REGULARITIES FOR SOLUTIONS TO THE L_p DUAL MINKOWSKI PROBLEM FOR UNBOUNDED CLOSED SETS

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Abstract. Recently, the L_p dual Minkowski problem for unbounded closed convex sets in a pointed closed convex cone was proposed and a weak solution to this problem was provided. In smooth setting, this problem is equivalent to solving the Dirichlet problem for a class of Monge-Ampère type equations.

In this paper, we show the existence, regularity and uniqueness of solutions to this Monge-Ampère type equation in the case $p \geq 1$ by studying variational properties for a family of Monge-Ampère functionals. Moreover, the existence and optimal global Hölder regularity in the case $p < 1$ and $q \geq n$ is also be discussed.

Keywords: *The L_p dual Minkowski problem, Monge-Ampère type equations, C-close sets.*

1. Introduction

The main purpose of this paper is to study the L_p dual Minkowski problem for unbounded convex sets in views of PDEs. Such type of problem is an analogue of the classical Minkowski type problem concerning convex bodies (compact convex sets with nonempty interiors) which has a long history and strong influence in convex geometry and PDEs. Examples of the Minkowski type problem concerning convex bodies include the classical Minkowski problem [42], the L_p Minkowski problem [36], the dual Minkowski problem [21], the L_p dual Minkowski problem [37] and so on.

The Minkowski type problem related to unbounded convex sets has also been studied by Chou-Wang [15], Pogorelov [40] and Urbas [48] for unbounded, complete and convex hypersurfaces two decades ago. An L_p version can be found in [23] by Huang-Liu. Recently, Schneider [43, 44] proposed the Minkowski problem for unbounded closed convex set in a closed convex cone. Soon, the corresponding L_p Minkowski problem, dual Minkowski problem and L_p dual Minkowski problem were proposed by Yang-Ye-Zhu [49], Li-Ye-Zhu [31] and Ai-Yang-Ye [1] respectively.

In the smooth setting, the L_p dual Minkowski problem for unbounded closed convex set in a closed convex cone [1] is equivalent to solving the Dirichlet problem of the

Monge-Ampère type equation

$$
\begin{cases}(-h)^{1-p}\det(\nabla^2 h + hI) = f[|\nabla h|^2 + h^2]^{\frac{n-q}{2}} \quad \text{in} \quad \Omega, \\ h = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}
$$

where Ω is an open convex set in S^{n-1}, f is a positive smooth function on $\overline{\Omega}$, h is the unknown function, I is the identity matrix, ∇h and $\nabla^2 h$ are the gradient and the Hessian of h on S^{n-1}. A weak solution to the Dirichlet problem (1.1) was provided in [1]. Thus, it is interesting to study the regularities of solutions to the Dirichlet problem (1.1).

In order to study the regularities, it is convenient to express the equation (1.1) in Euclidean space. According to Lemma 5.1, the problem (1.1) is equivalent to the following Dirichlet problem for the Monge-Ampère type equation in Euclidean space

$$
\begin{cases}\det(D^2 u) = g(x)(-u)^{p-1}\left[|Du|^2 + (x\cdot Du - u)^2\right]^{\frac{n-q}{2}} \quad \text{in} \quad U, \\ u = 0 \quad \text{on} \quad \partial U, \end{cases} \tag{1.2}
$$

where U is an open convex set in $\mathbb{R}^{n-1}$, g is a positive smooth function on $\overline{U}$ (see (5.1)), ∇u and $\nabla^2 u$ are the gradient and the Hessian of u on $\mathbb{R}^{n-1}$.

The problem (1.2) is a special case of the following Dirichlet problem for the Monge-Ampère equation which has been widely studied,

$$
\begin{cases}\det(D^2 u) = F(x, u, Du) \quad \text{in} \quad U \subset \mathbb{R}^n, \\ u = 0 \quad \text{on} \quad \partial U. \end{cases} \tag{1.3}
$$

The equation (1.3) was first studied by Pogorelov in [38]. When F is independent of Du and $F_u > 0$, Cheng-Yau obtained the existence and uniqueness of solutions to the equation (1.3) in [13]. Then, Caffarelli-Nirenberg-Spruck [10] and Krylov [30] obtained the smoothness of solution for the equation (1.3) up to the boundary under further regularity conditions for F. When F_u is not necessarily positive, Caffarelli-Nirenberg-Spruck [10] solved the equation (1.3) under the assumption of the existence of a subsolution. However, constructing such a subsolution is a difficult task. A different approach without constructing a subsolution was taken by Tso [46]. He used a variational approach for a family of Monge-Ampère functionals, which was introduced by Bakelman in [2, 3], to study such problems. Recently, the analogous variational approach was introduced by Tong-Yau [45] to study the solvability of the

Dirichlet problem

$$\begin{cases} \det(D^2 u) = \lambda (u^*)^{-k}(-u)^l \quad \text{in} \quad U, \\ u = 0 \quad \text{on} \quad \partial U, \end{cases}$$

where $\lambda \in \mathbb{R}$, $k > 0, l \geq 0$ and $u^* = x \cdot Du - u$.

Following the idea of [46, 45], we wish to find a variational structure for the Dirichlet problem (1.1), and use this to undertake a variational study for (1.1). An important ingredient in our variational approach is a Sobolev type inequality for q-volume (see Lemma 2.5). Let Ω be an open set with the smooth boundary in S^{n-1}, we call Ω is strictly convex domain in S^{n-1} if the cone $\widehat{\Omega} = \{\lambda x \mid x \in \Omega, \lambda > 0\}$ is a strictly convex domain in $\mathbb{R}^n$. The following is our first main result.

Theorem 1.1. *Let Ω be an open, bounded, smooth and strictly convex domain in S^{n-1}, f be a positive smooth function on $\overline{\Omega}$ and $p \geq 1$.*

- *(i) If $q > p$, then there exists a unique and non-zero solution $h \in C^\infty(\overline{\Omega})$ to the Dirichlet problem* (1.1)*.*
- *(ii) If $p = q$, then there exists a unique and non-zero λ such that the Dirichlet problem* (1.1) *with f replaced by λf admits a non-zero solution $h \in C^\infty(\overline{\Omega})$. Moreover, the solution is unique up to scaling by a positive constant.*
- *(iii) If $p > q \geq n$, then there exists a non-zero solution $h \in C^\infty(\overline{\Omega})$ to the Dirichlet problem* (1.1)*.*

The existence and uniqueness of smooth solutions to the L_p dual Minkowski problem (2.1) for convex bodies have been proved in [22] for $p > q$ and in [12] for $p = q \neq 0$. For the other case $p < q$, the uniqueness may fails [6, 26, 24, 25]. Thus, although the Monge-Ampère equations (1.1) and (2.1) differ from each other only by a negative sign, their solvability seems to be quite different.

It would be desirable to obtain the existence of solutions to the Dirichlet problem (1.1) in the case $p < 1$, but we have not been able to do this by the variational approach. In fact, the equation (1.1) becomes a singular Monge-Ampère function in the case $p < 1$, and the high order regularity of solutions to the equation (1.1) may fail up to boundary. In details, we get the following result.

Theorem 1.2. *Assume $p < 1$ and $q \geq n \geq 3$. Let Ω be an open, bounded, smooth and convex domain in S^{n-1}, $f \in C^\infty(\Omega) \cap C(\overline{\Omega})$ with $f > 0$. Then there exist a unique nontrivial solution $h \in C^\infty(\Omega) \cap C^{\frac{q-n+2}{q-p}}(\overline{\Omega})$ to the equation* (1.1) *with the following* *estimate*

$$|h(x)| \leq C(n,p,q,diam(\Omega),\sup f)[dist(x,\partial\Omega)]^{\frac{q-n+2}{q-p}} \tag{1.4}$$

for any $x \in \Omega$. Moreover, the exponent $\frac{q-n+2}{q-p}$ is optimal, i.e., for any $a \in (\frac{q-n+2}{q-p}, 1)$, there exist a bounded convex domain $\Omega \subset S^{n-1}$ such that the solution h of the equation (1.1) *satisfies $h \notin C^a(\overline{\Omega})$.*

The ideas for the proof of the above result comes from the study for the following Dirichlet problem

$$\begin{cases} \det(D^2u) = (u^*)^{-k}(-u)^l \quad \text{in} \quad U \subset \mathbb{R}^n, \\ u = 0 \quad \text{on} \quad \partial U, \end{cases} \tag{1.5}$$

where $l < 0$ and $k > 0$. When $k = 0$ and $l = -n-2$, Cheng-Yau [13] obtained the existence result for the equation (1.5). Then Le [33] extended the existence result to the case $l < 0$. When $l = -n-k-2$, the equation (1.5) was related to proper affine hyperspheres and Chen-Huang [12] showed the existence of solutions to the equation (1.5) in the space $C^\infty(\Omega) \cap C(\overline{\Omega})$ via the regularization method. Moreover, Le [33, 34] established the optimal global Hölder regularity of solutions.

The rest of the paper is organized as follows. In Section 2, we start with some preliminaries. The proofs of Theorem 1.1 are given in section 3. In section 4, the existence result and optimal global Hölder regularity for solutions to the equation (1.1) in the case $p < 1$ and $q \geq n$ are established. In the appendix, we establish some basic a priori estimates for the elliptic and parabolic Monge-Ampère equations.

2. Preliminaries

In this section, we collect the necessary background, preliminaries, and notations. More details can be found in [21, 37, 47] for convex bodies and in [49, 31, 43] for C-close convex sets.

2.1. **Convex bodies and their associated L_p dual Minkowski problem.** Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Denote by $\mathcal{K}_0^n$ the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. The support function $h : \mathbb{R}^n \to \mathbb{R}$ of a convex body K is defined as

$$h_K(x) = \max\{x \cdot y : y \in K\},$$

where $\cdot$ is the standard inner product in $\mathbb{R}^n$. The radial function ρ of $K \in \mathcal{K}_0^n$ is defined as

$$\rho_K(u) = \max\{\lambda > 0 : \lambda u \in K\}.$$

For a convex body K, its L_p surface area measure $S_p(K, \cdot)$ is defined by Lutwak [36],

$$S_p(K, \omega) = \int_{\nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} dx$$

for any Borel set $\omega \subset S^{n-1}$, where the set $\nu_K^{-1}(\omega)$ is the inverse image of ω under the Gauss map ν_K of K. If $p = 1$, it is just the surface area measure of K. Recently, Huang-LYZ in [21] proposed a fundamental family of geometric measures in the dual Brunn-Minkowski theory: the dual curvature measure which is defined by

$$\widetilde{C}_q(K, \omega) = \frac{1}{n} \int_{\alpha_K^*(\omega)} h_K^{-p}(\alpha_K(u)) \rho_K^q(u) du$$

for any Borel set $\omega \subset S^{n-1}$, where $\alpha_K^*(\omega)$ is the radial Gauss image of K given by

$$\alpha_K^*(\omega) = \{u \in S^{n-1} : u\rho_K(u) \in \nu_K^{-1}(\omega)\}.$$

Later, LYZ in [37] unified the L_p surface area measure and the dual curvature measure by introducing the L_p dual curvature measure

$$\widetilde{C}_{p,q}(K, \omega) = \frac{1}{n} \int_{\alpha_K^*(\omega)} h_K^{-p}(\alpha_K(u)) \rho_K^q(u) du$$

for any Borel set $\omega \subset S^{n-1}$. It is worth pointing out that the L_p-dual curvature measure becomes the L_p surface area measure for $q = n$ and the dual curvature measure for $p = 0$.

The following L_p dual Minkowski problem was posed in [37].

Problem 2.1. *For $p, q \in \mathbb{R}$, under what conditions on a non-zero finite Borel measure μ defined on S^{n-1}, can one find $K \in \mathcal{K}_0^n$ such that*

$$\mu = \widetilde{C}_{p,q}(K, \cdot)?$$

The L_p dual Minkowski problem becomes the L_p Minkowski problem for $q = n$ [36] and the dual Minkowski problem for $p = 0$ [21]. When the given measure μ has a density f, the L_p dual Minkowski problem is equivalent to solving the following Monge-Ampère type equation on S^{n-1}:

$$h^{1-p} \det(\nabla^2 h + hI) = f[|\nabla h|^2 + h^2]^{\frac{n-q}{2}}, \tag{2.1}$$

where f is a smooth function on S^{n-1}, h is the unknown function, I is the identity matrix, ∇h and $\nabla^2 h$ are the gradient and the Hessian of h on S^{n-1}.

2.2. **C-close convex sets and their associated L_p dual Minkowski problem.** A set $C \subseteq \mathbb{R}^n$ is said to be a closed convex cone, if C is closed and convex such that the interior of C is nonempty and $\lambda x \in C$ for all $x \in C$ and $\lambda \geq 0$. If $C \cap \{-x : x \in C\} = \{o\}$, then the closed convex cone C is called a pointed cone. For a pointed closed convex cone C, its polar cone is denoted by C° and defined by

$$C^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 0 \text{ for all } y \in C\}.$$

Let C be a pointed closed convex cone with nonempty interior and $A = C \backslash \mathbb{A}$ for any $\mathbb{A} \subsetneq C$. For a closed convex set $\mathbb{A} \subsetneq C$, if $0 < V_n(A) < \infty$, we call $\mathbb{A}$ a C-close set and A a C-coconvex set, while if A is bounded and nonempty, we call $\mathbb{A}$ a C-full set. Note that $o \notin \mathbb{A}$ if $\mathbb{A}$ is C-close or C-full.

Most concepts for convex bodies can be defined for C-close set (with slight or without changes). For example, the support function of a C-close set $\mathbb{A}$ can be defined by

$$h_C(\mathbb{A}, x) = \sup\{x \cdot y : y \in \mathbb{A}\}, \quad x \in \Omega_{C^\circ}, \tag{2.2}$$

where $\Omega_{C^\circ} = S^{n-1} \cap \operatorname{int} C^\circ$. Note that $o \notin \mathbb{A}$ and hence $-\infty < h_C(\mathbb{A}, x) < 0$ for any $x \in \Omega_{C^\circ}$. Let $\Omega_C = S^{n-1} \cap \operatorname{int} C$. The radial function of $\mathbb{A}$ is defined by

$$\rho_C(\mathbb{A}, u) = \sup\{r > 0 : ru \in C \backslash \mathbb{A}\}, \quad u \in \Omega_C.$$

At $u \in \Omega_C$, $\rho_C(\mathbb{A}, u)$ could be finite or ∞ depending on whether $\mathbb{A}$ intersects with ∂C at the direction u.

Lemma 2.2. *Assume that $\mathbb{A}$ is a C-full set, we have*

$$\max_{\Omega_{C^\circ}} |h_C(\mathbb{A}, \cdot)| = \min_{\Omega_C} \rho_C(\mathbb{A}, \cdot). \tag{2.3}$$

Proof. Assume that

$$\rho_C(\mathbb{A}, u_{min}) = \min_{\Omega_C} \rho_C(\mathbb{A}, \cdot), \quad h_C(\mathbb{A}, x_{min}) = \min_{\Omega_{C^\circ}} h_C(\mathbb{A}, \cdot).$$

On one hand, by the definition, we have

$$h_C(\mathbb{A}, x_{min}) \geq \rho_C(\mathbb{A}, u_{min}) u_{min} \cdot x_{min} \geq -\rho_C(\mathbb{A}, u_{min}).$$

Thus,

$$\max_{\Omega_{C^\circ}} |h_C(\mathbb{A}, \cdot)| \leq \min_{\Omega_C} \rho_C(\mathbb{A}, \cdot).$$

On the other hand, we have

$$|h_C(\mathbb{A}, x_{min})| \geq \rho_C(\mathbb{A}, x_{min}) \geq \min_{\Omega_C} \rho_C(\mathbb{A}, \cdot).$$

So, we complete the proof. □

Let $\mathbb{A}$ a C-close set. Assume $\partial\mathbb{A} \subsetneq \mathrm{int}C$ is smooth and strictly convex with $\lim_{x\to\partial\Omega_{C^\circ}} h_C(\mathbb{A}, x) = 0$. Clearly, $h_C(\mathbb{A}, x) \in (-\infty, 0)$ for all $x \in \Omega_{C^\circ}$ due to $o \notin \mathbb{A}$. In this case, $\partial\mathbb{A}$ can be determined by its radical function $\rho_C(\mathbb{A}, \cdot)$. If $x \in \Omega_{C^\circ}$ is the outer normal of $\partial\mathbb{A}$ at the point $u \in \partial\mathbb{A}$, then $u = h_C(\mathbb{A}, x)x + \nabla h_C(\mathbb{A}, x)$. This gives

$$\rho_C(\mathbb{A}, u) = \sqrt{|h_C(\mathbb{A}, x)|^2 + |\nabla h_C(\mathbb{A}, x)|^2}. \tag{2.4}$$

Moreover,

$$du = \rho_C^{-n}(-h_C)\det(\nabla^2 h_C + h_C I)dx. \tag{2.5}$$

Inspired by the L_p dual curvature measure introduced by LYZ for convex bodies [37], Ai-Yang-Ye [1] introduce the L_p dual curvature measure for a C-close set $\mathbb{A}$

$$\widetilde{C}_{p,q}(\mathbb{A}, \omega) = \frac{1}{n}\int_{\alpha^*_{\mathbb{A}}(\omega)} h_C^{-p}(\mathbb{A}, \alpha_{\mathbb{A}}(u))\rho_C^q(\mathbb{A}, u)du,$$

where ω is a Borel set in Ω_{C° and $\alpha^*_{\mathbb{A}}(\cdot)$ is the reverse radial Gauss image of $\mathbb{A}$ (see (4.6) in [31] for the definition). It is worth pointing out that $\widetilde{C}_{p,q}(\mathbb{A}, \cdot)$ is the L_p surface area measure for $q = n$ [49] and the q-th dual curvature measure for $p = 0$ [31]. Thus, the following L_p dual Minkowski problem for C-close sets is proposed in [1].

Problem 2.3. *For $p, q \in \mathbb{R}$, under what conditions on a nonzero finite Borel measure μ defined on Ω_{C°, can one find a C-close set $\mathbb{A}$ such that*

$$\mu = \widetilde{C}_{p,q}(\mathbb{A}, \cdot)?$$

Obviously, Problem 2.3 unifies the L_p Minkowski problem for C-close sets [49] and the dual Minkowski problem for C-close sets [31]. In particular, when the given measure μ has a density f, Problem 2.3 is equivalent to solving the Dirichlet problem (1.1) with $\Omega = \Omega_{C^\circ}$.

2.3. **The q-volume functional and the Sobolev type inequality.** If $\mathbb{A}$ is a C-full set, then $0 < \rho_C(\mathbb{A}, u) < \infty$ for any $u \in \mathbb{S}^{n-1} \cap \partial C$. The q-volume of $C \backslash \mathbb{A}$ or A is defined by

$$V_q(C\backslash\mathbb{A}) = \frac{1}{q}\int_{\Omega_C} \rho_C^q(\mathbb{A}, u)du. \tag{2.6}$$

When $q = n$, it is just the volume of A. Moreover, if $\partial\mathbb{A}$ is a smooth hypersurface, using (2.4) and (2.5), we have

$$V_q(C\backslash\mathbb{A}) = \frac{1}{q}\int_{\Omega_{C^\circ}} (|h_C|^2 + |\nabla h_C|^2)^{\frac{q-n}{2}}(-h_C)\det(\nabla^2 h_C + h_C I)dx. \tag{2.7}$$

Thus, we define the q-volume functional with respect with h_C

$$V_q(h_C) = \frac{1}{q}\int_{\Omega_{C^\circ}} (|h_C|^2 + |\nabla h_C|^2)^{\frac{q-n}{2}}(-h_C)\det(\nabla^2 h_C + h_C I)dx. \tag{2.8}$$

Now, we will calculate the first variation of V_q with respect with h_C. For convenience, we denote by $\Omega = \Omega_{C^\circ}$, $h(x) = h_C(\mathbb{A}, x)$ and $\rho(u) = \rho_C(\mathbb{A}, u)$.

Lemma 2.4. *Let $\mathbb{A}_t$ be a family of C-full sets with the support function $h(\cdot, t)$ satisfying $h(\partial\Omega, t) = 0$. We denote by $h = h(\cdot, 0)$ and $\varphi = \frac{d}{dt}\Big|_{t=0} h(\cdot, t)$. Then, the first variation of V_q at h with respect to φ is given by*

$$q \cdot \delta V_q(h)[\varphi] = -q\int_\Omega \rho^{q-n}\varphi\det(\nabla^2 h + hI)\ dx, \tag{2.9}$$

where $\rho = \sqrt{h^2 + |\nabla h|^2}$.

Proof. For convenience, we denote by $b = \nabla^2 h + hI$, $b_{ij} = h_{ij} + h\delta_{ij}$ and $(b^{ij}) = (b_{ij})^{-1}$. Then, using $(\log\det\, b)_k = b^{ij}b_{ijk}$, the first variation of V_q at h with respect to φ is given by

$$\begin{aligned}
& q \cdot \delta V_q(h)[\varphi] \\
= \ & q\frac{d}{dt}\Big|_{t=0} V_q(h(\cdot, t)) \\
= \ & -\int_\Omega \rho^{q-n}\varphi\det b\ dx - (q-n)\int_\Omega \rho^{q-n-2}(\nabla h \cdot \nabla\varphi + h\varphi)h\det b\ dx \\
& + \int_\Omega \rho^{q-n}(-h)b^{ij}(\varphi_{ij} + \varphi\delta_{ij})\det b\ dx.
\end{aligned}$$

Then, using the fact $(b^{ij}\det b)_j = 0$ and $h|_{\partial\Omega} = 0$, and integrating by parts give

$$\begin{aligned} & q \cdot \delta V_q(h)[\varphi] \\ = & -\int_\Omega \rho^{q-n}\varphi \det b\ dx - (q-n)\int_\Omega \rho^{q-n-2}(\nabla h\cdot\nabla\varphi + h\varphi)h\det b\ dx \\ & + \int_\Omega \rho^{q-n}(-h)\varphi b^{ij}\delta_{ij}\det b\ dx + \int_\Omega \rho^{q-n} h_j\varphi_i b^{ij}\det b\ dx \\ & +(q-n)\int_\Omega h\rho^{q-n-1}\rho_j\varphi_i b^{ij}\det b\ dx. \end{aligned}$$

Then, using $(b^{ij}\det b)_j = 0$ again and

$$b^{ij}\rho_j\varphi_i = \frac{1}{\rho}\nabla h\cdot\nabla\varphi, \quad b^{ij}\rho_j h_i = \frac{1}{\rho}|\nabla h|^2,$$

we have

$$\begin{aligned} & q \cdot \delta V_q(h)[\varphi] \\ = & -\int_\Omega \rho^{q-n}\varphi \det b\ dx - (q-n)\int_\Omega \rho^{q-n-2}h^2\varphi\det b\ dx \\ & + \int_\Omega \rho^{q-n}(-h)\varphi b^{ij}\delta_{ij}\det b\ dx + \int_\Omega \rho^{q-n} h_j\varphi_i b^{ij}\det b\ dx \\ = & -\int_\Omega \rho^{q-n}\varphi \det b\ dx - (q-n)\int_\Omega \rho^{q-n-2}h^2\varphi\det b\ dx \\ & + \int_\Omega \rho^{q-n}(-h)\varphi b^{ij}\delta_{ij}\det b\ dx - \int_\Omega \rho^{q-n} h_{ij}\varphi b^{ij}\det b\ dx \\ & -(q-n)\int_\Omega \rho^{q-n-1}\rho_i h_j\varphi b^{ij}\det b\ dx \\ = & -\int_\Omega \rho^{q-n}\varphi \det b\ dx - (q-n)\int_\Omega \rho^{q-n}\varphi\det b\ dx \\ & -(n-1)\int_\Omega \rho^{q-n}\varphi\det b\ dx \\ = & -q\int_\Omega \rho^{q-n}\varphi\det b\ dx. \end{aligned}$$

So, we complete the proof. □

In particular, for $q = 0$, we have

$$\delta\left(\int_\Omega \frac{(-h)\det(\nabla^2 h + hI)}{\rho^n}dx\right) = 0.$$

Thus,

$$\int_\Omega \frac{(-h)\det(\nabla^2 h + hI)}{\rho^n} dx = const.$$

Moreover, we have

Corollary 2.1. *If $\mathbb{A}$ is a C-full set with the support function h satisfying $h|_{\partial\Omega} = 0$, then we have*

$$\int_\Omega \frac{(-h)\det(\nabla^2 h + hI)}{\rho^n} dx = \mathrm{Area}(\Omega_C). \tag{2.10}$$

Proof. The equality (2.10) can be easily deduced by (2.6) and (2.7)

$$\int_\Omega \frac{(-h)\det(\nabla^2 h + hI)}{\rho^n} dx = \int_{\Omega_C} du = \mathrm{Area}(\Omega_C).$$

□

Using this corollary, we can easily deduce the following Sobolev type inequality for the q-volume functional.

Lemma 2.5. *Let $\mathbb{A}$ be a C-full set with the support function h satisfying $h|_{\partial\Omega} = 0$ and $q > 0$. Then,*

$$V_q(h) \geq \frac{\mathrm{Area}(\Omega_C)}{q} \parallel h \parallel^q_{C^0(\Omega)} .$$

Thus,

$$V_q(h) \geq \frac{\mathrm{Area}(\Omega_C)}{q\mathrm{Vol}(\Omega)} \int_\Omega |h|^q dx.$$

Proof. Using (2.3) and (2.10), we have

$$\begin{aligned} V_q(h) &= \frac{1}{q}\int_\Omega \rho^{q-n}(-h)\det(\nabla^2 h + hI)dx \\ &\geq \frac{\mathrm{Area}(\Omega_C)}{q}(\min_\Omega \rho)^q \\ &= \frac{\mathrm{Area}(\Omega_C)}{q} \parallel h \parallel^q_{C^0(\Omega)}, \end{aligned}$$

as claimed. □

3. The Dirichlet problem in the case $p \geq 1$

By Theorem 1.3 in [41] and Theorem 1.2 in [35], it is possible to obtain the existence of smooth solutions to the Dirichlet problem (1.1) for $p \geq 1$. We follow the ideas in [46] to find the variational functional of the Dirichlet problem (1.1). Then, we obtain the existence by using the corresponding parabolic gradient flow. In fact, the parabolic gradient flow method is widely used to prove the existence of smooth solutions to the Minkowski type problems, see [16, 17, 4, 27, 5, 12, 28, 7, 8, 9] and the references therein.

The argument of the Dirichlet problem (1.1) is divided into three cases:

(1) Subcritical case: $p < q$;

(2) Supercritical case: $p > q$;

(3) Critical case: $p = q$.

In this section, let Ω be an open, bounded, smooth and strictly convex domain in S^{n-1}, $f \in C^\infty(\overline{\Omega})$ with $f > 0$ and h_0 be the support function of a smooth C-full set with $h_0(\partial\Omega) = 0$.

3.1. **Subcritical case.**

3.1.1. *A parabolic gradient flow.* Since the original equation (1.1) becomes degenerate or singular at the boundary, we modify the original equation (1.1) by a perturbation

$$\begin{cases} [|\nabla h|^2 + h^2]^{\frac{q-n}{2}} \det(\nabla^2 h + hI) = (\varepsilon - h)^{p-1} f \quad \text{in} \quad \Omega, \\ h = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

The equation (3.1) is the Euler-Lagrange equation of the the functional

$$\mathcal{J}_\varepsilon(h) := V_q(h) - \frac{1}{p}\int_\Omega (\varepsilon - h)^p f(x)dx.$$

This fact can be easily seen by its variation

$$\delta\mathcal{J}_\varepsilon(h)[\varphi] = -\int_\Omega \varphi\Big[\rho^{q-n}\det(\nabla^2 h + hI) - (\varepsilon - h)^{p-1} f\Big]dx. \tag{3.2}$$

This variation (3.2) can be derived by (2.9).

In this subsection, we will study a gradient flow of the functional J_ε. In details, we consider the following parabolic equation with initial condition h_0:

(3.3)
$$\begin{cases} h_t - \log\det(\nabla^2 h + hI) + \dfrac{n-q}{2}\log[|\nabla h|^2 + h^2] = \log[(\varepsilon - h)^{1-p} f^{-1}] \quad \text{in} \quad \Omega_T, \\ h = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \\ h = h_0 \quad \text{on} \quad \Omega \times \{0\}, \end{cases}$$

where $\Omega_T = \Omega \times (0,T]$. By the first variation formula (3.2) of $\mathcal{J}_\varepsilon$, we can see that

Lemma 3.1. *$\mathcal{J}_\varepsilon$ is non-increasing along this flow* (3.3).

3.1.2. *The long time existence.* The short time existence can be guaranted by Theorem A in [46].

Theorem 3.2. *There exists a unique $T^\star$, $0 < T^\star \leq +\infty$, such that the flow* (3.3) *has a unique solution h which belongs to $C^{1,1}(\overline{\Omega_T}) \cap C^\infty(\overline{\Omega} \times (0,T])$ for all $T < T^\star$ and*

$$\mathcal{J}_\varepsilon(h(\cdot,t)) \leq \mathcal{J}_\varepsilon(h_0) \tag{3.4}$$

for all $t < T^\star$.

In order to get the long time existence, we first establish the a priori estimates. In this subsection, let $T < T^\star$ and $\mathbb{A}_t$ be a family of C-full sets with the support function $h(\cdot,t)$ which belongs to $C^{1,1}(\overline{\Omega_T}) \cap C^\infty(\overline{\Omega} \times (0,T])$ satisfying the flow (3.3). Moreover, by scaling, we choose h_0 such that $\mathcal{J}_\varepsilon(h_0) < \mathcal{J}_0(h_0) < 0$ for $q > p > 0$.

Lemma 3.3. *For $q > p \geq 1$, we have*

$$-C \leq h(x,t) < 0, \quad \forall\ (x,t) \in \Omega \times [0,T], \tag{3.5}$$

where C is independent of ε. Moreover,

$$\min_{\overline{\Omega}} \sqrt{|\nabla h|^2 + h^2}(\cdot,t) = \| h(\cdot,t) \|_{C^0(\overline{\Omega})} \geq \frac{1}{C}, \quad \forall\ t \in [0,T], \tag{3.6}$$

where C is independent of ε.

Proof. Using Lemma 2.5 and (3.4), we have

$$\mathcal{J}_\varepsilon(h_0) \geq \mathcal{J}_\varepsilon(h) \geq \frac{C}{q} \| h \|^q_{C^0(\overline{\Omega})} - \frac{C}{p} \| \varepsilon - h \|^p_{C^0(\overline{\Omega})} . \tag{3.7}$$

Hence, we can deduce if $q > p$

$$\| h \|_{C^0(\overline{\Omega})} \leq C.$$

Since $\mathcal{J}_\varepsilon(h_0) < \mathcal{J}_0(h_0) < 0$, it follows that

$$-C \geq \mathcal{J}_\varepsilon(h) \geq -\frac{1}{p}\int_\Omega (\varepsilon - h)^p f(x)dx.$$

Thus,

$$\frac{1}{p}\int_\Omega (\varepsilon - h)^p f(x)dx \geq C > 0,$$

which implies

$$\| h(\cdot, t) \|_{C^0(\overline{\Omega})} \geq \frac{1}{C}.$$

Then, (3.6) follows from the relation (2.3). So we complete the proof. $\square$

Theorem 3.4. *The flow* (3.3) *exists all the time for $q > p \geq 1$. Moreover, after choosing a subsequence, the flow converges to a non-zero and smooth solution to the equation* (3.1).

Proof. Using Lemma 5.1, we transform the flow (3.3) of h to that of u in $U \subset \mathbb{R}^{n-1}$

$$\begin{cases} \dfrac{u_t}{\sqrt{1+x^2}} - \log\det(D^2u) = -G(x,u,Du) \quad \text{in} \quad U \times (0,T], \\ u = 0 \quad \text{on} \quad \partial U \times [0,T], \\ u = u_0 \quad \text{on} \quad U \times \{0\}, \end{cases} \tag{3.8}$$

where

$$G(x,u,Du) = \frac{n-q}{2}\log[|Du|^2 + (x\cdot Du - u)^2] + \log[(\varepsilon\sqrt{1+|x|^2} - u)^{p-1}g].$$

Using the relation (5.2) and the inequality (3.6), we arrive

$$\Big[|Du|^2 + (x\cdot Du - u)^2\Big](x,t) \geq \frac{1}{C}, \quad \forall\ (x,t) \in U \times [0,T]. \tag{3.9}$$

Using (3.9) and the C^0 estimate (3.5), we obtain that the right part of the equation (3.8) satisfies

$$G(x,u,Du) \leq \frac{n-1}{2}\log(1+|Du|^2) + C, \quad \forall\ x \in \overline{U}.$$

Then, we can use Lemma 5.2 and Lemma 5.3 in Appendix to obtain the gradient estimates and C^2 estimates. Thus, we conclude that evolution equation (5.3) is uniformly parabolic on any finite time interval. Thus, the result of [29] and the standard parabolic theory show that the solution of (5.3) exists for all time. Using these estimates again, a subsequence of $h(\cdot,t)$ converges to a function h_∞. Since $\frac{d}{dt}\mathcal{J}_\varepsilon(h(\cdot,t)) \le 0$ for any $t > 0$ by Lemma 3.1, we have

$$\int_0^t \Big[-\frac{d}{dt}\mathcal{J}_\varepsilon(h(\cdot,t))\Big]dt = \mathcal{J}_\varepsilon(0) - \mathcal{J}_\varepsilon(t) \le \mathcal{J}_\varepsilon(0),$$

which implies that there exists a subsequence of times $t_j \to \infty$ such that

$$-\frac{d}{dt}\mathcal{J}_\varepsilon(h(\cdot,t_j)) \to 0 \quad \text{ as } \ t_j \to \infty.$$

That is to say

$$\begin{aligned}0 = \lim_{j\to+\infty} \quad & \int_\Omega \Big(\rho_j^{q-n}\det(\nabla^2 h_j + h_j I) - (\varepsilon - h_j)^{p-1} f\Big) \\ & \cdot\Big(\log\Big[\rho_j^{q-n}\det(\nabla^2 h_j + h_j I)\Big] - \log\Big[(\varepsilon - h_j)^{p-1} f\Big]\Big)dx,\end{aligned}$$

where we denote $h_j(x) = h(x,t_j)$ and $\rho_j(x) = \rho(x,t_j)$. From this and the a priori estimates of h_j, we conclude that h_∞ is a smooth solution to the equation (3.3). $\square$

3.1.3. *The Dirichlet problem.*

Theorem 3.5. *The Dirichlet problem* (1.1) *admits a unique and non-zero solution* $h \in C^\infty(\overline{\Omega})$ *for* $q > p \ge 1$. *Moreover, this solution is the minimum of the functional*

$$\mathcal{J}(h) := V_q(h) - \frac{1}{p}\int_\Omega (-h)^p f(x)dx.$$

Proof. Theorem 3.4 tells us that there exists a solution $h_\varepsilon \in C^\infty(\overline{\Omega})$ solving the Dirichlet problem (3.1). Using Lemma 5.1, we transform the Dirichlet problem (3.1) of h to that of u in $U \subset \mathbb{R}^{n-1}$. Then, there exists a solution $u_\varepsilon \in C^\infty(\overline{U})$ that solves the Dirichlet problem

(3.10)
$$\begin{cases}\det(D^2 u_\varepsilon) = [|Du_\varepsilon|^2 + (x\cdot Du_\varepsilon - u_\varepsilon)^2]^{\frac{n-q}{2}}(\varepsilon\sqrt{1+|x|^2} - u_\varepsilon)^{p-1} g \quad \text{in} \quad U,\\ u_\varepsilon = 0 \quad \text{on} \quad \partial U.\end{cases}$$

Using (3.5), we have

$$-C \le u_\varepsilon(x) < 0, \quad \forall\, x \in \Omega,$$

where the constant C is independent of ε. Using (3.9), the right hand term of the equation (3.10) satisfies

$$[|Du_\varepsilon|^2 + (x\cdot Du_\varepsilon - u_\varepsilon)^2]^{\frac{n-q}{2}}(\varepsilon\sqrt{1+|x|^2} - u_\varepsilon)^{p-1} g \le C[1 + |Du_\varepsilon|^2]^{\frac{n-1}{2}},$$

where the constant C is also independent of ε. The gradient estimate and interior estimates of all order can be deduced by Lemma 5.4 in Appendix. Thus, u_ε must converge along some subsequence to some $u \in C^\infty(U) \cap C^{0,1}(\overline{U})$ which is a solution to the following Dirichlet problem

$$\begin{cases} \det(D^2 u) = F(x, u, Du) \quad \text{in} \quad U, \\ u = 0 \quad \text{on} \quad \partial U, \end{cases} \tag{3.11}$$

where

$$F(x, u, Du) = [|Du|^2 + (x \cdot Du - u)^2]^{\frac{n-q}{2}} (-u)^{p-1} g.$$

By the argument of Theorem 1.3 in [41] and Theorem 1.2 in [35], we can derive that $u \in C^{2,\alpha}(\overline{U})$ for $p \geq 1$ and thus $u \in C^\infty(\overline{U})$ which is a solution to (3.11). Hence, the Dirichlet problem (1.1) admits a non-zero solution $h \in C^\infty(\overline{\Omega})$ for $q > p \geq 1$.

We follow the idea of the proof of Theorem 4.1 in [46] to show the uniqueness. Let us suppose that the Dirichlet problem (3.11) has two solutions u and v with $u - v$ being positive somewhere in U. Assume that the origin is contained in U and we define

$$u_\lambda(x) = u(\lambda^{-1} x)$$

for $\lambda > 1$ and let

$$\eta_\lambda(x) = \frac{-v(x)}{-u_\lambda(x)}.$$

There exist ε and $\lambda^\star$ such that

$$\eta_\lambda(x_\lambda) = \max_{\overline{U}} \eta_\lambda(x) \geq 1 + \varepsilon$$

for all $1 < \lambda \leq \lambda^\star$. At x_λ, we have

$$\frac{Dv}{v} = \frac{\lambda^{-1} Du(\lambda^{-1} x)}{u_\lambda}, \quad v_{ij} = \eta_\lambda (u_\lambda)_{ij} + (\eta_\lambda)_{ij} u_\lambda$$

and the matrix $(\eta_\lambda)_{ij}$ is non-positive definite. Therefore, we have at x_λ

$$\begin{aligned} F(x, v, Dv) &= \det(D^2 v) \\ &\geq \eta_\lambda^{n-1}(x) \lambda^{-2(n-1)} \det(D^2 u(\lambda^{-1} x)) \\ &= \eta_\lambda^{n-1}(x) \lambda^{-2(n-1)} F(\lambda^{-1} x, u(\lambda^{-1} x), Du(\lambda^{-1} x)). \end{aligned}$$

Letting $\lambda \to 1$, we conclude that $(1 + \varepsilon)^{p-q} \geq 1$ which yields a contradiction. Thus, the uniqueness follows.

Moreover, we know from the equality (3.7) that $\mathcal{J}(h) \to +\infty$ as $\| h \|_{C^0(\Omega)} \to +\infty$ for $q > p \geq 1$. This implies that the unique solution of the Dirichlet problem (1.1) is the minimum of the functional. $\square$

3.2. **Supercritical case.** We first solve the perturbed equation (3.1). Let

$$\mathcal{I}_\varepsilon(h) := V_q(h) - \frac{1}{p}\int_\Omega \Big[(\varepsilon - h)^p - \varepsilon^p\Big] f(x)dx,$$

which is just different from $\mathcal{J}_\varepsilon$ by a constant. Using Lemma 2.5, we have

$$\mathcal{I}_\varepsilon(h) \geq a \| h \|^q_{C^0(\Omega)} - b \| h \|^p_{C^0(\Omega)} - O(\varepsilon),$$

where a, b are constants depending on q, p and Ω. It follows that

$$\mathcal{I}_\varepsilon(h) \geq \frac{a}{2}\frac{p-q}{p}\Big(\frac{qa}{pb}\Big)^{\frac{q}{p-q}} \quad \text{for } \| h \|_{C^0(\Omega)} = \Big(\frac{qa}{pb}\Big)^{\frac{1}{p-q}}.$$

Set

$$\sigma := \Big(\frac{qa}{pb}\Big)^{\frac{1}{p-q}}, \quad \delta := \frac{a}{2}\frac{p-q}{p}\Big(\frac{qa}{pb}\Big)^{\frac{q}{p-q}},$$

and

$$\mathcal{C}_0 := \{h \in C^\infty(\overline{\Omega}) : \nabla^2 h + hI > 0 \text{ and } h < 0 \text{ in } \Omega,\ h|_{\partial\Omega} = 0\}. \tag{3.12}$$

By scaling of h, there exists $h_0, h_1 \in \mathcal{C}_0$ such that

$$\| h_0 \|_{C^0(\Omega)} < \sigma < \| h_1 \|_{C^0(\Omega)}, \quad \mathcal{I}_\varepsilon(h_0) < \delta, \quad \mathcal{I}_\varepsilon(h_1) < \delta.$$

Thus, the set

$$\mathcal{P} := \{\gamma : [0,1] \mapsto \mathcal{C}_0 : \| \gamma(0) \|_{C^0(\Omega)} < \sigma < \| \gamma(1) \|_{C^0(\Omega)},\ \mathcal{I}_\varepsilon(\gamma(0)) < \delta,\ \mathcal{I}_\varepsilon(\gamma(1)) < \delta\}$$

is nonempty and

$$c = \inf_{\gamma \in \mathcal{P}} \sup_{s \in [0,1]} \mathcal{I}_\varepsilon(\gamma(s)) \geq \delta > 0.$$

We will show that c is a critical value of $\mathcal{I}_\varepsilon$ which is attained by some $h \in \mathcal{C}_0$.

Theorem 3.6. *For $p > q \geq n$, the Dirichlet problem* (3.1) *admits a non-zero solution $h \in C^\infty(\overline{\Omega})$ with $\mathcal{I}_\varepsilon(h) = c$.*

Proof. The proof follows from a mountain-pass lemma as [46]. For $0 < \sigma < 1$, pick a path $\gamma \in \mathcal{P}$ such that

$$\mathcal{I}_\varepsilon(\gamma) = \sup_{s \in [0,1]} \mathcal{I}_\varepsilon(\gamma(s)) < c + \sigma.$$

Then, we have

Lemma 3.7. *If $q \geq n$, the parabolic equation* (3.3) *with initial data $h(x, 0, s) := \gamma(s)$ has a solution $\gamma(t, s) := h(x, t, s)$ for all $t \geq 0$.*

Proof. Since the equivalence of the flow (3.3) and the flow (3.8), it sufficient to prove the long time existence of the flow (3.8) for $q \geq n$. Let $v : B_r(0) \subset \Omega \to \mathbb{R}$ solving

$$\begin{cases} \det(D^2 v) = [|Dv|^2 + (x \cdot Dv - v)^2]^{\frac{n-q}{2}} \varepsilon^{p-1} \inf\limits_{B_r(0)} g \quad \text{in} \quad B_r(0), \\ v = 0 \quad \text{on} \quad \partial B_r(0). \end{cases}$$

Then, v is a supersolution of (3.8). Since $q \geq n$, the comparison principle (see Theorem 14.1 in [32]) tells us that it holds for any solution $u(x, t)$ to the equation (3.8)

$$|u(0, t)| \geq |v(0)|.$$

Thus, we have by the convexity of u

$$x \cdot Du - u \geq \inf_U [x \cdot Du - u] = -u(0, t) \geq |v(0)| > 0. \tag{3.13}$$

Thus,

$$|G_u| = \Big| \frac{q-n}{2} \frac{2(x \cdot Du - u)}{|Du|^2 + (x \cdot Du - u)^2} + \frac{1-p}{\varepsilon\sqrt{1+|x|^2} - u} \Big| \leq C.$$

Using the maximal principle to the evolution equation of u_t, we have $|u_t| \leq Ce^{Ct}$. Thus,

$$-C(T) \leq u(x, t) < 0, \quad \forall\ (x, t) \in \overline{U} \times [0, T]$$

and

$$G(x, u, Du) \leq \frac{n-1}{2} \log(1 + |Du|^2) + C(T), \quad \forall\ x \in \overline{U}.$$

The above two inequalities imply the assumptions (5.4) and (5.5) are satisfied. Hence, we can use Lemma 5.2 and Lemma 5.3 to conclude that the evolution equation (3.8) is uniformly parabolic on $[0, T]$. Thus, the result of [29] and the standard parabolic theory show that the solution of (3.8) exists for all time. □

Obviously, $\gamma(t, s)$ belongs to $\mathcal{P}$ for each $t \geq 0$. Now for each $s \in [0, 1]$, we define

$$t^*(s) := \sup\{t \geq 0 : \mathcal{I}_\varepsilon(\gamma(t, s)) \geq c - \sigma\}$$

and set $t^*(s) = 0$ if $\mathcal{I}_\varepsilon(\gamma(t, s)) < c - \sigma$ for all t.

Lemma 3.8. *There exists $s_0 \in [0, 1]$ such that $t^*(s_0) = +\infty$.*

Proof. We first show that t^* cannot have a uniform upper bound, say $T < +\infty$. Otherwise, there exists $T < +\infty$ such that

$$\gamma(T, s) \in \mathcal{P} \quad \text{and} \quad \sup_s \mathcal{I}_\varepsilon(\gamma(T, s)) \leq c - \sigma.$$

This will contradict with the definition of c. Therefore, there exists a sequence $\{s_k\}$ with

$$\lim_{k \to +\infty} s_k = s_0 \quad \text{and} \quad \lim_{k \to +\infty} t^*(s_k) = +\infty.$$

We want to prove that $t^*(s_0) = +\infty$. If this is not true, we claim $\gamma(t^*(s_0), s_0)$ is actually a solution of (3.1). Indeed, if $\frac{d}{dt}\mathcal{I}_\varepsilon$ does not vanish at $\gamma(t^*(s_0), s_0)$, we can find $t' > t^*(s_0)$ such that

$$\mathcal{I}_\varepsilon(\gamma(t', s_0)) < c - \sigma.$$

However,

$$\lim_{k \to +\infty} \mathcal{I}_\varepsilon(\gamma(t', s_k)) = \mathcal{I}_\varepsilon(\gamma(t', s_0))$$

implies that for k large enough

$$\mathcal{I}_\varepsilon(\gamma(t', s_k)) < c - \sigma.$$

Thus, $t^*(s_k) < t'$ which yields a contradiction by letting k go to infinity. Thus, $\gamma(t^*(s_0), s_0)$ is actually a solution of (3.1). So, $\gamma(t^*(s_0), s_0)$ also solves (3.3) with itself as initial datum. Hence, $t^*(s_0) = +\infty$ which is a contradiction. Therefore, $t^*(s_0) = +\infty$. □

It follows that there exists a long time solution $h(x, t) := \gamma(t, s_0)$ to the flow (3.3) such that for all $t > 0$

$$\mathcal{I}_\varepsilon(h(\cdot, t)) \geq c - \sigma.$$

Then, using the monotonicity of $\mathcal{I}_\varepsilon(h(\cdot, t))$ along the parabolic flow (3.3), for any $\sigma > 0$, we can choose T sufficiently large such that

$$\int_T^{+\infty} \Big(-\frac{d}{dt} I_\varepsilon(h(\cdot, t)) \Big) dt = \int_T^{+\infty} \int_\Omega (A - B)(\log A - \log B) dx dt \leq 2\sigma, \tag{3.14}$$

where

$$A = [|\nabla h|^2 + h^2]^{\frac{q-n}{2}} \det(\nabla^2 h + hI), \quad B = (\varepsilon - h)^{p-1} f.$$

Lemma 3.9. *We have for* $t \geq 0$

$$\| h(\cdot,t) \|_{C^0(\Omega)} < C. \tag{3.15}$$

Proof. Using (3.14) and the mean value theorem, we know that for every interval $[k, k+1]$ with k large enough, there exists $t_k \in [k, k+1]$ such that

$$\int_\Omega (A(t_k) - B(t_k))(\log A(t_k) - \log B(t_k))dx \leq 2\sigma. \tag{3.16}$$

From now on, we will fix such a t_k and assume all quantities are evaluated at the chosen time t_k and suppress the dependence on t_k.

Let $\alpha > 0$ be given by $e^{-\alpha} = 1 - \frac{p-q}{2p}$, and define $S \subset \Omega$ to be the set

$$S := \{x : |\log A(x) - \log B(x)| \leq \alpha\}.$$

Then, we get from (3.16)

$$\begin{aligned} 2\sigma &\geq \int_{\Omega\setminus S} (A-B)(\log A - \log B)dx \\ &\geq \alpha \int_{\Omega\setminus S} |A-B|dx \\ &\geq \alpha(1-e^{-\alpha}) \int_{\Omega\setminus S} Bdx \\ &\geq \alpha(1-e^{-\alpha})\varepsilon|\Omega \setminus S| \inf_\Omega f, \end{aligned} \tag{3.17}$$

where we used $|\frac{A}{B} - 1| \geq 1 - e^{-\alpha}$ for all $x \in \Omega \setminus S$ to get the last inequality. Thus,

$$|\Omega \setminus S| \leq \frac{2\sigma}{\alpha(1-e^{-\alpha})\varepsilon \inf_\Omega f}.$$

Choosing σ small enough such that

$$|\Omega \setminus S| \leq \frac{1}{2}|\Omega|. \tag{3.18}$$

Note that

$$\mathcal{J}_\varepsilon(h) = \int_S \left[\frac{(-h)A}{q} - \frac{(\varepsilon - h)B}{p}\right]dx + \int_{\Omega\setminus S} \left[\frac{(-h)A}{q} - \frac{(\varepsilon - h)B}{p}\right]dx. \tag{3.19}$$

We first estimate the first term in (3.19) by using the fact $\frac{A}{B} \geq e^{-\alpha}$ for all $x \in S$

$$\begin{aligned}
& \int_S \left[\frac{(-h)A}{q} - \frac{(\varepsilon - h)B}{p}\right] dx \\
\geq\ & \int_S \left[\frac{(-h)(A-B)}{q} + \frac{(p-q)(-h)B}{pq}\right] dx - C\varepsilon \parallel h \parallel_{C^0(\Omega)}^{p-1} \\
\geq\ & -\int_S \left[\frac{(1-e^{-\alpha})(-h)B}{q} + \frac{(p-q)(-h)B}{pq}\right] dx - C\varepsilon \parallel h \parallel_{C^0(\Omega)}^{p-1} \\
\geq\ & \frac{(p-q)}{2pq} \int_S (-h) B dx - C\varepsilon \parallel h \parallel_{C^0(\Omega)}^{p-1} \\
\geq\ & \frac{(p-q)}{2pq} \int_S (-h)^p f dx - C\varepsilon \parallel h \parallel_{C^0(\Omega)}^{p-1} . \qquad (3.20)
\end{aligned}$$

From the fact (see Page 439 in [46])

$$\lim_{R \to +\infty} \frac{1}{|U|} \inf \left| \left\{ x \in U : D^2 u \geq 0,\ u(x) \leq -R/2,\ \parallel u \parallel_{C^0(U)} = R \right\} \right| = 1,$$

Using the relation of h and u in Lemma 5.1, we also have

$$\lim_{R \to +\infty} \frac{1}{|\Omega|} \inf \left| \left\{ x \in \Omega : D^2 h + hI \geq 0,\ h(x) \leq -R/2,\ \parallel h \parallel_{C^0(\Omega)} = R \right\} \right| = 1.$$

Then, we conclude from the above fact and (3.18) that there exists $R_1 > 0$ such that

$$\Big| S \cap \{x \in \Omega : h(x) \leq -R_1/2\} \Big| \geq \frac{1}{4} |\Omega| \quad \text{for} \quad \parallel h \parallel_{C^0(\Omega)} \geq R_1.$$

Let $E := \{x \in \Omega : h(x) \leq - \parallel h \parallel_{C^0(\Omega)} /2\}$. If $\parallel h \parallel_{C^0(\Omega)} \geq R_1$, we have by (3.20)

$$\begin{aligned}
& \int_S \left[\frac{(-h)A}{q} - \frac{(\varepsilon - h)B}{p}\right] dx \\
\geq\ & \frac{p-q}{2pq} \int_{S \cap E} (-h)^p f dx - C\varepsilon \parallel h \parallel_{C^0(\Omega)}^{p-1} \\
\geq\ & \frac{(p-q) \inf_\Omega f}{2^{p+3} pq |\Omega|} \parallel h \parallel_{C^0(\Omega)}^{p} - C\varepsilon \parallel h \parallel_{C^0(\Omega)}^{p-1} . \qquad (3.21)
\end{aligned}$$

For the second term in (3.19), we obtain from (3.17)

$$\begin{aligned}
&\int_{\Omega\setminus S}\left[\frac{(-h)A}{q}-\frac{(\varepsilon-h)B}{p}\right]dx\\
\geq\ & \frac{1}{q}\int_{\Omega\setminus S}(-h)Adx-\frac{1}{p}\int_{\Omega\setminus S}(-h)Bdx-C\varepsilon\parallel h\parallel_{C^0(\Omega)}^{p-1}\\
\geq\ & -\frac{1}{p}\parallel h\parallel_{C^0(\Omega)}\int_{\Omega\setminus S}Bdx-C\varepsilon\parallel h\parallel_{C^0(\Omega)}^{p-1}\\
\geq\ & -\frac{2\sigma}{p\alpha(1-e^{-\alpha})}\parallel h\parallel_{C^0(\Omega)}-C\varepsilon\parallel h\parallel_{C^0(\Omega)}^{p-1}. \qquad (3.22)
\end{aligned}$$

Combining (3.21) and (3.22), it yields that

$$c+\sigma\geq\mathcal{I}_\varepsilon(h)\geq\frac{p-q}{2^{p+3}pq|\Omega|}\parallel h\parallel_{C^0(\Omega)}^{p}-\frac{\sigma}{p\alpha(1-e^{-\alpha})}\parallel h\parallel_{C^0(\Omega)}-C\varepsilon\parallel h\parallel_{C^0(\Omega)}^{p-1}.$$

Thus, $\parallel h(\cdot,t_k)\parallel_{C^0(\Omega)}\leq C$. We know from (3.14)

$$\int_T^{+\infty}\int_\Omega B(e^{h_t}-1)h_t dxdt\leq 2\sigma.$$

Using the simple facts $-\frac{x}{2}\leq(e^x-1)x$ for $x\leq-1$ and $\frac{1}{3}x^2\leq(e^x-1)x$ for $-1\leq x\leq 0$, we have

$$\begin{aligned}
&\frac{1}{p}\int_\Omega[\varepsilon-h(t)]^p f dx\\
=\ & \frac{1}{p}\int_\Omega[\varepsilon-h(t_k)]^p f dx-\int_{t_k}^t\int_\Omega Bh_t dxdt\\
\leq\ & C-\int_{t_k}^t\int_{\{h_t\leq-1\}}Bh_t dxdt-\int_{t_k}^t\int_{\{0>h_t>-1\}}Bh_t dxdt\\
\leq\ & C+2\int_{t_k}^t\int_{\{h_t\leq-1\}}Bh_t(e^{h_t}-1)dxdt-\int_{t_k}^t\int_{\{0>h_t>-1\}}Bh_t dxdt\\
\leq\ & C+4\sigma+\left(\int_{t_k}^t\int_{\{0>h_t>-1\}}Bdxdt\right)^{\frac{1}{2}}\left(\int_{t_k}^t\int_{\{0>h_t>-1\}}Bh_t^2 dxdt\right)^{\frac{1}{2}}\\
\leq\ & C+4\sigma+\left(\int_{t_k}^t\int_\Omega Bdxdt\right)^{\frac{1}{2}}\left(3\int_{t_k}^t\int_{\{0>h_t>-1\}}B(e^{h_t}-1)h_t dxdt\right)^{\frac{1}{2}}\\
\leq\ & C+4\sigma+\sqrt{6\sigma}\left(\int_{t_k}^t\int_\Omega Bdxdt\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\frac{\varepsilon}{p}\int_\Omega B(t)dx \le C + 4\sigma + \sqrt{6\sigma}\bigg(\int_{t_k}^t \int_\Omega B dx dt\bigg)^{\frac{1}{2}}.$$

Set

$$M_k = \max\bigg\{\int_\Omega B(t)dx : t \in [t_k, t_{k+1}]\bigg\}.$$

Then,

$$\frac{\varepsilon}{p}M_k \le C + 4\sigma + \sqrt{6\sigma}M_k^{\frac{1}{2}}.$$

This implies an upper bound on M_k if we choose σ small enough. Then, a uniform upper bound on $\| h(\cdot,t) \|_{C^0(\Omega)}$ follows from the convexity. □

From the gradient estimate (3.13) and the C^0 estimate (3.15), we obtain that the right part of the equation in (3.8) satisfies

$$G(x,u,Du) \le \frac{n-1}{2}\log(1+|Du|^2) + C, \quad \forall\ x \in \overline{U}.$$

Then, we can use Lemma 5.2 and Lemma 5.3 in Appendix to conclude that the evolution equation (5.3) is uniformly parabolic on any finite time interval. Thus, the result of [29] and the standard parabolic theory show that the solution of (5.3) exists for all time. By these estimates again, a subsequence of $h(\cdot,t)$ converges to a function h_σ satisfies $c \le \mathcal{I}_\varepsilon(h_\sigma) \le c+\sigma$. Let $\sigma \to 0$, h_σ converges to a function h that satisfies $\mathcal{I}_\varepsilon(h) = c$. □

Theorem 3.10. *The Dirichlet problem* (1.1) *admits a non-zero solution $h \in C^\infty(\overline{\Omega})$ for $p > q \ge n$.*

Proof. Using Theorem 3.6, there exists a solution h_ε with $\mathcal{I}_\varepsilon(h_\varepsilon) = c$ solving (3.1). Hence,

$$\mathcal{I}_\varepsilon(h_\varepsilon) = \frac{1}{q}\int_\Omega (-h_\varepsilon)(\varepsilon - h_\varepsilon)^{p-1} f dx - \frac{1}{p}\int_\Omega [(\varepsilon - h_\varepsilon)^p - \varepsilon^p] f dx.$$

Since $\delta \le \mathcal{I}_\varepsilon(h_\varepsilon) \le C$, we have

$$C^{-1} \le \| h_\varepsilon \|_{C^0(\Omega)} \le C,$$

where the constant C is independent of ε. Following the same argument in Theorem 3.5, h_ε must converge along some subsequence to a non-zero function $h \in C^{2,\alpha}(\overline{\Omega})$ and thus $h \in C^\infty(\overline{\Omega})$ which is a solution to the Dirichlet problem (1.1). □

3.3. **Critical case.** In this subsection, we use the variational method to study the solvability of the equation (1.1) in the case $p = q \geq 1$.

Theorem 3.11. *Let $p \geq 1$, there exists a constant $\lambda > 0$ such that the Dirichlet problem*

$$\begin{cases} \det(\nabla^2 h + hI) = \lambda f(-h)^{p-1}[|\nabla h|^2 + h^2]^{\frac{n-p}{2}} \quad in \quad \Omega, \\ h = 0 \quad on \quad \partial\Omega \end{cases} \tag{3.23}$$

admits a non-zero solution $h \in C^\infty(\overline{\Omega})$. Moreover, if (λ_1, h_1) is another pair of such solution, then $\lambda = \lambda_1$ and there exists a constant c such that $h_1 = ch$.

Proof. We divide the proof into three steps.

Step 1: Existence of solutions to the equation (3.23) in the case $p > 1$.

Let $s \in (1, p)$, we consider a family of equations depending on s:

$$\begin{cases} \det(\nabla^2 h + hI) = \lambda f(-h)^{s-1}[|\nabla h|^2 + h^2]^{\frac{n-p}{2}} \quad \text{in} \quad \Omega, \\ h = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.24}$$

where λ is an invariant defined by

$$\lambda(\Omega) := \inf_{h \in \mathcal{C}_0} \left\{ \frac{pV_p(h)}{\int_\Omega (-h)^p f(x)dx} \right\}$$

and see (3.12) for the definition of $\mathcal{C}_0$. Clearly, $\lambda(\Omega) > 0$ by Lemma 2.5.

Note that (3.24) is the Euler-Lagrange equation of the functional

$$\mathcal{J}_s(h) := V_p(h) - \frac{\lambda}{s} \int_\Omega (-h)^s f(x)dx.$$

By Theorem 3.5, the equation (3.24) admits a unique solution h_s which minimizes the functional $\mathcal{J}_s(h)$. Now we consider the sequence of rescaled solutions

$$v_s := \frac{h_s}{\|h_s\|_{C^0(\Omega)}}.$$

Then,

$$\|h_s\|_{C^0(\Omega)}^{p-s} = \frac{\lambda \int_\Omega (-v_s)^s f(x)dx}{pV_p(v_s)}.$$

By Hölder inequality and the definition of λ,

$$
\begin{aligned}
\|h_s\|_{C^0(\Omega)}^{p-s} &\leq \frac{\int_\Omega (-v_s)^s f(x)dx}{\int_\Omega (-v_s)^p f(x)dx} \\
&\leq \frac{\left(\int_\Omega (-v_s)^p f(x)dx\right)^{\frac{s}{p}} \left(\int_\Omega f dx\right)^{\frac{p-s}{p}}}{\int_\Omega (-v_s)^p f(x)dx} \\
&= \frac{\left(\int_\Omega f dx\right)^{\frac{p-s}{p}}}{\left(\int_\Omega (-v_s)^p f(x)dx\right)^{\frac{p-s}{p}}}.
\end{aligned}
$$

Thus

$$
\|h_s\|_{C^0(\Omega)} \leq A^{-\frac{1}{p}} \left(\int_\Omega f dx\right)^{\frac{1}{p}},
$$

where $A = \inf\{\int_\Omega (-h)^p f(x)dx : h \in \mathcal{C}_0, \|h\|_{C^0(\Omega)} = 1\}$ is a constant depending on p, f and Ω.

According to the definition of λ, there exists a function $\tilde{h} \in \mathcal{C}_0$ such that $\|\tilde{h}\|_{C^0(\Omega)} = 1$ and

$$
\left(\frac{\lambda \int_\Omega (-\tilde{h})^p f(x)dx}{pV_p(\tilde{h})}\right)^{\frac{1}{p-s}} \geq \frac{1}{2}.
$$

Then

$$
a := \left(\frac{\lambda \int_\Omega (-\tilde{h})^s f(x)dx}{pV_p(\tilde{h})}\right)^{\frac{1}{p-s}} \geq \frac{1}{2}.
$$

On one hand, we know that from the equation (3.24)

$$
\mathcal{J}_s(h_s) = \frac{s-p}{sp}\lambda \int_\Omega (-h_s)^s f(x)dx.
$$

On the other hand,

$$
\begin{aligned}
\mathcal{J}_s(h_s) &\leq \mathcal{J}_s(a\tilde{h}) \\
&= \frac{a^p}{p} pV_p(\tilde{h}) - \frac{\lambda a^s}{s} \int_\Omega (-\tilde{h})^s f(x)dx \\
&= \frac{s-p}{sp} a^s \lambda \int_\Omega (-\tilde{h})^s f(x)dx \\
&\leq \frac{s-p}{sp} 2^{-s} \lambda \int_\Omega (-\tilde{h})^p f(x)dx < 0.
\end{aligned}
$$

It implies that

$$
\|h_s\|_{C^0(\Omega)}^s \int_\Omega f(x)dx \geq \int_\Omega (-h_s)^s f(x)dx \geq 2^{-s} A.
$$

Thus,

$$\|h_s\|_{C^0(\Omega)} \geq 2^{-1}\left(\frac{A}{\int_\Omega f(x)dx}\right)^{\frac{1}{s}} \geq 2^{-1}\left(\frac{A}{\int_\Omega f(x)dx}\right).$$

Therefore, h_s has a uniform upper bound and a uniform positive lower bound. Then, by the argument in the proof of Theorem 3.5, h_s converges to a smooth solution of the equation (3.23) in the case $p > 1$.

Step 2: Existence of solutions to the equation (3.23) in the case $p = 1$.

Let $p_\epsilon = 1 + \epsilon$. From Step 1, we know that there exist constants λ_ϵ and a non-zero function h_ϵ satisfying the equation

$$\begin{cases} \det(\nabla^2 h + hI) = \lambda_\epsilon f(-h)^{p_\epsilon - 1}[|\nabla h|^2 + h^2]^{\frac{n-p_\epsilon}{2}} \quad \text{in} \quad \Omega, \\ h = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

There is no loss of generality in assuming $\|h_\epsilon\|_{C^0} = 1$. It is clear that $\lambda_\epsilon \leq C$ uniformly in ϵ. Then, using the same argument in the proof of Step 1, we conclude that there is a nonzero solution $h \in C^\infty(\bar{\Omega})$ to the equation (3.23) in the case $p = 1$.

Step 3: Uniqueness of solutions to the equation (3.23).

Suppose that (λ_1, h_1) and (λ_2, h_2) are two pairs which solve the equation (3.23). Then, using Lemma 5.1, the transform of h_i $(i = 1, 2)$, denoted by u_i, satisfies the following equation

$$\begin{cases} \det(D^2 u_i) = \lambda_i g(x)(-u_i)^{p-1}[|Du_i|^2 + (x \cdot Du_i - u_i)^2]^{\frac{n-p}{2}} \quad \text{in} \quad U, \\ u_i = 0 \quad \text{on} \quad \partial U. \end{cases} \tag{3.25}$$

Without loss of generality, we can assume that $\lambda_1 \leq \lambda_2$. Since u_i is convex in U,

$$\frac{\partial u_i}{\partial \nu} > 0 \quad \text{on} \quad \partial U,$$

where ν is the unit outward normal to ∂U. Thus, for some $t > 0$ small, we have

$$0 \leq t(-u_2) \leq -u_1 \quad \text{on} \quad \overline{U}.$$

Thus,

$$t_0 := \sup\{t > 0 : t(-u_2) \leq -u_1 \text{ on } \overline{U}\} > 0.$$

Note that any scaling (λ_2, tu_2) also solves equation (3.25) for any $t > 0$, therefore we can replace u_2 by its scalings $t_0 u_2$. We have $u_2 - u_1 \geq 0$ in U. We can divide the proof into two cases.

Case 1: $u_2 \equiv u_1$ in U, it is easy to check that $\lambda_1 = \lambda_2$.

Case 2: there exists a point $x_0 \in U$ such that $u_2 - u_1 > 0$ at x_0. We can apply similar method in the proof of uniqueness in Theorem 3.5 to show that $\lambda_1 \geq \lambda_2$. Then our theorem is now proved. $\square$

4. The Dirichlet problem in the case $p < 1$

In this section, we establish the existence and optimal global Hölder regularity for solutions to the Dirichlet problem (1.1) in the case $p < 1$. Using Lemma 5.1, we only need to consider the equation (1.2).

First, we introduce the following comparison principle.

Lemma 4.1. *Let $U \subset \mathbb{R}^{n-1}$ be a bounded convex domain, $p < 1$ and $q \geq n$. Assume that $u, v \in C^2(U) \cap C^0(\overline{U})$ are convex functions such that*

$$0 \geq u \geq v \quad on \quad \partial U,$$

and

$$\det(D^2u) \leq g(x)(-u)^{p-1}\left[|Du|^2 + (x \cdot Du - u)^2\right]^{\frac{n-q}{2}} \quad in \quad U,$$
$$\det(D^2v) \geq g(x)(-v)^{p-1}\left[|Dv|^2 + (x \cdot Dv - v)^2\right]^{\frac{n-q}{2}} \quad in \quad U.$$

Then $u \geq v$ in U.

Proof. Assume $u - v$ attains its minimum value at $x_0 \in U$ with $u(x_0) < v(x_0) < 0$, then

$$Du(x_0) = Dv(x_0), \quad D^2u(x_0) \geq D^2v(x_0),$$

which implies at x_0

$$\left[|Dv|^2 + (x \cdot Dv - v)^2\right]^{\frac{n-q}{2}} > \left[|Du|^2 + (x \cdot Du - u)^2\right]^{\frac{n-q}{2}}$$

and

$$(-v)^{p-1}\left[|Dv|^2 + (x \cdot Dv - v)^2\right]^{\frac{n-q}{2}} \leq (-u)^{p-1}\left[|Du|^2 + (x \cdot Du - u)^2\right]^{\frac{n-q}{2}}.$$

Thus we easily find $(-v)^{p-1}(x_0) \leq (-u)^{p-1}(x_0)$, which contradicts $u(x_0) < v(x_0)$ if $p < 1$. $\square$

Lemma 4.2. *Let $U \subset \mathbb{R}^{n-1}$ be a bounded convex domain, $\epsilon > 0$, $p < 1$ and $q \geq n$. Assume that $u \in C^2(U) \cap C^0(\overline{U})$ is a convex solution to the equation*

$$\begin{cases} \det(D^2u) = g(x)(\epsilon - u)^{p-1}\left[|Du|^2 + (x \cdot Du - u)^2\right]^{\frac{n-q}{2}} \quad in \quad U, \\ u = 0 \quad on \quad \partial U. \end{cases}$$

There exists a constant $\epsilon_0(n,p,q,g)$ *such that*

$$\|u\|_{C^0(\overline{U})} \geq c(n,p,q,\inf g)\,(|U|^*)^{\frac{1}{q-p}}$$

if $\epsilon < \epsilon_0(n,p,q,g)$, *where* $|U|^* := \min\{|U|^2, |U|^{\frac{n+q-2}{n-1}}\}$.

Proof. The proof is similar to that of Lemma 2.3 in [33]. There is no loss of generality in assuming U is normalized, i.e., there exists a constant R such that

$$B_R \subset U \subset B_{(n-1)R}.$$

Let $s = \|u\|_{C^0(U)}$ and $v = \frac{u}{s}$, then $v \in C^2(U)\cap C^0(\overline{U})$ is a convex solution to

$$\begin{cases} \det(D^2 v) = s^{1-q} g(x)(\epsilon - sv)^{p-1}\left[|Dv|^2 + (x\cdot Dv - v)^2\right]^{\frac{n-q}{2}} \quad \text{in} \quad U, \\ v = 0 \quad \text{on} \quad \partial U \end{cases}$$

with $\|v\|_{C^0(\overline{U})} = 1$. It follows that

$$\begin{aligned} \frac{g(x)}{(s+\epsilon)^{1-p}} &\leq \det(D^2 v) s^{q-1}\left[|Dv|^2 + (x\cdot Dv - v)^2\right]^{\frac{q-n}{2}} \\ &\leq s^{q-1}\det(D^2 v)\left[\left(1+R^2/2\right)|Dv|^2 + 2\right]^{\frac{q-n}{2}} \end{aligned}$$

for any $x \in B_{R/2}$. Integrating both sides over $B_{R/2}$ and using area formula, we have

$$\begin{aligned} \frac{\int_{B_{R/2}} g(x)dx}{(s+\epsilon)^{1-p}} &\leq s^{q-1}\int_{B_{R/2}} \det(D^2 v)\left[\left(1+R^2/2\right)|Dv|^2 + 2\right]^{\frac{q-n}{2}} dx \\ &= s^{q-1}\int_{Dv(B_{R/2})} \left[\left(1+R^2/2\right)|y|^2 + 2\right]^{\frac{q-n}{2}} dy. \end{aligned} \tag{4.1}$$

Note that $v \in C^0(\overline{U})$ is convex with $v = 0$ on ∂U, it is easy to see that

$$|Dv(x)| \leq \frac{|v(x)|}{\operatorname{dist}(x,\partial U)} \leq \frac{2}{R} \tag{4.2}$$

for any $x \in B_{R/2}$. Substituting (4.2) into (4.1) yields

$$s^{q-1}(s+\epsilon)^{1-p} \geq \frac{\int_{B_{R/2}} g(x)dx}{(4+\frac{4}{R^2})^{\frac{q-n}{2}}(\frac{2}{R})^{n-1}|B_1|} \geq c_1(n,p,q,\inf g)|U|^*, \tag{4.3}$$

which implies that

$$s^{q-p}2^{1-p}\max\{1, \epsilon_0/s\}^{1-p} \geq c_1(n,p,q,\inf g)|U|^*. \tag{4.4}$$

By choosing $\epsilon_0 = \left(\frac{c_1(n,p,q,\inf g)|U|^*}{2^{2-p}}\right)^{\frac{1}{q-p}}$, we obtain the conclusion of the lemma 3.1. $\square$

Now, we construct supersolutions to the equation (1.2) with optimal global Hölder regularity, which is similar to [33, 34].

Lemma 4.3. *Let $p<1$ and $q\geq n\geq 3$, $U\subset\mathbb{R}^{n-1}$ be a bounded convex domain with $0\in\partial U$ and $U\subset\{x=(x',x_{n-1})\subset\mathbb{R}^{n-1}:x_{n-1}>0\}$. Then there exists a constant $C=C(n,p,q,diam(U),\sup g)$ such that the following function*

$$v_a(x)=x_{n-1}^a(|x'|^2-C) \tag{4.5}$$

is smooth, convex and satisfies

$$\begin{cases}\det(D^2v_a)(-v_a)^{1-p}\left[|Dv_a|^2+(x\cdot Dv_a-v_a)^2\right]^{\frac{q-n}{2}}\geq g(x) \quad in \quad U,\\ v_a\leq 0 \quad on \quad \partial U.\end{cases}$$

Here $a=\frac{q-n+2}{q-p}\in(0,1)$.

Proof. For $x=(x',x_{n-1})$, we denote $r=|x'|$, then $v_a=x_{n-1}^a(r^2-C)$ and

$$\begin{aligned}(v_a)_r &= 2rx_{n-1}^a,\\ (v_a)_{rr} &= 2x_{n-1}^a,\\ (v_a)_{x_{n-1}} &= ax_{n-1}^{a-1}(r^2-C),\\ (v_a)_{x_{n-1}x_{n-1}} &= a(a-1)x_{n-1}^{a-2}(r^2-C),\\ (v_a)_{x_{n-1}r} &= 2rax_{n-1}^{a-1}.\end{aligned}$$

In suitable coordinate systems, such as cylindrical in x', the Hessian of v_a has the following form

$$D^2v_a=\begin{pmatrix}\frac{(v_a)_r}{r} & 0 & \cdots & 0 & 0\\ 0 & \frac{(v_a)_r}{r} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (v_a)_{rr} & (v_a)_{x_{n-1}r}\\ 0 & 0 & \cdots & (v_a)_{x_{n-1}r} & (v_a)_{x_{n-1}x_{n-1}}\end{pmatrix}.$$

We have

$$\begin{aligned}\det(D^2v_a) &= \left(\frac{(v_a)_r}{r}\right)^{n-3}\left((v_a)_{x_{n-1}x_{n-1}}(v_a)_{rr}-(v_a)^2_{x_{n-1}r}\right)\\ &= 2^{n-2}x_{n-1}^{an-a-2}\left((a-a^2)C-(a+a^2)r^2\right).\end{aligned}$$

It follows that v_a is smooth and convex in U provided by $C >> \mathrm{diam}^2(U)$. Then

$$\begin{aligned}
& |Dv_a|^2 + (x \cdot v_a - v_a)^2 \\
= \;& ((v_a)^2_{x_{n-1}} + (v_a)^2_r) + (x_{n-1}(v_a)_{x_{n-1}} + r(v_a)_r - v_a)^2 \\
= \;& x_{n-1}^{2a-2}\left(a^2(C-r^2)^2 + 4r^2x_{n-1}^2\right) + x_{n-1}^{2a}[(1+a)r^2 + (1-a)C]^2 \\
\geq \;& x_{n-1}^{2a-2}a^2(C-r^2)^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \det(D^2v_a)(-v_a)^{1-p}\left[|Dv_a|^2 + (x\cdot Dv_a - v_a)^2\right]^{\frac{q-n}{2}} \\
\geq \;& 2^{n-2}x_{n-1}^{a(q-p)-q+n-2}\left((a-a^2)C - (a+a^2)r^2\right)(C-r^2)^{q-n+1-p}a^{q-n} \\
\geq \;& 2^{n-2}(a+a^2)a^{q-n}\left(\frac{a-a^2}{a+a^2}C - r^2\right)^{q-n+2-p} \\
\geq \;& g(x)
\end{aligned}$$

if we choose $a = \frac{q-n+2}{q-p}$ and $C = C(n,p,q,\sup g)(1+\mathrm{diam}^2(U))$ so large. $\square$

Lemma 4.4. *Let $q \geq n \geq 3$ and $p < 1$. For any $a \in [\frac{q-n+2}{q-p}, 1)$, we denote by*

$$b = \frac{q-1}{q-p}, \quad s = \frac{b}{1-a}.$$

Let $U = \{(x', x_{n-1}) \subset \mathbb{R}^{n-1} : |x'| < 1, 0 < x_{n-1} < (1-|x'|^2)^s\}$. Then there exists a constant $C = C(n,p,q,\inf g)$ such that the following function

$$w(x) = Cx_{n-1} - Cx_{n-1}^a(1-|x'|)^b \tag{4.6}$$

is smooth and satisfies

$$\begin{cases} \det(D^2w)(-w)^{1-p}\left[|Dw|^2 + (x\cdot Dw - w)^2\right]^{\frac{q-n}{2}} \leq g(x) \quad \textit{in} \quad U, \\ w = 0 \quad \textit{on} \quad \partial U. \end{cases}$$

Proof. Denote $r = |x'|$. As in the proof of Lemma 4.3, we know that $w = Cx_{n-1} - Cx_{n-1}^a(1-r^2)^b$ and

$$\begin{aligned}
w_r &= 2Cbx_{n-1}^a(1-r^2)^{b-1}r, \\
w_{rr} &= 2Cbx_{n-1}^a(1-r^2)^{b-2}[1-(2b-1)r^2], \\
w_{x_{n-1}} &= C - Cax_{n-1}^{a-1}(1-r^2)^b, \\
w_{x_{n-1}r} &= 2Cabx_{n-1}^{a-1}(1-r^2)^{b-1}r, \\
w_{x_{n-1}x_{n-1}} &= -Ca(a-1)x_{n-1}^{a-2}(1-r^2)^b.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\det(D^2w) &= \left(\frac{w_r}{r}\right)^{n-3}\left(w_{x_{n-1}x_{n-1}}w_{rr} - w^2_{x_{n-1}r}\right)\\ &= 2^{n-2}C^{n-1}ab^{n-2}x_{n-1}^{an-a-2}(1-r^2)^{(b-1)(n-1)}\left(1-a+(1-2b-a)r^2\right)\\ &\leq 2^{n-2}C^{n-1}ab^{n-2}x_{n-1}^{an-a-2}(1-r^2)^{(b-1)(n-1)}(1-a+|1-2b-a|).\end{aligned}$$

Hence,

$$\begin{aligned}&|Dw|^2+(x\cdot w-w)^2\\ =\;& 4C^2b^2x_{n-1}^{2a}(1-r^2)^{2b-2}r^2+C^2(1-ax_{n-1}^{a-1}(1-r^2)^b)^2\\ &+(x_{n-1}w_{x_{n-1}}+rw_r-w)^2\\ \leq\;& 4C^2b^2x_{n-1}^{2a}(1-r^2)^{2b-2}+2C^2+2C^2a^2x_{n-1}^{2a-2}(1-r^2)^{2b}\\ &+C^2x_{n-1}^{2a}(1-r^2)^{2b-2}\left((1-a)(1-r^2)+2br^2\right)^2\\ \leq\;& 2C^2+C^2x_{n-1}^{2a-2}(1-r^2)^{2b-2}\left(12b^2+4a^2+2-4a\right)\\ \leq\;& C^2x_{n-1}^{2a-2}(1-r^2)^{2b-2}\left(12b^2+4a^2+4-4a\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\det(D^2w)(-w)^{1-p}\left[|Dw|^2+(x\cdot Dw-w)^2\right]^{\frac{q-n}{2}}\\ \leq\;& 2^{n-2}C^{q-p}ab^{n-2}x_{n-1}^{a(q-p)-q+n-2}(1-r^2)^{b(q-p)-(q-1)}(12b^2+4a^2+4-4a)^{\frac{q-n}{2}}\\ &\cdot(1-a+|1-2b-a|)\\ \leq\;& g(x)\end{aligned}$$

if we choose a suitable $C=C(n,p,q,\inf g)$. Thus we complete the proof. $\square$

Theorem 4.5. *Let U be a bounded, open and convex domain in $\mathbb{R}^{n-1}$, $p<1$ and $q\geq n\geq 3$. There exist a unique nontrivial convex solution $u\in C^\infty(U)\cap C^0(\overline{U})$ to the equation* (1.2) *with the estimate*

$$|u(x)|\leq C(n,p,q,diam(U),\sup g)\left(dist(x,\partial U)\right)^{\frac{q-n+2}{q-p}} \text{ for any } x\in U. \tag{4.7}$$

Moreover, the exponent $\frac{q-n+2}{q-p}$ is optimal, i.e., for any $a\in(\frac{q-n+2}{q-p},1)$, there exists a bounded convex domain $U\subset\mathbb{R}^{n-1}$ such that the solution of the equation (1.2) *satisfies $u\notin C^a(\overline{U})$.*

Proof. We divide our proof into three steps.

Step 1: We show the estimate (4.7) holds.

Let $u \in C^\infty(U) \cap C^0(\overline{U})$ be the unique nontrivial convex solution to the equation (1.2), z be an arbitrary point in U, z_0 be a point in ∂U such that $|z - z_0| = \operatorname{dist}(z, \partial U)$. Suppose that the supporting hyperplane $l_{z_0} := \{x \in \mathbb{R}^{n-1} \mid \mathbf{n} \cdot (x - z_0) = 0\}$ to ∂U at z_0, where $\mathbf{n}$ is the inner normal unit vector to ∂U at z_0. Then

$$U \subset \{x \in \mathbb{R}^{n-1} \mid \mathbf{n} \cdot (x - z_0) \geq 0\}.$$

Define a function

$$v(x) = [\mathbf{n} \cdot (x - z_0)]^a (|x - \mathbf{n} \cdot x|^2 - C),$$

where $a = \frac{q-n+2}{q-p}$, C is a large constant to be determined later. By translation and rotation of coordinates, we can assume that $\mathbf{n} = (0, \cdots, 0, 1)$, $z_0 = 0$, then $v = x_{n-1}^a (|x'|^2 - C)$. According to Lemma 4.3, we can choose a suitable constant C such that v is a subsolution to equation (1.2). Using Lemma 4.1, we have

$$|u(z)| \leq |v(z)| \leq C|z - z_0|^a = C\left(\operatorname{dist}(z, \partial U)\right)^a \tag{4.8}$$

By the convexity of u, we easily obtain $u \in C^{\frac{q-n+2}{q-p}}(\overline{U})$.

Step 2: We prove the existence and uniqueness of solutions to the equation (1.2).

Let U_ϵ be a sequence of open, bounded, smooth and strictly convex domains in $\mathbb{R}^{n-1}$ such that $U_\epsilon \to U$ in the Hausdorff distance. Consider the following Monge-Ampère equation

$$\begin{cases} \det(D^2 u_\epsilon) = g(x)(\epsilon - u_\epsilon)^{p-1} \left[|Du_\epsilon|^2 + (x \cdot Du_\epsilon - u_\epsilon)^2\right]^{\frac{n-q}{2}} \quad \text{in} \quad U_\epsilon, \\ u_\epsilon = 0 \quad \text{on} \quad \partial U_\epsilon, \end{cases} \tag{4.9}$$

where $\epsilon < \epsilon_0$, which is given in Lemma 4.2. From Theorem 7.1 in [10], there exists a unique convex solution $u_\epsilon \in C^\infty(\overline{U_\epsilon})$ to the equation (4.9). Lemma 4.2 implies that there exists a constant $c(p, q, n, \inf g)$ such that

$$\|u_\epsilon\|_{C^0(\overline{U})} \geq c(p, q, n, \inf g) \left(|U_\epsilon|^*\right)^{\frac{1}{q-p}}.$$

We now apply the same argument in Step 1 to obtain that

$$|u_\epsilon|(x) \leq C(n, p, q, \operatorname{diam}(U), \sup g) \left(\operatorname{dist}(x, \partial U_\epsilon)\right)^{\frac{q-n+2}{q-p}} \text{ for any } x \in U_\epsilon.$$

It follows that u_ϵ is uniformly bounded in $C^{\frac{q-n+2}{q-p}}(\overline{U_\epsilon})$. We can choose a subsequence of u_ϵ that uniformly converges to a limit $u \in C^0(\overline{U})$ which satisfies $u = 0$ on ∂U and

$$\|u\|_{C^0(\overline{U})} \geq c(p, q, n, \inf g) \left(|U|^*\right)^{\frac{1}{q-p}}.$$

According to Lemma 1.2.3 in [20], we know that u is actually an Aleksandrov solution of (1.2).

For any $\delta \in (0, \|u\|_{C^0(\overline{U})})$, let $U_\delta = \{x \in U : u(x) \leq -\delta\}$, which is convex with nonempty interior. Note that

$$-C_\delta \leq u \leq -\delta, \quad c_\delta \leq |Du|^2 + (x \cdot Du - u)^2 \leq C_\delta \quad \text{in} \quad U_\delta,$$

if we choose δ small enough. Thus the Monge-Ampère measure M_u, which is the weak limit of $\det D^2 u_\epsilon$, satisfies

$$0 < c_\delta \leq M_u \leq C_\delta < \infty \quad \text{in} \quad U_\delta.$$

Therefore u is strictly convex in U_δ and $u \in C^{1,\alpha}(U_\delta)$ by Theorem 5.4.10 and Theorem 5.4.8 in [20].

For any $x_0 \in \overline{U_\delta}$ and $p_0 \in \partial u(x_0)$, we know that there exists a constant t_0 such that $\Sigma_{t_0} := \{u(x) < l_0(x) = u(x_0) + p_0 \cdot (x - x_0) + t_0\} \subset\subset U$ by the similar method in the proof of Theorem 1.1 in [11]. Then by Pogorelov's interior estimates (Theorem 17.19 in [19]), we know

$$(l_0 - u)|D^2 u| \leq C(n, |u|_{C^{0,1}(\Sigma_{t_0})}, \delta) \quad \text{in} \quad \Sigma_{t_0}.$$

It implies $|D^2 u|_{U_\delta} \leq C_\delta$ and the equation is uniformly elliptic in U_δ. Using Evans-Krylov's estimates [18, 30], we have

$$\|u\|_{C^{k,\alpha}(\overline{U_\delta})} \leq C(\delta, k).$$

We conclude $u \in C^\infty(U)$. Moreover, it is easy to obtain the uniqueness of solution to (1.2) by the comparison principle.

Step 3: We show the optimality of the exponent $\frac{q-n+2}{q-p}$.

Indeed, for any $a \in (\frac{q-n+2}{q-p}, 1)$, we choose U and the function w as in Lemma 4.4. It follows that w is a supersolution to the equation (1.2). We show that $w \geq u$ in U. Note that $w = 0 \geq u$ on ∂U. If $w - u$ attains its minimum value on $\overline{U}$ at $y \in U$ with $w(y) < u(y) < 0$, then $Dw(y) = Du(y)$ and $D^2 w(y) \geq D^2 u(y)$. It follows that at y

$$(-w)^{1-p} \left[|Dw|^2 + (x \cdot Dw - w)^2\right]^{\frac{q-n}{2}} \leq (-u)^{1-p} \left[|Du|^2 + (x \cdot Du - u)^2\right]^{\frac{q-n}{2}},$$

which contradicts $w(y) < u(y) < 0$.

For $x = (0, x_{n-1}) \in U$, we have

$$|u(x)| \geq |w(x)| = C\left(x_{n-1}^a - x_{n-1}\right) \geq \frac{C}{2} x_{n-1}^a = \frac{C}{2}\left(\operatorname{dist}(x, \partial U)\right)^a \tag{4.10}$$

by assuming $x_{n-1} < \log_{\frac{1}{2}}(1-a)$, which implies the optimality of the exponent. □

Proof of Theorem 1.2. The theorem can be easily obtained by Theorem 4.5 and Lemma 5.1. □

5. Appendix

5.1. **The Monge-Ampère equation in Euclidean space.** We transfer the Monge-Ampère equation (1.1) on $\Omega \subset S^{n-1}$ to a Euclidean Monge-Ampère equation on $U \subset \mathbb{R}^{n-1}$. For $e \in S^{n-1}$, we consider the restriction of a solution h of (1.1) to the hyperplane $e^\perp$ tangent to S^{n-1} at e, i.e.

$$u(x) = h(x+e).$$

We consider $\pi : e^\perp \to S^{n-1}$ defined by

$$\pi(x) = \frac{1}{\sqrt{1+|x|^2}}(x+e).$$

Thus,

$$u(x) = \sqrt{1+|x|^2}h(\pi(x)).$$

Let ∇, $\overline{\nabla}$ and D be the standard Levi-Civita connections in S^{n-1}, $\mathbb{R}^n$, and $e^\perp = \mathbb{R}^{n-1}$.

Lemma 5.1. *The Dirichlet problem* (1.1) *of h is equivalent to the following Dirichlet problem of u*

$$\begin{cases} \det(D^2 u) = g(x)(-u)^{p-1}\Big[|Du|^2 + (x\cdot Du - u)^2\Big]^{\frac{n-q}{2}} \quad \text{in} \quad U \subset \mathbb{R}^{n-1}, \\ u = 0 \quad \text{on} \quad \partial U, \end{cases}$$

where $U = \pi^{-1}(\Omega)$ and

$$g(x) = f(\pi(x))(1+|x|^2)^{-\frac{n+p}{2}}. \tag{5.1}$$

Proof. Note that

$$tu(x) = h(tx+te).$$

Differentiating both sides of the above equation with t and x respectively, we obtain

$$u(x) = \sum_{i=1}^n x^i \cdot \overline{\nabla}_i h + \overline{\nabla}_n h$$

and

$$tD_i u(x) = t\overline{\nabla}_i h(tx+te).$$

Thus, we have by letting $t = \frac{1}{\sqrt{1+|x|^2}}$

$$u(x) = \sum_{i=1}^n x^i \cdot \overline{\nabla}_i h(\pi(x)) + \overline{\nabla}_n h(\pi(x))$$

and

$$D_i u(x) = \overline{\nabla}_i h(\pi(x)).$$

Therefore (see also Page 500 in [14]),

$$|h(\pi(x))|^2 + |\nabla h(\pi(x))|^2 = |\overline{\nabla} h(\pi(x))|^2 = |Du|^2 + (x \cdot Du - u)^2. \tag{5.2}$$

On the other hand, we have (see (2.4) in [14])

$$(1+|x|^2)^{\frac{n+1}{2}} \det(D^2 u(x)) = \det(\nabla^2 h(\pi(x)) + h(\pi(x)) I).$$

Thus,

$$\det(D^2 u) = f(\pi(x))(1+|x|^2)^{-\frac{n+p}{2}}(-u)^{p-1}\Big[|Du|^2 + (x \cdot Du - u)^2\Big]^{\frac{n-q}{2}}.$$

□

5.2. **The a priori estimates for solutions to the parabolic Monge-Ampère equation.** Let U be an open, bounded, smooth and strictly convex domain in $\mathbb{R}^n$. We denote by

$$\mathcal{C}_0 = \{u \in C^\infty(\overline{U}) : D^2 u > 0,\ u|_{\partial U} = 0\}.$$

We consider the initial-boundary problem of the type

$$\begin{cases} u_t - \log\det(D^2 u) = -g(x, u, Du) \quad \text{in} \quad U \times (0, T], \\ u = 0 \quad \text{on} \quad \partial U \times [0, T], \\ u = u_0 \quad \text{on} \quad U \times \{0\}, \end{cases} \tag{5.3}$$

where $g(x, u, Du) = \log f(x, u, Du)$ and $u_0 \in \mathcal{C}_0$ satisfies the compatibility condition

$$\det(D^2 u_0) = f(x, u_0, Du_0) \quad \text{on} \quad \partial U.$$

Let $u \in C^4(U \times (0, T)) \cap C^2(\overline{U} \times [0, T])$ be a solution to (5.3), and suppose further that

$$-K \le u(x, t) < 0, \quad \forall\ (x, t) \in \overline{U} \times [0, T] \tag{5.4}$$

and

$$0 < f(x, u(x), p) \le C(1 + p^2)^{\frac{n}{2}}, \quad \forall\ x \in \overline{U}, \tag{5.5}$$

where C is a positive constant depending only on K.

Now, we will establish the a priori estimates for solutions to the initial-boundary problem (5.3).

Lemma 5.2. *Let $u \in C^4(U \times (0,T)) \cap C^2(\overline{U} \times [0,T])$ be a solution to* (5.3) *satisfying the assumptions* (5.4) *and* (5.5). *Then, we have*

$$|Du|(x,t) \leq C, \quad \forall\ (x,t) \in \overline{U} \times [0,T].$$

Proof. Using the condition (5.5) and following the same argument in section 7 in [10], there exists a convex subsolution $\underline{u} \in C^2(\overline{U})$

$$\begin{cases} \det(D^2\underline{u}) \geq C(1+|D\underline{u}|^2)^{\frac{n}{2}} & \text{in} \quad U, \\ \underline{u} = 0, \quad \text{on} \quad \partial U. \end{cases}$$

Set $\underline{v} = \mu\underline{u} + u_0$. For large μ, it is easy to show that $\underline{v}$ also satisfies the above inequality with the same boundary value. Since

$$\underline{v} = 0 \text{ on } \partial U \times [0,T), \quad \underline{v} \leq u_0 \text{ on } U \times \{0\},$$

we have by maximum principle $\underline{v} \leq u$ in $\overline{U} \times [0,T]$, it follows that

$$0 \leq \frac{\partial u}{\partial \nu} \leq \frac{\partial \underline{v}}{\partial \nu} \quad \text{on} \quad \partial U,$$

where ν is the unit outer vector of ∂U. Due to the convexity of u, we have

$$|Du|_{C^0(\overline{U}\times[0,T])} \leq \Big|\frac{\partial \underline{v}}{\partial \nu}\Big|_{C^0(\partial U)},$$

which completes the proof. □

Based on the above gradient estimate, we can follow the same arguments in Step 1 and Step 3 in Appendix [46] to obtain the a priori estimates for u_t. Then, we follow almost the same argument in Section 7 in [10] to get the global second order estimates of u for the variable x.

Lemma 5.3. *Let $u \in C^4(U \times (0,T)) \cap C^2(\overline{U} \times [0,T])$ be a solution to* (5.3) *satisfying the assumptions* (5.4) *and* (5.5). *Then, we have*

$$|u_t(x,t)| + |D^2u(x,t)| \leq C, \quad \forall\ (x,t) \in \overline{U} \times [0,T].$$

5.3. **The a priori estimates for solutions to the Monge-Ampère equation.** We consider the a priori estimates of to solutions of the Dirichlet problem

$$\begin{cases} \det(D^2u) = f(x,u,Du) & \text{in} \quad U \subset \mathbb{R}^n, \\ u = 0, \quad \text{on} \quad \partial U. \end{cases} \tag{5.6}$$

Let $u \in C^4(U) \cap C^2(\overline{U})$ be a solution to (5.3), and suppose further that

$$-K \leq u(x) < 0, \quad \forall\ x \in U$$

and

$$0 < f(x, u(x), D^2 u(x)) \leq C(1 + |D^2 u(x)|^2)^{\frac{n}{2}}, \quad \forall\, x \in U,$$

where C is a positive constant depending only on K.

Lemma 5.4. *We have*

(1) The gradient estimate

$$|Du|(x) \leq C, \quad \forall\, x \in \overline{U},$$

where C is a positive constant depending only on K.

(2)The high order estimates

$$\| u \|_{C^{k,\alpha}(U')} \leq C, \quad \forall\, U' \subset\subset U,$$

where C is a positive constant depending only on K, $d(U', \partial U)$, $\inf_{U'} f$, the bounds on f and its derivatives on U'.

(3) If $f(x, u(x), Du(x)) \geq \eta > 0$, we have

$$\| u \|_{C^{k,\alpha}(\overline{U})} \leq C,$$

where C is a positive constant depending only on K and f.

Proof. The gradient estimate can be deduced by Lemma 5.2 in Appendix. By Pogorelov's interior estimates [19, 39] and Evans-Krylov estimates [18, 30], we have the interior high order estimates. The global high order estimates from Theorem 7 in [10]. □

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