have

$$(2.29) \quad \operatorname{Re}\langle (P - z)u, (1 + \delta(h)(A + A^*))u \rangle \leq \|(P - z)u\| \|1 + \delta(h)(A + A^*)\| \|u\|$$

On the other hand, we can see thanks to Proposition 2.8 that

$$\operatorname{Re}\langle (P - z)u, (1 + \delta(h)(A + A^*))u \rangle \geq C g(h) \|u\|^2 - \operatorname{Re}\langle z \langle u, (1 + \delta(h)(A + A^*))u \rangle \rangle$$

But because $1 + \delta(h)(A + A^*)$ is positive for δ_0 small enough, we can simplify it to

$$(2.30) \quad \operatorname{Re}\langle (P - z)u, (1 + \delta(h)(A + A^*))u \rangle \geq C g(h) \|u\|^2 - \operatorname{Re}\langle z \rangle \|1 + \delta(h)(A + A^*)\| \|u\|^2$$

We now use that for δ_0 small

$$\|1 + \delta(h)(A + A^*)\| \leq 1 + 2\delta(h)\alpha^{-\frac{1}{2}} \leq 2$$

Therefore (2.29) becomes

$$(2.31) \quad \operatorname{Re}\langle (P - z)u, (1 + \delta(h)(A + A^*))u \rangle \leq 2 \|(P - z)u\| \|u\|$$

and (2.30) becomes :

$$(2.32) \quad \operatorname{Re}\langle (P - z)u, (1 + \delta(h)(A + A^*)u) \rangle \geq Cg(h) \|u\|^2 - 2 \operatorname{Re}(z) \|u\|^2.$$

We can combine (2.31) and (2.32) to have

$$\|(P - z)u\| \geq \frac{C}{2}g(h) \|u\| - \operatorname{Re}(z) \|u\|.$$

Finally, taking $0 < c_0 < \frac{C}{2}$ and noting $c_1 = \frac{C}{2} - c_0 > 0$, we have for $\operatorname{Re} z \leq c_0g(h)$

$$\|(P - z)u\| \geq c_1g(h) \|u\|.$$

And we can now deduce the second part of Theorem 1 from that, following the same sketch of proof than in [31].

By recalling $f_{\mathbf{m}}(x, v, y) = \chi_{\mathbf{m}}(x)e^{-(f(x, v, y) - f(\mathbf{m}))/h}$, as we know from (1.10) that $e^{-f/h} \in \operatorname{Ker} \mathcal{O} \cap \operatorname{Ker} Y$, we obtain :

$$P(f_{\mathbf{m}}) = H_0(f_{\mathbf{m}}) = hv \cdot \nabla \chi_{\mathbf{m}} e^{-(f - f(\mathbf{m}))/h} = O(e^{-c_{\mathbf{m}}/h})$$

with $c_{\mathbf{m}} = \inf_{\operatorname{supp} \nabla \chi_{\mathbf{m}}} f - f(\mathbf{m}) > 0$ (because $\chi \equiv 1$ near \mathbf{m}).

Moreover, since the $(f_{\mathbf{m}})_{\mathbf{m} \in \mathcal{U}^{(0)}}$ are orthonormal, we actually have :

$$(2.33) \quad \forall u \in F_h, \|Pu\| = O(e^{-c_f/h}) \|u\|$$

where $c_f = \min_{\mathbf{m} \in \mathcal{U}^{(0)}} c_{\mathbf{m}} > 0$. Furthermore, (2.33) is also true replacing P by P^* because $P^*(f_{\mathbf{m}}) = -H_0(f_{\mathbf{m}})$.

We denote by Π the projector on F_h . Let $z \in \{\operatorname{Re} z \leq c_0g(h)\}$ such that $|z| \geq c'_0g(h)$ with $0 < c'_0 \leq c_1$, and $u \in D(P)$

$$\begin{aligned} \|(P - z)u\|^2 &= \|(P - z)(\Pi + \operatorname{Id} - \Pi)u\|^2 \\ &= \|(P - z)(\operatorname{Id} - \Pi)u\|^2 + \|(P - z)\Pi u\|^2 \\ &\quad + 2 \operatorname{Re}\langle (P - z)(\operatorname{Id} - \Pi)u, (P - z)\Pi u \rangle \end{aligned}$$

One has

$$\begin{aligned} \|(P - z)(\operatorname{Id} - \Pi)u\|^2 &\geq c_1^2g(h)^2 \|(\operatorname{Id} - \Pi)u\|^2, \\ \|(P - z)\Pi u\|^2 &\geq (\|P\Pi u\| - \|z\Pi u\|)^2 \geq \|z\Pi u\| (\|z\Pi u\| - 2\|P\Pi u\|). \end{aligned}$$

Using that $|z| \geq c'_0g(h) \geq c'_0e^{-c/(2h)} \geq c'_0e^{-c/h}$ with $c < c_f$ thanks to (1.12), we have using (2.33)

$$\|(P - z)\Pi u\|^2 \geq \frac{|z|^2}{2} \|\Pi u\|^2$$

We can also see, studying each term in the scalar product :

$$\begin{aligned} \operatorname{Re}\langle (P - z)(\operatorname{Id} - \Pi)u, (P - z)\Pi u \rangle &= \operatorname{Re} \left(\langle P(\operatorname{Id} - \Pi)u, P\Pi u \rangle - z\langle (\operatorname{Id} - \Pi)u, P\Pi u \rangle \right. \\ &\quad \left. - \bar{z}\langle P(\operatorname{Id} - \Pi)u, \Pi u \rangle \right) \\ &\leq (1 + |z|) \|(\operatorname{Id} - \Pi)u\| \|\Pi u\| O(e^{-c_f/h}) \\ &\leq \left(\|u\|^2 + |z|^2 \|\Pi u\|^2 + \|(\operatorname{Id} - \Pi)u\|^2 \right) O(e^{-c_f/h}) \end{aligned}$$

hence

$$\begin{aligned} \|(P - z)u\|^2 &\geq c_1^2 g(h)^2 \|(\operatorname{Id} - \Pi)u\|^2 + \frac{|z|^2}{3} \|\Pi u\|^2 \\ &\quad + (\|u\|^2 + \|(\operatorname{Id} - \Pi)u\|^2) O(e^{-c_f/h}) \\ &\geq \frac{c_0^2}{3} g(h)^2 \|u\|^2 + (\|u\|^2 + \|(\operatorname{Id} - \Pi)u\|^2) O(e^{-c_f/h}) \\ &\geq \frac{c_0^2}{4} g(h)^2 \|u\|^2 \end{aligned}$$

for h small enough, using (1.12). It leads to

$$(2.34) \quad \|(P - z)u\| \geq \frac{c'_0}{2} g(h) \|u\|.$$

By using the same arguments for P^* we have the same result for it (the key point is that $e^{-f/h}$ is in the kernel of \mathcal{O} , H_0 and Y hence it also is in P^* 's one). It just remains to show that $P - z$ is surjective in order to obtain the resolvent estimate, we show it the classical way, by showing that $\operatorname{Ran}(P - z)$ is closed and dense.

Let $u_n \in D(P)$ and $v \in L^2$ such that $(P - z)u_n \rightarrow v$ therefore $((P - z)u_n)_{n \in \mathbb{N}}$ is Cauchy and so is $(u_n)_{n \in \mathbb{N}}$ thanks to (2.34), hence there exists $u \in L^2$ such that $u_n \rightarrow u$. Because the convergence is also true in \mathcal{D}' , we have that $(P - z)u = v$ in \mathcal{D}' , and since $v \in L^2$, so is $(P - z)u$, thus $u \in D(P)$ and $\operatorname{Ran}(P - z)$ is closed. Now to show that $\operatorname{Ran}(P - z)$ is dense, we use (2.34) for P^* and so $\operatorname{Ker}(P^* - \bar{z}) = \{0\}$.

All this leads to the resolvent estimate :

$$(2.35) \quad \|(P - z)^{-1}\| \leq \frac{2}{c'_0 g(h)}.$$

Hence, P has no spectrum in $\{\operatorname{Re} z \leq c_0 g(h)\} \cap \{|z| \geq c'_0 g(h)\}$.

We will show that on $\{\operatorname{Re} z \leq c_0 g(h)\}$, P has at most $n_0 = \dim F_h$ eigenvalues. By denoting $D_\varepsilon = D(0, \varepsilon g(h))$, let us denote

$$\Pi_0 = \frac{1}{2i\pi} \int_{\partial D_\varepsilon} (z - P)^{-1} dz$$

the projector on the small eigenvalues. We start by proving the following lemma

Lemma 2.9. *There exists $C > 0$ such that $\|P\Pi_0\| \leq C \varepsilon^2 g(h)$*

Proof.

$$P\Pi_0 = \frac{1}{2i\pi} \int_{\partial D_\varepsilon} P(z - P)^{-1} dz = \frac{1}{2i\pi} \int_{\partial D_\varepsilon} z(z - P)^{-1} dz,$$

hence

$$\|P\Pi_0\| \leq C \frac{(\varepsilon g(h))^2}{c'_0 g(h)}$$

thanks to (2.35). □

We first prove that $\dim \text{Ran } \Pi_0 \leq n_0$.

By contradiction, let us suppose $F_h^\perp \cap \text{Ran } \Pi_0 \neq \emptyset$ and so let us take $u \in F_h^\perp \cap \text{Ran } \Pi_0$ of norm one. Since $u \in \text{Ran } \Pi_0$, by Lemma 2.9, $\|Pu\| \leq C \varepsilon^2 g(h)$, but because $u \in F_h^\perp$ we can use Theorem 1 and so $\|Pu\| \geq c_1 g(h)$. Taking ε low enough, we have the contradiction we aimed for and thus, $\dim \text{Ran } \Pi_0 \leq n_0$.

For the converse inequality, taking $\varepsilon = \frac{c_1}{2}$, we have

$$\Pi_0 - \text{Id} = \frac{1}{2i\pi} \int_{\partial D_\varepsilon} z^{-1} (z - P)^{-1} P dz$$

and therefore

$$\begin{aligned} \varepsilon_{\mathbf{m}} &= \Pi_0 f_{\mathbf{m}} - f_{\mathbf{m}} = \frac{1}{2i\pi} \int_{\partial D_\varepsilon} \underbrace{(z - P)^{-1}}_{=O(g(h)^{-1})} \underbrace{P(f_{\mathbf{m}})}_{=O(e^{-c_f/h})} \frac{dz}{z} \\ (2.36) \quad &= O(g(h)^{-1} e^{-c_f/h}) = O(e^{-\frac{c_f}{2h}}) \end{aligned}$$

using (2.35) and (2.33) and the hypothesis (1.12).

Let us suppose $\sum_{\mathbf{m} \in \mathcal{U}^{(0)}} a_{\mathbf{m}} \Pi_0 f_{\mathbf{m}} = 0$ with $\sum_{\mathbf{m} \in \mathcal{U}^{(0)}} |a_{\mathbf{m}}|^2 = 1$, since $\Pi_0 f_{\mathbf{m}} = f_{\mathbf{m}} + \varepsilon_{\mathbf{m}}$, we have $\forall \mathbf{m}' \in \mathcal{U}^{(0)} \sum_{\mathbf{m} \in \mathcal{U}^{(0)}} a_{\mathbf{m}} (\delta_{\mathbf{m}, \mathbf{m}'} + \langle \varepsilon_{\mathbf{m}}, f_{\mathbf{m}'} \rangle) = 0$ and so $\forall \mathbf{m}, a_{\mathbf{m}} = O(e^{-c/h})$ which is in contradiction with $\sum |a_{\mathbf{m}}|^2 = 1$. We deduce that $\dim \text{Ran } \Pi_0 \geq n_0$ and hence, with what we already showed, $\dim \text{Ran } \Pi_0 = n_0$.

And so we can say we have

$$\sigma(P) \cap \{\text{Re } z \leq c_0 g(h)\} = \{\lambda_{\mathbf{m}}(h), \mathbf{m} \in \mathcal{U}^{(0)}\} \subset D(0, \frac{c_1}{2} g(h)).$$

It only remains to show that $\lambda_{\mathbf{m}}(h) = O(e^{-c/h})$, noticing that $\text{Ran } \Pi_0$ is P -stable and that $(\Pi_0 f_{\mathbf{m}})_{\mathbf{m} \in \mathcal{U}^{(0)}}$ is one of its basis,

$$\|P\Pi_0 f_{\mathbf{m}}\| = \|\Pi_0 P f_{\mathbf{m}}\| = O(\|P f_{\mathbf{m}}\|) = O(e^{-c_f/h}).$$

therefore, $P|_{\text{Ran } \Pi_0} = O(e^{-c_f/h})$ hence $\sigma(P|_{\text{Ran } \Pi_0}) \subset D(0, C e^{-c_f/h})$.

3. SHARP QUASIMODES

We now want to have a better view on the small eigenvalues of P . For this purpose, we are going to build sharp quasimodes, and so we are following the steps of [1, section 3&4]. Their Theorem does not apply here because our operator P does not satisfy the hypothesis they labeled (Harmo) as explained in the last paragraph before the statements. We therefore have to rewrite the proof using tricks to avoid that necessity. As in this reference, given $\mathbf{s} \in \mathcal{U}^{(1)}$ we look for an approximate solution to the equation $P\tilde{u} = 0$ in a neighborhood U of \mathbf{s} under the form

$$\tilde{u} = ue^{-(f-f(\mathbf{m}))/h}$$

where we recall $e^{-(f-f(\mathbf{m}))/h} \in \text{Ker } P$ with $f(x, v, y) = \frac{V(x)}{2} + \frac{|v|^2 + y^2}{4}$. And we set

$$u(x, v, y) = \int_0^{\ell(x, v, y, h)} \zeta(s/\tau) e^{-s^2/2h} ds$$

where the function $\ell \in \mathcal{C}^\infty(U)$ has a classical expansion $\ell \sim \sum_{j \geq 0} h^j \ell_j$ in $\mathcal{C}^\infty(U)$. Here,

ζ denotes a fixed smooth even function equal to 1 on $[-1, 1]$ and supported in $[-2, 2]$, and $\tau > 0$ is a small parameter which will be fixed later. The object of this section is to construct the function ℓ . In the following, we will use X instead of (x, v, y) to simplify the equations.

We see that our operator P can be written as in [1] :

$$(3.1) \quad P = -h \operatorname{div} \circ A \circ h \nabla + \frac{1}{2}(b \cdot h \nabla + h \operatorname{div} \circ b) + c$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma \operatorname{Id} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} v \\ -\partial_x V - \nu y v \\ \nu(|v|^2 - dh) \end{pmatrix} \quad \text{and} \quad c = \gamma \left(\frac{|v|^2}{4} - h \frac{d}{2} \right)$$

Since P is of the form (3.1), we can apply [1, Lemma 3.1], and we get

$$(3.2) \quad P(ue^{-(f-f(\mathbf{m}))/h}) = h(w + r)e^{-(f-f(\mathbf{m})+\frac{\ell^2}{2})/h}$$

with r vanishing around \mathbf{s} and

$$(3.3) \quad w = (b + 2A\nabla f) \cdot \nabla \ell + \ell A \nabla \ell \cdot \nabla \ell - h \operatorname{div} A \nabla \ell.$$

In the following, we will consider $\gamma, \nu > 0$ fixed and $\mathbf{s} = 0$. Under these hypothesis, w can be expressed in powers of h , $w \sim \sum_{j \geq 0} h^j w_j$.

Foreshadowing the suitable estimates we will need in the end, we want to solve $w = O(X^4 + hX^2 + h^2)$ in order to have

$$\|P(ue^{-(f-f(\mathbf{m}))/h})\| = O(h^2)\sqrt{\lambda_{\mathbf{m}}},$$

this order is the lowest that will give us precise results on the low lying eigenvalues $\lambda_{\mathbf{m}}$, hence this choice. Thus we decide to take $\ell = \ell_0 + h\ell_1$, which gives us $w = w_0 + hw_1 + O(h^2)$ with

$$(3.4) \quad \begin{cases} w_0 = (b^0 + 2A\nabla f) \cdot \nabla \ell_0 + A\nabla \ell_0 \cdot \nabla \ell_0 \ell_0, \\ w_1 = (b^0 + 2A(\nabla f + \ell_0 \nabla \ell_0)) \cdot \nabla \ell_1 + A\nabla \ell_0 \cdot \nabla \ell_0 \ell_1 + R_1, \end{cases}$$

where $b = b^0 + hb^1$ and $R_1 = b^1 \cdot \nabla \ell_0 - \operatorname{div} A\nabla \ell_0$. As in [1] we call eikonal equation $w_0 = 0$ and transport equation $w_1 = 0$, we now are going to solve the eikonal equation up to the fourth order, and the transport equation up to the second one.

3.1. Solving the eikonal equation. Unlike in [1], the outgoing manifolds of the flow passing through the saddle point are not a Lagrangian ones that project nicely on the X -space. Therefore, we need to find an other way to solve that equation, we will consider homogeneous polynomials to simplify it following [13, Remark 2.3.9].

We introduce \mathcal{P}_{hom}^j the set of homogeneous polynomial of degree j and we consider

$$(3.5) \quad \ell_0 = \sum_{j=0}^3 \ell_{0,j},$$

with $\ell_{0,j} \in \mathcal{P}_{hom}^j$. In the following, we will need to have $\ell_0(\mathbf{s}) = 0$ therefore, we need to set $\ell_{0,0} = 0$. We also denote $\ell_{0,1}(X) = \xi \cdot X$ for a certain $\xi \in \mathbb{R}^{2d+1}$ to be determined. Thanks to (3.5) and (3.4), we have that w_0 also has a similar development

$$w_0 = \sum_{j=0}^3 w_{0,j} + O(X^4), \quad w_{0,j} \in \mathcal{P}_{hom}^j \quad \text{with} \quad w_{0,0} = A\xi \cdot \xi \ell_{0,0} = 0.$$

As in [1], we denote by H and B the matrix $\operatorname{Hess}_{\mathbf{s}} f$ and $db^0(\mathbf{s})$ respectively. We also denote

$$\Lambda = 2HA + B^T = \begin{pmatrix} 0 & -\operatorname{Hess}_{\mathbf{s}} V & 0 \\ \operatorname{Id} & \gamma \operatorname{Id} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and noticing that at the order 1, we have $\nabla f(X) \sim HX$ and $b^0(X) \sim BX$, $w_{0,1} = 0$ becomes

$$(3.6) \quad \Lambda \xi \cdot X + (A\xi \cdot \xi) \xi \cdot X = 0,$$

with unknown ξ and must be true for any X near 0. Taking ξ an eigenvector of Λ associated with a negative eigenvalue will solve the equation, we will just have to chose the right vector on $\mathbb{R}\xi$.

Recalling we set ourselves on \mathbf{s} a critical point of order 1 of V , therefore, there exists $\eta > 0$, $\xi_v \in \mathbb{R}^d \setminus \{0\}$ such that $\operatorname{Hess}_{\mathbf{s}} V \xi_v = -\eta \xi_v$.

We look for an eigenvector $\xi_1 = \begin{pmatrix} \xi_x \\ \xi_v \\ 0 \end{pmatrix}$ of Λ associated to an eigenvalue $-\mu < 0$, thus they satisfy :

$$\Lambda \xi_1 = \begin{pmatrix} \eta \xi_v \\ \xi_x + \gamma \xi_v \\ 0 \end{pmatrix} = -\mu \begin{pmatrix} \xi_x \\ \xi_v \\ 0 \end{pmatrix}$$

Wich leads to the system :

$$\begin{cases} \eta \xi_v = -\mu \xi_x \\ \xi_x + \gamma \xi_v = -\mu \xi_v \end{cases}$$

Hence μ is the positive solution of $\mu^2 + \gamma\mu - \eta = 0$, i.e.

$$(3.7) \quad \mu = \frac{1}{2}(-\gamma + \sqrt{\gamma^2 + 4\eta}) > 0,$$

and we get

$$\xi_x = -\frac{\eta}{\mu} \xi_v.$$

Going back to (3.6), we look for a ξ that solves this equation, we are going to look for one of the type $\xi = t\xi_1$, in other words, we have to solve :

$$(3.8) \quad t^2(A\xi_1 \cdot \xi_1) - \mu = 0$$

We thus have $t = \sqrt{\frac{\mu}{A\xi_1 \cdot \xi_1}} = \frac{1}{|\xi_v|} \sqrt{\frac{\mu}{\gamma}}$ (with $A\xi_1 \cdot \xi_1 \neq 0$ because ξ_v is an eigenvector, hence it is not null).

We now notice that for $j \in \{2, 3\}$, we have

$$(3.9) \quad w_{0,j} = \mathcal{L}l_{0,j} + R_{0,j},$$

with $\mathcal{L} = \Upsilon X \cdot \nabla + \mu$ an endomorphism of \mathcal{P}_{hom}^j , $\Upsilon = \Lambda^T + 2A\Pi_\xi$, $\Pi_\xi(X) = X \cdot \xi \xi$ and where $R_{0,j}$ is a smooth function of $l_{0,k}$ and $\nabla l_{0,k}$ for $k < j$.

Solving (3.9) by homogeneous polynomial is a technique we take from [9, Chapter 3], although we will not solve it up to $O(X^\infty)$. One can show that if $\sigma(\Upsilon) \subset \{\text{Re } z \geq 0\}$, then $\sigma(\Upsilon X \cdot \nabla) \subset \{\text{Re } z \geq 0\}$, see [1, Lemma A.1] where they only consider $\{\text{Re } z > 0\}$, but there is no difficulty expanding the result to $\{\text{Re } z \geq 0\}$ either by continuity or just by doing the same proof. Because $\mu > 0$, if we show that $\sigma(\Upsilon) \subset \{\text{Re } z \geq 0\}$, we will have that \mathcal{L} is invertible and so we will solve $w_0 = O(X^4)$.

In a basis of eigenvectors of Λ with ξ being the first one, we have that Λ is diagonal with only its first entry negative : $-\mu$ to be exact, the other ones are positive. In that same base, $2\Pi_\xi A$ is also diagonal and its first entry is exactly 2μ , the other ones being non negative, therefore Υ^T and thus Υ has no eigenvalue with negative real part. Conclusion :

$$w_0 = O(X^4)$$

3.2. Solving the transport equation. The transport equation is much simpler to solve after having solved the eikonal one. Taking $\ell_1 = \ell_{1,0} + \ell_{1,1}$ with $\ell_{1,j} \in \mathcal{P}_{hom}^j$, we have $w_1 = w_{1,0} + w_{1,1} + O(X^2)$, $w_{1,j} \in \mathcal{P}_{hom}^j$ and

$$\begin{cases} w_{1,0} = \mu \ell_{1,0} + b^1 \cdot \xi - \operatorname{div} A \nabla \ell_{0,2}, \\ w_{1,1} = \mathcal{L} \ell_{1,1} + R_{1,1}, \end{cases}$$

with $R_{1,1}$ a smooth function of ℓ_0 , $\ell_{1,0}$ and their derivatives up to the second order. The first equation is easily solved ($\mu \neq 0$) and the second is solved using the same method as for $w_{0,2}$ and $w_{0,3}$.

After this we now have

$$(3.10) \quad P(ue^{-(f-f(\mathbf{m}))/h}) = hO(X^4 + hX^2 + h^2)e^{-(f-f(\mathbf{m})+\frac{\ell_0^2}{2})/h}.$$

We now have to state [1, Lemma 3.3]'s result and adapt [1, Lemma 4.1]'s proof as we don't have their Lagrangian manifold and its generating function which directly proves this lemma, let us recall the result :

Lemma 3.1. *We have*

$$(3.11) \quad \det \operatorname{Hess}_{\mathbf{s}} \left(f + \frac{\ell_0^2}{2} \right) = -\det H.$$

and hence, recalling that $\mathbf{s} \in \mathcal{U}^{(0)}$

$$\operatorname{Hess}_{\mathbf{s}} \left(f + \frac{\ell_0^2}{2} \right) > 0.$$

Thus, around \mathbf{s} ,

$$(3.12) \quad X - \mathbf{s} \in \xi^\perp \implies f(X) > f(\mathbf{s}).$$

Proof. We first observe that :

$$\operatorname{Hess}_{\mathbf{s}} \left(f + \frac{\ell_0^2}{2} \right) = H + \Pi_\xi$$

We thus have that (3.11) is equivalent to

$$\det E = -1,$$

where $E = \operatorname{Id} + H^{-1}\Pi_\xi$. We first observe that ξ^\perp is stable by E and that $E|_{\xi^\perp} = \operatorname{Id}$. On the other hand, one has

$$E\xi \cdot \xi = \|\xi\|^2 (1 + H^{-1}\xi \cdot \xi).$$

But, $H(2A + H^{-1}B^T)\xi = \Lambda\xi = -\mu\xi$ gives $(2A + H^{-1}B^T)\xi \cdot \xi = -\mu H^{-1}\xi \cdot \xi$.

Looking at the skew-adjoint part of P and using (1.9), we obtain

$$h \operatorname{div} b - b \cdot \nabla f = 0,$$

identifying the first term in the classical expansion we get

$$b^0 \cdot \nabla f = 0,$$

and knowing that $\nabla f(X) \sim HX$ and $b^0(X) \sim BX$ it follows that $B^T H$ is antisymmetric and so is $H^{-1}B^T = H^{-1}(B^T H)H^{-1}$ because H is symmetric. Hence we have

$$H^{-1}\xi \cdot \xi = -\frac{2}{\mu}A\xi \cdot \xi = -2$$

using (3.8), which leads to $E\xi \cdot \xi = -\|\xi\|^2$. Taking a basis adapted to ξ^\perp completed with ξ we can easily compute $\det E$ and obtain the aimed result.

For (3.12), around \mathbf{s} ,

$$f(X) = f(\mathbf{s}) + \frac{1}{2}H(X - \mathbf{s}) \cdot (X - \mathbf{s}) + O(|X - \mathbf{s}|^3)$$

But, over ξ^\perp , $H = H + \Pi_\xi = \text{Hess}_s(f + \frac{\ell_0^2}{2}) > 0$ by what we have done just before. □

4. SIMPLER CASE OF A TWO WELLS FUNCTION

In the following, we will restrain our study to a potential V being a two-wells function, in other words, a function satisfying $\mathcal{U}^{(0)} = \{\underline{\mathbf{m}}, \widehat{\mathbf{m}}\}$ and $\mathcal{U}^{(1)} = \{\mathbf{s}\}$ (see fig. 4.1). It is a much simpler case that allows us not to consider many geometric constructions but still have interesting results and see how the ℓ we built is useful in these findings, see [1, Definition 1.3] for a complete description of the geometric construction on a more general case.

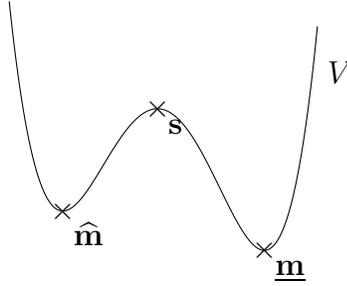


FIGURE 4.1. Representation of a typical two-wells Morse function

In this configuration we know from Theorem 1 that P has exactly two low lying eigenvalues, among which 0 that we decide to associate to $\underline{\mathbf{m}}$, the other one still not precisely known is associated to $\widehat{\mathbf{m}}$. This choice will appear to be relevant later on, when the exact form of the eigenvalue will be explicit. This is why we consider two wells and so we will focus on the second smallest eigenvalue, the aim of this section is to prove Theorem 2.

We define several sets following the description of [1], for $\tau, \delta > 0$:

$$\begin{aligned} \mathcal{B}_{\tau, \delta} &= \{f \leq f(\mathbf{s}) + \delta\} \cap \{X \in \mathbb{R}^{2d+1}, |\xi \cdot (X - \mathbf{s})| \leq \tau\} \\ \mathcal{E}_{\tau, \delta} &= \{f \leq f(\mathbf{s}) + \delta\} \setminus \mathcal{C}_{\tau, \delta} \end{aligned}$$

where $\mathcal{C}_{\tau,\delta}$ denotes the connected component of $\mathcal{B}_{\tau,\delta}$ containing \mathbf{s} . We note $E_{\tau,\delta}^+$ the connected component of $E_{\tau,\delta}$ containing $\widehat{\mathbf{m}}$ and $E_{\tau,\delta}^-$ its complement in $E_{\tau,\delta}$. One can show that for τ_0, δ_0 small enough, for every $\tau \in]0, \tau_0], \delta \in]0, \delta_0]$, we have $\underline{\mathbf{m}} \in E_{\tau,\delta}^-$.

These are useful to define the following cutoff properly

$$\chi_\ell(X) = \begin{cases} +1 & \text{for } X \in E_{4\tau,4\delta}^+ \\ -1 & \text{for } X \in E_{4\tau,4\delta}^- \end{cases}$$

and

$$\chi_\ell(X) = C_h^{-1} \int_0^{\ell(X)} \zeta(r/\tau) e^{-\frac{r^2}{2h}} dr \quad \text{for } X \in \mathcal{C}_{4\tau,4\delta},$$

where $C_h = \frac{1}{2} \int_{-\infty}^{+\infty} \zeta(r/\tau) e^{-\frac{r^2}{2h}} dr$, $\zeta \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ is even and satisfies $\zeta = 1$ on $[-1, 1]$ and $\zeta = 0$ outside $[-2, 2]$.

We notice by the way that $\exists \beta > 0, C_h^{-1} = \sqrt{\frac{2}{\pi h}} (1 + O(e^{-\beta/h}))$. χ_ℓ is indeed a smooth function because on the border of $\mathcal{C}_{4\tau,4\delta}$ we have for h small $\ell \sim \ell_0 \sim \xi \cdot (X - \mathbf{s}) = 4\tau$ but $\zeta(\cdot/\tau)$ is vanishing passing 2τ . To have a cutoff defined properly on \mathbb{R}^d , we introduce

$$\theta(X) = \begin{cases} 1 & \text{for } X \in \{f \leq f(\mathbf{s}) + \delta\} \\ 0 & \text{for } X \in \mathbb{R}^d \setminus \{f \leq f(\mathbf{s}) + 2\delta\} \end{cases}$$

and smooth between these sets.

Hence, we have $\theta\chi_\ell \in \mathcal{C}_c^\infty(\mathbb{R}^{2d+1}, [-1, 1])$ and $\text{supp } \theta\chi_\ell \subset \{f \leq f(\mathbf{s}) + 2\delta\}$.

Definition 4.1. For $\tau > 0$ and then $\delta, h > 0$ small enough, we define the quasimodes

$$\begin{cases} \psi_{\underline{\mathbf{m}}}(X) = 2e^{-\frac{f(X)-f(\underline{\mathbf{m}})}{h}} \\ \psi_{\widehat{\mathbf{m}}}(X) = \theta(X)(\chi_\ell(X) + 1)e^{-\frac{f(X)-f(\widehat{\mathbf{m}})}{h}} \end{cases}$$

And in the same time, we define the normalized quasimodes

$$\varphi_{\underline{\mathbf{m}}} = \frac{\psi_{\underline{\mathbf{m}}}}{\|\psi_{\underline{\mathbf{m}}}\|} \quad \varphi_{\widehat{\mathbf{m}}} = \frac{\psi_{\widehat{\mathbf{m}}}}{\|\psi_{\widehat{\mathbf{m}}}\|}$$

For shortness, we write $D_{X^*} = |\det \text{Hess}_{X^*}(f)|^{1/2}$ for $X^* \in \mathcal{U}$. We recall that $\gamma, \nu > 0$ are fixed.

Proposition 4.2. For $\tau > 0$ and then $\delta > 0$ small enough, there exists $C > 0$ such that for every $\mathbf{m}, \mathbf{m}' \in \mathcal{U}^{(0)} = \{\widehat{\mathbf{m}}, \underline{\mathbf{m}}\}$ and $h > 0$ small,

- i) $\langle \varphi_{\mathbf{m}}, \varphi_{\mathbf{m}'} \rangle = \delta_{\mathbf{m}, \mathbf{m}'} + O(e^{-C/h})$,
- ii) $\langle P \varphi_{\mathbf{m}}, \varphi_{\mathbf{m}} \rangle = h e^{-2\tilde{S}(\mathbf{m})/h} \frac{\mu(\mathbf{s})}{2\pi} \frac{D_{\mathbf{m}}}{D_{\mathbf{s}}} (1 + O(h))$
- iii) $\|P \varphi_{\mathbf{m}}\|^2 = O(h^4) \langle P \varphi_{\mathbf{m}}, \varphi_{\mathbf{m}} \rangle$
- iv) $\|P^* \varphi_{\mathbf{m}}\|^2 = O(h) \langle P \varphi_{\mathbf{m}}, \varphi_{\mathbf{m}} \rangle$

where $\tilde{S}(\widehat{\mathbf{m}}) = f(\mathbf{s}) - f(\widehat{\mathbf{m}})$ and $\tilde{S}(\underline{\mathbf{m}}) = +\infty$.

Remark 4.3. We have built our constant μ to be positive and thus $-\mu$ is the eigenvalue of Λ while [1] did it the other way, hence the lack of absolute value in *ii*).

Proof. Noticing f attains its absolute minimum at \mathbf{m} on $\text{supp } \psi_{\mathbf{m}}$, by using a Laplace method on $\psi_{\mathbf{m}}$ we obtain

$$(4.1) \quad \|\psi_{\mathbf{m}}\| = 2(\pi h)^{\frac{2d+1}{4}} D_{\mathbf{m}}^{-1/2} (1 + O(h)).$$

Let us now prove *i*). By definition, $\forall \mathbf{m} \in \mathcal{U}^{(0)}$, $\langle \varphi_{\mathbf{m}}, \varphi_{\mathbf{m}} \rangle = 1$. Computing $\langle \varphi_{\widehat{\mathbf{m}}}, \varphi_{\underline{\mathbf{m}}} \rangle$, using Cauchy-Schwarz inequality and noticing that $f \geq f(\widehat{\mathbf{m}})$ on $\text{supp } \varphi_{\widehat{\mathbf{m}}}$, we have

$$\langle \varphi_{\widehat{\mathbf{m}}}, \varphi_{\underline{\mathbf{m}}} \rangle = \frac{1}{\|\psi_{\underline{\mathbf{m}}}\|} \langle 2e^{-(f-f(\underline{\mathbf{m}}))/h}, \varphi_{\widehat{\mathbf{m}}} \rangle_{L^2(\text{supp } \varphi_{\widehat{\mathbf{m}}})} = \frac{1}{\|\psi_{\underline{\mathbf{m}}}\|} O(e^{-(f(\widehat{\mathbf{m}})-f(\underline{\mathbf{m}}))/h}),$$

which implies *i*) using (4.1) and recalling $\underline{\mathbf{m}}$ is the lone global minimum of f .

Recalling $P\psi_{\underline{\mathbf{m}}} = P^*\psi_{\underline{\mathbf{m}}} = 0$ we only have to prove *ii*), *iii*) and *iv*) for $\mathbf{m} = \widehat{\mathbf{m}}$. Using the calculus done in [1] of $\langle P\psi_{\widehat{\mathbf{m}}}, \psi_{\widehat{\mathbf{m}}} \rangle$ we have

$$\langle P\psi_{\widehat{\mathbf{m}}}, \psi_{\widehat{\mathbf{m}}} \rangle = h^2 C_h^{-2} \int_{\mathcal{C}_{4\tau, 4\delta}} \theta^2 \zeta(\ell/\tau)^2 A \nabla \ell \cdot \nabla \ell e^{-2(f + \frac{\ell^2}{2} - f(\widehat{\mathbf{m}}))/h} + O(e^{-2(\tilde{S}(\widehat{\mathbf{m}}) + \delta)/h}).$$

According to how we built ℓ and Lemma 3.1,

$$\left(f + \frac{\ell_0^2}{2}\right)(\mathbf{s}) = f(\mathbf{s}), \quad \nabla \left(f + \frac{\ell_0^2}{2}\right)(\mathbf{s}) = 0 \quad \text{and} \quad \text{Hess}_{\mathbf{s}} \left(f + \frac{\ell_0^2}{2}\right) > 0.$$

With another Laplace method, considering $-2(f + \frac{\ell_0^2}{2} - f(\widehat{\mathbf{m}}))$ as the phase function and \mathbf{s} as the global minima, we have

$$\langle P\psi_{\widehat{\mathbf{m}}}, \psi_{\widehat{\mathbf{m}}} \rangle = \frac{2h}{\pi} (\pi h)^{\frac{2d+1}{2}} (A \nabla \ell_0 \cdot \nabla \ell_0)(\mathbf{s}) \left(\det \text{Hess}_{\mathbf{s}} \left(f + \frac{\ell_0^2}{2}\right) \right)^{-1/2} e^{-2\tilde{S}(\widehat{\mathbf{m}})/h} (1 + O(h)).$$

Using that

$$(A \nabla \ell_0 \cdot \nabla \ell_0)(\mathbf{s}) = \mu = \mu(\mathbf{s}) \quad \text{and} \quad \det \text{Hess}_{\mathbf{s}} \left(f + \frac{\ell_0^2}{2}\right) = D_{\mathbf{s}}^2$$

thanks to (3.8) and (3.11), we have

$$\langle P\psi_{\widehat{\mathbf{m}}}, \psi_{\widehat{\mathbf{m}}} \rangle = \frac{2h}{\pi} (\pi h)^{\frac{2d+1}{2}} \mu D_{\mathbf{s}}^{-1} e^{-2\tilde{S}(\widehat{\mathbf{m}})/h} (1 + O(h)).$$

and (4.1) is enough to conclude.

Let us now prove *iii*). Using [1]'s work on $\|P\psi_{\widehat{\mathbf{m}}}\|^2$, we have

$$\|P\psi_{\widehat{\mathbf{m}}}\|^2 = \|P(\chi_{\ell} e^{-(f-f(\widehat{\mathbf{m}}))/h})\|_{L^2(\text{supp } \theta)}^2 + O(e^{-2(\tilde{S}(\widehat{\mathbf{m}}) + \delta)/h}),$$

and we recall (3.10) with the constant C_h

$$P(\chi_{\ell} e^{-(f-f(\widehat{\mathbf{m}}))/h}) = \sqrt{h} O(X^4 + hX^2 + h^2) e^{-(f-f(\widehat{\mathbf{m}}) + \frac{\ell^2}{2})/h}$$

and thus, since we are on $\text{supp } \theta \subset \mathcal{C}_{4\tau, 4\delta}$, we obtain with another Laplace method

$$\|P\psi_{\widehat{\mathbf{m}}}\|^2 = O(h^4) \langle P\psi_{\widehat{\mathbf{m}}}, \psi_{\widehat{\mathbf{m}}} \rangle.$$

Now let us move to *iv*). We notice P^* satisfies (3.1) with $-b$ instead of b therefore, it satisfies an equation similar to (3.2), with a w^* slightly different from w but with $w_{0,0}^* = w_{0,0} = 0$ and $w_{0,1}^* = O(X)$, it leads to

$$P^*(\chi_\ell e^{-(f-f(\mathbf{m}))/h}) = \sqrt{h}O(X)e^{-(f-f(\mathbf{m})+\frac{\ell^2}{2})/h},$$

and hence

$$\|P^*\psi_{\widehat{\mathbf{m}}}\|^2 = O(h)\langle P\psi_{\widehat{\mathbf{m}}}, \psi_{\widehat{\mathbf{m}}}\rangle.$$

□

We can now complete the proof of Theorem 2.

From Proposition 4.2 we denote

$$\tilde{\lambda} = \langle P\varphi_{\widehat{\mathbf{m}}}, \varphi_{\widehat{\mathbf{m}}}\rangle = he^{-2\tilde{S}(\widehat{\mathbf{m}})/h} \frac{\mu(\mathbf{s})}{2\pi} \frac{D_{\widehat{\mathbf{m}}}}{D_{\mathbf{s}}}(1 + O(h)).$$

Recalling $f(x, v, y) = \frac{1}{2}\left(\frac{V(x)}{2} + \frac{|v|^2 + y^2}{2}\right)$ we have that

$$2\tilde{S}(\widehat{\mathbf{m}}) = 2(f(\mathbf{s}) - f(\widehat{\mathbf{m}})) = V(\mathbf{s}) - V(\widehat{\mathbf{m}}) = S(\widehat{\mathbf{m}})$$

and

$$\frac{D_{\widehat{\mathbf{m}}}}{D_{\mathbf{s}}} = \frac{|\det \text{Hess}_{\widehat{\mathbf{m}}} f|^{1/2}}{(\det \text{Hess}_{\mathbf{s}} f)^{1/2}} = \frac{2^{-(2d+1)} |\det \text{Hess}_{\widehat{\mathbf{m}}} V|^{1/2}}{2^{-(2d+1)} (\det \text{Hess}_{\mathbf{s}} V)^{1/2}} = \frac{|\det \text{Hess}_{\widehat{\mathbf{m}}} V|^{1/2}}{(\det \text{Hess}_{\mathbf{s}} V)^{1/2}}$$

hence

$$\tilde{\lambda} = he^{-S(\widehat{\mathbf{m}})/h} \frac{\mu(\mathbf{s})}{2\pi} \frac{|\det \text{Hess}_{\widehat{\mathbf{m}}} V|^{1/2}}{(\det \text{Hess}_{\mathbf{s}} V)^{1/2}}(1 + O(h)).$$

We thus have

$$\|P\varphi_{\widehat{\mathbf{m}}}\| = O\left(h^2\sqrt{\tilde{\lambda}}\right) \quad \text{and} \quad \|P^*\varphi_{\widehat{\mathbf{m}}}\| = O\left(\sqrt{h\tilde{\lambda}}\right).$$

Let us recall $\Pi_0 = \frac{1}{2i\pi} \int_{\partial D} (z - P)^{-1} dz$ where $D = D(0, \frac{c_1}{2}g(h))$, recalling $g(h)$ is defined in (1.11), we also denote $u_1 = \Pi_0 \varphi_{\widehat{\mathbf{m}}}$ and notice that $u_0 = \Pi_0 \varphi_{\underline{\mathbf{m}}} = \varphi_{\underline{\mathbf{m}}}$.

And so there exists $c > 0$, such that for $j, k \in \{0, 1\}$

$$\langle u_j, u_k \rangle = \delta_{j,k} + O(e^{-c/h}) \quad \text{and} \quad \langle Pu_j, u_k \rangle = \delta_{j,k} \delta_{j,1} \tilde{\lambda} + O(g(h)^{-1} h^{\frac{5}{2}} \tilde{\lambda}),$$

having computed, following (2.36)

$$\begin{aligned} \langle Pu_1, u_1 \rangle &= \langle P\varphi_{\widehat{\mathbf{m}}}, \varphi_{\widehat{\mathbf{m}}}\rangle + \langle P(\Pi_0 \varphi_{\widehat{\mathbf{m}}} - \varphi_{\widehat{\mathbf{m}}}), \varphi_{\widehat{\mathbf{m}}}\rangle + \langle P\Pi_0 \varphi_{\widehat{\mathbf{m}}}, \Pi_0 \varphi_{\widehat{\mathbf{m}}} - \varphi_{\widehat{\mathbf{m}}}\rangle \\ &= \tilde{\lambda} + O\left(\|\Pi_0 \varphi_{\widehat{\mathbf{m}}} - \varphi_{\widehat{\mathbf{m}}}\| \|P^*\varphi_{\widehat{\mathbf{m}}}\| + \|\Pi_0 \varphi_{\widehat{\mathbf{m}}} - \varphi_{\widehat{\mathbf{m}}}\| \|P\varphi_{\widehat{\mathbf{m}}}\|\right) \\ &= \tilde{\lambda} + O(g(h)^{-1} h^2 (\sqrt{h} + h^2) \tilde{\lambda}) \\ &= \tilde{\lambda} (1 + O(g(h)^{-1} h^{\frac{5}{2}})) \\ &= \tilde{\lambda} (1 + O(\sqrt{h})) \end{aligned}$$

We then see that for h small enough, (u_0, u_1) is a basis of $\text{Ran } \Pi_0$, by the Gram-Schmidt process we orthonormalize the previous basis into a basis $e = (e_0, e_1)$ (notice that in fact $e_0 = \varphi_{\underline{\mathbf{m}}}$), and we have $e_j = u_j + O(e^{-c/h})$. We can easily compute the matrix of $P|_{\text{Ran } \Pi_0}$ in this basis

$$\text{Mat}_e P|_{\text{Ran } \Pi_0} = (\langle P e_j, e_k \rangle)_{0 \leq j, k \leq 1} = \begin{pmatrix} 0 & 0 \\ 0 & \langle P e_1, e_1 \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\lambda}(1 + O(\sqrt{h})) \end{pmatrix}.$$

Seeing its eigenvalues, we conclude Theorem 2.

APPENDIX A. SOME TECHNICAL RESULTS

A.1. **Proof of Proposition 1.1.** The idea is to mimic the proof of [14, Theorem 15.1].

Let $h, \nu, \gamma > 0$ be fixed. To show that P admits a maximal accretive extension, it is first necessary to show that it admits an accretive extension, this comes from the skew-adjointness of H_0 and Y , as well as from the positivity of \mathcal{O} . It therefore remains to show the maximal side, for that we use the criterion which tells us that P is maximal accretive if $T = P + \gamma(h/2 + 1) \text{Id}$ has a dense image.

Let $f \in L^2(\mathbb{R}^{2d+1})$ such that

$$\forall u \in \mathcal{C}_c^\infty(\mathbb{R}^{2d+1}), \langle f, Tu \rangle = 0.$$

We then must show that $f = 0$. As P is real, we can assume also is f . We split $Y = \underbrace{h(|v|^2 - dh) \partial_y - yv \cdot \partial_v}_{=Y_1} - \underbrace{\frac{y}{2} dh}_{=Y_0}$ where we can see Y_1 is a homogeneous differential

operator of order 1 and Y_0 is a mere \mathcal{C}^∞ function. We want to apply the standard hypoellipticity theorem for Hörmander operators to use our solution as test function,

so let's verify the hypothesis : $P = -\sum_{j=1}^k X_j^2 + X_0 + a(x, v, y)$ with $k = d$, $X_j = h \partial_{v_j}$,

$$a(x, v, y) = \gamma \frac{|v|^2}{4} - \gamma \frac{dh}{2} - \nu \frac{y}{2} dh \text{ and}$$

$$X_0 = H_0 + \nu Y_1 = v \cdot h \partial_x - \partial_x V \cdot h \partial_v + \nu h (|v|^2 - dh) \partial_y - yv \cdot \partial_v$$

Therefore, the Lie brackets are :

$$\begin{aligned} [X_j, X_0] &= h^2 \partial_{x_j} + \nu h^2 (2v_j \partial_y - y \partial_{v_j}) \\ [X_j, [X_j, X_0]] &= 2\nu h^3 \partial_y \end{aligned}$$

Which ensures that $(X_j)_{j \geq 0}$ is bracketgenerating. By taking back the computation of [14, p219] with a modified $\zeta : \zeta_k(x, v, y) = \tilde{\zeta} \left(\frac{x}{k_1} \right) \hat{\zeta} \left(\frac{|v|^2 + y^2}{k_2} \right)$ (where $\tilde{\zeta}, \hat{\zeta} \in \mathcal{C}_c^\infty$, are cutoffs around 0 and $k = (k_1, k_2) \in (\mathbb{R}_+^*)^2$), we get :

$$\begin{aligned} (A.1) \quad & \gamma h^2 \|\partial_v(\zeta_k f)\|^2 + \frac{\gamma}{4} \|\zeta_k v f\|^2 + \gamma \|\zeta_k f\|^2 \\ & + \int \zeta_k f^2 H_0(\zeta_k) + \nu \int f Y(\zeta_k^2 f) = \gamma h^2 \|\partial_v \zeta_k |f|\|^2. \end{aligned}$$

Helffer doesn't use the expression of ζ_k before this, the result is true for any compactly supported function. For the following computations, C will denote a positive constant that might change from line to line.

Noting $C(k_1) \simeq \sup_{|x| \leq 2k_1} |\nabla V|$ we have :

$$\begin{aligned} - \int \zeta_k f^2 H_0(\zeta_k) &\leq \frac{C}{k_1} \|\zeta_k v f\| \|f\| + \frac{C(k_1)}{k_2} \|\zeta_k v f\| \|f\| \\ &\leq \frac{\gamma}{16} \|\zeta_k v f\|^2 + \frac{C}{k_1^2} \|f\|^2 + \frac{\gamma}{16} \|\zeta_k v f\|^2 + \frac{C(k_1)^2}{k_1^2} \|f\|^2 \end{aligned}$$

And since $|\partial_{v_j} \zeta_k| \leq C \frac{v_j}{k_2}$, we have $\gamma h^2 \|\nabla_v \zeta_k\| f \leq \frac{C}{\sqrt{k_2}} \|f\|$. Plugging these two into (A.1) we get

$$\gamma \|\zeta_k f\|^2 + \frac{\gamma}{8} \|\zeta_k v f\|^2 + \nu \int f Y(\zeta_k^2 f) \leq C \left(\frac{1}{k_1^2} + \frac{C(k_1)^2}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2.$$

It only remains to study the last integral, we have by integration by parts :

$$\begin{aligned} \int f Y_1(\zeta_k^2 f) &= \int f (Y_1(\zeta_k) \zeta_k f + Y_1(\zeta_k f) \zeta_k) \\ \int \zeta_k f Y_1(\zeta_k f) &= \frac{1}{2} \int Y_1((\zeta_k f)^2) = -\frac{h}{2} \int y v \cdot \partial_v ((\zeta_k f)^2) \\ \int f Y(\zeta_k^2 f) &= \int f^2 \zeta_k Y(\zeta_k) + \frac{dh}{2} \int y (\zeta_k f)^2 \end{aligned}$$

Now with the expression of ζ_k , we have $Y(\zeta_k) = -dh \left(h \partial_y + \frac{y}{2} \right) (\zeta_k)$, and thus :

$$- \int f Y(\zeta_k^2 f) = h^2 d \int \zeta_k f^2 \partial_y \zeta_k \leq \frac{C}{\sqrt{k_2}} \|\zeta_k f\| \|f\| \leq \frac{\gamma}{2\nu} \|\zeta_k f\|^2 + \frac{C}{k_2} \|f\|^2$$

We finally get :

$$\frac{\gamma}{2} \|\zeta_k f\|^2 \leq \frac{\gamma}{8} \|\zeta_k v f\|^2 + \frac{\gamma}{2} \|\zeta_k f\|^2 \leq C \left(\frac{1}{k_1^2} + \frac{C(k_1)^2}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2$$

And by taking the limit $k_2 \rightarrow +\infty$ and then $k_1 \rightarrow +\infty$, we obtain $\|f\| = 0$. □

A.2. Proof of Lemma 2.6. By simple computations, we recall (2.2) :

$$\begin{aligned} H_0 &= v \cdot \delta_x - \partial_x V \cdot \delta_v, \\ Y &= (|v|^2 - dh) \delta_y - y v \cdot \delta_v. \end{aligned}$$

For (2.9), this leads to

$$\begin{aligned} H_i^* H_j \Pi_\rho &= -H_i v_j \delta_{x_j} \Pi_\rho = (-v_i v_j \delta_{x_i} \delta_{x_j} + h \partial_{x_i} V \partial_{v_i} (v_j) \delta_{x_j}) \Pi_\rho \\ &= (-v_i v_j \delta_{x_i} \delta_{x_j} + \delta_{i,j} h \partial_{x_i} V \delta_{x_j}) \Pi_\rho. \end{aligned}$$

For (2.10), recalling that Y is skew-adjoint and thanks to $-\delta_y = \delta_y^* - y$, we first set

$$\begin{aligned} Y^* &= (|v|^2 - dh) (\delta_y^* - y) + y v \cdot \delta_v \\ &= (|v|^2 - dh) \delta_y^* - y (|v|^2 - dh) + y v \cdot \delta_v. \end{aligned}$$

We then get

$$\begin{aligned} Y^*Y\Pi_\rho &= Y^*(|v|^2 - dh)\delta_y\Pi_\rho \\ &= ((|v|^2 - dh)^2\delta_y^*\delta_y - ((|v|^2 - dh)^2 - v \cdot h \partial_v(|v|^2 - dh))y\delta_y)\Pi_\rho \\ &= ((|v|^2 - dh)^2\delta_y^*\delta_y - ((|v|^2 - dh)^2 - 2h|v|^2)y\delta_y)\Pi_\rho \end{aligned}$$

Obtaining the last two equations (2.11) and (2.12) is pretty straightforward, we just write

$$\begin{aligned} H_i^*Y\Pi_\rho &= -H_i(|v|^2 - dh)\delta_y\Pi_\rho \\ &= (-v_i(|v|^2 - dh)\delta_{x_i}\delta_y + \partial_{x_i} Vh \partial_v(|v|^2 - dh)\delta_y)\Pi_\rho \\ &= (-v_i(|v|^2 - dh)\delta_{x_i}\delta_y + 2v_i h \partial_{x_i} V\delta_y)\Pi_\rho, \end{aligned}$$

and

$$\begin{aligned} Y^*H_i\Pi_\rho &= -Yv_i\delta_{x_i}\Pi_\rho \\ &= (-v_i(|v|^2 - dh)\delta_{x_i}\delta_y + yv \cdot h \partial_v(v_i)\delta_{x_i})\Pi_\rho \\ &= (-v_i(|v|^2 - dh)\delta_{x_i}\delta_y + v_i h y\delta_{x_i})\Pi_\rho \end{aligned}$$

□

A.3. Some resolvent estimates.

Lemma A.1. *One has the following estimates*

$$\|(h\alpha + d\Delta_{\frac{v}{2}})(h\alpha + dB)^{-1}\| \leq 1$$

and

$$\|(h\alpha + 2d\nu^2 hN_y)(h\alpha + dB)^{-1}\| \leq 1$$

Proof. For any $u \in L^2$, since $N_y \geq 0$, one has

$$\begin{aligned} \|(h\alpha + d\Delta_{\frac{v}{2}})^{\frac{1}{2}}(h\alpha + dB)^{-\frac{1}{2}}u\|^2 &= \langle (h\alpha + d\Delta_{\frac{v}{2}})(h\alpha + dB)^{-\frac{1}{2}}u, (h\alpha + dB)^{-\frac{1}{2}}u \rangle \\ &\leq \langle (h\alpha + dB)(h\alpha + dB)^{-\frac{1}{2}}u, (h\alpha + dB)^{-\frac{1}{2}}u \rangle \\ &\leq \|u\|^2 \end{aligned}$$

Consequently, denoting $T = (h\alpha + d\Delta_{\frac{v}{2}})^{\frac{1}{2}}(h\alpha + dB)^{-\frac{1}{2}}$, one has $\|T\|_{L^2 \rightarrow L^2} \leq 1$. Moreover, since $\Delta_{\frac{v}{2}}$ and N_y commute, one has $(h\alpha + d\Delta_{\frac{v}{2}})(h\alpha + dB)^{-1} = T^*T$ and the first estimate follows immediately. The second is identical.

□

Lemma A.2. *There exists $C > 0$ such that for all $s > 0$, one has the following resolvent estimates*

$$\|(h\alpha + d\Delta_{\frac{v}{2}})^{-s}\| + \|(h\alpha + 2d\nu^2 hN_y)^{-s}\| \leq Ch^{-s}\alpha^{-s}$$

and hence

$$\|\Delta_{\frac{v}{2}}(h\alpha + d\Delta_{\frac{v}{2}})^{-1}\| + \nu^2 h \|N_y(h\alpha + 2d\nu^2 hN_y)^{-1}\| \leq C$$

Proof. The first equation is straightforward when noticing $\Delta_{\frac{\nu}{2}}$ and N_y are non negative. The second equation can be proved using functional calculus considering $x \mapsto \frac{x}{h\alpha + cx}$. Let us remark that we can compute the constant, being 2 for the first equation, and $\frac{3}{2d}$ for the second one, but since the exact form of these constant will not be useful, we will keep C . □

Lemma A.3. *There exists $C > 0$ such that for all $h > 0$ and $\nu > 0$ one has*

$$\|\delta_{x_i}(h\alpha + d\Delta_{\frac{\nu}{2}})^{-\frac{1}{2}}\| + \nu\sqrt{h}\|\delta_y(h\alpha + 2d\nu^2hN_y)^{-\frac{1}{2}}\| \leq C$$

Proof. This estimate is obtained by taking the adjoint and using the spectral theorem. More precisely, for any $u \in L^2$, one has

$$\begin{aligned} \|\delta_{x_i}(h\alpha + d\Delta_{\frac{\nu}{2}})^{-\frac{1}{2}}u\|^2 &= \langle \delta_{x_i}^* \delta_{x_i} (h\alpha + d\Delta_{\frac{\nu}{2}})^{-\frac{1}{2}}u, (h\alpha + d\Delta_{\frac{\nu}{2}})^{-\frac{1}{2}}u \rangle \\ &\leq \langle \Delta_{\frac{\nu}{2}} (h\alpha + d\Delta_{\frac{\nu}{2}})^{-\frac{1}{2}}u, (h\alpha + d\Delta_{\frac{\nu}{2}})^{-\frac{1}{2}}u \rangle \\ &\leq \|\Delta_{\frac{\nu}{2}} (h\alpha + d\Delta_{\frac{\nu}{2}})^{-1}\| \|u\|^2 \\ &\leq C \|u\|^2 \end{aligned}$$

by the previous lemma. The same arguments give the estimate on $\delta_y(h\alpha + 2d\nu^2hN_y)^{-\frac{1}{2}}$. □

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