ON THE MAXIMUM OF THE POTENTIAL OF A GENERAL TWO-DIMENSIONAL COULOMB GAS

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Abstract. We determine the leading order of the maximum of the random potential associated to a two-dimensional Coulomb gas for general β and general confinement potential, extending the result of [LLZ23, Theorem 1]. In the case $\beta = 2$, this corresponds to the (centered) log-characteristic polynomial of either the Ginibre random matrix ensemble for $V(x) = \frac{|x|^2}{2}$ or a more general normal matrix ensemble. The result on the leading order asymptotics for the maximum of the log-characteristic polynomial is new for random normal matrices.

We rely on connections with the classical obstacle problem and the theory of Gaussian Multiplicative Chaos. We make use of a new concentration result for fluctuations of $C^{1,1}$ linear statistics which may be of independent interest.

1. Introduction

1.1. **The Model.** We are interested in studying the two-dimensional Coulomb gas with general confinement potential. This is an interacting particle system with point masses $X_N = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{2N}$ distributed according to the Gibbs measure

$$d\mathbb{P}_{N,\beta}(X_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N^V(X_N)}\, dX_N, \tag{1.1}$$

where $\mathcal{H}_N^V$ is given by

$$\mathcal{H}_N^V(X_N) = \frac{1}{2}\sum_{i \neq j} \mathsf{g}(x_i - x_j) + N \sum_{i=1}^{N} V(x_i). \tag{1.2}$$

g is the logarithmic kernel

$$\mathsf{g}(x) = -\log|x| \tag{1.3}$$

and V assumed to grow sufficiently fast at infinity. $\beta > 0$ denotes the inverse temperature, which we take to be order one.

From Frostman [Fro35] (see [ST97]), if the potential V is lower semicontinuous, bounded below and satisfies the growth condition

$$\liminf_{|x| \to +\infty} \frac{V(x)}{\log|x|} > 1 \tag{1.4}$$

then the continuous approximation of the Hamiltonian

$$\mathcal{I}_V(\mu) = \frac{1}{2}\int\int \mathsf{g}(x-y)\ d\mu(x)d\mu(y) + \int V(x)\ d\mu(x) \tag{1.5}$$

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has a unique, compactly supported minimizer μ_V (called the *equilibrium measure*) among the set of probability measures on $\mathbb{R}$, characterized by the Euler-Lagrange equation

$$\begin{cases} \mathsf{g} * \mu_V + V = c_V & \text{on } \Sigma \\ \mathsf{g} * \mu_V + V \geq c_V & \text{otherwise,} \end{cases} \tag{1.6}$$

where Σ denotes the *support* of μ_V. If Σ is connected, we will say we are in the *one-cut* regime; otherwise, Σ may consist of multiple connected components, which is usually called the *multi-cut* regime. c_V is a fixed constant depending on the potential V. In the following, we denote the corresponding *effective potential* by

$$\zeta_V := \mathsf{g} * \mu_V + V - c_V \tag{1.7}$$

and will review some important properties in §1.3. We also denote

$$\mathfrak{h}_0 := \mathsf{g} * \mu_V \tag{1.8}$$

to match the notation of [LLZ23]. Furthermore, if we assume that V is continuous, then the empirical measures μ_N given by

$$\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \tag{1.9}$$

converge almost surely to μ_V under $\mathbb{P}_{N,\beta}$ (cf [BAZ98]).

With this understanding of the leading order behavior, one can split the Hamiltonian and write (cf [AS21, Lemma 2.1] or [LS18, Lemma 2.2])

$$\mathcal{H}_N^V(X_N) = N^2 \mathcal{I}_V(\mu_V) + 2N \sum_{i=1}^N \zeta_V(x_i) + F_N(X_N, \mu_V)$$

where $F_N(X_N, \mu_V)$ is a next-order energy defined by

$$F_N(X_N, \mu_V) = \frac{1}{2} \int\int_{\Delta^c} \mathsf{g}(x-y) \left(\sum_{i=1}^N \delta_{x_i} - N\mu_V \right)(x) \left(\sum_{i=1}^N \delta_{x_i} - N\mu_V \right)(y) \tag{1.10}$$

for a configuration of points $X_N \subseteq \mathbb{R}^N$ and $\Delta \subseteq \mathbb{R}^2$ denotes the *diagonal*

$$\Delta := \{(x,y) \in \mathbb{R}^2 : x = y\}. \tag{1.11}$$

This allows us to rewrite (1.1) as

$$d\mathbb{P}_{N,\beta}(X_N) = \frac{1}{K_{N,\beta}} \exp\left(-\beta \left(F_N(X_N, \mu_V) + 2N \sum_{i=1}^N \zeta_V(x_i) \right) \right) \, dX_N \tag{1.12}$$

where $K_{N,\beta}$ is the *next-order partition function*

$$K_{N,\beta} = \int_{\mathbb{R}^{2N}} \exp\left(-\beta \left(F_N(X_N, \mu_V) + 2N \sum_{i=1}^N \zeta_V(x_i) \right) \right) \, dX_N. \tag{1.13}$$

To match the discussion in [LLZ23], we will take this as our definition of the two-dimensional Coulomb gas going forward, and simply refer to $F_N(X_N, \mu_V)$ as the energy of the system.

1.2. **Main Result.** We are interested in studying the maximum of the *Coulomb gas potential* generated by a point configuration X_N and background measure μ_V given by

$$\mathrm{Pot}_N(z) = \int \log|z-x| \left(\sum \delta_{x_i} - N\mu_V\right)(x) \tag{1.14}$$

over a closed disk of radius r centered at $x \in \Sigma$, which we denote by $\mathsf{D}(x,r)$. We have chosen to match the notation of [LLZ23] to emphasize the connection with their result. Note that if the Coulomb gas (1.1) corresponds to the eigenvalues of a random matrix model, then this potential corresponds to the (centered) log-characteristic polynomial of the matrix model. This has been studied in [Lam20] for the Ginibre random matrix model, and the authors in [LLZ23] studied the two-dimensional Coulomb gas potential in the case $V(x) = \frac{x^2}{2}$; we are interested in extending the law of large numbers that they prove for Pot_N in that regime to general V. We accomplish this in the following theorem.

Theorem 1. *Let V satisfy assumptions **(A1)**-**(A3)**, given in §1.3. Suppose additionally that Σ is one-cut and that $V \in C^5(\mathbb{R}^2)$. Let $r > 0$ be such that $\mathsf{D}(x,r) \subseteq \Sigma$ and $\mathsf{D}(x,r) \cap \partial\Sigma = \emptyset$. Then, we have*

$$\frac{1}{\log N} \max_{z \in \mathsf{D}(x,r)} \mathrm{Pot}_N(z) \to \frac{1}{\sqrt{\beta}} \tag{1.15}$$

as $N \to +\infty$, with the convergence in probability.

The leading order asymptotics above agree with those already known for the Ginibre ensemble ([Lam20, Theorem 1.1]) and general β with quadratic potential ([LLZ23, Theorem 1.1]). The result is new for the log-characteristic polynomial of normal random matrices, and for the general Coulomb gas. Importantly, in line with the approach used in [LLZ23], we extend the scope of *nondeterminantal* tools for studying such interacting particle systems.

In order to prove Theorem 1, we need to establish the following bound on fluctuations of linear statistics for test functions with a small amount of regularity, which may be of independent interest. The additional assumption $V \in C^5$ is needed for the main theorem, but is not necessary for our fluctuation result. For a test function ξ, we let $\mathrm{Fluct}_N(\xi)$ denote the random variable

$$\mathrm{Fluct}_N(\xi) = \int \xi(x) \left(\sum \delta_{x_i} - N\mu_V\right)(x). \tag{1.16}$$

ξ^Σ denotes the unique bounded harmonic extension of a test function ξ outside of Σ.

Proposition 1. *Let $\xi \in C^{1,1}(\mathbb{R}^2)$ and suppose $\Delta\xi \in C^{0,\alpha}(\Sigma)$ for some $\alpha > 0$. Suppose also that there is a constant $C > 0$ such that*

$$|\xi(x)| \le C(\log|x| + 1).$$

Suppose that Σ is one-cut. Then

$$\begin{aligned} &\mathbb{E}\left[\exp\left(-\beta t N \mathrm{Fluct}_N(\xi)\right)\right] \lesssim \\ &\exp\left(\beta N^2 t^2 \max\left(1, \|\Delta\xi\|_{C^{0,\alpha}(\Sigma)} + \|\xi\|_{C^{1,1}(U)}\right)^2 + \max(1,\beta) N|t| \left(\|\Delta\xi\|_{C^{0,\alpha}(\Sigma)} + \|\xi\|_{C^{1,1}(U)}\right)\right). \end{aligned} \tag{1.17}$$

If Σ is multi-cut, the same control holds if we assume in addition that

$$\int_{\partial\Sigma_i} \nabla\xi^\Sigma \cdot \widehat{n} = 0 \tag{1.18}$$

for every connected component Σ_i of Σ.

We obtain Proposition 1 from the more general Proposition 3.1, which we state and prove in Section §3. As the next subsection makes clear, this is enough regularity to control the fluctuations of $\mathfrak{h}_0$ at order 1, which is the key technical difficulty in extending [LLZ23, Theorem 1].

This is an improvement in required regularity over the CLT results (see in particular [BBNY19, Theorem 1.2], [LS18, Theorem 1], [Ser23, Theorem 3]) and the order one fluctuation bound of [Ser23, Theorem 1], which require at least $C^{2,1}$ regularity on the test function ξ. It should be noted that some assumption on ξ beyond Lipschitz regularity is necessary; [PG23] shows that concentration of measure happens at order $\sqrt{N}$, and discusses optimality ([PG23, Proposition 5.1]). It is unclear if one could improve our regularity assumptions with the current methods; [RV07, Theorem 1] at least suggests that we should be able to obtain order one fluctuations as soon as the test function is C^1.

1.3. **Assumptions.** We will need to make some assumptions on $V(x)$ to guarantee a sufficiently regular $\partial\Sigma$ and to guarantee sufficient regularity on $\mathfrak{h}_0$. As discussed before, we first need some regularity and growth on the potential V. This is given by the following.

(A1) -Growth and Regularity: $V \in C^{2,\alpha}(\mathbb{R}^2)$ for some $\alpha > 0$, and

$$\liminf_{|x|\to+\infty} \frac{V(x)}{\log|x|} > 1.$$

This guarantees the existence of the equilibrium measure μ_V (see [ST97]), and gives us enough regularity to assume that the second derivative of V is uniformly bounded in the *droplet* $\Sigma = \operatorname{supp}(\mu_V)$.

Next, it is observed in [Ser15, Section 2.5] that the Euler-Lagrange equation (1.6) for $\mathfrak{h}_0$ can be interpreted as an *obstacle problem* for $\mathfrak{h}_0$, namely

$$\min\left((-\Delta)\mathfrak{h}_0, \mathfrak{h}_0 - (c_V - V)\right) = 0. \tag{1.19}$$

With this interpretation, we can draw on much of the theory of the classical obstacle problem. First, we will need the following *nondegeneracy assumption*.

(A2) -Nondegeneracy: There exists a constant $\lambda > 0$ such that

$$\Delta V \geq \lambda$$

in the *coincidence set* $\{\mathfrak{h}_0 - (c_V - V) = 0\}$.

This guarantees that $\Sigma = \{\mathfrak{h}_0 - (c_V - V) = 0\}$, and thus that ζ_V given by (1.7) is strictly positive in Σ^c.

The boundary of the coincidence set, which here is given by $\partial\Sigma$, is known as the *free boundary*. Points on the free boundary are either *regular* or *singular* (see [Caf98]), and at regular points the regularity of the free boundary is well understood. In particular, it is shown in [Caf77] that the free boundary is as regular as the obstacle at all *regular points*. Thus, we make the following assumption.

(A3) -Regularity of the Free Boundary: All points of $\partial\Sigma$ are regular in the sense of [Caf98]. Furthermore, $\partial\Sigma$ is a finite union of $C^{2,\alpha}$ curves.

This is conjectured to be generic in the work of Schaeffer [Sch74], and is shown for dimensions $d \leq 4$ (in particular, this is generic for $d = 2$) in the recent work [FROS20, Theorem 1.1]. It is shown in [Caf77] that $\partial\Sigma$ is $C^{1,\gamma}$ (see also [Caf98, Theorem 7]) in a neighborhood of all regular points; we need the additional regularity of the second derivative to complete the analysis in Proposition 1.

With assumptions **(A1)-(A3)**, we have the following regularity on $\mathfrak{h}_0$, which we use repeatedly. The result dates back to [Fre72]; we use the version stated in [Caf98] (and proven in [CK80]).

Proposition 1.1 ([Caf98]; Theorem 2; [CK80])**.** *Suppose V satisfies **(A1)-(A3)**. Then, there exists a set $U \supset \Sigma$ such that $\mathfrak{h}_0 \in C^{1,1}(U)$.*

1.4. **Connection with Literature.** This note is in direct response to the article [LLZ23], wherein the authors prove a law of large numbers for the maximum of the potential of a two-dimensional Coulomb gas with quadratic confinement potential. Their result extended the earlier work of [Lam20], which proved this convergence for the *Ginibre ensemble*, a nonHermitian random matrix ensemble composed of i.i.d. complex standard Gaussian entries. The eigenvalue density for this ensemble corresponds to a 2d Coulomb gas at inverse temperature $\beta = 2$ with confinement potential $V(x) = \frac{|x|^2}{2}$ (cf [For10]). A related question, the asymptotics of the moments of the characteristic polynomial for such matrices, was examined in [WW19]. [LLZ23] thus is an extension of this result to general inverse temperature $\beta > 0$, and we in turn extend that result to general potential V which yield a *one-cut* equilibrium measure.

A key ingredient in understanding the potential of the two-dimensional Coulomb gas is a fine understanding of the behavior of fluctuations of linear statistics at small scales. This question was initially studied in [RV07], and similar questions were asked for generalizations called *normal matrix models* in [AHM11], [AHM15] and [AKS23]. This question is still of interest for Coulomb gases with general potential due to various connections with mathematical physics, in particular the fractional quantum Hall effect, cf [STG99]. Approaches to the Coulomb gas using an *electric energy* interpretation have yielded a fruitful understanding of the behavior of Coulomb gases at small scales in two dimensions in [Leb17], and in higher dimensions and varying temperature regimes in [AS21]. This led to generalizations of the fluctuation results described above in [LS18] and [Ser23]. A related study was also accomplished in [BBNY17] and [BBNY19].

The behavior of the lower bound relies on the theory of Gaussian Multiplicative chaos, as discussed in [CFLW21]. These measures were introduced by Kahane in [Kah85], and are a family of random fractal measures associated to a generalized log-correlated Gaussian field X defined formally by

$$d\mu^\gamma \text{``}=\text{''} \frac{e^{\gamma X(x)}}{\mathbb{E} e^{\gamma X(x)}}\, dx$$

for parameters $\gamma > 0$. Since many random matrix ensembles and, more generally, Coulomb gases behave asymptotically like log-correlated fields, one expects the weak convergence of measures

$$d\mu_N^\gamma := \frac{e^{\gamma \mathrm{Pot}_N(x)}}{\mathbb{E} e^{\gamma \mathrm{Pot}_N(x)}} \tag{1.20}$$

to μ^γ associated to an appropriate limiting Gaussian field. One reason that these results are useful is that the measures μ^γ are primarily supported on so-called “thick points" where the field X is large. Thus, these kinds of convergence results can be used to obtain information about extreme values of Pot_N.

This convergence has been established for the GUE in [CFLW21], and for the GOE and GSE in [Kiv21]. The proof of the lower bound for Pot_N in [LLZ23, Theorem 1], which extends to our case, relies on a convergence as in (1.20) for regularizations of Pot_N at scales $\epsilon \downarrow 0$.

It is expected that this convergence holds without regularization, although that question is currently open.

Related questions for the one-dimensional log-gas have also seen extensive study. This is an interacting particle system given by (1.2) on the real line, and corresponds to the eigenvalue distributions of certain classical *Hermitian* matrix ensembles (namely the GOE, GUE and GSE in $\beta = 1$, 2 and 4 respectively). Central limit theorems for fluctuations of linear statistics of the log-gas go back to [Joh98] and have subsequently been generalized in [BG13], [BG22], [Shc13], [Shc14], [BLS18], [BL18], [Lam21], [BMP22] and [Pei24]. Questions regarding the maximum of the potential field have also seen significant study, for instance in the works [ABB17], [PZ18], [CMN18] and [PZ22].

1.5. **Proof Structure and Outline of Paper.** In Section 2, we introduce some important terminology and review key elements of the proof of [LLZ23, Theorem 1]. We describe how almost all of the necessary steps transfer immediately to our model. As we discuss in Section 2, the entirety of [LLZ23] generalizes immediately once one can show the truncation error estimate

$$\mathbb{P}_{N,\beta}\left(|\mathrm{Fluct}_N(g)| \geq (\log N)^{0.8}\right) \leq \exp\left(-\frac{1}{2}(\log N)^{1.5}\right) \tag{1.21}$$

for a function analogous to the g found in [LLZ23, Proposition 3.2].

As discussed in [LLZ23, Appendix A], this follows immediately from showing that both the exponential moments of $\mathrm{Fluct}_N(\mathfrak{h}_0)$ and $\mathrm{Fluct}_N(g - c\mathfrak{h}_0)$ are typically order 1, where c is chosen to make $\Delta(g - c\mathfrak{h}_0)$ mean zero. Since $\mathfrak{h}_0 \in C^{1,1}(U)$ (Proposition 1.1) and $\Delta\mathfrak{h}_0 = \mathsf{c_d}\mu_V = \Delta V \in C^{0,\alpha}(\Sigma)$, we can apply Proposition 1 to directly find this control for $\mathfrak{h}_0$. Despite the possible lack of regularity of g at points in Σ, we show that since $\Delta(g - c\mathfrak{h}_0)$ is mean zero and sufficiently regular we can still invert (3.1) and obtain the requisite fluctuation control. This allows us to establish (1.21) in our model, leading to Theorem 1.

Section 3 is then devoted to the proof of Propositions 3.1 and 1, which uses Johansson's method [Joh98], the transport approach of [LS18] and a detailed analysis of the resulting expansion with an eye towards minimizing the requisite regularity of our test functions.

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2. Proof of the Main Theorem

2.1. **Infrastructure from the Quadratic Case.** In this section, we discuss how the proof of [LLZ23, Theorem1] goes through without issue as soon as the fluctuations of $\mathfrak{h}_0$ are shown to be order 1 on the level of exponential moments.

2.1.1. [LLZ23, §2]*, Preliminaries on Coulomb Gases.* This section quotes important results on local energy laws and fluctuations of linear statistics from [AS21] and [Ser23], in addition to providing a useful description and quoting necessary results on comparison of partition functions for perturbations of μ_V by transport. All of the results other than [LLZ23, Lemma 2.8] are already stated for general measure μ_V with density in $C^3(\Sigma)$ that satisfies

$$\frac{3}{4} \leq \mu_V \leq \frac{3}{2}.$$

The choice of constants are arbitrary, and instead can be rephrased for $0 < \lambda \leq \mu_V \leq \Lambda$, where the constants would then depend on λ and Λ. This assumption is guaranteed by **(A2)**, since

$$(1.6) \implies \mu_V = \frac{\Delta V}{\mathsf{c_d}} \geq \lambda > 0$$

and is $C^3(\Sigma)$ on $\Sigma = \mathrm{supp}(\mu_V)$.

The one proposition that makes use of specifically the constant and radial nature of the equilibrium measure in the case $V(x) = \frac{|x|^2}{2}$ is [LLZ23, Lemma 2.8]; a careful reading of their arguments, however show that this is only used to prove the fluctuation control [LLZ23, Corollary A.8] and yields the critical [LLZ23, Proposition 3.2], which we instead here prove by appeal to Proposition 3.1.

2.1.2. [LLZ23, §3]*, Upper bound for Law of Large Numbers.* This section is devoted to the proof of

$$\lim_{N\to\infty} \mathbb{P}_{N,\beta}\left(\max_{z\in\mathsf{D}(x,r)} \mathrm{Pot}_N(z) \geq \frac{\alpha \log N}{\sqrt{\beta}}\right) = 0 \tag{2.1}$$

for all $\alpha > 1$ and fixed r with $\mathsf{D}(x,r) \subseteq \Sigma$. The first key step observes that

$$\int \Delta \log|x - z|\, dx = 2\pi,$$

which prevents the authors from using the theorems from [LLZ23, §2] which require $\int \Delta f = 0$, with f sufficiently smooth. Thus, the authors consider instead

$$\varphi_{z,\epsilon} = \rho_\epsilon * \log|x - z| - g,$$

where ρ_ϵ denotes the standard mollifier and g is a solution to

$$\Delta g = 2\pi\chi,$$

with $\int \chi = 1$ and $\chi \in C_0^\infty(\mathsf{D}(z,r')$ with $\mathsf{D}(z,r') \subseteq \mathsf{D}(x,r)$. The proof then uses this function to prove (2.1) for a regularization of Pot_N, the proof of which does not require that μ_V be the specific equilibrium measure for $V(x) = \frac{|x|^2}{2}$. The conclusion for Pot_N then follows by comparison, using [LLZ23, Proposition 3.2], which follows from [LLZ23, Corollary A.8] and states that

$$\log \mathbb{E}\left[e^{t\mathrm{Fluct}_N[g]}\right] = O(t + t^2) \tag{2.2}$$

for an implied constant dependent only on β and V. We will establish (2.2) by application of Proposition 3.1 at the end of this section.

2.1.3. [LLZ23, §3]*, Lower bound for Law of Large Numbers and Gaussian Multiplicative Chaos.* This section is devoted to the proof of

$$\lim_{N\to\infty} \mathbb{P}_{N,\beta}\left(\sqrt{\beta}\sup_{z\in\mathcal{U}} \mathrm{Pot}_N(z) \geq \alpha \log\frac{1}{\epsilon(N)}\right) = 0 \tag{2.3}$$

for $\alpha < 2$, $\mathcal{U}$ a fixed small ball and $\epsilon(N) \gg N^{-1/2}$, from which the lower bound follows. The method uses the theory of Gaussian multiplicative chaos; (2.3) is deduced from the convergence of the measures

$$\mu_k^\gamma := \frac{e^{\gamma\sqrt{\beta}\mathrm{Fluct}_N[\varphi_{z,e^{-k}}]}}{\mathbb{E}\left[e^{\gamma\sqrt{\beta}\mathrm{Fluct}_N[\varphi_{z,e^{-k}}]}\right]}$$

to GMC_γ; this result is established by the theory built in [CFLW21, Section 3] and [LOS18, Section 2], coupled with the comparison of partition functions along mesoscopic perturbations with $\int \Delta\phi = 0$ drawn on in [LLZ23, §2]. In particular, it doesn't require that μ_V be the equilibrium measure for $V(x) = \frac{|x|^2}{2}$.

2.2. Proof of Main Theorem.

Proof of Theorem 1. We see that it is sufficient to establish (2.2). Let c be an order 1 constant such that

$$\int \Delta(g - c\mathfrak{h}_0) = 0$$

and let $\xi = g - c\mathfrak{h}_0$, which is also at least Lipschitz on all of $\mathbb{R}^2$ and satisfies $|\xi(x)| \leq C(\log|x| + 1)$ (this follows from the compact support of χ and μ_V). Consider the equation

$$\begin{cases} \mathrm{div}\,(\psi\mu_V) = -\frac{1}{\mathsf{c_d}}\Delta\xi & \text{in } \Sigma \\ \psi \cdot \widehat{n} = 0 & \text{on } \partial\Sigma \\ \psi = \psi^\perp & \text{in } U \setminus \Sigma \end{cases}$$

where $\psi^\perp$ is a $C^{1,\alpha}$ vector field satisfying $\psi \cdot \nabla\zeta_V = 0$ in $U \setminus \Sigma$. The interior equation is well-posed since $\int_\Sigma \Delta\xi = 0$, and $\psi \in C^{1,\alpha}(\Sigma)$ because χ and μ_V are both α-Hölder continuous densities in Σ. The vector field $\psi^\perp$ can be constructed by hand as in the proof of [LS18, Lemma 3.4]; one considers an extension of the tangential component ψ on Σ (which here is just ψ) and subtracts off the projection of $\widetilde{\psi}$ onto $\nabla\zeta_V$.

This transport solves (3.1). Integrating by parts and using $\psi \cdot \nabla\zeta_V = 0$ we find

$$\begin{aligned}\psi(x) \cdot \nabla\zeta_V(x) + \xi(x) - \int \nabla\mathsf{g}(x-y) \cdot \psi(y)\, d\mu_V(y) \\ = \xi - \int_\Sigma \mathsf{g}(x-y)\left(-\frac{1}{\mathsf{c_d}}\Delta\xi\right) = \xi - \mathsf{g} * \left(-\frac{1}{\mathsf{c_d}}\Delta\xi\right)\end{aligned}$$

since $\Delta\xi$ is only supported in Σ. However, $\xi - \mathsf{g} * \left(-\frac{1}{\mathsf{c_d}}\Delta\xi\right)$ grows at most logarithmically at ∞ and

$$\Delta\left(\xi - \mathsf{g} * \left(-\frac{1}{\mathsf{c_d}}\Delta\xi\right)\right) = \Delta\xi - \Delta\xi = 0$$

on all of $\mathbb{R}^2$. Thus, by Liouville's theorem this is constant and so

$$\psi(x) \cdot \nabla\zeta_V(x) + \xi(x) - \int \nabla\mathsf{g}(x-y) \cdot \psi(y)\, d\mu_V(y) = c_\xi.$$

Therefore we can apply Proposition 3.1 to find

$$\mathbb{E}\left[e^{t\mathrm{Fluct}_N[g - c\mathfrak{h}_0]}\right] = e^{O(t+t^2)}. \tag{2.4}$$

Next, $\mathfrak{h}_0 \in C^{1,1}(U)$ and satisfies $\mathfrak{h}_0 \leq C(\log|x| + 1)$; granted Proposition 1 then we also have

$$\mathbb{E}\left[e^{t\mathrm{Fluct}_N[\mathfrak{h}_0]}\right] = e^{O(t+t^2)}. \tag{2.5}$$

Now, (2.4) and (2.5) are comparable so we find

$$\mathbb{E}\left[e^{t\mathrm{Fluct}_N[g]}\right] = e^{O(t+t^2)}$$

by Hölder, establishing (2.2) for our case and proving Theorem 1. $\square$

3. Main Technical Estimate

The goal of this section is to prove Proposition 1. We first establish the following.

Proposition 3.1. *Let $\xi \in C^{0,1}(\mathbb{R}^2) \cap C^{2,\alpha}(\Sigma)$ for some $\alpha > 0$. Suppose also that there is a constant $C > 0$ such that*

$$|\xi(x)| \le C(\log|x| + 1).$$

Suppose further that there exists a Lipschitz vector field ψ and a constant c_ξ such that

$$\psi(x) \cdot \nabla \zeta_V(x) + \xi(x) - \int \nabla \mathsf{g}(x-y) \cdot \psi(y)\ d\mu_V(y) = c_\xi \tag{3.1}$$

in an open neighborhood $U \supset \Sigma$. Then,

$$\mathbb{E}\left[\exp\left(-\beta t N \mathrm{Fluct}_N(\xi)\right)\right] \lesssim \exp\left(\beta N^2 t^2 \left(\left(\|\psi\|_{L^\infty} + \|\psi\|^2_{C^{0,1}}\right)\left(1 + \|\xi\|_{C^{0,1}}\right)\right) + \max(1,\beta) N |t| \|\psi\|_{C^{0,1}}\right), \tag{3.2}$$

where the constant depends only on V, C and U. In particular, if $|t| \lesssim \frac{1}{N}$ fluctuations are typically order 1.

We first truncate ξ so that we only need to examine the fluctuations on the given open neighborhood U of Σ.

Proposition 3.2. *Let ξ be a measurable function such that*

$$|\xi(x)| \le C(\log|x| + 1)$$

for some $C > 0$. Let $\eta \in C_c^\infty$ be a test function such that

$$\begin{cases} \eta \equiv 1 & \text{in } U \\ 0 \le \eta \le 1 & \text{everywhere} \\ \|\eta\|_{C^\infty} \lesssim 1. \end{cases}$$

Then, there exists a constant $K > 0$ such that for all $|t| \le K\beta N$,

$$\mathbb{E}\exp\left(t \mathrm{Fluct}[\xi(1-\eta)]\right) = \exp(o(t)). \tag{3.3}$$

Proof. We show that the fluctuations of $\xi(1-\eta)$ should be small simply due to the fact that there are not many points outside of U. The approach is borrowed from [LS18] and uses the expansion of partition functions in terms of ζ_V. First, observe that

$$\left| \int \xi(1-\eta) \left(\sum \delta_{x_i} - N\mu_V\right) \right| = \left| \sum_{x_i \notin U} \xi(1-\eta)(x_i) \right| \le \frac{1}{K} \sum_{x_i \notin U} \zeta_V(x_i)$$

because $V \gtrsim (1+\epsilon)\log|x|$ (**A1**) and $\mathsf{g} * \mu_V + \log|x| \to 0$ as $|x| \to 0$ implies $\zeta_V \gtrsim \log|x| \gtrsim |\xi(x)|$ as $|x| \to +\infty$. Now,

$$\mathbb{E}\exp\left(t \int \xi(1-\eta)\left(\sum \delta_{x_i} - N\mu_V\right)\right) \le \mathbb{E}\exp\left(\frac{|t|}{K}\sum_{i=1}^N \zeta(x_i)\right).$$

Choose $t = \pm K\beta N$. Then,

$$\mathbb{E}\exp\left(\pm K\beta N \int \xi(1-\eta)\left(\sum \delta_{x_i} - N\mu_V\right)\right) \le \mathbb{E}\exp\left(\beta N \sum \zeta(x_i)\right).$$

Using [LS15, (4.12)] we find

$$\log \mathbb{E} \exp \left(\beta N \sum \zeta(x_i) \right) = \log K_{N,\beta} \left(\mu_V, \frac{1}{2}\zeta \right) - \log K_{N,\beta}(\mu_V, \zeta) = o(N).$$

Hölder then gives

$$\mathbb{E} \exp \left(t \int \xi (1-\eta) \left(\sum \delta_{x_i} - N\mu_V \right) \right) = \exp(o(t))$$

for $|t| \le K\beta N$. □

We turn to controlling the fluctuations of ξ in U. The idea is to use Johansson's method [Joh98] coupled with a transport approach as in [LS18]. We opt to use the Taylor expansion approach of [BLS18] and [Pei24] due to the ease with which it allows us to relax the regularity of ξ. First, we expand the fluctuations along a transport.

Proposition 3.3. *Let ξ be a measureable test function. Then,*

$$\mathbb{E}_{\mathbb{P}_{N,\beta}} \left[\exp\left(-\beta t N \mathrm{Fluct}_N(\xi) \right) \right] = e^{\beta t N^2 \int \xi \, d\mu_V} \frac{Z_{N,\beta}^{V_t}}{Z_{N,\beta}}, \tag{3.4}$$

and

$$Z_{N,\beta}^{V_t} = \int \exp \left(-\beta \left(\sum_{i \neq j} \frac{1}{2} \mathsf{g}(x_i - x_j) + N \sum V_t(x_i) \right) \right) \, dX_N.$$

Furthermore, we can expand

$$\mathbb{E}_{\mathbb{P}_{N,\beta}} \left[\exp\left(-\beta t N \mathrm{Fluct}_N(\xi) \right) \right] = e^{T_0} \mathbb{E}_{\mathbb{P}_{N,\beta}} \left[\exp\left(T_1 + T_2 \right) \right], \tag{3.5}$$

where

$$T_0 = -\beta N^2 \left(\frac{1}{2} \iint (\mathsf{g}(\phi_t(x) - \phi_t(y)) - \mathsf{g}(x-y)) \, d\mu_V(x) d\mu_V(y) + \int (V_t \circ \phi_t - V)(x) \, d\mu_V(x) \right)$$

(3.6)

$$+ N \int \log\det D\phi_t(x) \, d\mu_V(x) + t\beta N^2 \int \xi(x) \, d\mu_V(x)$$

(3.7)

$$T_1 = -\beta N \int \left(\int (\mathsf{g}(\phi_t(x) - \phi_t(y)) - \mathsf{g}(x-y)) \, d\mu_V(y) + (V_t \circ \phi_t - V)(x) + \log\det D\phi_t \right) d\mathrm{fluct}_N(x)$$

(3.8)

$$T_2 = -\frac{\beta}{2} \int \int_{\Delta^c} (\mathsf{g}(\phi_t(x) - \phi_t(y)) - \mathsf{g}(x-y)) \, d\mathrm{fluct}_N(y) d\mathrm{fluct}_N(x).$$

Proof. With the change of variables $y_i = \phi_t(x_i)$ and $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$ the map $\mathrm{Id} + t\psi$ we obtain

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(-\beta t N \mathrm{Fluct}_N(\xi)\right)\right] = \frac{\exp\left(\beta t N^2 \int \xi \, d\mu_V\right)}{Z_{N,\beta}}$$
$$\int \exp\left(-\beta\left(\sum_{i\neq j} \frac{1}{2}\mathsf{g}(\phi_t(x_i) - \phi_t(x_j)) + N\sum_{i=1}^N V_t(\phi_t(x_i))\right) + \sum_i \log\det D\phi_t(x_i)\right) dX_N$$
$$= \exp\left(-\beta t N^2 \int \xi \, d\mu_V\right) \mathbb{E}_{\mathbb{P}_{N,\beta}}\Bigg[\exp\Bigg(-\beta\Bigg(\frac{1}{2}\sum_{i\neq j}(\mathsf{g}(\phi_t(x_i) - \phi_t(x_j)) - \mathsf{g}(x_i - x_j)) +$$
$$N\sum_{i=1}^N (V_t \circ \phi_t - V)(x_i)\Bigg) + \sum_{i=1}^N \log\det D\phi_t(x_i)\Bigg)\Bigg].$$

Writing

$$\mathrm{fluct}_N := \sum_{i=1}^N \delta_{x_i} - N\mu_V$$

we find

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(-\beta t N \mathrm{Fluct}_N(\xi)\right)\right] = e^{T_0}\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(T_1 + T_2\right)\right]$$

with

$$T_0 = -\beta N^2\Bigg(\frac{1}{2}\iint (\mathsf{g}(\phi_t(x) - \phi_t(y)) - \mathsf{g}(x-y)) \, d\mu_V(x) d\mu_V(y) +$$
$$\int (V_t \circ \phi_t - V)(x) \, d\mu_V(x)\Bigg) + N\int \log\det D\phi_t(x) \, d\mu_V(x) + t\beta N^2 \int \xi(x) \, d\mu_V(x)$$
$$T_1 = -\beta N \int \left(\int (\mathsf{g}(\phi_t(x) - \phi_t(y)) - \mathsf{g}(x-y)) \, d\mu_V(y) + (V_t\circ\phi_t - V)(x)\right) d\mathrm{fluct}_N(x)$$
$$+ \int \log\det D\phi_t \, d\mathrm{fluct}_N(x)$$
$$T_2 = -\frac{\beta}{2}\iint_{\Delta^c} (\mathsf{g}(\phi_t(x) - \phi_t(y)) - \mathsf{g}(x-y)) \, d\mathrm{fluct}_N(y) d\mathrm{fluct}_N(x).$$

□

First, we control T_0.

Proposition 3.4 (Control of T_0)**.** *Suppose ψ is Lipschitz and let T_0 be as in (3.6). Then,*

$$|T_0| \lesssim \beta N^2 t^2 \left(\|\psi\|^2_{C^{0,1}} + \|\xi\|_{C^{0,1}}\|\psi\|_{L^\infty}\right) + N|t|\|\psi\|_{C^{0,1}}. \tag{3.9}$$

In particular, if we take $t \sim \frac{1}{N}$, T_0 is order 1.

Proof. This follows from the definition of ϕ_t and a first order Taylor expansion. We first rewrite

(3.10)
$$\frac{1}{2}\iint -\log|x + t\psi(x) - y - t\psi(y)| \, d\mu_V(x)d\mu_V(y) - \frac{1}{2}\iint -\log|x-y| \, d\mu_V(x)d\mu_V(y) =$$
$$\frac{1}{2}\iint -\log\left|\frac{x-y}{|x-y|} + t\frac{\psi(x)-\psi(y)}{|x-y|}\right| \, d\mu_V(x)d\mu_V(y)$$

and then Taylor expand the logarithm to obtain

$$-t \iint \frac{x-y}{|x-y|^2} \cdot (\psi(x)-\psi(y))\ d\mu_V(x) d\mu_V(y) + O\left(t^2 \|\psi\|_{C^{0,1}}^2\right).$$

Next,

$$\int (V(x+t\psi(x)) - V(x))\ d\mu_V(x) + t\int \xi(x+t\psi(x))\ d\mu_V(x) - t\int \xi(x)\ d\mu_V(x) = \\ t\int \nabla V(x)\cdot\psi(x)\ d\mu_V(x) + O\left(t^2\|\psi\|_{L^\infty}^2 + t^2\|\xi\|_{C^{0,1}}\|\psi\|_{L^\infty}\right). \tag{3.11}$$

Since $\nabla\left(\int -\log|x-y|\ d\mu_V(y) + V(x)\right) = 0$ on $\mathrm{supp}(\mu_V)$ we have via differentiation and symmetry that

$$-t \iint \frac{x-y}{|x-y|^2} \cdot (\psi(x)-\psi(y))\ d\mu_V(x) d\mu_V(y) + t\int \nabla V(x)\cdot\psi(x)\ d\mu_V(x) = 0.$$

Finally, $\|\log\det D\phi_t\|_{L^\infty} \lesssim |t|\|\psi\|_{C^{0,1}}$. Combining all of these with the definition of T_0 in (3.6) yields the result. □

Next, we will control T_1 by choosing a transport ψ that causes T_1 to vanish at order t. We will accomplish this by inverting (3.1). First, we need the following careful estimate since the second derivatives of g are not bounded.

Lemma 3.5 (Careful Taylor Expansion)**.** *Let $|t| = o_N(1)$ and suppose that ψ is Lipschitz. For any $|y-x| \geq \epsilon$ and large enough N,*

$$\left|\mathsf{g}((x-y)+t(\psi(x)-\psi(y))) - \mathsf{g}(x-y) - t\nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))\right| \\ \leq Ct^2\|\psi\|_{C^{0,1}}^2$$

with constant independent of ϵ and β.

Proof. With $|y-x| \geq \epsilon$ we have enough smoothness of g to Taylor expand; the quantity on the left hand side is the function minus its first Taylor polynomial, whose remainder is given by the integral remainder

$$\sum_{i,j} v_i v_j \int_0^1 (1-s)(\partial_{i,j}\mathsf{g})(\vec{a}+s\vec{v})\ ds,$$

where we have introduced the shorthand $\vec{a} = x-y$ and $\vec{v} = t(\psi(x)-\psi(y))$. Computing directly we have

$$\partial_{i,j}\mathsf{g}(x) = \frac{-1}{|x|^2}\mathbf{1}_{i=j} + \frac{2x_i x_j}{|x|^4}$$

and so the absolute value of the remainder is given by

$$\left| -\int_0^1 (1-s)\frac{|\vec{v}|^2}{|\vec{a}+s\vec{v}|^2}\ ds + \sum_{i,j}\int_0^1 \frac{2(1-s)(\vec{a}+s\vec{v})_i(\vec{a}+s\vec{v})_j v_i v_j}{|\vec{a}+s\vec{v}|^4}\ ds \right| \\ \lesssim \int_0^1 (1-s)\frac{|\vec{v}|^2}{|\vec{a}+s\vec{v}|^2}\ ds + \int_0^1 2(1-s)\frac{(\vec{v}\cdot(\vec{a}+s\vec{v}))^2}{|\vec{a}+s\vec{v}|^4}\ ds \\ \lesssim \int_0^1 (1-s)\frac{|\vec{v}|^2}{|\vec{a}+s\vec{v}|^2}\ ds.$$

with constant independent of ϵ. Next, since $\|t\psi\|_{C^{0,1}} = o_N(1)$, we have for large enough N that $|\vec{a}+c\vec{v}| \geq \frac{1}{2}|\vec{a}|$. Thus,

$$\begin{aligned}|\mathsf{g}((x-y)+t(\psi(x)-\psi(y))) - \mathsf{g}(x-y) - t\nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))| \\ \leq Ct^2\frac{|\psi(x)-\psi(y)|^2}{|x-y|^2} \leq Ct^2\|\psi\|^2_{C^{0,1}}\end{aligned}$$

with constant independent of ϵ, as desired. □

Proposition 3.6 (Control of T_1)**.** *Suppose ψ is Lipschitz and let T_1 be as in (3.7). Suppose ψ solves (3.1). Then,*

$$|T_1| \lesssim \beta N^2 t^2\left(1+\|\xi\|_{C^{0,1}}\right)\left(\|\psi\|_{L^\infty}+\|\psi\|^2_{C^{0,1}}\right) + |t|N\|\psi\|_{C^{0,1}}. \tag{3.12}$$

In particular, if $|t| \sim \frac{1}{N}$, T_1 is order 1.

Proof. Let's start with

$$-\beta N\int\left(\int(\mathsf{g}(\phi_t(x)-\phi_t(y))-\mathsf{g}(x-y))\,d\mu_V(y)+(V_t\circ\phi_t-V)(x)\right)\,d\mathrm{fluct}_N(x).$$

We can rewrite the integrand via Taylor expansion using Lemma 3.5 as

$$\begin{aligned}\int \mathsf{g}(\phi_t(x)-\phi_t(y))\,d\mu_V(y)+V(\phi_t(x))+t\xi(\phi_t(x))-\int\mathsf{g}(x-y)\,d\mu_V(y)-V(x) \\ = t\left[\int\nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))\,d\mu_V(y)+\nabla V(x)\cdot\psi(x)+\xi(x)\right] \\ + O\left(t^2(1+\|\xi\|_{C^{0,1}})\left(\|\psi\|_{L^\infty}+\|\psi\|^2_{C^{0,1}}\right)\right)\end{aligned}$$

where we have taken $\epsilon\downarrow 0$ in the error terms in the expansion of g, since the O is independent of ϵ. Note that the expression in brackets is equivalent to (3.1), and in particular is constant by assumption (so its fluctuations are zero). Bounding the fluctuation measure by $\sim N$ we find

$$\begin{aligned}\left|-\beta N\int\left(\int(\mathsf{g}(\phi_t(x)-\phi_t(y))-\mathsf{g}(x-y))\,d\mu_V(y)+(V_t\circ\phi_t-V)(x)\right)\,d\mathrm{fluct}_N(x)\right| \\ \lesssim \beta N^2t^2\left(1+\|\xi\|_{C^{0,1}}\right)\left(\|\psi\|_{L^\infty}+\|\psi\|^2_{C^{0,1}}\right).\end{aligned} \tag{3.13}$$

Finally, we use again that $\|\log\det D\phi_t\|_{L^\infty} \lesssim t\|\psi\|_{C^{0,1}}$ and control the fluctuation measure again by $\sim N$ to bound

$$\left|\int\log\det D\phi_t\,d\mathrm{fluct}_N(x)\right| \lesssim |t|N\psi\|_{C^{0,1}},$$

as desired. □

We conclude by a careful analysis of T_2. The argument is again a Taylor expansion, appealing also now to the commutator type estimates of [RS22].

Proposition 3.7 (Control of T_2)**.** *Let ψ be Lipschitz, and suppose T_2 is as in (3.8). Then,*

$$\|T_2\| \lesssim \beta N^2t^2\|\psi\|^2_{C^{0,1}} + \beta|t|N\|\psi\|_{C^{0,1}} + \beta|t|\|\psi\|_{C^{0,1}}\left(F_N(X_N,\mu_V)+\frac{N}{2}\log N\right).$$

Proof. We Taylor expand as before, writing

$$\begin{aligned}\mathsf{g}(\phi_t(x)-\phi_t(y))-g(x-y)&=-t\frac{x-y}{|x-y|^2}\cdot(\psi(x)-\psi(y))+O\left(t^2\|\psi\|^2_{C^{0,1}}\right)\\&=t\nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))+O\left(t^2\|\psi\|^2_{C^{0,1}}\right).\end{aligned}$$

The error term is immediately controlled by $\beta N^2t^2\|\psi\|^2_{C^{0,1}}$ by controlling the fluctuation measure by $\sim N$. For the main term, we use the commutator estimate of [RS22, Theorem 1.1]:

$$\left|\int\nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))\;d\mathrm{fluct}_N(x)d\mathrm{fluct}_N(y)\right|\lesssim\|\psi\|_{C^{0,1}}\left(F_N(X_N,\mu_V)+\frac{N}{2}\log N+N\right),$$

which establishes the result. □

Coupling all of this together yields Proposition 3.1.

Proof of Proposition 3.1. Coupling Propositions 3.2, 3.3, 3.4, 3.6 and 3.7 we find

(3.14)
$$\begin{aligned}\mathbb{E}\left[\exp\left(-\beta tN\mathrm{Fluct}_N(\xi)\right)\right]\lesssim\exp\Big(&\beta N^2t^2\left(\|\psi\|_{L^\infty}+\|\psi\|^2_{C^{0,1}}+\|\xi\|_{C^{0,1}}\|\psi\|_{L^\infty}+\|\xi\|_{C^{0,1}}\|\psi\|^2_{C^{0,1}}\right)\\&+\max(1,\beta)N|t|\|\psi\|_{C^{0,1}}\Big)\mathbb{E}\left[\exp\left(\beta|t|\|\psi\|_{C^{0,1}}\left(F_N(X_N,\mu_V)+\frac{N}{2}\log N\right)\right)\right].\end{aligned}$$

Using the expansion of the partition function from [SS15] or proceeding as in [LS18, Lemma 2.15], we find

$$\mathbb{E}\exp\left(t\left(F_N+\frac{N}{2}\log N\right)\right)\le e^{C\frac{|t|}{\beta}N}$$

for any $|t|\le\frac{\beta}{2}$. Inserting this into (3.14) and simplifying somewhat, we find

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\beta tN\mathrm{Fluct}_N(\xi)\right)\right]\lesssim\\&\qquad\exp\left(\beta N^2t^2\left(\left(\|\psi\|_{L^\infty}+\|\psi\|^2_{C^{0,1}}\right)\left(1+\|\xi\|_{C^{0,1}}\right)\right)+\max(1,\beta)N|t|\|\psi\|_{C^{0,1}}\right),\end{aligned}$$

as desired. □

Constructing a transport as in [LS18, Lemma 3.4] allows us to invert (3.1) and obtain a concentration result with less regularity on the test function ξ. This is Proposition 1.

Proof of Proposition 1. We can apply Proposition 3.1 as soon as we can invert (3.1). This can be accomplished using the transport map introduced in [LS18, Theorem 3.4], which solves

$$\begin{cases}\mathrm{div}\,(\mu_V\psi)=-\frac{1}{\mathsf{c_d}}\Delta\xi & \text{in }\Sigma_i\\ \psi\cdot\vec{n}=\frac{1}{\mathsf{c_d}\mu_V}[\nabla\xi^\Sigma]\cdot\vec{n} & \text{on }\partial\Sigma_i\\ \psi=\left(\xi^\Sigma-\xi\right)\frac{\nabla\zeta_V}{|\nabla\zeta_V|^2}+\psi^\perp & \text{in }\Sigma^c\end{cases}\tag{3.15}$$

on each Σ_i, where $\psi^\perp$ is orthogonal to $\nabla\zeta_V$ and is chosen to make the transport map continuous at the boundary in the tangential direction (it is continuous in the normal direction automatically by the behavior of $\nabla\zeta_V$ at the interface). $[\nabla\xi^\Sigma]$ denotes the jump in the harmonic extension across the interface of the droplet.

In the one-cut regime, no additional assumptions are needed to guarantee a solution since $\int_{\partial\Sigma}\nabla\xi^\Sigma\cdot\widehat{n}=0$; this can be seen by integrating by parts in B_R with $R\to+\infty$ and using classical gradient estimates on harmonic functions. In the multicut regime this may not be true on every connected component, hence the additional assumption (1.18) as in [LS18, Lemma 3.4].

Let us discuss briefly why the transport map above inverts our equation (3.1). Recall that we want to solve

$$\psi \cdot \nabla \zeta_V + \xi - \int \nabla \mathsf{g}(x-y) \cdot \psi \mu_V(y) = c_\xi$$

for some constant c_ξ. In Σ, $\nabla \zeta_V = 0$; integrating by parts in y and substituting (3.15) inside Σ and on $\partial\Sigma$ we find

$$\begin{aligned}\xi(x) - \int \nabla \mathsf{g}(x-y) \cdot \psi \mu_V(y) &= \xi(x) - \int_\Sigma \mathsf{g}(x-y) \mathrm{div}\,(\psi\mu_V)(y) + \int_{\partial\Sigma} \mathsf{g}(x-y)\psi\mu_V(y) \cdot \widehat{n} \\ &= \xi(x) - \int_\Sigma \mathsf{g}(x-y) \frac{-\Delta\xi}{\mathsf{c_d}}(y) + \int_{\partial\Sigma} \mathsf{g}(x-y)[\nabla \xi^\Sigma] \cdot \widehat{n}.\end{aligned}$$

Outside, $\psi \cdot \nabla\zeta_V = \xi^\Sigma - \xi$, so

$$\psi \cdot \nabla \zeta_V + \xi - \int \nabla \mathsf{g}(x-y) \cdot \psi\mu_V(y) = \xi^\Sigma(x) - \int_\Sigma \mathsf{g}(x-y)\frac{-\Delta\xi}{\mathsf{c_d}}(y) + \int_{\partial\Sigma} \mathsf{g}(x-y)[\nabla\xi^\Sigma] \cdot \widehat{n}.$$

Let $w(x)$ denote

$$w(x) = \begin{cases} \xi(x) - \int_\Sigma \mathsf{g}(x-y)\frac{-\Delta\xi}{\mathsf{c_d}}(y) + \int_{\partial\Sigma} \mathsf{g}(x-y)[\nabla\xi^\Sigma]\cdot\widehat{n} & \text{if } x \in \Sigma \\ \xi^\Sigma(x) - \int_\Sigma \mathsf{g}(x-y)\frac{-\Delta\xi}{\mathsf{c_d}}(y) + \int_{\partial\Sigma} \mathsf{g}(x-y)[\nabla\xi^\Sigma]\cdot\widehat{n} & \text{if } x \in \Sigma^c. \end{cases} \tag{3.16}$$

By classical results on single-layer potentials (cf. [DL90, Sec. II.3], [Fol95, Chapter 3] and [SS18, Appendix A] for a review) $\int_{\partial\Sigma} \mathsf{g}(x-y)[\nabla\xi^\Sigma]\cdot\widehat{n}$ is harmonic in Σ and Σ^c. Furthermore, for $x \in \Sigma$

$$\Delta\left(\xi(x) - \int_\Sigma \mathsf{g}(x-y)\frac{-\Delta\xi}{\mathsf{c_d}}(y)\right) = \Delta\xi - \Delta\xi = 0$$

and for $x \in \Sigma^c$,

$$\Delta\left(\xi^\Sigma(x) - \int_\Sigma \mathsf{g}(x-y)\frac{-\Delta\xi}{\mathsf{c_d}}(y)\right) = 0 - 0 = 0$$

so w is harmonic in Σ and Σ^c. Since $\nabla\mathsf{g}$ is locally integrable, $\int_\Sigma \mathsf{g}(x-y)\frac{-\Delta\xi}{\mathsf{c_d}}(y)$ has a continuous normal derivative across the interface $\partial\Sigma$. Using classical formulae for the jump in the normal component of the gradient of a single layer potential (cf. [SS18, Theorem A.1], [DL90])] we have

$$\partial_{\widehat{n},\mathrm{out}} w - \partial_{\widehat{n},\mathrm{in}} w = [\nabla\xi^\Sigma]\cdot\widehat{n} - [\nabla\xi^\Sigma]\cdot\widehat{n} = 0$$

where $\partial_{\widehat{n},\mathrm{out}} w$ and $\partial_{\widehat{n},\mathrm{in}} w$ denote the normal derivatives of w taken from outside and inside Σ, respectively. So, no divergence is created at the boundary when we make the piecewise definition (3.16) for $w(x)$. As a result, $w(x)$ is harmonic on $\mathbb{R}^2$ and is bounded by $O(\log|x|)$ as $|x| \to +\infty$ and is therefore constant by Liouville's theorem. Hence, the map ψ given in (3.15) solves (3.1).

Finally, since $\Delta\xi \in C^{0,\alpha}(\Sigma)$ we have the Schauder estimate $\|\psi\|_{C^{1,\alpha}(\Sigma)} \lesssim \|\Delta\xi\|_{C^{0,\alpha}(\Sigma)}$. In $U \setminus \Sigma$ the map is Lipschitz away from the boundary; we also know that it is continuous across to the boundary by choice of the boundary condition and so we have the estimate

$$\|\psi\|_{C^{0,1}(U)} \lesssim \|\xi\|_{C^{0,1}(U)} + \|\Delta\xi\|_{C^{0,\alpha}(\Sigma)}$$

Substituting this into (3.2) yields the estimate (1.17). □

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