STATIONARY ENTRANCE CHAINS AND APPLICATIONS TO RANDOM WALKS

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Abstract. For a Markov chain Y with values in a Polish space, consider the *entrance chain* obtained by sampling Y at the moments when it enters a fixed set A from its complement A^c. Similarly, consider the *exit chain*, obtained by sampling Y at the exit times from A^c to A. We use the method of inducing from ergodic theory to study invariant measures of these two types of Markov chains in the case when the initial chain Y has a known invariant measure. We give explicit formulas for invariant measures of the entrance and exit chains under certain recurrence-type assumptions on A and A^c, which apply even for transient chains. Then we study uniqueness and ergodicity of these invariant measures assuming that Y is topologically recurrent, topologically irreducible, and weak Feller.

We give applications to random walks in $\mathbb{R}^d$, which we regard as "stationary" Markov chains started under the Lebesgue measure. We are mostly interested in dimension one, where we study the Markov chain of overshoots above the zero level of a random walk that oscillates between $-\infty$ and $+\infty$. We show that this chain is ergodic, and use this result to prove a central limit theorem for the number of level crossings for random walks with zero mean and finite variance of increments.

1. Introduction

Let $S = (S_n)_{n\geq 0}$ be a non-degenerate random walk in $\mathbb{R}^d$, where $d \in \mathbb{N}$. That is $S_n = S_0 + X_1 + \ldots + X_n$ for $n \in \mathbb{N}$, where $X_1, X_2, \ldots$ are independent identically distributed increments and S_0 is the starting point that is independent of the increments.

For now consider the case $d = 1$ and assume that the random walk S *oscillates*, that is $\limsup S_n = -\liminf S_n = +\infty$ a.s. as $n \to \infty$. Then either $\mathbb{E}X_1 = 0$ or $\mathbb{E}X_1$ does not exist; in particular, in the latter case S can be transient. Define the *crossing times* of the zero level by $\mathcal{T}_0 := 0$ and

$$\mathcal{T}_n := \inf\{k > \mathcal{T}_{n-1} : S_{k-1} < 0, S_k \geq 0 \text{ or } S_{k-1} \geq 0, S_k < 0\}, \qquad n \in \mathbb{N},$$

and let

$$\mathcal{O}_n := S_{\mathcal{T}_n}, \qquad n \in \mathbb{N}, \tag{1}$$

be the corresponding *overshoots*. It is easy to show, using that the $\mathcal{T}_n$'s are stopping times, that the sequence $\mathcal{O} := (\mathcal{O}_n)_{n\geq 1}$ is a Markov chain.

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This paper was motivated by our interest in stationarity and stability properties of the Markov chain of overshoots $\mathcal{O}$. In [34] we essentially showed that the measure

$$\pi(dx) := \big[\mathbb{1}_{[0,\infty)}(x)\mathbb{P}(X_1 > x) + \mathbb{1}_{(-\infty,0)}(x)\mathbb{P}(X_1 \le x)\big]\lambda(dx), \qquad x \in \mathcal{Z}, \tag{2}$$

is invariant for this chain, where $\mathcal{Z}$ the minimal topologically closed subgroup of $(\mathbb{R}, +)$ that contains the topological support of the distribution of X_1 and λ is the normalized Haar measure on $(\mathcal{Z}, +)$. Thus, $\mathcal{Z}$ and λ are either $\mathbb{R}$ equipped with the Lebesgue measure or a multiple of $\mathbb{Z}$ with the counting measure. Note that π is finite if and only if $\mathbb{E}|X_1| < \infty$, in which case $\mathbb{E}X_1 = 0$ by the assumption of oscillation.

We found the measure π in [34], deriving it informally in a special case using an ergodic averaging argument and assuming that the chain of overshoots has an invariant distribution. Then we proved invariance of π under general assumptions using quite a complicated ad-hoc argument based on time-reversibility, clarified and generalized in the present paper. The same approach of deriving (or even guessing) and then proving invariance was used in a number of other works concerning stability of certain related Markov chains, e.g. in [6, 27, 37]. In all these examples this approach neither explains the form of the invariant measures nor shows how to find them. Moreover, the uniqueness of invariant measures has to be established separately – for example, in [34, Section 3] we did this only under additional assumptions on the distribution of X_1, ensuring the convergence in total variation.

This paper presents a unified approach to *finding* invariant measures and proving their uniqueness and ergodicity, which applies in a much more general context than level-crossings of one-dimensional random walks in [34]. Our method is built on *inducing*, a basic tool of ergodic theory, introduced by S. Kakutani in 1943. In order to proceed to a general setting, note that the chain of overshoots has a periodic structure since its values at consecutive steps have different signs. Therefore, it suffices to consider the non-negative Markov chain $O = (O_n)_{n\ge1}$ of *overshoots at up-crossings* defined by $O_n := \mathcal{O}_{2n-\mathbb{1}(S_0<0)}$. We will also consider the sequence $U = (U_n)_{n\ge1}$ of *undershoots at up-crossings* given by $U_n := \mathcal{U}_{2n-\mathbb{1}(S_0<0)}$, where $\mathcal{U}_n := S_{\mathcal{T}_n - 1}$. This latter sequence turns out to be a Markov chain, but this fact is far less intuitive since $\mathcal{T}_n - 1$ are not stopping times. The chain U played an important role in the proof of invariance of π presented in [34].

Observe that the Markov chain of overshoots O at up-crossings above the zero level is obtained by sampling the one-dimensional random walk S at the moments it enters the set $[0, \infty)$ from $(-\infty, 0)$. Similarly, for any Markov chain[1] Y with values in a Polish space $\mathcal{X}$ we can consider the *entrance Markov chain*, denoted by $Y^{\rangle A}$, obtained by sampling Y at the moments of entry into an arbitrary fixed Borel set A from its complement A^c. We also consider the *exit Markov chain*, denoted by $Y^{A^c\rangle}$, obtained by sampling Y at the exit times from A^c to A; the Markov property of this sequence is not obvious and we refer to Lemma 2.1 for its proof. In this notation, we have $O = S^{\rangle A}$ and $U = S^{A^c\rangle}$ for $A = [0, \infty) \cap \mathcal{Z}$.

We will show (Theorems 3.1 and 4.1) that if Y has an invariant σ-finite measure μ, then the entrance chain $Y^{\rangle A}$ and the exit chain $Y^{A^c\rangle}$ have respective the invariant measures

$$\mu_A^{entr}(dx) = \mathbb{P}_x(\hat{Y}_1 \in A^c)\mu(dx) \text{ on } A \quad \text{and} \quad \mu_{A^c}^{exit}(dx) = \mathbb{P}_x(Y_1 \in A)\mu(dx) \text{ on } A^c, \tag{3}$$

[1] All Markov chains considered in this paper are time-homogeneous.

where $\hat{Y}$ is a Markov chain *dual* to Y relative to μ and satisfying $\mathbb{P}_x(Y_0 = x) = 1 = \mathbb{P}_x(\hat{Y}_0 = x)$ for every $x \in \mathcal{X}$, provided that Y and $\hat{Y}$ visit both sets A and A^c infinitely often $\mathbb{P}_x$-a.s. for $\mu_{A^c}^{exit}$-a.e. x and for μ_A^{entr}-a.e. x. In particular, these assumptions are satisfied if Y is recurrent starting under μ and $\mu_{A^c}^{exit}(A^c) > 0$, that is Y can get from A^c to A. In this case the chains $Y^{\rangle A}$ and $Y^{A^c\rangle}$ are recurrent and also ergodic if so is Y started under μ. However, we stress that our results also apply when Y is *transient*.

Note that formulas (3) are symmetric in the sense that their right-hand sides interchange if we swap the chain Y and the set A with the dual chain $\hat{Y}$ and the complement set A^c. The reason is that the exit chain $Y^{A^c\rangle}$ of Y from A^c to A turns out to be dual to the entrance chain $\hat{Y}^{\rangle A^c}$ of $\hat{Y}$ into A^c from A relative to the measure $\mu_{A^c}^{exit}$ (Proposition 4.1). This immediately implies that $\mu_{A^c}^{exit}$ is invariant for the exit chain $Y^{A^c\rangle}$. This in turn yields invariance of μ_A^{entr} for the entrance chain $Y^{\rangle A}$ by swapping Y and A with $\hat{Y}$ and A^c. The described duality between the entrance and the exit chains explains the need to consider the latter ones.

Our further result concerns ergodicity and uniqueness of the invariant measures for the entrance and exit chains. In Theorem 3.2 we show that under the topological assumptions of recurrence, irreducibility, and the weak Feller property of the chain Y and, essentially, non-emptiness of the interiors of the sets A and A^c, the questions of existence of an invariant measure, its ergodicity and uniqueness (up to a constant factor) in the class of locally finite Borel measures have the same answer for each of the three chains Y, $Y^{\rangle A}$, $Y^{A^c\rangle}$.

Next we consider applications of the general results described above to random walks on $\mathbb{R}^d$ (see Section 5). In this case the normalized Haar measure λ on the minimal closed subgroup $\mathcal{Z}$ of $(\mathbb{R}^d, +)$ that contains the support of X_1 is invariant for the random walk S. This explains why we need to use the results of infinite ergodic theory. Since the dual of S relative to λ is $-S$, the first formula in (3) reads

$$\lambda_A^{entr}(dx) = \mathbb{P}(X_1 \in x - A^c)\lambda(dx), \qquad x \in A.$$

If the random walk S is topologically recurrent on $\mathcal{Z}$, the Haar measure λ is known to be ergodic and unique locally finite invariant measure of S, and then so is the measure λ_A^{entr} for the entrance chain $S^{\rangle A}$ when $\lambda(A) > 0$, $\lambda(A^c) > 0$, and $\lambda(\partial A) = 0$ (Theorem 5.1).

To give a concrete example, consider the orthant $A = \{x \in \mathbb{R}^d : x \geq 0\}$, where and below the inequalities between points in $\mathbb{R}^d$ are understood coordinate-wise. Assume that S hits the interiors of A and $-A$ a.s. when starting at $S_0 = 0$; in dimension one this assumption is equivalent to oscillation of S (which admits transience of S). Then λ_A^{entr} can be written as

$$\pi_+(dx) := (1 - \mathbb{P}(X_1 \leq x))\lambda(dx), \qquad x \in \mathcal{Z} \cap [0, \infty)^d. \tag{4}$$

In particular, for $d = 1$ this means that π_+ is invariant for the chain O. Combining this with an analogous result for the chain of overshoots at down-crossings of zero yields the stated invariance of the measure π for the chain $\mathcal{O}$ (Corollary 5.1). This invariant measure is unique and ergodic when S is topologically recurrent (in particular, this settles the question of uniqueness of π in dimension $d = 1$, only partially answered in [34]).

Our interest in stationarity of overshoots and level-crossings of one-dimensional random walks was motivated as follows. First, the overshoots are related to the *local times* of random walks. Perkins [40] defined[2] the local time of S at zero (at time n) as $\sum_{k=1}^{L_n} |\mathcal{O}_k|$, where

$$L_n := \max\{k \geq 0 : \mathcal{T}_k \leq n\} \tag{5}$$

denotes the *number of zero-level crossings* of the walk by time n. Then [40] proved a limit theorem for the local time, assuming that the walk has zero mean and finite variance. From this result and ergodicity of the Markov chain $\mathcal{O}$ (established in Theorem 5.2), we obtain a limit theorem for the number of level crossings L_n (Theorem 6.1).

Second, the chain O appeared in the study of the probabilities that the *integrated random walk* $(S_1 + \ldots + S_n)_{n\geq 1}$ stays positive for a long time; see Vysotsky [50, 51]. The main idea of the approach of [50, 51] is in a) splitting the trajectory of the walk into consecutive "cycles" between the up-crossing times; and b) using that for certain distributions of increments, e.g. when the distribution $\mathbb{P}(X_1 \in \cdot | X_1 > 0)$ is exponential, the overshoots $(O_n)_{n\geq 1}$ are i.i.d. regardless of the starting point S_0. This paper was originally motivated by the question whether this approach can be extended to general distributions of increments but starting S so that O remains stationary.

Third, the level-crossings define the dynamics of the so-called *switching random walks*. This a special type of Markov chains with the transition probabilities of the form $P(x, dy) = P_{\operatorname{sign} x}(dy - x)$ for $x \neq 0$ and $P(0, dy) = \alpha P_+(dy) + (1 - \alpha)P_-(dy)$, where P_+ and P_- are two probability distributions on $\mathbb{R}$ and $\alpha \in [0, 1]$. Such chains, introduced by Kemperman [26] under the name of *oscillating random walks*, were also studied e.g. by Borovkov [6], Brémont [7], and Vo [49]. In the antisymmetric case $P_+(dy) = P_-(-dy)$, the absolute value of such chain form the other Markov chain, called a *reflected random walks*. Chains of this type received a lot of attention, see Peigné and Woess [38] for references and generalizations. Invariant distributions for reflected and switching random walks were known in some cases, see [6] and [38]. We will generalize these results and clarify connections to the classical stationary distributions[3] of the renewal theory in a separate note [52], which is based on the ideas of this paper.

We are not aware of any works concerning the entrance and exit Markov chains in any generality. We are also not aware of any applications of inducing in the problems related to level-crossings of one-dimensional random walks, and the idea to regard them as "stationary" processes starting from the Haar measure is new in this context. Here the classical and universal tool is the Wiener–Hopf factorization, which does not yield much for our problem. In particular, this factorization was used by Baxter [2], Borovkov [6], and Kemperman [26], which are all closely related to the questions considered in our paper. For the higher dimensional generalisations, we believe that our formulas for the invariant measures, such as (4), are the only explicit results available.

Finally, let us briefly describe possible applications of general results of Sections 3 and 4 to reversible Markov chains, which is a wide class of chains with a known invariant distribution. For such chains, formula (3) for μ_A^{entr} is particularly simple since we can take $\hat{Y}_1 = Y_1$. To verify the recurrence assumptions of Theorems 3.1 and 4.1, one can use the following

[2]There is no canonical definition of local times of random walks, see Csörgő and Révész [13] and Mijatović and Uribe Bravo [33] for other versions.

[3]These distributions have the same form as π_+ in $d = 1$, as discussed in [34, Sections 2.1 and 2.2].

results. A simple necessary and sufficient condition for recurrence of countable reversible Markov chains is due to Lyons [29]; criteria for recurrence of general chains are given in Menshikov et al. [31, Section 2.5]. Conditions for recurrence of a set for general transient Markov chains on a countable state space can be found e.g. in Bucy [8] and Murdoch [35]. For transient chains, there is one example with a particularly simple characterization of recurrent sets: by Gantert et al. [19, Theorem 1.7], a planar simple random walk conditioned on never hitting the origin visits any infinite subset of $\mathbb{Z}^2$ infinitely often a.s.

1.1. **Structure of the paper.** In Section 2 we carefully define the entrance and exit sequences and prove their Markov property. In Section 3 we study stationarity of these chains using the idea of inducing from ergodic theory – in Section 3.1 we provide a self-contained setup needed to apply inducing in the context of Markov chains; in Section 3.2 we show the use of inducing in finding invariant measures for the entrance and exit chains sampled from a general recurrent Markov chain; and in Section 3.3 we study existence and uniqueness of these invariant measures for the specific class of recurrent weak Feller chains on Polish spaces. In Section 4 drop the assumption of recurrence and explore the duality between the entrance and exit chains and its role in proving invariance of the measures defined in (3). The rest of the paper concerns applications of the general results of Sections 3 and 4 to random walks. In Section 5, where we study the entrance chains sampled from random walks in $\mathbb{R}^d$, including the chains of overshoots in dimension one. In Section 6 we prove a limit theorem for the number of level-crossings. The Appendix contains some relevant basics of infinite ergodic theory.

2. Entrance and exit Markov chains

This this section we set up the basic notation and show that the a Markov chain sampled at the exit times from a set is again a Markov chain.

Throughout this paper $(\mathcal{X}, \mathcal{F})$ will be a measurable space. For a measure μ on $(\mathcal{X}, \mathcal{F})$ and a non-empty set $A \in \mathcal{F}$, by μ_A we denote the measure on $(A, \mathcal{F}_A)$ given by $\mu_A := \mu|_{\mathcal{F}_A}$, where $\mathcal{F}_A := \{B \subset A : B \in \mathcal{F}\}$. If $\mathcal{X}$ is a metric space, we always equip it with the Borel σ-algebra $\mathcal{B}(\mathcal{X})$ and refer to measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ as Borel measures on $\mathcal{X}$ (for example, in this case μ_A is a Borel measure on A).

Throughout this paper $Y = (Y_n)_{n \ge 0}$ will be a time-homogeneous Markov chain taking values in $\mathcal{X}$. By saying this, we assume that Y is defined on some generic probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and Y has a probability transition kernel P on $(\mathcal{X}, \mathcal{F})$ under $\mathbb{P}$. To simplify the notation, it is convenient to assume that $(\Omega, \mathcal{A})$ is also equipped with a family of probability measures $\{\mathbb{P}_x\}_{x \in \mathcal{X}}$ such that: Y is a Markov chain with the transition kernel P under $\mathbb{P}_x$ and $\mathbb{P}_x(Y_0 = x) = 1$ for every $x \in \mathcal{X}$, and the function $x \mapsto \mathbb{P}_x(Y \in B)$ is measurable for any set $B \in \mathcal{F}^{\otimes \mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Such a family of measures always exists for any probability kernel on $\mathcal{X}$ when $(\Omega, \mathcal{A})$ is the canonical space $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{F}^{\otimes \mathbb{N}_0})$ and Y is its identity mapping by the Ionescu Tulcea extension theorem (Kallenberg [24, Theorem 6.17]).

Furthermore, for any measure ν on $(A, \mathcal{F}_A)$, where $A \in \mathcal{F}$ is non-empty, denote $\mathbb{P}_\nu := \int_A \mathbb{P}_x(\cdot)\nu(dx)$. Then Y_0 has "distribution" ν under $\mathbb{P}_\nu$, in which case we say that Y *starts under* ν. Although ν is not necessarily a probability, we prefer to (ab)use probabilistic notation and terminology as above, instead of using respective notions of general measure theory. Denote by $\mathbb{E}_x$ and $\mathbb{E}_\nu$ respective expectations (Lebesgue integrals) over $\mathbb{P}_x$ and $\mathbb{P}_\nu$. We say that a measure ν on $(A, \mathcal{F}_A)$, where $A \in \mathcal{F}$, is *invariant* for Y if $\int_A P(x, \cdot)\nu(dx) = \nu$.

Define the entrance times of Y to a set $A \in \mathcal{X}$ from A^c by $T_0^{\rangle A} := 0$ and

$$T_n^{\rangle A} := \inf\{k > T_{n-1}^{\rangle A} : Y_{k-1} \in A^c, Y_k \in A\}$$

for $n \in \mathbb{N}$, where $\inf_{\varnothing} := \infty$ by convention. The respective positions of Y when entering A from A^c and exiting from A^c to A are denoted by

$$Y_n^{\rangle A} := Y_{T_n^{\rangle A}} \quad \text{and} \quad Y_n^{A^c\rangle} := Y_{T_n^{\rangle A}-1}$$

for $n \in \mathbb{N}$, where we put $Y_\infty := \dagger$ and denote by $\dagger$ the "cemetery" state, that is an additional point that does not belong to $\mathcal{X}$. These variables are random elements of $(A_\dagger, \mathcal{F}_A^\dagger)$ and $(A_\dagger^c, \mathcal{F}_{A^c}^\dagger)$, respectively, where for any $B \in \mathcal{F}$ we define $B_\dagger := B \cup \{\dagger\}$ and $\mathcal{F}_B^\dagger := \mathcal{F}_B \cup \{C \cup \{\dagger\} : C \in \mathcal{F}_B\}$, and write $A_\dagger^c$ for $(A^c)_\dagger$. Put $Y^{\rangle A} := (Y_n^{\rangle A})_{n \geq 1}$ and $Y^{A^c\rangle} := (Y_n^{A^c\rangle})_{n \geq 1}$.

To identify for which initial values of Y all entrance times $T_n^{\rangle A}$ are finite, put

$$N_A(Y) := \{x \in \mathcal{X} : \mathbb{P}_x(Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}) = 1\}, \tag{6}$$

where "i.o." stands for "infinitely often"; we will write N_A in short when the reference to Y is unambiguous. This set is measurable. It is absorbing for Y, in the sense that

$$\mathbb{P}_x(Y_1 \in N_A) = 1, \qquad x \in N_A. \tag{7}$$

Indeed, for every $x \in N_A$ we have

$$1 = \mathbb{P}_x(Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}) = \int_{\mathcal{X}} \mathbb{P}_x(Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}|Y_1 = y)\mathbb{P}_x(Y_1 \in dy).$$

Hence $\mathbb{P}_y(Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}) = 1$, i.e. $y \in N_A$ for $\mathbb{P}_x(Y_1 \in \cdot)$-a.e. y. This proves (7).

Furthermore, define the *exit sets* for Y:

$$B_{ex}(Y) := \{x \in B : \mathbb{P}_x(Y_1 \notin B) > 0\}, \qquad B \in \mathcal{F}. \tag{8}$$

These sets are measurable. We will write $A_{ex}^c(Y)$ or simply A_{ex}^c in short instead of $(A^c)_{ex}(Y)$.

We will refer to the sequences $Y^{\rangle A}$ and $Y^{A^c\rangle}$ respectively as *entrance* and *exit* Markov chains. This is justified by the following result.

Lemma 2.1. *Let Y be a Markov chain taking values in a measurable space $(\mathcal{X}, \mathcal{F})$, and let $A \in \mathcal{F}$. Then for every $x_0 \in \mathcal{X}$, the entrance sequence $Y^{\rangle A}$ and the exit sequence $Y^{A^c\rangle}$ are time-homogeneous Markov chains under $\mathbb{P}_{x_0}$ and their transition probabilities are given by*

$$P_A^{entr}(x, dy) := \mathbb{P}_x(Y_1^{\rangle A} \in dy), \qquad x \in A, y \in A \cup \{\dagger\}, \tag{9}$$

$$P_{A^c}^{exit}(x, dy) := \int_A \mathbb{P}_z(Y_1^{A^c\rangle} \in dy)\mathbb{P}_x(Y_1 \in dz|Y_1 \in A), \quad x \in A_{ex}^c, y \in A_{ex}^c \cup \{\dagger\}, \tag{10}$$

and

$$P_{A^c}^{exit}(\dagger, \{\dagger\}) := 1, \quad P_A^{entr}(\dagger, \{\dagger\}) := 1.$$

If $x_0 \in N_A$, these chains take values in the sets $A \cap N_A$ and $A_{ex}^c \cap N_A$, respectively.

It is convenient to extend the transition kernel of the exit chain to the whole of A^c, say, by putting $P^{exit}_{A^c}(x,\{\dagger\}) := 1$ for $x \in A^c \setminus A^c_{ex}$. Thus, we can formally regard $Y^{A^c\rangle}$ as a Markov chain either on $A^c_\dagger$ or on $A^c_{ex} \cap N_A$, and we regard $Y^{\rangle A}$ as a Markov chain either on $A_\dagger$ or on $A \cap N_A$. We say that a measure ν on $(A, \mathcal{F}_A)$ is *proper* for the entrance chain $Y^{\rangle A}$ if $\nu(A \setminus N_A) = 0$. Similarly, a measure ν on $(A^c, \mathcal{F}_{A^c})$ is *proper* for the exit chain $Y^{A^c\rangle}$ if $\nu(A^c \setminus (A^c_{ex} \cap N_A)) = 0$. We will be interested only in proper invariant measures of these chains.

Proof. It is clear that P^{entr}_A is a probability kernel on $(A_\dagger, \mathcal{F}^\dagger_A)$. Its restriction to $(A \cap N_A, \mathcal{F}_{A\cap N_A})$ is also a probability kernel since $\mathbb{P}_x(T^{\rangle A}_1 < \infty) = 1$ for every $x \in N_A$ and the set N_A is absorbing for Y by (7). Similarly, $P^{exit}_{A^c}$ is a probability kernel on $(A^c_\dagger, \mathcal{F}^\dagger_{A^c})$ and its restriction to $(A^c_{ex} \cap N_A, \mathcal{F}_{A^c_{ex}\cap N_A})$ is a probability kernel too. This implies the last claim of the lemma.

Fix an $x_0 \in \mathcal{X}$. Since the entrance times $T^{\rangle A}_n$ are increasing stopping times with respect to Y, it follows from the equality $Y^{\rangle A} = Y_{T^{\rangle A}}$ and a standard argument based on the strong Markov property of Y under $\mathbb{P}_{x_0}$ that $Y^{\rangle A}$ is an $A_\dagger$-valued Markov chain under $\mathbb{P}_{x_0}$. The formula for its transition kernel is evident. However, the Markov property of the exit sequence $Y^{A^c\rangle}$ is not evident since $(T^{\rangle A}_n - 1)_{n\ge 1}$ are not stopping times.

To prove that $Y^{A^c\rangle}$ is Markov chain under $\mathbb{P}_{x_0}$ with values in $A^c_{ex} \cup \{\dagger\}$ and the transition kernel $P^{exit}_{A^c}$ (in short, $P^{ex}_{A^c}$), it suffices to show that for any integer $n \ge 2$ and measurable sets $B_1, B_2, \ldots \subset A^c_{ex} \cup \{\dagger\}$,

$$\mathbb{P}_{x_0}(Y^{A^c\rangle}_1 \in B_1, \ldots, Y^{A^c\rangle}_n \in B_n) = \int_{B_1} \mathbb{P}_{x_0}(Y^{A^c\rangle}_1 \in dx_1) \int_{B_2} P^{ex}_{A^c}(x_1, dx_2) \ldots \int_{B_n} P^{ex}_{A^c}(x_{n-1}, dx_n).$$

The proof is by induction. Denote $\bar{B}_1 := B_1 \setminus \{\dagger\}$. Let $n = 2$, then

$$\begin{aligned}
\mathbb{P}_{x_0}(Y^{A^c\rangle}_1 \in \bar{B}_1, Y^{A^c\rangle}_2 \in B_2) &= \sum_{k=1}^\infty \mathbb{P}_{x_0}(T^{\rangle A}_1 = k, Y_{k-1} \in \bar{B}_1, Y^{A^c\rangle}_2 \in B_2) \\
&= \sum_{k=1}^\infty \mathbb{P}_{x_0}\big(T^{\rangle A}_1 > k-1, Y_{k-1} \in \bar{B}_1, Y_k \in A, Y^{A^c\rangle}_2 \in B_2\big) \\
&= \sum_{k=1}^\infty \int_{\bar{B}_1} \mathbb{P}_{x_0}(T^{\rangle A}_1 > k-1, Y_{k-1} \in dx_1) \\
&\qquad \times \mathbb{P}_{x_0}\big(Y_k \in A, Y^{A^c\rangle}_2 \in B_2 \big| Y_{k-1} = x_1, T^{\rangle A}_1 > k-1\big).
\end{aligned}$$

By the Markov property of Y, for $\mathbb{P}_{x_0}(Y_{k-1} \in \cdot)$-a.e. $x_1 \in \bar{B}_1$ and every $k \ge 1$ it is true that

$$\begin{aligned}
\mathbb{P}_{x_0}\big(Y_k \in A, Y^{A^c\rangle}_2 \in B_2 \big| Y_{k-1} = x_1, T^{\rangle A}_1 > k-1\big) &= \mathbb{P}_{x_1}(Y_1 \in A, Y^{A^c\rangle}_2 \in B_2) \\
&= \int_A \mathbb{P}_z(Y^{A^c\rangle}_1 \in B_2) \mathbb{P}_{x_1}(Y_1 \in dz).
\end{aligned}$$

On the other hand, from definition (10) of $P^{ex}_{A^c}$ we see that for every $x_1 \in \bar{B}_1$,

$$\int_A \mathbb{P}_z(Y_1^{A^c\rangle} \in B_2)\mathbb{P}_{x_1}(Y_1 \in dz) = \mathbb{P}_{x_1}(Y_1 \in A)P^{ex}_{A^c}(x_1, B_2). \tag{11}$$

Putting everything together, we obtain

$$\begin{aligned}\mathbb{P}_{x_0}(Y_1^{A^c\rangle} \in \bar{B}_1, Y_2^{A^c\rangle} \in B_2) &= \sum_{k=1}^{\infty} \int_{\bar{B}_1} \mathbb{P}_{x_0}(T_1^{\rangle A} > k-1, Y_{k-1} \in dx_1)\mathbb{P}_{x_1}(Y_1 \in A)P^{ex}_{A^c}(x_1, B_2) \\ &= \int_{\bar{B}_1} \mathbb{P}_{x_0}(Y_1^{A^c\rangle} \in dx_1) \int_{B_2} P^{ex}_{A^c}(x_1, dx_2). \end{aligned}\tag{12}$$

It remains to notice that we can replace $\bar{B}_1$ by B_1 using that $B_1 \setminus \bar{B}_1 = \{\dagger\}$ and

$$\mathbb{P}_{x_0}(Y_1^{A^c\rangle} = \dagger, Y_2^{A^c\rangle} \in B_2) = \mathbb{P}_{x_0}(T_1^{\rangle A} = \infty)\mathbb{1}_{B_2}(\dagger) = \int_{\{\dagger\}} \mathbb{P}_{x_0}(Y_1^{A^c\rangle} \in dx_1) \int_{B_2} P^{ex}_{A^c}(x_1, dx_2).$$

This proves the basis of induction.

To prove the inductive step, we proceed exactly as above and arrive at

$$\begin{aligned}&\mathbb{P}_{x_0}(Y_1^{A^c\rangle} \in \bar{B}_1, \dots, Y_{n+1}^{A^c\rangle} \in B_{n+1}) \\ &= \sum_{k=1}^{\infty} \int_{\bar{B}_1} \mathbb{P}_{x_0}(T_1^{\rangle A} > k-1, Y_{k-1} \in dx_1) \int_A \mathbb{P}_z(Y_1^{A^c\rangle} \in B_2, \dots, Y_n^{A^c\rangle} \in B_{n+1})\, \mathbb{P}_{x_1}(Y_1 \in dz).\end{aligned}$$

Using the assumption of induction for the integrand under $\int_A$, we get

$$\int_A \mathbb{P}_z(Y_1^{A^c\rangle} \in B_2, \dots, Y_n^{A^c\rangle} \in B_{n+1})\, \mathbb{P}_{x_1}(Y_1 \in dz) = \int_A \mathbb{P}_{x_1}(Y_1 \in dz) \int_{B_2} f(x_2)\mathbb{P}_z(Y_1^{A^c\rangle} \in dx_2),$$

where f is a non-negative measurable function on B_2 given by

$$f(x_2) := \int_{B_3} P^{ex}_{A^c}(x_2, dx_3) \dots \int_{B_{n+1}} P^{ex}_{A^c}(x_n, dx_{n+1}).$$

We claim that for any $x_1 \in \bar{B}_1$ and any non-negative measurable function g on B_2,

$$\int_A \mathbb{P}_{x_1}(Y_1 \in dz) \int_{B_2} g(x_2)\mathbb{P}_z(Y_1^{A^c\rangle} \in dx_2) = \mathbb{P}_{x_1}(Y_1 \in A) \int_{B_2} g(x_2)P^{ex}_{A^c}(x_1, dx_2). \tag{13}$$

Indeed, for indicator functions g this holds by definition (10) of $P^{ex}_{A^c}$; cf. (11). Hence, (13) holds for simple functions (i.e. finite linear combinations of indicator functions) by additivity of the three integrals in (13). Finally, since any non-negative measurable function g can be represented as pointwise limit of a pointwise non-decreasing sequence of simple functions, equality (13) follows from the monotone convergence theorem.

Putting everything together and applying (13) with $g = f$ establishes the inductive step exactly as we obtained (12) applying (11) in the case $n = 2$ and then replacing $\bar{B}_1$ by B_1. □

3. Invariance by inducing for recurrent chains

In this section we study stationarity of general entrance and exit Markov chains using the methods of infinite ergodic theory. Our main results here concern recurrent chains.

3.1. **Setup and notation.** Let Y be a Markov chain on a measurable space $(\mathcal{X}, \mathcal{F})$. Denote by $\mathrm{P}^Y_\nu := \mathbb{P}_\nu(Y \in \cdot)$ the "distribution" on the measurable space of sequences $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{F}^{\otimes \mathbb{N}_0})$ of Y started under a measure ν, and denote by E^Y_ν the "expectation"with respect to P^Y_ν.

For the rest of Section 3.1 we assume that μ a σ-finite non-zero invariant measure of Y.

Let θ be the (one-sided) *shift* operator on $\mathcal{X}^{\mathbb{N}_0}$ defined by $\theta : (x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots)$. This is a measure preserving transformation of the σ-finite measure space $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{F}^{\otimes \mathbb{N}_0}, \mathrm{P}^Y_\mu)$. For a set $C \in \mathcal{F}^{\otimes \mathbb{N}_0}$, consider the *first hitting time* T_C of C and the *induced shift* θ_C defined by

$$T_C(x) := \inf\{n \in \mathbb{N} : \theta^n x \in C\},\ x \in \mathcal{X}^{\mathbb{N}_0}, \quad \text{and} \quad \theta_C(x) := \theta^{T_C(x)} x,\ x \in C \cap \{T_C < \infty\},$$

where $\inf_\varnothing := \infty$ by convention. These mappings are measurable.

The powerful idea of ergodic theory is that the induced shift θ_C is a measure preserving transformation of the induced space $(C, (\mathcal{F}^{\otimes \mathbb{N}_0})_C, (\mathrm{P}^Y_\mu)_C)$, under certain recurrence-type assumptions on Y and C, e.g. as in Lemmas A.1 and A.2 in the Appendix (where we also review the relevant notions of ergodic theory). Below we introduce the definitions needed to apply these general results of ergodic theory in the context of Markov chains. We also refer the reader to Kaimanovich [23, Section 1] for a brief account of relevant results on invariant Markov shifts, and to Foguel [18] for a detailed one.

Denote by $C_B := \{x \in \mathcal{X}^{\mathbb{N}_0} : (x_0, \ldots, x_{k-1}) \in B\}$ the cylindrical set with a base $B \in \mathcal{F}^{\otimes k}$, where $k \geq 1$, and put $\tau_B := T_{C_B}$. An invariant measure μ of the Markov chain Y is called *recurrent* if for every set $A \in \mathcal{F}$ such that $\mu(A) < \infty$, we have $\mathbb{P}_x(\tau_A(Y) < \infty) = 1$ for μ-a.e. $x \in A$. It follows easily from invariance of μ that this definition is equivalent to $\mathbb{P}_x(\{Y_n \in A \text{ i.o.}\}) = 1$ for μ-a.e. $x \in A$. Following Kaimanovich [23], we say that an invariant measure μ of Y is *transient* if for every $A \in \mathcal{F}$ such that $\mu(A) < \infty$, we have $\mathbb{P}_x(\{Y_n \in A \text{ i.o.}\}) = 0$ for μ-a.e. $x \in A$. We stress that the latter condition can be violated when $\mu(A) = \infty$. There is a usual transience–recurrence dichotomy, see Lemma 3.1.e below.

Furthermore, we say that μ is *ergodic* if the shift θ is ergodic (and θ is P^Y_μ-preserving). We say that μ is *irreducible* if every invariant set of Y is μ-trivial, that is for any $A \in \mathcal{F}$, the equality $\mathbb{P}_x(Y_1 \in A) = \mathbb{1}_A(x)$ for μ-a.e. x implies that either $\mu(A) = 0$ or $\mu(A^c) = 0$; this shall not be confused with the notion of *ψ-irreducibility* of Markov chains.

Let us give necessary and sufficient conditions for recurrence and ergodicity of Y.

Lemma 3.1. *Let Y be a Markov chain that takes values in a measurable space $(\mathcal{X}, \mathcal{F})$ and has a σ-finite invariant measure μ on $(\mathcal{X}, \mathcal{F})$. The following statements hold true.*

a) μ is recurrent for Y iff the shift θ on $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{F}^{\otimes \mathbb{N}_0}, \mathrm{P}^Y_\mu)$ is conservative.

b) μ is recurrent for Y iff there exists a sequence of sets $\{B_n\}_{n \geq 1} \subset \mathcal{F}$ such that $\mathcal{X} = \cup_{n \geq 1} B_n \bmod \mu$, and $\mathbb{P}_{\mu_{B_n}}(\tau'_{B_n}(Y) = \infty) = 0$ and $\mu(B_n) < \infty$ for every $n \geq 1$.

c) μ is recurrent for Y if for some $k \geq 1$ there exists a set $B \in \mathcal{F}^{\otimes k}$ such that $\mathbb{P}_\mu(\tau_B(Y) = \infty) = 0$ and $\mathbb{P}_\mu((Y_1, \ldots, Y_k) \in B) < \infty$.

d) μ is ergodic and recurrent for Y iff it is irreducible and recurrent for Y.

e) If μ is irreducible for Y, then it is either recurrent for Y or transient for Y.

Proof. a) For the direct implication, note that since μ is σ-finite, $\mathcal{X}^{\mathbb{N}_0}$ can be exhausted by countably many cylindrical sets C_{B_n} with bases $B_n \in \mathcal{B}(\mathcal{X})$ of finite measure. Each set has measure $\mathrm{P}^Y_\mu(C_{B_n}) = \mu(B_n) < \infty$ and is recurrent for θ by recurrence of μ for Y. Then θ is conservative by Lemma A.3. For the reverse implication, every measurable cylindrical set C_B is recurrent for θ by conservativity of θ, hence μ is recurrent.

b) The shift θ is conservative by Lemma A.3 since the sets C_{B_n} of finite measure P_μ^Y exhaust $\mathcal{X}^{\mathbb{N}_0}$. Then μ is recurrent for Y by Condition a.

c) This follows as above using that the shift θ is conservative by Lemma A.3 since $\mathrm{P}_\mu^Y(\tau_{C_B} = \infty) = 0$ and $\mathrm{P}_\mu^Y(C_B) = \mathbb{P}_\mu((Y_1, \dots, Y_k) \in B) < \infty$.

d) The direct implication holds since every θ-invariant cylindrical set C_B with one-dimensional base $B \in \mathcal{F}$ is P_μ^Y-trivial. The reverse one is in [23, Proposition 1.7].

e) This is stated in [23, Theorem 1.2]. Since neither formal proof nor exact reference are given there, let us comment that this claim follows from the considerations by Foguel [18, Chapter II]. In more detail, we have $\mathcal{X} = C \cup D$, where C and D are respectively *conservative* and *dissipative* parts, which are disjoint and measurable. It follows from irreducibility of Y that $\mathbb{P}_x(Y_1 \in C) = \mathbb{1}_C(x) \mod \mu$; see [18, p. 17]. Then either $C = X \mod \mu$, in which case Y is recurrent by [18, Eq. (2.4)], or $D = X \mod \mu$, in which case Y is transient by repeating the argument after [18, Eq. (2.4)] (for any $B \in \mathcal{F}$ such that $\mu(B) < \infty$, take $f = \mathbb{1}_B$ and $u = \mathbb{1}_{B_M}$ with $B_M := \big\{x \in B : \sum_{k=0}^\infty \frac{d}{d\mu} \mathbb{P}_{\mu_B}(Y_k \in \cdot) \le M\big\}$, and let $M \to \infty$). $\square$

3.2. **General recurrent Markov chains.** The proof of our first result shows that the method of inducing allows us to *compute* invariant measures of the Markov chains mentioned.

Theorem 3.1. *Let Y be a Markov chain that takes values in a measurable space $(\mathcal{X}, \mathcal{F})$ and has a σ-finite recurrent invariant measure μ on $(\mathcal{X}, \mathcal{F})$. Let $A \in \mathcal{F}$ be a set such that $\mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A) > 0$. Then the measures*

$$\mu_A^{entr} := \int_{A^c} \mathbb{P}_x(Y_1 \in \cdot)\mu(dx) \text{ on } (A, \mathcal{F}_A) \quad \text{and} \quad \mu_{A^c}^{exit}(dx) := \mathbb{P}_x(Y_1 \in A)\mu(dx) \text{ on } (A^c, \mathcal{F}_{A^c}) \tag{14}$$

are proper, recurrent, and invariant for the entrance chain $Y^{\rangle A}$ and the exit chain $Y^{A^c\rangle}$, respectively. They are ergodic if μ is irreducible, and in this case

$$\mu(B) = \int_A \mathbb{E}_x\Bigg[\sum_{k=0}^{T_1^{\rangle A}-1} \mathbb{1}(Y_k \in B)\Bigg] \mu_A^{entr}(dx), \qquad B \in \mathcal{F}. \tag{15}$$

Moreover, equality (15) *holds true when*

$$\mathbb{P}_{\mu_A}(\tau_{A^c}(Y) = \infty) = 0 \quad \text{and} \quad \mathbb{P}_{\mu_{A^c}}(\tau_A(Y) = \infty) = 0. \tag{16}$$

The measure μ_A^{entr} has a simpler form if the chain Y has a dual relative to μ, see Section 4. Equation (15) is a particular case of *Kac's formula* of Lemma A.4.

Proof. Put $C := C_{A^c \times A}$. The measure-preserving the shift θ on $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{F}^{\otimes \mathbb{N}_0}, \mathrm{P}_\mu^Y)$ is conservative by Lemma 3.1.a. Then $\mathrm{P}_\mu^Y(C \setminus \{\theta^k \in C \text{ i.o.}\}) = 0$ by Halmos' recurrence theorem; see [1, Theorem 1.1.1]. Hence, by definitions of the set N_A and the measures $\mu_{A^c}^{exit}$ and μ_A^{entr},

$$0 = \mathbb{P}_\mu(Y_0 \in A^c \setminus N_A, Y_1 \in A) + \mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A \setminus N_A) = \mu_{A^c}^{exit}(A^c \setminus N_A) + \mu_A^{entr}(A \setminus N_A).$$

Thus, the measures $\mu_{A^c}^{exit}$ and μ_A^{entr} are proper for the chains $Y^{A^c\rangle}$ and $Y^{\rangle A}$, respectively.

Because θ is conservative and the cylindrical set C with the two-dimensional base $A^c \times A$ satisfies $\mathrm{P}^Y_\mu(C) = \mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A) > 0$, the induced shift θ_C is a measure preserving transformation of the induced space $(C, (\mathcal{F}^{\otimes \mathbb{N}_0})_C, (\mathrm{P}^Y_\mu)_C)$ by Lemma A.2. Then for any measurable set $B \subset A^c \times A$,

$$\begin{aligned}(\mathrm{P}^Y_\mu)_C(C_B) &= \mathrm{P}^Y_\mu\big(C \cap \{\theta_C \in C_B\} \cap \{\theta^k \in C \text{ i.o.}\}\big)\\ &= \mathbb{P}_\mu\big((Y_0, Y_1) \in A^c \times A, \theta_C(Y) \in C_B, Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}\big),\end{aligned}$$

hence

$$\mathbb{P}_\mu((Y_0, Y_1) \in B) = \mathbb{P}_\mu\big((Y_0, Y_1) \in A^c \times A, (Y_2^{A^c\rangle}, Y_2^{\rangle A}) \in B\big).$$

Taking $B = A^c \times B_1$, where $B_1 \subset A$ is a measurable set, the above implies that

$$\begin{aligned}\mu^{entr}_A(B_1) &= \mathbb{P}_\mu\big((Y_0, Y_1) \in A^c \times A, Y_2^{\rangle A} \in B_1\big)\\ &= \int_{A^c} \mu(dx_0) \int_A \mathbb{P}_{x_0}(Y_2^{\rangle A} \in B_1 | Y_1 = x_1)\mathbb{P}_{x_0}(Y_1 \in dx_1)\\ &= \int_{A^c} \mu(dx_0) \int_A \mathbb{P}_{x_1}(Y_1^{\rangle A} \in B_1)\mathbb{P}_{x_0}(Y_1 \in dx_1)\\ &= \int_A P^{entr}_A(x_1, B_1)\mu^{entr}_A(dx_1), \end{aligned} \tag{17}$$

where in the third equality we used the Markov property of Y and in the last one we used formula (9) for the transition kernel P^{entr}_A. Thus, μ^{entr}_A is invariant for $Y^{\rangle A}$.

Similarly, let us take $B = B_0 \times A$, where $B_0 \subset A^c$ is an arbitrary measurable set. Then

$$\begin{aligned}\mu^{exit}_{A^c}(B_0) &= \mathbb{P}_\mu\big((Y_0, Y_1) \in A^c \times A, Y_2^{A^c\rangle} \in B_0\big)\\ &= \int_{A^c} \mu(dx_0) \int_A \mathbb{P}_{x_1}(Y_1^{A^c\rangle} \in B_0)\mathbb{P}_{x_0}(Y_1 \in dx_1)\\ &= \int_{A^c_{ex}} \mathbb{P}_{x_0}(Y_1 \in A)\mu(dx_0) \int_A \mathbb{P}_{x_1}(Y_1^{A^c\rangle} \in B_0)\mathbb{P}_{x_0}(Y_1 \in dx_1 | Y_1 \in A)\\ &= \int_{A^c} P^{exit}_{A^c}(x_0, B_0)\mu^{exit}_{A^c}(dx_0), \end{aligned} \tag{18}$$

where in the last equality we used formula (10) for the transition kernel $P^{exit}_{A^c}$ of the entrance chain $Y^{A^c\rangle}$. Thus, $\mu^{exit}_{A^c}$ is invariant for $Y^{A^c\rangle}$.

For $i \in \{0, 1\}$, define the mappings

$$\psi_i(x) := (x_i, \theta_C(x)_i, (\theta_C)^2(x)_i, \ldots), \qquad x \in C \cap \{\theta^k \in C \text{ i.o.}\},$$

from their common domain to $(A^c)^{\mathbb{N}}$ and $A^{\mathbb{N}}$, respectively. These mapping are measurable. The entrance chain $Y^{\rangle A}$ starts from $Y_1^{\rangle A}$, which is Y_1 on the event $\{Y_0 \in A^c, Y_1 \in A\}$. Moreover, $\psi_1(Y) = Y^{\rangle A}$ on $\{Y_0 \in A^c, Y_1 \in A, Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}\}$. Therefore, since for any measurable set $B \subset A$ it is true that

$$\mu^{entr}_A(B) = \mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in B) = \mathrm{P}^Y_\mu\big(x \in C : x_1 \in B\big),$$

we see that $(\mathrm{P}^Y_\mu)_C \circ \psi_1^{-1}$ is the law on $(A^{\mathbb{N}}, \mathcal{F}_A^{\otimes \mathbb{N}})$ of $Y^{\rangle A}$ with $Y_1^{\rangle A}$ distributed according to μ_A^{entr}. Denote this law by P.

This representation of P and the fact that $\mathrm{P}^Y_\mu(C_{A^c \times B} \cap \{T_{C_{A^c\times B}} = \infty\}) = 0$, which holds by conservativity of θ on $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{F}^{\otimes \mathbb{N}_0}, \mathrm{P}^Y_\mu)$, imply that the measure μ_A^{entr} is recurrent for $Y^{\rangle A}$.

The shift θ_* on $A^{\mathbb{N}}$ is measure preserving on $(A^{\mathbb{N}}, (\mathcal{F}_A)^{\otimes \mathbb{N}}, \mathrm{P})$. If $E \in (\mathcal{F}_A)^{\otimes \mathbb{N}}$ is its invariant set, that is $\theta_*^{-1} E = E$ mod P, then $\psi_1^{-1}(\theta_*^{-1} E) = \psi_1^{-1} E$ mod $(\mathrm{P}^Y_\mu)_C$. Note that $\psi_1(\theta_C(x)) = \theta_*(\psi_1(x))$ for every $x \in C$, hence $\psi_1^{-1}(\theta_*^{-1}E) = \theta_C^{-1}(\psi_1^{-1}E)$, and therefore $\theta_C^{-1}(\psi_1^{-1}E) = \psi_1^{-1}E$ mod $(\mathrm{P}^Y_\mu)_C$, which means that $\psi_1^{-1}E$ is an invariant set for the induced shift θ_C on C. This set is $(\mathrm{P}^Y_\mu)_C$-trivial, and thus E is P-trivial, because θ_C is ergodic when the shift θ on $(\mathcal{X}^{\mathbb{N}}, \mathcal{F}^{\otimes\mathbb{N}}, \mathrm{P}^Y_\mu)$ is ergodic and conservative; see Aaronson [1, Proposition 1.5.2]. This establishes ergodicity of the invariant measure μ_A^{entr} of the entrance chain $Y^{\rangle A}$ when μ is an ergodic invariant measure of Y. It remains to use that μ is ergodic when it is recurrent and irreducible; see Lemma 3.1.d.

Similarly, the law of the exit chain $Y^{A^c \rangle}$ with $Y_1^{A^c\rangle}$ following μ_A^{exit} is $(\mathrm{P}^Y_\mu)_C \circ \psi_0^{-1}$. Ergodicity and recurrence of μ_A^{exit} for $Y^{A^c\rangle}$ follow exactly as above.

To prove equality (15), we will use Kac's formula of Lemma A.4 with C substituted for A. To show that this result applies, we shall prove that τ_C is finite P^Y_μ-a.e. This holds true by [1, Proposition 1.2.2] in the case when μ is irreducible (hence ergodic), while under assumptions (16) we argue as follows. Define $N_0 := \mathcal{X}$ and for any $k \in \mathbb{N}$,

$$N_k := \bigcup_{B \in \{A,\, A^c\}} \Big\{ x \in N_{k-1} \cap B : \mathbb{P}_x(Y_{\tau_{B^c}(Y)} \in N_{k-1}, \tau_{B^c}(Y) < \infty) = 1 \Big\}.$$

Clearly, $N_0 \supset N_1 \supset N_2 \supset \ldots$, and it follows from the strong Markov property of Y that when started from an $x \in N_k$, this chain crosses from A to A^c and from A^c to A at least k times in total $\mathbb{P}_x$-a.s. Then $\tau_C(Y)$ is finite $\mathbb{P}_x$-a.s. for every $x \in N_3$, and it suffices to show that

$$\mu(N_k^c) = 0, \qquad k \in \mathbb{N}. \tag{19}$$

This follows by induction since $\mu(N_1^c) = 0$ by (16), and for any $k \geq 2$,

$$\sum_{B \in \{A,\, A^c\}} \int_{N_{k-1} \cap B} \mathbb{P}_x(Y_{\tau_{B^c}(Y)} \notin N_{k-1}) \mu(dx) \leq \int_{N_{k-1}} \sum_{n=1}^\infty \mathbb{P}_x(Y_n \notin N_{k-1}) \mu(dx) \leq \sum_{n=1}^\infty \mu(N_{k-1}^c)$$

by μ-invariance of Y. Hence $\mu(N_{k-1}^c) = 0$ implies that the integrand under the first integral is zero for μ-a.e. x, which is equivalent to $\mu(N_k^c) = 0$.

Thus, Lemma A.4 applies. We have

$$\begin{aligned}\int_A \mathbb{E}_x \Bigg[\sum_{k=0}^{T_1^{\rangle A} - 1} \mathbb{1}(Y_k \in B) \Bigg] \mu_A^{entr}(dx) &= \int_{A^c} \mu(dx_0) \int_A \mathbb{E}_{x_1} \Bigg[\sum_{k=0}^{T_C(Y)} \mathbb{1}(Y_k \in B) \Bigg] \mathbb{P}_{x_0}(Y_1 \in dx_1) \\ &= \int_{A^c} \mathbb{E}_{x_0} \Bigg[\sum_{k=1}^{T_C(Y)} \mathbb{1}(Y_k \in B, Y_1 \in A) \Bigg] \mu(dx_0), \end{aligned} \tag{20}$$

where in the first equality we used the definition of μ_A^{entr} and the fact that $T_1^{\rangle A} = T_C(Y) + 1$ on $\{Y_0 \in A\}$, and in the second equality we used the Markov property of Y. Finally, we obtain (15) by

$$\int_A \mathbb{E}_x\Bigg[\sum_{k=0}^{T_1^{\rangle A}-1} \mathbb{1}(Y_k \in B)\Bigg]\mu_A^{entr}(dx) = \int_C \Bigg[\sum_{k=1}^{T_C(x)} \mathbb{1}(\theta^k x \in C_B)\Bigg]\mathrm{P}_\mu^Y(dx) = \mathrm{P}_\mu^Y(C_B), \tag{21}$$

where in the second equality we applied Lemma A.4 after shifting the summation indices by one using invariance of $(\mathrm{P}_\mu^Y)_C$ under the induced shift θ_C. $\square$

3.3. **Weak Feller recurrent Markov chains.** In this section we give a topological counterpart to Theorem 3.1, assuming throughout that $\mathcal{X}$ is a metric space.

We first give topological versions of the ergodic-theoretic definitions from Section 3.1. We say that a Markov chain Y on $\mathcal{X}$ is *topologically irreducible* if $\mathbb{P}_x(\tau_G(Y) < \infty) > 0$ for every $x \in \mathcal{X}$ and every non-empty open set $G \subset \mathcal{X}$. We say that Y is *topologically recurrent* if $\mathbb{P}_x(\tau_G(Y) < \infty) = 1$ for every non-empty open set $G \subset \mathcal{X}$ and *every* $x \in G$. We warn that the Markov chains literature often defines topological recurrence by taking every $x \in \mathcal{X}$ instead of every $x \in G$; see Lemma 3.3 below regarding equivalence of these definition.

The chain Y is called *weak Feller* if its transition probability $\mathbb{P}_x(Y_1 \in \cdot)$ is weakly continuous in x. Equivalently, the mapping $x \mapsto \mathbb{E}_x f(Y_1)$ is continuous on $\mathcal{X}$ for any continuous bounded function $f : \mathcal{X} \to \mathbb{R}$.

A Borel measure on $\mathcal{X}$ is called *locally finite* if every point of $\mathcal{X}$ admits an open neighbourhood of finite measure. Such measures are finite on compact sets. Also, they are σ-finite if $\mathcal{X}$ is separable. Indeed, every separable metric space has the Lindelöf property (see Engelking [16, Corollary 4.1.16]), i.e. its every open cover contains a countable subcover, therefore $\mathcal{X}$ can be represented as a countable union of open balls of finite measure.

Our main result on weak Feller chains is as follows.

Theorem 3.2. *Let Y be a topologically irreducible topologically recurrent weak Feller Markov chain that takes values in a separable metric space $\mathcal{X}$. Let $A \subset \mathcal{X}$ be a Borel set such that $\mathbb{P}_x(Y_1 \in \mathrm{Int}(A)) > 0$ for some $x \in \mathrm{Int}(A^c)$. Then the mapping $\mu \mapsto \mu_A^{entr}$ (resp., $\mu \mapsto \mu_{A^c}^{exit}$), defined in* (14)*, is a bijection between the sets of locally finite invariant Borel measures of the chain Y on $\mathcal{X}$ and the entrance chain $Y^{\rangle A}$ on A (resp., the exit chain $Y^{A^c\rangle}$ on A^c).*

The role of the condition $\mathbb{P}_x(Y_1 \in \mathrm{Int}(A)) > 0$ for an $x \in \mathrm{Int}(A^c)$ is to exclude the case where the chain Y can enter $\mathrm{Int}\, A$ from its complement only through ∂A.

The main use of Theorem 3.2 is when the initial chain Y is known to have a unique (up to a multiplicative constant) locally finite invariant measure. This is the case for recurrent random walks on $\mathbb{R}^d$, which we study below in Section 5. It remarkable that under the assumptions of Theorem 3.2, the chain Y may have two non-proportional invariant measures even if the space $\mathcal{X}$ is compact; see Skorokhod [44, Example 1] and also a simpler example by Carlsson [9, Theorem 1], where the assumptions are satisfied by Lemma 3.3 below.

The question of whether a weak Feller chain has a (non-zero) locally finite invariant measure was studied by Lin [28, Theorem 5.1] and Skorokhod [44, Theorem 3]; the approach of [44] was similar to the one used here. They showed that under assumptions of Theorem 3.2, the answer is positive when $\mathcal{X}$ is a locally finite Polish space. The case of non-locally compact spaces was studied by Szarek [46]. For existence and uniqueness results on invariant measures under much stronger assumptions on Y, such as strong Feller or Harris properties or ψ-irreducibility, see Foguel [18, Chapters IV and VI] and Meyn and Tweedie [32].

Before proceeding to the proof of Theorem 3.2, we give two simple auxiliary results.

Lemma 3.2. *Let Y be a topologically irreducible weak Feller Markov chain that takes values in a metric space $\mathcal{X}$ and has a non-zero invariant Borel measure μ. Then μ is strictly positive on every non-empty open set, and μ is locally finite if and only if it is finite on some non-empty open set.*

Proof. The necessary condition is trivial. To prove the sufficient one, assume that G is a non-empty open subset of $\mathcal{X}$ satisfying $\mu(G) < \infty$. By topological irreducibility of Y, for any $x \in \mathcal{X}$ there exists an $n = n(x) \geq 1$ such that $\mathbb{P}_x(Y_n \in G) > 0$. It follows by a simple inductive argument that the n-step transition probability $\mathbb{P}_x(Y_n \in \cdot)$ is weakly continuous in x. Indeed, for any continuous bounded function $f : \mathcal{X} \to \mathbb{R}$, we have

$$\mathbb{E}_x f(Y_n) = \int_{\mathcal{X}} \mathbb{E}_y f(Y_{n-1}) \mathbb{P}_x(Y_1 \in dy), \qquad x \in \mathcal{X}$$

by the Chapman–Kolmogorov equation. The integrand is a continuous bounded function by assumption of induction, and so is the integral since Y is weak Feller.

Then there is an open neighbourhood U_x of x such that $\mathbb{P}_y(Y_n \in G) \geq \frac{1}{2}\mathbb{P}_x(Y_n \in G)$ for every $y \in U_x$. By invariance of μ, this gives

$$\infty > \mu(G) = \int_{\mathcal{X}} \mathbb{P}_y(Y_n \in G)\mu(dy) \geq \int_{U_x} \mathbb{P}_y(Y_n \in G)\mu(dy) \geq \frac{1}{2}\mathbb{P}_x(Y_n \in G)\mu(U_x), \tag{22}$$

implying finiteness of $\mu(U_x)$, as required.

Lastly, assume that there is a non-empty open subset G of $\mathcal{X}$ satisfying $\mu(G) = 0$. Then from invariance of μ it follows that $\mathbb{P}_y(Y_1 \in G) = 0$ for every $y \in G^c$. Since $\mu(\mathcal{X}) > 0$, we can pick an $x \in G^c$, and then take $n = n(x) \geq 1$ to be a minimal number such that $\mathbb{P}_x(Y_n \in G) > 0$. This leads to a contradiction by

$$\mathbb{P}_x(Y_n \in G) = \int_{G^c} \mathbb{P}_y(Y_1 \in G)\mathbb{P}_x(Y_{n-1} \in dy) = 0.$$

□

Lemma 3.3. *Let Y be a topologically irreducible topologically recurrent weak Feller Markov chain that takes values in a metric space $\mathcal{X}$. Then*

$$\mathbb{P}_x(\tau_G(Y) < \infty) = 1 \text{ for every } x \in \mathcal{X} \text{ and non-empty open } G \subset \mathcal{X}. \tag{23}$$

Proof. As in the proof of Lemma 3.2, by topological irreducibility and weak Feller property of Y we can find an open neighbourhood U of x such that $\inf_{y \in U} \mathbb{P}_y(\tau_G(Y) < \infty) > 0$. The claim now follows by topological recurrence and the strong Markov property of the chain Y, which returns to U $\mathbb{P}_x$-a.s. □

Proof of Theorem 3.2. First of all, the sets of invariant measures of the chains $Y^{\rangle A}$ and $Y^{A^c\rangle}$ are always non-empty since they contain the zero measures.

Let μ be a non-zero locally finite Borel invariant measure of the Markov chain Y. Since $\mathbb{P}_x(Y_1 \in \mathrm{Int}(A)) > 0$ for an $x \in \mathrm{Int}(A^c)$, by weak Fellerness of Y there is an open neighbourhood $U \subset \mathrm{Int}(A^c)$ of x such that $\mathbb{P}_y(Y_1 \in \mathrm{Int}(A)) > 0$ for every $y \in U$. We have $\mu(U) > 0$ by Lemma 3.2, therefore $\mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A) \ge \mathbb{P}_\mu(Y_0 \in U, Y_1 \in \mathrm{Int}(A)) > 0$. Hence the measures μ_A^{entr} and $\mu_{A^c}^{exit}$ are non-zero, and they are locally finite by $\mu_A^{entr} \le \mu_A$ and $\mu_{A^c}^{exit} \le \mu_{A^c}$. Furthermore, μ is σ-finite as a locally finite measure on a separable metric space (cf. Engelking [16, Corollary 4.1.16]). By choosing an open set G in (23) of finite measure, we conclude that μ is recurrent for Y by Lemma 3.1.c. Therefore, Theorem 3.1 applies, the measures μ_A^{entr} and $\mu_{A^c}^{exit}$ are invariant for the respective chains $Y^{\rangle A}$ and $Y^{A^c\rangle}$.

We first consider the mapping $\mu \mapsto \mu_A^{entr}$. Is is injective by (15). To prove its surjectivity, let ν be a locally finite non-zero Borel invariant measure of the entrance chain $Y^{\rangle A}$ on A. Consider the Borel measure

$$\mu_1(B) := \mathbb{E}_\nu \left[\sum_{k=0}^{T_1^{\rangle A}-1} \mathbb{1}(Y_k \in B) \right], \qquad B \in \mathcal{B}(\mathcal{X}). \tag{24}$$

It follows from the strong Markov property of Y combined with the equalities

$$\mathbb{P}_y(\tau'_A(Y) \le \tau'_{\mathrm{Int}(A)}(Y) < \infty) = \mathbb{P}_y(\tau'_{A^c}(Y) \le \tau'_{\mathrm{Int}(A^c)}(Y) < \infty) = 1, \qquad y \in \mathcal{X},$$

that $T_1^{\rangle A}$ is finite $\mathbb{P}_\nu$-a.s. Then

$$\mathbb{P}_{\mu_1}(Y_1 \in B) = \int_{\mathcal{X}} \mathbb{P}_y(Y_1 \in B)\mu_1(dy) = \int_{\mathcal{X}} \sum_{k=0}^{\infty} \mathbb{P}_y(Y_1 \in B)\mathbb{P}_\nu(Y_k \in dy, T_1^{\rangle A} > k),$$

hence

$$\mathbb{P}_{\mu_1}(Y_1 \in B) = \sum_{k=0}^{\infty} \mathbb{P}_\nu(Y_{k+1} \in B, T_1^{\rangle A} \ge k+1) = \mathbb{E}_\nu \left[\sum_{k=1}^{T_1^{\rangle A}} \mathbb{1}(Y_k \in B) \right] = \mu_1(B), \tag{25}$$

where in the last equality we used that $Y_1^{\rangle A} = Y_{T_1^{\rangle A}}$ by the definition of the entrance chain and that ν is invariant for this chain by the assumption. Thus, μ_1 is invariant for Y.

Furthermore, for any Borel set $B \subset A$, we have

$$\begin{aligned}\int_{A^c} \mathbb{P}_y(Y_1 \in B)\mu_1(dy) &= \sum_{k=0}^{\infty} \int_{A^c} \mathbb{P}_y(Y_1 \in B)\mathbb{P}_\nu(Y_k \in dy, T_1^{\rangle A} > k) \\ &= \sum_{k=0}^{\infty} \mathbb{P}_\nu(Y_{k+1} \in B, T_1^{\rangle A} = k+1) \\ &= \mathbb{P}_\nu(Y_1^{\rangle A} \in B) = \nu(B). \end{aligned} \tag{26}$$

By the assumption we have $\mathbb{P}_x(Y_1 \in \mathrm{Int}(A)) > 0$ for some $x \in \mathrm{Int}(A^c)$, and it follows that there exists an open set $G \subset \mathrm{Int}(A)$ such that $\nu(G) < \infty$ and $\mathbb{P}_x(Y_1 \in G) > 0$. Indeed, since the measure ν is locally finite and the separable metric space $\mathcal{X}$ has the Lindelöf property (by [16, Corollary 4.1.16]), there is a countable cover of $\mathrm{Int}(A)$ by open sets of finite measure ν. At least one of these sets must satisfy the requirement, otherwise $\mathbb{P}_x(Y_1 \in \mathrm{Int}(A)) = 0$ by sub-additivity.

By the weak Feller property of Y, we can find an open set U_x such that $x \in U_x \subset \mathrm{Int}(A^c)$ and $\mathbb{P}_y(Y_1 \in G) \ge \frac{1}{2}\mathbb{P}_x(Y_1 \in G)$ for every $y \in U_x$. By (26) and exactly the same argument as in (22), this gives $\mu_1(U_x) < \infty$. Hence the measure μ_1 on $\mathcal{X}$ is locally finite by Lemma 3.2, and so the mapping $\mu \mapsto \mu_A^{entr}$ is surjective.

Now consider the mapping $\mu \mapsto \mu_{A^c}^{exit}$. To prove its surjectivity, let ν^{exit} be a locally finite non-zero Borel invariant measure of the exit chain $Y^{A^c\rangle}$ on A^c. Then the Borel measure $\nu := \int_{A^c_{ex}} \mathbb{P}_y(Y_1 \in \cdot | Y_1 \in A)\nu_0(dy)$ on A is invariant for the entrance chain $Y^{\rangle A}$ from A^c to A, and the measure μ_1 introduced in (24) is invariant for the chain Y. Moreover, we have the following equality of Borel measures on A^c:

$$\begin{aligned}\mathbb{P}_y(Y_1 \in A)\mu_1(dy) &= \sum_{k=0}^{\infty} \mathbb{P}_y(Y_1 \in A)\mathbb{P}_\nu(Y_k \in dy, T_1^{\rangle A} > k) \\ &= \mathbb{P}_\nu(Y_{T_1^{\rangle A}-1} \in dy) = \mathbb{P}_\nu(Y_1^{A^c\rangle}) = \nu_0(dy), \qquad y \in A^c. \qquad (27)\end{aligned}$$

Then, if $x \in A^c$ is such that $\mathbb{P}_x(Y_1 \in \mathrm{Int}(A)) > 0$, by weak Fellerness of Y and local finiteness of ν_0 we can choose an open set U such that $x \in U \subset \mathrm{Int}(A^c)$, $\nu_0(U)$ is finite, and $\mathbb{P}_y(Y_1 \in \mathrm{Int}(A)) \ge \frac{1}{2}\mathbb{P}_x(Y_1 \in \mathrm{Int}(A))$ for every $y \in U$. By (27), this gives

$$\mu_1(U) = \int_U \frac{\nu_0(dy)}{\mathbb{P}_y(Y_1 \in A)} \le \int_U \frac{\nu_0(dy)}{\mathbb{P}_y(Y_1 \in \mathrm{Int}(A))} \le \frac{2\nu_0(U)}{\mathbb{P}_x(Y_1 \in \mathrm{Int}(A))} < \infty,$$

hence the measure μ_1 on $\mathcal{X}$ is locally finite by Lemma 3.2. So the mapping $\mu \mapsto \mu_{A^c}^{exit}$ is surjective. Also, by the equality $\nu = \int_{A^c} \mathbb{P}_y(Y_1 \in \cdot)\mu_1(dy)$ of measures on A, ν is locally finite since μ_1 is so, as we proved earlier. Combined with injectivity of the mapping $\mu \mapsto \mu_A^{entr}$, this implies injectivity of the mapping $\mu \mapsto \mu_{A^c}^{exit}$. □

4. Invariance by duality

In this section we study invariant measures of entrance and exit chains derived from a Markov chain that is no longer assumed to be recurrent. Instead, we need to make additional assumptions in terms of the dual chain. We present our proofs in probabilistic notation but essentially we employ inducing for *invertible* measure preserving two-sided Markov shifts.

Recall that probability transition kernels P and $\hat{P}$ on $(\mathcal{X}, \mathcal{F})$ are *dual* relative to a σ-finite measure μ on $(\mathcal{X}, \mathcal{F})$ if

$$\mu(dx)P(x, dy) = \mu(dy)\hat{P}(y, dx), \qquad x, y \in \mathcal{X}. \qquad (28)$$

This equality of measures on $(\mathcal{X} \times \mathcal{X}, \mathcal{F} \otimes \mathcal{F})$ is called the *detailed balance condition*. It implies, by integration in x or in y, that the measure μ is *invariant* for both P and $\hat{P}$.

If $\mathcal{X}$ is a Polish space, then any transition kernel on $\mathcal{X}$ with a σ-finite invariant measure μ always has a dual kernel $\hat{P}$ relative to μ. Indeed, if μ is a probability measure, then this claim is nothing but the disintegration theorem combined with existence of regular conditional distributions for probability measures on Polish spaces; see Kallenberg [24, Theorems 6.3, 6.4, A1.2] or Aaronson [1, Theorem 1.0.8]. This easily extends to σ-finite measures by σ-additivity. Note that if $\hat{P}'$ is another transition kernel dual to P relative to μ, then $\hat{P}'(x,\cdot) = \hat{P}(x,\cdot)$ for μ-a.e. x by Lemma 4.7 in Chapter 2 in Revuz [42].

Two Markov chains Y and $\hat{Y}$ on $\mathcal{X}$ are *dual* relative to μ if so are their transition kernels. In other words, we have the *time-reversal equality*

$$\mathbb{P}_\mu((Y_0, Y_1) \in B) = \mathrm{P}^{\hat{Y}}_\mu(y : (y_1, y_0) \in B), \qquad B \in \mathcal{F} \otimes \mathcal{F},$$

where $y = (y_0, y_1, \ldots) \in \mathcal{X}^{\mathbb{N}_0}$ and the r.h.s. refers to a realisation of $\hat{Y}$ as the identity mapping on the canonical space $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{F}^{\otimes \mathbb{N}_0}, \mathrm{P}^{\hat{Y}}_\mu)$. More generally, for any $k \geq 1$ we have

$$\mathbb{P}_\mu((Y_0, \ldots, Y_k) \in B) = \mathrm{P}^{\hat{Y}}_\mu((y_k, \ldots, y_0) \in B), \qquad B \in \mathcal{F}^{\otimes(k+1)}. \tag{29}$$

We now state the main result of the section.

Theorem 4.1. *Let Y be a Markov chain that takes values in a Polish space $\mathcal{X}$ and has a σ-finite invariant Borel measure μ. Then there exists a Markov chain $\hat{Y}$ with values in $\mathcal{X}$ that is dual to Y relative to μ. Furthermore, let $A \in \mathcal{B}(\mathcal{X})$ be a set such that*

$$\mathrm{P}^Y_{\mu_A}(\tau_{A^c} = \infty) = \mathrm{P}^Y_{\mu_{A^c}}(\tau_A = \infty) = \mathrm{P}^{\hat{Y}}_{\mu_A}(\tau_{A^c} = \infty) = \mathrm{P}^{\hat{Y}}_{\mu_{A^c}}(\tau_A = \infty) = 0. \tag{30}$$

Then the measures μ^{entr}_A and $\mu^{exit}_{A^c}$, defined in (14), *are proper and invariant for the entrance chain $Y^{\rangle A}$ and the exit chain $Y^{A^c\rangle}$, respectively; and we have*

$$\mu^{entr}_A(dx) = \hat{P}(x, A^c)\mu(dx), \qquad x \in A, \tag{31}$$

where $\hat{P}$ denotes the transition kernel of $\hat{Y}$. Moreover, Kac's formula (15) *holds true.*

We will prove Theorem 4.1 as an easy corollary to the following duality result. Recall that $N_A(Y)$, defined in (6), is the set of all points starting from where Y visits both sets A and A^c infinitely often.

Proposition 4.1. *Let Y and $\hat{Y}$ be Markov chains with values in a measurable space $(\mathcal{X}, \mathcal{F})$ that are dual relative to a σ-finite measure μ. Let $A \in \mathcal{F}$ be a set such that*

$$\mu\big(N_A(Y) \Delta N_A(\hat{Y})\big) = 0. \tag{32}$$

Then the exit chain $Y^{A^c\rangle}$ and the entrance chain $\hat{Y}^{\rangle A^c}$ are dual relative to the measure

$$\tilde{\mu}^{exit}_{A^c}(B) := \mu^{exit}_{A^c}(B \cap N_A(Y)), \qquad B \in \mathcal{F}^\dagger_{A^c}.$$

Likewise, the chains $Y^{\rangle A}$ and $\hat{Y}^{A\rangle}$ are dual relative to the measure

$$\tilde{\mu}^{entr}_A(B) := \mu^{entr}_A(B \cap N_A(Y)), \qquad B \in \mathcal{F}^\dagger_A.$$

Moreover, it is true that

$$\mathbb{P}_{\tilde{\mu}^{entr}_A}(Y^{\rangle A^c}_1 \in \cdot) = \tilde{\mu}^{entr}_{A^c} \qquad \text{and} \qquad \mathbb{P}_{\tilde{\mu}^{entr}_{A^c}}(Y^{\rangle A}_1 \in \cdot) = \tilde{\mu}^{entr}_A. \tag{33}$$

In the special case when Y is a one-dimensional oscillating random walk S on $\mathcal{X} = \mathcal{Z}$ and $A = [0, \infty) \cap \mathcal{Z}$, we can write (33) as $\mathbb{P}_{\pi_+}(O^\downarrow_1 \in \cdot) = \pi_-$ and $\mathbb{P}_{\pi_-}(O_1 \in \cdot) = \pi_+$, where $O^\downarrow_1 := \mathcal{O}_{2 - \mathbb{1}(S_0 \geq 0)}$ is the first overshoot at down-crossing of zero, $\pi_+ := \pi|_A$ and $\pi_- := \pi|_{A^c}$ with π is defined in (2). These equalities were proved in [34, Remark 2.2].

Moreover, if $\mathcal{Z}=\mathbb{R}$, we can complement the second duality in Proposition 4.1 by a surprising representation of the transition probabilities of the chains $S^{\rangle A}$ and $-S^{A^c\rangle}$ (i.e. O and $-U$) as products of two transition probabilities that are *reversible* relative to λ_A^{entr} (i.e. π_+); see [34, Section 2.4].

Proof of Proposition 4.1. To stress that the measures μ_A^{entr} and $\mu_{A^c}^{exit}$ are defined in (14) in terms of the chain Y, we us write $\mu_{A,Y}^{entr}$ and $\mu_{A^c,Y}^{exit}$. For any measurable set $B \subset A$,

$$\mu_A^{entr}(B)=\int_{A^c}\mathbb{P}_x(Y_1\in B)\mu(dx)=\mathbb{P}_\mu(Y_0\in A^c,Y_1\in B)=\mathrm{P}_\mu^{\hat{Y}}(y_0\in B,y_1\in A^c), \qquad (34)$$

hence $\mu_{A,Y}^{entr}(dx)=\hat{P}(x,A^c)\mu(dx)$ for $x\in A$. Thus $\mu_{A,Y}^{entr}=\mu_{A,\hat{Y}}^{exit}$, that is the entrance measure of Y into A from A^c is the exit measure of $\hat{Y}$ exiting from A to A^c. Then we also have $\tilde{\mu}_{A,Y}^{entr}=\tilde{\mu}_{A,\hat{Y}}^{exit}$ by (32). Therefore, the second duality stated follows from the first one applied to $\hat{Y}$ and A in place of Y and A^c.

To prove the first duality, we need to check the detailed balance condition

$$\tilde{\mu}_{A^c}^{exit}(dx)P_{A^c}^{exit}(x,dy)=\tilde{\mu}_{A^c}^{exit}(dy)\hat{P}_{A^c}^{entr}(y,dx), \qquad x,y\in A_\dagger^c, \qquad (35)$$

where $P_{A^c}^{exit}$ and $\hat{P}_{A^c}^{entr}$ denote the transition kernels of the chains $Y^{A^c\rangle}$ and $\hat{Y}^{\rangle A^c}$, and recall that by convention, $P_{A^c}^{exit}(x,\{\dagger\})=1$ for $x\in A^c\setminus A_{ex}^c(Y)$. We will use the simplified notation $N=N_A(Y)$ and $\hat{N}=N_A(\hat{Y})$.

If $x=\dagger$, then the l.h.s. of (35) is zero by the definition of $\tilde{\mu}_{A^c}^{exit}$ and the r.h.s. of (35) is zero since $\hat{P}_{A^c}^{entr}(y,\{\dagger\})=0$ for every $y\in A^c\cap\hat{N}$ (by Lemma 2.1) and $\tilde{\mu}_{A^c}^{exit}$ is supported on $A^c\cap\hat{N}$ by assumption (32). Thus, equality (35) is satisfied when $x=\dagger$. Similarly, (35) is true when $y=\dagger$, in which case the l.h.s. is zero since $\tilde{\mu}_{A^c}^{exit}$ is supported on $A_{ex}^c(Y)\cap N$. Thus, we need to establish (35) only for $x,y\in A^c$. By the definition of $\tilde{\mu}_{A^c}^{exit}$, this amounts to showing that for any measurable sets $B_1,B_2\subset A^c$,

$$\int_{B_1\cap N}P_{A^c}^{exit}(x,B_2)\mathbb{P}_x(Y_1\in A)\mu(dx)=\int_{B_2\cap N}\hat{P}_{A^c}^{entr}(y,B_1)\mathbb{P}_y(Y_1\in A)\mu(dy). \qquad (36)$$

By formula (10) for the transition kernel $P_{A^c}^{exit}$ and the absorbing property (7) of N,

$$\begin{aligned}\text{LHS }(36)&=\int_{B_1\cap N}\mu(dx)\int_A\mathbb{P}_z(Y_1^{A^c\rangle}\in B_2)\mathbb{P}_x(Y_1\in dz)\\&=\sum_{k,m=1}^{\infty}\mathbb{P}_\mu\big((Y_n)_{n=0}^{k+m+1}\in(B_1\cap N)\times A^k\times(A^c)^{m-1}\times(B_2\cap N)\times A\big).\end{aligned}$$

In the last line, we can replace N by $\hat{N}$ on both occasions using assumption (32) and invariance of μ for Y. Next we apply duality relation (29) to obtain that

$$\begin{aligned}\text{LHS (36)} &= \sum_{k,m=1}^{\infty} \mathrm{P}_{\mu}^{\hat{Y}}\big((y_n)_{n=0}^{k+m+1} \in A \times (B_2 \cap \hat{N}) \times (A^c)^{m-1} \times A^k \times (B_1 \cap \hat{N})\big),\\ &= \int_A \mu(dx) \int_{B_2\cap\hat{N}} \sum_{k,m=1}^{\infty} \mathrm{P}_z^{\hat{Y}}\big((y_n)_{n=0}^{k+m-1} \in (A^c)^{m-1} \times A^k \times (B_1 \cap \hat{N})\big)\, \mathrm{P}_x^{\hat{Y}}(y_1 \in dz),\end{aligned}$$

and noting that the sum in the last line is $\hat{P}_{A^c}^{entr}(z, B_1)$ by (9) and the absorbing property (7) of $\hat{N}$, we arrive at

$$\text{LHS (36)} = \mathbb{E}_{\mu}^{\hat{Y}}[\mathbb{1}_A(y_0)\mathbb{1}_{B_2\cap\hat{N}}(y_1)\hat{P}_{A^c}^{entr}(y_1, B_1)].$$

By duality of Y and $\hat{Y}$ with respect to μ, this gives the required equality

$$\text{LHS (36)} = \mathbb{E}_{\mu}[\mathbb{1}_A(Y_1)\mathbb{1}_{B_2\cap\hat{N}}(Y_0)\hat{P}_{A^c}^{entr}(Y_0, B_1)] = \text{RHS (36)},$$

where in the last equality we replaced $\hat{N}$ by N using (32).

It remain to establish (33), where the second equality follows from the first one by swapping A and A^c. It suffices prove the first equality only on measurable subsets of A^c. For any such set B, by the definitions of μ_A^{entr} and $\tilde{\mu}_A^{entr}$, we have

$$\mathbb{P}_{\tilde{\mu}_A^{entr}}(Y_1^{\rangle A^c} \in B) = \int_{A^c} \mu(dx) \int_{A\cap N} \mathbb{P}_z(Y_1^{\rangle A^c} \in B)\mathbb{P}_x(Y_1 \in dz).$$

Then, arguing as in the proof of (36),

$$\begin{aligned}\mathbb{P}_{\tilde{\mu}_A^{entr}}(Y_1^{\rangle A^c} \in B) &= \sum_{k=1}^{\infty} \mathbb{P}_{\mu}\big((Y_n)_{n=0}^{k+1} \in A^c \times (A\cap N)^k \times (B\cap N)\big)\\ &= \sum_{k=1}^{\infty} \tilde{\mathbb{P}}_{\mu}\big((\hat{Y}_n)_{n=0}^{k+1} \in (B\cap\hat{N}) \times (A\cap\hat{N})^k \times A^c\big)\\ &= \int_{B\cap\hat{N}} \mu(dx) \int_{A\cap\hat{N}} \tilde{\mathbb{P}}_z(\tau_{A^c}(\hat{Y}) < \infty)\,\tilde{\mathbb{P}}_x(\hat{Y}_1 \in dz)\\ &= \int_{B\cap\hat{N}} \mu(dx)\tilde{\mathbb{P}}_x(\hat{Y}_1 \in A) = \tilde{\mu}_{A^c}^{entr}(B).\end{aligned}$$

□

Proof of Theorem 4.1. Let P be the transition kernel of the chain Y. It is invariant with respect to the σ-finite measure μ. Because $\mathcal{X}$ is a Polish space, there exists a transition kernel $\hat{P}$ on $\mathcal{X}$ that is dual to P relative to μ. Then there exists a dual chain $\hat{Y}$ with the transition kernel $\hat{P}$.

We already proved equality (31), cf. (34) above. We claim from (30) is follows that both sets $N_A(Y)$ and $N_A(\hat{Y})$ are of full measure μ. This implies that the measure $\mu_{A^c}^{exit}$ is proper for the chain $Y^{A^c\rangle}$ and it equals $\tilde{\mu}_{A^c}^{exit}$ restricted to $\mathcal{F}_{A^c}$. Hence $\mu_{A^c}^{exit}$ is invariant for $Y^{A^c\rangle}$ since so is $\tilde{\mu}_{A^c}^{exit}$ by Proposition 4.1. By the same reasoning, μ_A^{entr} is invariant for $Y^{\rangle A}$.

We next prove that $N_A(Y)$ and $N_A(\hat{Y})$ are of full measure μ. By symmetry, it suffices to do this only for the first set. We have $N_A(Y) = \cap_{k=1}^{\infty} N_k$, hence $\mu(N_A(Y)^c) = \mu(\cup_{k=1}^{\infty} N_k^c) = 0$ by (19), which followed only from (16) and invariance of μ for Y.

Finally, we prove Kac's formula (15). Writing its r.h.s. using (20), we get

$$\begin{aligned}\int_A \mathbb{E}_x\Bigg[\sum_{k=0}^{T_1^{\rangle A}-1} \mathbb{1}(Y_k \in B)\Bigg]\mu_A^{entr}(dx) &= \mathbb{E}_\mu\Bigg[\sum_{k=1}^{T_C(Y)} \mathbb{1}(Y_0 \in A^c, Y_1 \in A, Y_k \in B)\Bigg]\\ &= \sum_{k=1}^{\infty} \mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A, Y_k \in B, T_C(Y) \geq k)\\ &= \sum_{k=1}^{\infty} \mathbb{P}_\mu\Big(\hat{Y}_0 \in B, \tau_{A\times A^c}(\hat{Y}) = k-1\Big) = \mu(B).\end{aligned}$$

□

5. Applications to random walks in $\mathbb{R}^d$

In this section we apply the ideas developed in Sections 3 and 4 to random walks in $\mathbb{R}^d$. In particular, we answer our initial questions on stationarity properties of the chain of overshoots of a one-dimensional random walk over the zero level.

Recall that $\mathcal{Z}$ denotes the minimal topologically closed subgroup of $(\mathbb{R}^d, +)$ that contains the topological support of the distribution of X_1. We assume throughout that $\mathcal{Z}$ has full dimension and $S_0 \in \mathcal{Z}$. We call $\mathcal{Z}$ the *state space* of the walk S. Denote by λ the Haar measure on $\mathcal{Z}$ normalized such that $\lambda(Q) = 1$, where $Q := \{x \in \mathcal{Z} : 0 \leq x < 1\}$ and we always mean that inequalities between points in $\mathbb{R}^d$ hold coordinate-wisely. Clearly, λ is invariant for the walk S on $\mathcal{X} = \mathcal{Z}$.

We say that a Borel set $A \subset \mathcal{Z}$ is *massive* for the random walk S if $\mathbb{P}_x(\tau_A(S) < \infty) = 1$ for λ-a.e. $x \in \mathcal{Z}$. Since $-A$ is massive for S if and only if A is massive for $-S$, and the random walk $-S$ is dual to S relative to the measure λ (see [34, Eq. (2.24)]), from Theorem 4.1 we immediately obtain the following result.

Theorem 5.1. *Assume that the sets A, $-A$, A^c, $-A^c$ are massive for a random walk S on its state space $\mathcal{Z}$, where $\mathcal{Z} \subset \mathbb{R}^d$ and $d \geq 1$. Then the measures $\mathbb{P}(X_1 \in x - A^c)\lambda(dx)$ on A and $\mathbb{P}(X_1 \in A - x)\lambda(dx)$ on A^c are invariant for the entrance chain $S^{\rangle A}$ and exit chain $S^{A^c\rangle}$, respectively.*

Remark 5.1. If $\mathcal{Z} = \mathbb{Z}^d$ with $d \geq 3$, $\mathbb{E}X_1 = 0$ and $\mathbb{E}\|X_1\|^2 < \infty$, then the assumptions on $-A$ and $-A^c$ in Theorem 5.1 are not required since by Uchiyama [48], a set is massive for such S whenever it is massive for a simple random walk, which is self-dual. We do not know if such reduction is possible for arbitrary S.

For a particular example of A, consider the orthants in $\mathbb{R}^d$. We have the following result, which we prove below after further comments.

Corollary 5.1. *Put $\tau_\pm := \tau_{\pm(0,\infty)^d}(S)$ and assume that $\mathbb{P}_0(\tau_\pm < \infty) = 1$. Then the measures π_+ (defined in (4)) and*

$$\pi_-(dx) := (1 - \mathbb{P}(X_1 > x))\lambda(dx), \qquad x \in \mathcal{Z} \cap (-\infty, 0)^d,$$

are invariant for the chains of entrances of S into $[0,\infty)^d$ and $(-\infty,0)^d$, respectively. Moreover, for $d = 1$, the measure π (defined in (2)) is invariant for the chain of overshoots $\mathcal{O}$.

The assumptions of the corollary imply that every coordinate of X_1 has either zero mean or no expectation. In dimension one τ_+ and τ_- are the first strict ascending and descending ladder times of the random walk S when $S_0 = 0$. Both quantities are finite a.s. if and only if S *oscillates*, that is $\limsup S_n = -\liminf S_n = +\infty$ a.s. as $n \to \infty$. Then a.s. finiteness of τ_+ and τ_- is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}_0(S_n > 0) = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}_0(S_n < 0) = +\infty;$$

cf. Feller [17, Theorems XII.2.1 and XII.7.2]. This equivalence remains valid in dimension $d = 2$; see Greenwood and Shaked [20, Corollary 3].

One can verify massiveness of a general set $A \subset \mathcal{Z}$ for a random walk S using the following results. If S is topologically recurrent, then any Borel set of positive measure λ is massive for S, as follows (Aaronson [1, Proposition 1.2.2]) from ergodicity and recurrence of λ for S (see Lemma 5.1 below). If S is transient (i.e. not topologically recurrent), no set of finite measure can be massive. For walks on $\mathcal{Z} = \mathbb{Z}^d$ with $d \geq 3$ satisfying $\mathbb{E}X_1 = 0$ and $\mathbb{E}\|X_1\|^2 < \infty$, there is a necessary and sufficient condition for massiveness of a set, called Wiener's test, stated in terms of capacity, by Itô and McKean [22] and Uchiyama [48]. Easily verifiable sufficient conditions for massiveness in $d = 3$ are due to Doney [15]. For example, any straight "line" in $\mathbb{Z}^3$ is massive. Under the above assumptions, a set is massive for every such a walk if it is massive for a simple random walk, and so this is a property of a set rather than of a walk. We are not aware of any explicit results for random walks with a general distribution of increments apart from partial results of Greenwood and Shaked [20] for convex cones with the apex at the origin. Based on the estimates of Green's function in Uchiyama [47, Section 8], it appears that such results whould be fully analogous to the ones for walks on $\mathcal{Z} = \mathbb{Z}^d$ if $\mathbb{E}X_1 = 0$, $\mathbb{E}\|X_1\|^2 < \infty$, and the distribution of X_1 has density with respect to the Lebesgue measure. The case of heavy-tailed random walks on $\mathbb{Z}^d$, including transient walks in dimensions $d \in \{1, 2\}$, is considered by Bendikov and Cygan [3, 4].

Proof of Corollary 5.1. For the non-negative orthant $A = [0,\infty)^d$, we have

$$\{X_1 \in x - A^c\} = \{X_1 \in (x - A)^c\} = \{X_1 \notin x - A\} = \{X_1 \not\leq x\}.$$

Since the complement of each of the orthants $\pm(0,\infty)^d$ contains the other one, the result on π_+ follows from Theorem 5.1 once we show that both orthants are massive. Equivalently, that $\tau_{x\pm(0,\infty)^d}(S)$ are finite $\mathbb{P}_0$-a.s. for every $x \in \mathcal{Z}$. We have $\tau_{x+(0,\infty)^d}(S) \leq \tau_{x+(0,\infty)^d}(H)$, where H is a random walk on $\mathcal{Z}$ defined by $H_n := S_{\tau_n}$, where $\tau_0 := 0$ and $\tau_n := \inf\{k > \tau_{n-1} : S_k > S_{\tau_{n-1}}\}$ for $n \in \mathbb{N}$. Then every $\tau_{x+(0,\infty)^d}(H)$ is finite $\mathbb{P}_0$-a.s. since every coordinate of H_n tends to ∞ as $n \to \infty$ by $H_1 > 0$. Hence $\tau_{x+(0,\infty)^d}(S)$ is finite $\mathbb{P}_0$-a.s. and the same applies to $\tau_{x-(0,\infty)^d}(S)$.

The result on π_- is analogous. For $d=1$, from invariance of π_+ and π_- it follows that π is invariant for the sub-chain $(\mathcal{O}_{2n})_{n\geq 1}$. Invariance of π for the full chain $\mathcal{O}$ is by (33). $\square$

We will present our uniqueness results after establishing the following lemma.

Lemma 5.1. *Let S be a topologically recurrent random walk on $\mathbb{R}^d$. Then λ is the unique (up to multiplication by constant) locally finite Borel invariant measure of S on $\mathcal{Z}$, and λ is recurrent and ergodic for S.*

Recall that topological recurrence of S by definition means that $\mathbb{P}_0(S_n \in G \text{ i.o.}) = 1$ for every open neighbourhood G of 0. For such random walks, this equality is in fact true for every non-empty $G \subset \mathcal{Z}$ open in the relative topology of $\mathcal{Z}$; see Revuz [42, Proposition 3.4]. Combined with the results of Chung and Fuchs [10, Theorems 1, 3 and 4], this gives that topological recurrence of S is equivalent to

$$\limsup_{r\to 1-} \int_{[-a,a]^d} \frac{1}{\mathrm{Re}(1 - r\mathbb{E}e^{it\cdot X_1})} dt = \infty \quad \text{for all } a > 0;$$

the limit is always finite for $d \geq 3$. The limit commutes with the integral if $d=1$ (Ornstein [36, Theorem 4.1]) or $\mathcal{Z} = \mathbb{Z}^d$ (Spitzer [45, Theorem 8.2]). In particular, for $d=1$ this integral diverges when $\mathbb{E}X_1 = 0$, and it may also diverge for arbitrarily heavy-tailed X_1 (Shepp [43]). In dimension $d=2$, S is topologically recurrent on $\mathcal{Z}$ if $\mathbb{E}X_1 = 0$ and $\mathbb{E}\|X_1\|^2 < \infty$ (Chung and Lindvall [12]). For more general results on recurrence of random walks on locally compact abelian metrizable groups, see Revuz [42, Chapters 3.3 and 3.4].

Proof. The uniqueness is by Proposition I.45 in Guivarc'h et al. [21], which states that the right Haar measure on a locally compact Hausdorff topological group G with countable base is a unique invariant Radon Borel measure for any topologically recurrent right random walk on G such that no proper closed subgroup of G contains the support of the distribution of increments of the walk.

To infer ergodicity, we first note that uniqueness of invariant measure implies irreducibility of S starting under λ. In fact, if there is a λ-non-trivial invariant set $A \in \mathcal{B}(\mathcal{Z})$ of S, then the locally finite measure $\mathbb{1}_A\lambda$ is invariant for S, which contradicts the uniqueness. From topological recurrence of S and Lemma 3.1.b applied to any sequence of bounded open sets B_n that cover $\mathcal{Z}$, we see that λ is recurrent for S. Then λ is ergodic for S by Lemma 3.1.d. $\square$

Theorem 5.2. *Let S be any topologically recurrent random walk on $\mathbb{R}^d$, and let $A \subset \mathcal{Z}$ be any λ-non-trivial Borel set with $\lambda(\partial A) = 0$. Then λ_A^{entr} and $\lambda_{A^c}^{exit}$ are ergodic, recurrent, and unique (up to multiplication by constant) locally finite Borel invariant measures of the respective chains $S^{\rangle A}$ and $S^{A^c\rangle}$ on $\mathcal{Z}$.*

Corollary 5.2. *If a one-dimensional random walk S is topologically recurrent, then the chains of overshoots O, $O^{\downarrow}$, and $\mathcal{O}$ (where $O_n^{\downarrow} := \mathcal{O}_{2n-\mathbb{1}(S_0\geq 0)}$ for $n \geq 1$) are ergodic and recurrent starting respectively under their unique invariant measures π_+, π_-, and π on $\mathcal{Z}$.*

Proofs. The transition probability of S is weak Feller by $\mathbb{P}_x(S_1 \in \cdot) = \mathbb{P}(x + X_1 \in \cdot)$. Since $\tau_{\mathrm{Cl}(A)}(S)$ is finite $\mathbb{P}_\lambda$-a.e. by ergodicity of S starting under λ, it follows that $\mathbb{P}_\lambda(S_{\tau_{\mathrm{Cl}(A)}(S)} \in$

$\partial A) = 0$ by $\lambda(\partial A) = 0$. Hence $\mathbb{P}_x(S_{\tau_{\mathrm{Cl}(A)}(S)} \in \mathrm{Int}(A)) = 1$ for λ-a.e. $x \in \mathrm{Int}(A^c)$. Therefore, $\mathbb{P}_x(S_1 \in \mathrm{Int}(A)) > 0$ for some $x \in \mathrm{Int}(A^c)$ because A is λ-non-trivial and $\lambda(\partial A) = 0$. Thus, the assumptions of Theorem 3.2 are satisified. Combined with Lemma 5.1, it implies Theorem 5.2.

The corollaries on π_+ and π_- follow directly from Theorem 5.2 by $\pi_+ = \lambda^{entr}_{[0,\infty)}$ and $\pi_- = \lambda^{entr}_{(-\infty,0)}$. Furthermore, if π is not ergodic, then by Lemma 3.1.d there is a π-non-trivial Borel set $A \subset \mathcal{Z}$ that is invariant for $\mathcal{O}$, i.e. $\mathbb{P}_x(\mathcal{O}_1 \in A) = \mathbb{1}_A(x)$ for π-a.e. $x \in \mathcal{Z}$. Then $A \cap [0,\infty)$ is a π_+-non-trivial invariant set for O or $A \cap (-\infty, 0)$ is a π_--non-trivial invariant set for $O^{\downarrow}$, which contradicts ergodicity of π_+ and π_-. A similar argument yields uniqueness of π. □

Finally, let us comment on stability of the "distribution" of the entrance chain into A. This question makes a probabilistic sense only if the measure λ^{entr}_A is finite and therefore can be normalized to be a probability. For example, this is the case when $d = 1$, $A = [0,\infty)$, $\mathbb{E}X_1 = 0$ or when S is topologically recurrent on $\mathcal{Z}$, A is bounded, and $d \in \{1, 2\}$. In the former case, the question of stability was studied in our paper [34]. In the latter case, it is reasonable to restrict the attention to convex and compact sets A. These are intervals when $d = 1$, considered in [34, Section 5.1]. It appears that convergence results in dimension $d = 2$ can be obtained using exactly the same approach as in [34].

6. The number of level-crossings for one-dimensional random walks

Throughout this section we assume that the random walk S is one-dimensional.

6.1. Limit theorem.

The main result of this section a limit theorem for L_n, the number of zero-level crossings of S by time n, defined in (5). We will prove it combining Theorem 5.2 on ergodicity of the chain of overshoots with a limit theorem for local times of random walks by Perkins [40].

Theorem 6.1. *For any random walk S such that $\mathbb{E}X_1 = 0$ and $\sigma^2 := \mathbb{E}X_1^2 \in (0,\infty)$, we have*

$$\lim_{n\to\infty} \mathbb{P}_x\left(\frac{\sigma L_n}{\mathbb{E}|X_1|\sqrt{n}} \le y\right) = 2\Phi(y) - 1, \qquad x \in \mathcal{Z},\, y \ge 0,$$

where Φ denotes the distribution function of a standard normal random variable.

This weak convergence was first proved by Chung [11] under the additional assumption $\mathbb{E}X_1^3 < \infty$. Maruyama [30, Theorem 3] claimed it under $\mathbb{E}X_1^2 < \infty$ but it appears that his prove actually assumes that $\mathbb{E}X_1^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$; indeed, the third equality in [30, Eq. (3.6)] seems to rely on the argument used after [30, Eq. (3.1)]. In the early 1980s, A.N. Borodin obtained limit theorems of more general type for additive functionals of consecutive steps of random walks; see [5, Chapter V] and references therein. However, his method limited by the assumption that the distribution of increments of the walk is either aperiodic integer-valued or has a square-integrable characteristic function, and hence absolutely continuous (by [25, Theorem 11.6.1]).

Our proof rests on the following auxiliary result, the law of large numbers for the chain $\mathcal{O}$. In this form it does *not* follow directly from ergodicity of $\mathcal{O}$ (stated in Corollary 5.2) since

Birkhoff's ergodic theorem implies convergence of the time averages only for π-a.e. $x \in \mathcal{Z}$ rather than for all x.

Proposition 6.1. *Let S be any random walk such that $\mathbb{E}X_1 = 0$ and $\sigma^2 := \mathbb{E}X_1^2 \in (0, \infty)$. Then for* every *$x \in \mathcal{Z}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\mathcal{O}_k| = \frac{1}{\mathbb{E}|X_1|} \int_{\mathcal{Z}} |y| \pi(dy) = \frac{\sigma^2}{2\mathbb{E}|X_1|}, \quad \mathbb{P}_x\text{-}a.s. \tag{37}$$

Proof of Theorem 6.1. Denote by ℓ_0 the local time at 0 at time 1 of a standard Brownian motion. By Lévy's theorem, ℓ_0 has the same distribution as the absolute value of a standard normal random variable. Combining this result with Theorem 1.3 by Perkins [40] and accounting for the 1/2 in the definition of the Brownian local time in [40], we get

$$\lim_{n\to\infty} \mathbb{P}_x\left(\frac{2}{\sigma\sqrt{n}} \sum_{k=1}^{L_n} |\mathcal{O}_k| \le y \right) = 2\Phi(y) - 1, \qquad x = 0,\ y \ge 0; \tag{38}$$

since Perkins's definition of crossing times is slightly different from the one of ours, his result shall be applied to the random walk $-S/\sigma$. On the other hand, by Proposition 6.1,

$$\lim_{n\to\infty} \frac{1}{L'_n} \sum_{k=1}^{L_n} |\mathcal{O}_k| = \frac{\sigma^2}{2\mathbb{E}|X_1|}, \quad \mathbb{P}_x\text{-a.s.}, \qquad x \in \mathcal{Z}, \tag{39}$$

where $L'_n := L_n + \mathbb{1}(L_n = 0)$ and we used the fact that $\mathbb{P}_x(\lim_{n\to\infty} L_n = \infty) = 1$, which holds true since S oscillates. Rewriting equality (38) using the identity $\frac{1}{\sqrt{n}} = \frac{L'_n}{\sqrt{n}} \cdot \frac{1}{L'_n}$ and then combining it with (39) yields the assertion of Theorem 6.1 for $x = 0$ by Slutsky's theorem.

Furthermore, the results of Perkins actually imply (by Perkins [41]) that equality (38) remains valid, although this is not stated in [40, Theorem 1.3], if we replace $x = 0$ by $x_n \in \mathcal{Z}$ for any sequence $(x_n)_{n\ge1} \subset \mathcal{Z}$ such that $\lim_{n\to\infty} x_n/\sqrt{n} = 0$. In particular, we can take $x_n \equiv x$ for an arbitrary $x \in \mathcal{Z}$, which yields Theorem 6.1 in full by the above argument.

Let us explain in detail this extension of (38). For $x = 0$, Theorem 1.3 of Perkins [40] is an immediate corollary to his Lemma 3.2 and Corollary 2.2. Our extension of (38) follows in exactly the same way if we let x in Lemma 3.2 be the *nearstandard* point in $^*\mathbb{R}$, the field of *nonstandard real numbers*, that corresponds to the sequence $(x_n)_{n\ge1}$, in which case $st(x) = {}^\circ x = 0$, i.e. the *standard part* of x is 0. We referred to Cutland [14] to digest the unusual notation and concepts of nonstandard analysis, which were used in [40] with no explanation. □

Proof of Proposition 6.1. Denote $h := \inf\{z \in \mathcal{Z} : z > 0\}$; then either $\mathcal{Z} = h\mathbb{Z}$ if $h > 0$ or $\mathcal{Z} = \mathbb{R}$ if $h = 0$. One can easily check that for π_+ (defined in (4)),

$$\int_{\mathcal{Z}\cap[0,\infty)} y\pi_+(dy) = \int_h^\infty (y - h/2)\mathbb{P}(X_1 > y)dy = \int_0^\infty (y - h/2)\mathbb{P}(X_1 > y)dy \tag{40}$$

and, similarly,

$$-\int_{\mathcal{Z}\cap(-\infty,0)} y\pi_-(dy) = \int_0^\infty (y + h/2)\mathbb{P}(-X_1 > y)dy. \tag{41}$$

Using that $\mathbb{E}X_1 = 0$ and integrating the above equality by parts, we find that the *probability* measure $\pi/\mathbb{E}|X_1|$ has the first absolute moment $\sigma^2/(2\mathbb{E}|X_1|)$. Therefore, by Birkhoff's ergodic theorem and ergodicity of the chain of overshoots $\mathcal{O}$ asserted in Corollary 5.2, the convergence in (37) holds true for π-a.e. $x \in \mathcal{Z}$. We need to prove this for every $x \in \mathcal{Z}$.

Denote by $\operatorname{supp}\pi$ the topological support of π and by N the set of points $x \in \operatorname{supp}\pi$ that satisfy (37). We clearly have $N = \operatorname{supp}\pi$ in the lattice case $h > 0$, where $\mathcal{Z}$ is discrete. In the non-lattice case $h = 0$, so far we only have that N is dense in $\operatorname{supp}\pi$. This is because N has full measure π, hence N has full Lebesgue measure $\lambda|_{\operatorname{supp}\pi}$, as readily seen from definition (2) of π. In order to prove (37), we need to show that $N = \operatorname{supp}\pi$, since the chain $\mathcal{O}$ hits the support of π (which is a closed interval, possibly infinite) at the first step regardless of the starting point. Our argument goes as follows.

Consider the random walk $S' := (S'_n)_{n\ge 0}$, where $S'_n = X_1 + \ldots + X_n$ for $n \ge 1$, starting at $S'_0 := 0$. Then $\mathbb{P}_x(S \in \cdot) = \mathbb{P}\big((x+S'_0, x+S'_1, \ldots) \in \cdot\big)$. For real y_1, y_2, define the functions

$$g(y_1,y_2) := \mathbb{1}(y_1<0, y_2\ge 0 \text{ or } y_1\ge 0, y_2<0), \qquad f(y_1,y_2) := |y_2|g(y_1,y_2).$$

We claim that for any $x \in \operatorname{supp}\pi$ and $\varepsilon \in (0,1)$, there exists a $y \in N$ such that

$$\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^n f(y+S'_{k-1},y+S'_k)}{\sum_{k=1}^n g(y+S'_{k-1},y+S'_k)} - \frac{\sum_{k=1}^n f(x+S'_{k-1},x+S'_k)}{\sum_{k=1}^n g(x+S'_{k-1},x+S'_k)}\right| \le \varepsilon, \quad \mathbb{P}\text{-a.s.} \tag{42}$$

This will imply that $x \in N$ and hence prove Proposition 6.1, since

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{\sum_{k=1}^n f(y+S'_{k-1},y+S'_k)}{\sum_{k=1}^n g(y+S'_{k-1},y+S'_k)} = \frac{\sigma^2}{2\mathbb{E}|X_1|}\right) = \mathbb{P}_y\left(\lim_{n\to\infty}\frac{1}{L'_n}\sum_{k=1}^{L_n}|\mathcal{O}_k| = \frac{\sigma^2}{2\mathbb{E}|X_1|}\right) = 1,$$

where $L'_n = L_n + \mathbb{1}(L_n = 0)$ and the last equality holds by definition of the set N and the fact that $\mathbb{P}_y(\lim_{n\to\infty} L_n = \infty) = 1$, which is true because S oscillates. Thus, it remains to prove inequality (42).

From the identity $\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1}{b_1}\big(1 - \frac{a_2}{a_1}\cdot\frac{b_1}{b_2}\big)$ for $a_1, a_2, b_1, b_2 > 0$ and the inequality $\big|1 - \frac{a}{b}\big| < 2|a-1| + 2|b-1|$ for $a > 0, b > \frac12$, we see that (42) will follow if we show that for any $x \in \operatorname{supp}\pi$ and $\varepsilon \in (0, \sigma^2/(2\mathbb{E}|X_1|))$, there exists a $y \in N$ such that $\mathbb{P}$-a.s.,

$$\limsup_{n\to\infty}\left[\left|\frac{\sum_{k=1}^n f(x+S'_{k-1},x+S'_k)}{\sum_{k=1}^n f(y+S'_{k-1},y+S'_k)} - 1\right| + \left|\frac{\sum_{k=1}^n g(x+S'_{k-1},x+S'_k)}{\sum_{k=1}^n g(y+S'_{k-1},y+S'_k)} - 1\right|\right] \le \frac{\varepsilon\mathbb{E}|X_1|}{\sigma^2}. \tag{43}$$

For any $\delta > 0$, integer $k \ge 1$, and any $y \in N$ such that $|x-y| \le \delta$, we have

$$|g(x+S'_{k-1},x+S'_k) - g(y+S'_{k-1},y+S'_k)| \le \mathbb{1}(|y+S'_{k-1}| \le \delta \text{ or } |y+S'_k| \le \delta)$$

and

$$\begin{aligned}&|f(x+S'_{k-1},x+S'_k) - f(y+S'_{k-1},y+S'_k)| \\ &\qquad\qquad \le \delta g(y+S'_{k-1},y+S'_k) + (|y+S'_k|+\delta)\mathbb{1}(|y+S'_{k-1}| \le \delta \text{ or } |y+S'_k| \le \delta).\end{aligned}$$

This gives

$$\left|\frac{\sum_{k=1}^n g(x+S'_{k-1},x+S'_k)}{\sum_{k=1}^n g(y+S'_{k-1},y+S'_k)} - 1\right| \le \frac{\sum_{k=1}^n\big[\mathbb{1}(|y+S'_{k-1}|\le\delta) + \mathbb{1}(|y+S'_k|\le\delta)\big]}{\sum_{k=1}^n g(y+S'_{k-1},y+S'_k)} \tag{44}$$

and

$$\left|\frac{\sum_{k=1}^n f(x+S'_{k-1}, x+S'_k)}{\sum_{k=1}^n f(y+S'_{k-1}, y+S'_k)} - 1\right|$$
$$\leq \frac{\sum_{k=1}^n \big[\delta g(y+S'_{k-1}, y+S'_k) + (|X_k|+2\delta)\mathbb{1}(|y+S'_{k-1}|\leq\delta) + 2\delta\mathbb{1}(|y+S'_k|\leq\delta)\big]}{\sum_{k=1}^n f(y+S'_{k-1}, y+S'_k)}. \quad (45)$$

By Lemma 3.2, the topologically recurrent random walk S on $\mathcal{Z}=\mathbb{R}$ is recurrent and ergodic starting under the Lebesgue measure λ. By Condition a in Section 3.1, recurrence of S starting under λ implies conservativity of the measure preserving one-sided shift θ on $(\mathbb{R}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{R}^{\mathbb{N}_0}), \mathrm{P}^S_\lambda)$. Therefore we can apply Hopf's ratio ergodic theorem (see the Appendix) to the ratios on the r.h.s.'s of (44) and (45). Let us explain in details, say, why

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{\sum_{k=1}^n g(y+S'_{k-1}, y+S'_k)}{\sum_{k=1}^n f(y+S'_{k-1}, y+S'_k)} = \frac{\mathbb{E}|X_1|}{\sigma^2/2}\right) = 1, \qquad \lambda\text{-a.e. } y. \quad (46)$$

Indeed, consider the functions on $\mathbb{R}^{\mathbb{N}_0}$ defined by $G(z) := g(z_0, z_1)$ and $F(z) := f(z_0, z_1)$ for $z=(z_0, z_1, \dots)\in\mathbb{R}^{\mathbb{N}_0}$. Both functions are non-negative, non-zero, and P^S_λ-integrable by

$$\mathrm{E}^S_\lambda G = \int_{\mathbb{R}} \mathbb{E}_{z_0} g(S_0, S_1)\lambda(dz_0) = \int_{-\infty}^0 \mathbb{P}(z_0+X_1\geq 0)dz_0 + \int_0^\infty \mathbb{P}(z_0+X_1<0)dz_0 = \mathbb{E}|X_1|$$

and

$$\begin{aligned}\mathrm{E}^S_\lambda F &= \int_{\mathbb{R}} \mathbb{E}_{z_0}[|S_1| g(S_0, S_1)]\lambda(dz_0)\\ &= \int_{-\infty}^0 \mathbb{E}[(z_0+X_1)\mathbb{1}(z_0+X_1\geq 0)]dz_0 - \int_0^\infty \mathbb{E}[(z_0+X_1)\mathbb{1}(z_0+X_1<0)]dz_0\\ &= \int_0^\infty \mathbb{E}[(|X_1|-z_0)\mathbb{1}(|X_1|>z_0)]dz_0 = \mathbb{E}|X_1|^2/2,\end{aligned}$$

where the last equality follows from Fubini's theorem. Finally, we have

$$\begin{aligned}&\mathrm{P}^S_\lambda\left(\limsup_{n\to\infty}\left|\frac{\sum_{k=0}^{n-1} G\circ\theta^k}{\sum_{k=0}^{n-1} F\circ\theta^k} - \frac{\mathrm{E}^S_\lambda G}{\mathrm{E}^S_\lambda F}\right| \neq 0\right)\\ &\qquad = \int_{\mathbb{R}} \mathbb{P}\left(\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^n g(y+S'_{k-1}, y+S'_k)}{\sum_{k=1}^n f(y+S'_{k-1}, y+S'_k)} - \frac{\mathbb{E}|X_1|}{\sigma^2/2}\right|\neq 0\right)\lambda(dy),\end{aligned}$$

hence equality (46) follows from Hopf's ratio ergodic theorem.

Similarly to (46), for every $\delta>0$, for λ-a.e. y the sum of the ratios on the r.h.s.'s of (44) and (45) converges $\mathbb{P}$-a.s. as $n\to\infty$ to

$$c(\delta) := \frac{\delta\mathbb{E}|X_1| + 2\delta(\mathbb{E}|X_1|+2\delta) + 4\delta^2}{\sigma^2/2} + \frac{4\delta}{\mathbb{E}|X_1|}.$$

Denote by N_δ the set of y where this $\mathbb{P}$-a.s. convergence holds true. Choose a $\delta>0$ such that $c(\delta) < \varepsilon\mathbb{E}|X_1|/\sigma^2$. The Borel set $N\cap N_\delta$ has full measure $\lambda|_{\operatorname{supp}\pi}$ and hence is dense in supp π. Therefore we can pick a $y \in N \cap N_\delta$ that satisfies $|x - y| \le \delta$. Then inequality (43) follows from (44) and (45), as required. □

6.2. **Stationarity of level-crossings.** Define the first up-crossing time of zero by

$$T := \inf\{k \ge 1 : S_{k-1} < 0, S_k \ge 0\}.$$

and the *numbers of up and down-crossings* of an arbitrary level $a \in \mathcal{Z}$ by time $n \ge 1$ by

$$L_n^{\uparrow}(a) := \sum_{i=0}^{n-1} \mathbb{1}(S_i < a, S_{i+1} \ge a), \qquad L_n^{\downarrow}(a) := \sum_{i=0}^{n-1} \mathbb{1}(S_i \ge a, S_{i+1} < a).$$

Recall that the measures π_+ and π_-, defined in (4), have the total mass $\mathbb{E}|X_1|/2$ each when $\mathbb{E}X_1 = 0$; put $\pi'_\pm := 2\pi_\pm/\mathbb{E}|X_1|$. We have the following rather surprising result.

Proposition 6.2. *For any non-degenerate random walk S satisfying $\mathbb{E}X_1 = 0$ and any $a \in \mathcal{Z}$, we have*

$$\mathbb{E}_{\pi'_+} L_T^{\uparrow}(a) = \mathbb{E}_{\pi'_-} L_T^{\uparrow}(a) = \mathbb{E}_{\pi'_+} L_T^{\downarrow}(a) = \mathbb{E}_{\pi'_-} L_T^{\downarrow}(a) = 1.$$

Thus, the expected number of up-crossings by the time T does not depend on the level if S is started under π'_+ or π'_- (i.e. at stationarity of either chain O or $O^{\downarrow}$), and therefore equals 1 since $L_T^{\uparrow}(0) = 1$ by the definition of T. In the particular case when S is a symmetric simple random walk, this is a well-known fact (see e.g. Feller [17, Section XII.2, Example b]) since here $\pi'_+ = \delta_0$ and $L_T^{\uparrow}(a) + L_T^{\downarrow}(a)$ is the local time of the walk at level a.

Proof. We use Kac' formula (48) for the measure-preserving shift θ on $(\mathcal{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathcal{Z}^{\mathbb{N}_0}), \mathrm{P}_\lambda^S)$ and $A = \{x \in \mathcal{Z}^{\mathbb{N}_0} : x_0 < 0, x_1 \ge 0\}$. For the up-crossings, take $B = \{x \in \mathcal{Z}^{\mathbb{N}_0} : x_0 < a, x_1 \ge a\}$ and use that T is the first entrance time of S into $[0, \infty)$. By the same computation as in (21), this gives $\pi_+(\mathcal{Z} \cap [0, \infty)) = \mathbb{E}_{\pi_+} L_T^{\uparrow}(a)$. Similarly, take $A = \{x \in \mathcal{Z}^{\mathbb{N}_0} : x_0 \ge 0, x_1 < 0\}$ to get $\pi_-(\mathcal{Z} \cap (-\infty, 0)) = \mathbb{E}_{\pi_-} L_T^{\uparrow}(a)$. For the down-crossings, consider $B = \{x \in \mathcal{Z}^{\mathbb{N}_0} : x_0 \ge a, x_1 < a\}$. □

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Appendix A. Induced transformations in infinite ergodic theory

Here we present some relevant basic results on inducing for measure preserving transformations of infinite measure spaces; see Aaronson [1, Chapter 1] for an introduction. To our surprise, we failed to find straightforward references to the results needed.

Let T be a measure preserving transformation of a measure space $(X, \mathcal{F}, m)$. For any set $A \in \mathcal{F}$, consider the *first hitting time* τ_A of A and the *induced mapping* T_A defined by

$$\tau_A(x) := \inf\{n \ge 1 : T^n x \in A\},\ x \in X \quad \text{and} \quad T_A(x) := T^{\tau_A(x)} x,\ x \in \{\tau_A < \infty\}. \tag{47}$$

All these mappings are measurable.

We say that a set $A \in \mathcal{F}$ is *recurrent* for T if τ_A is finite m-a.e. on A, that is $A \subset \cup_{k\ge 1} T^{-k}A \bmod m$, where mod m means true possibly except for a m-zero set. The transformation T is *ergodic* if its invariant σ-algebra $\mathcal{I}_T := \{A \in \mathcal{F} : T^{-1}A = A \bmod m\}$ is m-trivial, i.e. for every $A \in \mathcal{I}_T$ either $m(A) = 0$ or $m(A^c) = 0$.

The following statement essentially is [1, Proposition 1.5.3] (which is stated under slightly different assumptions but its proof works unchanged).

Lemma A.1. *Let T be a measure preserving transformation of a measure space $(X, \mathcal{F}, m)$, and $A \in \mathcal{F}$ be any set recurrent for T such that $0 < m(A) < \infty$. Then the induced mapping T_A is a measure preserving transformation of the induced space $(A, \mathcal{F}_A, m_A)$.*

To relax the condition $m(A) < \infty$, we need additional assumptions. We say that T is *conservative* if every measurable subset of X is recurrent for T. The following result is Corollary 1.1 in Pène and Thomine [39] (they formally assume that X is a Polish space but never used this in the proof of [39, Proposition 0.1]).

Lemma A.2. *Let T be a measure preserving conservative transformation of a σ-finite measure space $(X, \mathcal{F}, m)$, and $A \in \mathcal{F}$ be any set with $m(A) > 0$. Then T_A is a measure preserving conservative transformation of the induced space $(A, \mathcal{F}_A, m_A)$.*

We now present conditions for conservativity.

Lemma A.3. *A measure preserving transformation T of a σ-finite measure space $(X, \mathcal{F}, m)$ is conservative iff there exists a sequence of sets $\{A_k\}_{k\ge 1} \subset \mathcal{F}$, all of finite measure and recurrent for T, such that $X = \cup_{k\ge1} A_k \bmod m$. In particular, this holds if $X = \cup_{k\ge 1} T^{-k}A \bmod m$, i.e. $\tau_A < \infty$ m-a.e., for some measurable set A of finite measure.*

Proof. The direct implication in the first assertion is trivial. For the reverse one, assume that there is a set $A \in \mathcal{F}$ of positive measure that is not recurrent for T. Then so is $A' := A \setminus \cup_{n=1}^{\infty} T^{-n}A$. Pick a $k \ge 1$ such that $m(A_k \cap A') > 0$. By Lemma A.1, the induced mapping T_{A_k} is measure preserving on the induced space $(A_k, \mathcal{F}_{A_k}, m_{A_k})$ of finite measure. This mapping is conservative by Poincaré's recurrence theorem, hence $A_k \cap A'$ is a recurrent set for T_{A_k}, hence it is recurrent for T, which is a contradiction. $\square$

The last result, known as Kac's formula, concerns reversing the inducing.

Lemma A.4. *Let T be a conservative measure preserving transformation of a σ-finite measure space $(X, \mathcal{F}, m)$, and let $A \in \mathcal{F}$ be any set such that $X = \cup_{k\ge 1} T^{-k}A \bmod m$. Then*

$$m(B) = \int_A \left[\sum_{k=0}^{\tau_A(x)-1} \mathbb{1}(T^k x \in B) \right] m(dx), \qquad B \in \mathcal{F}. \tag{48}$$

Proof. Denote the r.h.s. of (48) by $\mu(B)$. By monotonicity and σ-finiteness of m, it suffices to check equality $m = \mu$ only on sets of finite measure m. For any measurable $B \subset X$,

$$\begin{aligned}\mu(B) &= \int_A \left[\sum_{n=1}^{\infty} \mathbb{1}(\tau_A(x) = n) \times \sum_{k=0}^{\tau_A(x)-1} \mathbb{1}(T^k x \in B) \right] m(dx) \\ &= \int_A \left[\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \mathbb{1}(T^k x \in B, \tau_A(x) = n) \right] m(dx) \\ &= \sum_{k=0}^{\infty} m(A \cap T^{-k} B \cap \{\tau_A > k\}), \end{aligned} \tag{49}$$

and therefore, assuming that $m(B) < \infty$, we get

$$\mu(B) = \sum_{k=0}^{\infty} m(A \cap T^{-k} B \setminus \cup_{n=1}^{k} T^{-n} A) = m(A \cap B) + \sum_{k=1}^{\infty} m(A \cap T^{-1} B'_{k-1}),$$

where $B'_k := T^{-k} B \setminus \cup_{n=0}^{k} T^{-n} A$ for $k \geq 0$. The set $T^{-1} B'_k$ has finite measure and it is a disjoint union of $A \cap T^{-1} B'_k$ and B'_{k+1}, hence $m(A \cap T^{-1} B'_k) = m(B'_k) - m(B'_{k+1})$. Then the sequence $m(B'_k)$ is decreasing, and

$$\mu(B) = m(A \cap B) + m(B'_0) - \lim_{k \to \infty} m(B'_k) = m(B) - \lim_{k \to \infty} m(B'_k). \tag{50}$$

It remains to show that the limit in the above formula is zero.

For any integer $N \geq 1$, denote $B^{(N)} := B \cap (\cup_{n=1}^{N} T^{-n} A)$. Notice that for any $k \geq N$, we have $\{\tau_{B^{(N)}} \leq k - N\} \subset \{\tau_A \leq k\}$, hence

$$T^{-k}(B^{(N)}) \setminus \cup_{n=0}^{k} T^{-n} A \subset \{k - N < \tau_{B^{(N)}} \leq k\}, \qquad k \geq N.$$

Then for $k \geq N$,

$$\begin{aligned} m(B'_k) &= m\big(T^{-k}(B \setminus \cup_{n=1}^{N} T^{-n} A) \setminus \cup_{n=1}^{k} T^{-n} A\big) + m\big(T^{-k}(B^{(N)}) \setminus \cup_{n=1}^{k} T^{-n} A\big) \\ &\leq m\big(T^{-k}(B \setminus \cup_{n=1}^{N} T^{-n} A)\big) + N \sup_{n > k-N} m(\tau_{B^{(N)}} = n) \\ &= m(B \setminus \cup_{n=1}^{N} T^{-n} A) + N \sup_{n > k-N} m(T^{-n}(B^{(N)}) \setminus \cup_{i=1}^{n-1} T^{-i} B^{(N)}). \end{aligned}$$

The first term in the last line can be made as small as necessary by choosing N to be large enough, and the second term vanishes as $k \to \infty$ for any fixed N by Remark to Proposition 1.5.3 in [1]. $\square$

Finally, we recall the following classical result; see Zweimüller [53].

Hopf's ratio ergodic theorem. *Let T be a conservative ergodic measure preserving transformation of a σ-finite measure space $(X, \mathcal{F}, m)$. Then for any functions $f, g \in L^1(X, \mathcal{F}, m)$ with non-zero $g \geq 0$,*

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} f \circ T^k}{\sum_{k=0}^{n-1} g \circ T^k} = \frac{\int_X f dm}{\int_X g dm}, \quad m\text{-a.e.}$$

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