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EQUICONTRACTIVE WEAK SEPARATION PROPERTY ON THE LINE DOES NOT IMPLY CONVEX FINITE TYPE CONDITION

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ABSTRACT. Let $\{S_1, S_2, \ldots, S_n\}$ be an iterated function system on $\mathbb{R}$ with attractor K. It is known that if the iterated function system satisfies the weak separation property and $K = [0, 1]$ then the iterated function system also satisfies the convex finite type condition. We show that the condition $K = [0, 1]$ is necessary. That is, we give two examples of iterated function systems on $\mathbb{R}$ satisfying weak separation condition, and $0 < \dim_H(K) < 1$ such that the IFS does not satisfy the convex finite type condition.

1. INTRODUCTION

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a set of linear contractions. It is well know [3, 4, 7] that there exists a unique non-empty compact set K such that $K = \cup S_i(K)$. In this case we call $\{S_1, S_2, \ldots, S_n\}$ an *iterated function system (IFS)* and we call K the *attractor* of the IFS.

In this paper we will assume $S_i : \mathbb{R} \to \mathbb{R}$. We further assume the maps are equicontractive in the positive direction. That is, there exists an $0 < r < 1$ such that for all i, $S_i(x) = rx + d_i$.

One common question in the study of IFSs is, under what conditions can we exactly compute the Hausdorff dimension of K.

The first such condition is called the *strong separation property (SSP)*. We say $\mathcal{S}$ satisfies the SSP if

$$\min_{i \neq j} \left(\inf_{\substack{x \in S_i(K), \\ y \in S_j(K)}} |x - y| \right) > 0.$$

Under this condition all images of K are very well separated. The dimension can be computed exactly as the unique $s > 0$ such that $\sum_{i=1}^n r_i^s = 1$. In our restricted setting, where all contraction ratios are the same, this gives the dimension as $-\frac{\log(n)}{\log(r)}$.

The next such properties are the *open set condition, (OSC)* and the *convex open set condition ($OSC_{\rm co}$)*. We say $\mathcal{S}$ satisfies the OSC if there exists an non-empty open set V such that $S_i(V) \subset V$ for all i, and $S_i(V) \cap S_j(V) = \emptyset$ for all $i \neq j$. We say $\mathcal{S}$ satisfies $\mathrm{OSC}_{\rm co}$ if it satisfies OSC with $V = \mathrm{int}(\mathrm{hull}(K))$. It is easy to see that if K satisfies SSP then it satisfies OSC.

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Thi next conditions considered is the *weak separation property (WSP)*. We will define this for the equicontractive case only. Care needs to be taken when adapting this to a non-equicontractive setting, or higher dimensions. Let $\sigma, \tau \in \Sigma^k = \{1, 2, \ldots, n\}^k$. That is $\sigma = s_1 s_2 \ldots s_k$ and $\tau = t_1 t_2 \ldots t_k$ where $s_i, t_i \in \{1, 2, \ldots, n\}$. We write $S_\sigma = S_{s_1} \circ S_{s_2} \circ \cdots \circ S_{s_k}$, and similarly for S_τ. We see that $S_\sigma^{-1} \circ S_\tau(x) = x + a_{\sigma,\tau}$ for some real number $a_{\sigma,\tau}$. We say $\mathcal{S}$ satisfies the WSP if

$$\inf_{\substack{\sigma,\tau \in \Sigma^k \\ k \geq 0 \\ a_{\sigma,\tau} \neq 0}} |a_{\sigma,\tau}| > 0.$$

It is clear that if $\mathcal{S}$ satisfies $\mathrm{OSC}_{\mathrm{co}}$, then it satisfies WSP. In this case the above infimum would be bounded below by $|K|$, the diameter of K. A stronger statement is true. It is known that $\mathcal{S}$ satisfies OSC if and only if it satisfies WSP and there are no exact overlaps. That is, there do not exist $\sigma \neq \tau \in \Sigma^*$ such that $S_\sigma = S_\tau$. (Here $\Sigma^* = \cup \Sigma^k$.) See for example [2, Theorem 4.2.11].

We can think of SSP as saying images of K do not overlap, OSC as saying images of K do not overlap in a meaningful way, and WSP as saying images of K either overlap exactly, or overlap in a way that isn't too close. The last condition, finite type condition can be thought of as saying that there are only a finite number of ways images of K can overlap with each other.

The last separation conditions to be discussed are the *finite type condition (FTC)* and the *convex finite type condition ($\mathit{FTC}_{\mathrm{co}}$)*. We will define this for the equicontractive case. Let V be a non-empty open set such that $S_i(V) \subset V$. We say $\sigma, \tau \in \{1, 2, \ldots, n\}^k$ are neighbours if $S_\sigma(V) \cap S_\tau(V) \neq \emptyset$. We define the neighbourhood type of S_σ as

$$N(S_\sigma) = \{S_\sigma^{-1} \circ S_\tau : S_\tau \text{ is a neighbour of } S_\sigma\}.$$

We say $\mathcal{S}$ satisfies the FTC if there exists a open set V for which there are a finite number of neighbourhood types. We say $\mathcal{S}$ satisfies the $\mathrm{FTC}_{\mathrm{co}}$ if it satisfies the FTC with $V = \mathrm{int}(\mathrm{hull}(K))$.

It is possible to extend the definition of (convex) finite type condition to (convex) generalized finite type condition. This allows for contractions that are not equicontractive, or logarithmically commensurate. This level of generalization is not needed for this paper. We refer the reader to [6, 8, 9]. The key point needed is that if $\mathcal{S}$ satisfies (convex) finite type condition then it satisfies (convex) generalized finite type condition. It is shown in [6] that convex generalized finite type condition is equivalent to finite neighbour condition, which again we will not define here.

It is clear that if $\mathcal{S}$ satisfies $\mathrm{FTC}_{\mathrm{co}}$, then it satisfies WSP. The stronger statement is true. It was shown in [10] that if $\mathcal{S}$ satisfies FTC, then it satisfies WSP. It was asked in [8] if the converse is also true. That is, if $\mathcal{S}$ satisfies WSP then is it true that it satisfy general finite type condition.

Despite the stronger conditions on FTC than WSP, it is not clear if in fact this conditions in fact give rise to different IFS. There are examples known in $\mathbb{R}^d$ with $d \geq 2$ which satisfy WSP but not FTC [8, 11]. These examples have the attractor K within a hyperplane of $\mathbb{R}^d$, and use rotations around this hyperplane to ensure that the example does not satisfy FTC. When these maps are renormalized so that they are maps from the hyperplane to the hyperplane, then these IFSs again satisfy FTC. In the other direction, it is shown that if $\mathcal{S}$ satisfies the WSP and $K = [0, 1]$ then $\mathcal{S}$ satisfies the $\mathrm{FTC}_{\mathrm{co}}$. Initially this was proved in [5] with the added restriction that the IFS was equicontractive, and all contractions were in the positive orientation. This proof was later extended in [6] to allow for non-equicontractive maps, and contractions with negative orientation. The proof in [6] still required $K = [0, 1]$. More precisely,

Corollary 1.1 (Corollary 4.6 of [6])**.** *Suppose the IFS $\mathcal{S}$ has self-similar set $[0, 1]$. Then then following are equivalent:*

(1) *$\mathcal{S}$ satisfies the weak separation property;*
(2) *$\mathcal{S}$ satisfies the finite neighbour condition;*
(3) *$\mathcal{S}$ satisfies the convex generalized finite type condition;*
(4) *There exists some $c > 0$ such that for any $0 < \alpha < 1$, words $\sigma, \tau \in \Lambda_\alpha$ and $z, w \in \{0, 1\}$, either $S_\sigma(z) = S_\tau(w)$ or $|S_\sigma(z) - S_\tau(w)| > c\alpha$.*

It was then asked if this was true if we relax the hypothesis.

In this paper we give an two examples of IFSs $\mathcal{S}$ with $[0, 1] \neq K \subset [0, 1]$ and where (1) and (4) are true, but (2) and (3) are not. The first example, given in Section 2 has the advantage that it is simpler to construct. It has the property that it satisfies WSP, OSC, FTC and does not satisfy OSC$_{co}$ nor FTC$_{co}$.

It is worth noting that if $\{S_1, S_2, \dots, S_n\}$ satisfies the OSC, then $\{S_1, S_1', S_2, \dots, S_n\}$ with $S_1 = S_1'$ will satisfy FTC, but will not satisfy OSC. This is because there is an exact overlap of level one cylinders of the maps. This is a trivial reason, and no of interest as an example.

The second example, given in Section 3 is a more complicated construction. It has the property that it satisfies WSP, FTC, and does not satisfy OSC, OSC$_{co}$ nor FTC$_{co}$. There is an exact overlap of level two cylinders, but no exact overlap of level one cylinders.

Both examples are equicontractive and satisfy $0, 1 \in K \subset [0, 1]$.

2. Construction of first example

2.1. Overview of IFS.

The constructed IFS will be of the form

$$\mathcal{S} = \{S_1, S_2, S_3\}$$

where

$$\begin{aligned} S_1(x) &= x/7 \\ S_2(x) &= x/7 + a \\ S_3(x) &= x/7 + \frac{6}{7} \end{aligned}$$

Here a is chosen such that the IFS satisfies the WSP and does not satisfy FTC$_{co}$. One choice gives an approximate value of a as

$$a \approx \frac{0.9482520978}{7} \approx 0.1354645854 \tag{1}$$

A precise description of a will be given in Section 2.6.

2.2. **Constructing the IFS.**

As we wish to show that the IFS does not satisfy $\mathrm{FTC}_{\mathrm{co}}$, and $0, 1 \in K \subset [0, 1]$ we may assume the open set in the definition of FTC is $(0, 1)$. Let $\sigma^{(1)} = 1$ and $\tau^{(1)} = 2$. We see that $S_{\sigma^{(1)}}(0) < S_{\tau^{(1)}}(0) < S_{\sigma^{(1)}}(1)$ if and only if $a \in J_1 := (0, 1/7)$. We see in this case that $N(S_{\sigma^{(1)}}) = \{\mathrm{Id}, S_{\sigma^{(1)}}^{-1} \circ S_{\tau^{(1)}}\}$ is a non-trivial neighbourhood type with two elements.

Assume what have a $\sigma^{(n)}$, $\tau^{(n)}$ and J_n such that $S_{\sigma^{(n)}}(0) < S_{\tau^{(n)}}(0) < S_{\sigma^{(n)}}(1)$ if and only if $a \in J_n$. We will extend these to a $\sigma^{(n+1)}$, $\tau^{(n+1)}$ and J_{n+1} such that these properties continue to hold. Further, we will ensure that for all $a \in J_{n+1}$ and all $\sigma \neq \tau$ of length n that $|S_\sigma^{-1} \circ S_\tau| \geq 4/7$. This will show that for $a \in \cap J_n$ that the IFS satisfies the WSP. As the J_n are open sets, isn't immediately clear that $\cap J_n$ is non-empty. This will be addressed later.

We extend $\sigma^{(n)}, \tau^{(n)}$ and J_n in one of two different ways at each step to ensure these properties continue to hold. We then exploit the fact that we have an infinite number of choices to ensure that $\cap J_n$ is non-empty, and such that for $a \in \cap J_n$ we have that $N(S_{\sigma^{(n)}})$ are all distinct.

Assume we have such a $\sigma^{(n)}, \tau^{(n)}$ and J_n. We see that both $S_{\sigma^{(n)}}(0)$ and $S_{\tau^{(n)}}(0)$ are linear functions in a. Further, for $\sigma^{(n)} = s_1 s_2 \dots s_n$, we see that the slope of this function is $\sum_{i : s_i = 1} \frac{1}{7^i}$. We create $\sigma^{(n)}$ and $\tau^{(n)}$ by extending $\sigma^{(n-1)}$ and $\tau^{(n-1)}$, which in turn were initially extended from $\sigma^{(1)} = 1$ and $\tau^{(1)} = 2$. (We occasionally reverse the roles of $\sigma^{(k)}$ and $\tau^{(k)}$ so we do not know which one was an extension of $\sigma^{(1)}$.) As one of $\sigma^{(n)}$ and $\tau^{(n)}$ has initial term 1 and one of them has initial term 2 we see $S_{\sigma^{(n)}}(0) - S_{\tau^{(n)}}(0)$ is a non-constant linear function with respect to a. Denote this as $T_n(a) = S_{\sigma^{(n)}}(0) - S_{\tau^{(n)}}(0)$.

We will choose J_{n+1} in one of two ways. The first option is to choose J_{n+1} such that $a \in J_{n+1}$ if and only if $S_{\sigma^{(n)}3}(0) < S_{\tau^{(n)}1}(0) < S_{\sigma^{(n)}3}(1)$. In this case we would set $\sigma^{(n+1)} = \sigma^{(n)}3$ and $\tau^{(n+1)} = \tau^{(n)}1$.

The second option is to choose J_{n+1} such that $a \in J_{n+1}$ if and only if $S_{\tau^{(n)}2}(0) < S_{\sigma^{(n)}3}(0) < S_{\tau^{(n)}2}(1)$. In this case we would set $\sigma^{(n+1)} = \tau^{(n)}2$ and $\tau^{(n+1)} = \sigma^{(n)}3$.

We see on J_n that $T_n(a) = S_{\tau^{(n)}}(0) - S_{\sigma^{(n)}}(0)$ is a non-constant linear function whose image is $(0, 1/7^n)$.

To show the existence of an J_{n+1} for the first option, let $J_{n+1} \subset J_n$ such that $T_n(J_{n+1}) = (6/7^{n+1}, 1/7^n)$. Note that $S_{\sigma^{(n+1)}}(0) = S_{\sigma^{(n)}3}(0) = S_{\sigma^{(n)}}(0) + 6/7^{n+1}$ and $S_{\tau^{(n+1)}}(0) = S_{\tau^{(n)}}(0)$. Hence, on this range $T_{n+1}(a) = S^{\tau^{(n)}1}(0) - S^{\sigma^{(n)}3}(0) = S_{\tau^{(n)}}(0) - S^{\sigma^{(n)}}(0) - 6/7^{n+1}$ has image $(0, 1/7^{n+1})$. This proves that $a \in J_{n+1}$ if and only if $S_{\sigma^{(n+1)}}(0) < S_{\tau^{(n+1)}}(0) < S_{\sigma^{(n+1)}}(1)$ as required.

The proof the existence of J_{n+1} for the second option is similar, by letting $J_{n+1} \subset J_n$ be such that $T_n(J_{n+1}) = (4/7^{n+1}, 5/7^{n+1})$.

We alternate between these options in a non-periodic way.

2.3. **The IFS satisfies WSP.**

From the above we note that if J_{n+1} is chosen from the first option, we have for all $a \in J_{n+1}$ we have $|S_{\sigma^{(n)}}^{-1} \circ S_{\tau^{(n)}}(0)| \geq 6/7$. If instead J_{n+1} is chosen as the second option, we have $|S_{\sigma^{(n)}}^{-1} \circ S_{\tau^{(n)}}(0)| \geq 4/7$. By noting $J_{n+1} \subset J_n \subset J_{n-1} \subset \cdots \subset J_1$ we see for all $a \in J_{n+1}$ and all $k \leq n$ we have $|S_{\sigma^{(k)}}^{-1} \circ S_{\tau^{(k)}}(0)| \geq 4/7$. Lastly we see for all $\sigma' \neq \tau'$ with $|\sigma'| = |\tau'| \leq n$ that either $|S_{\sigma'}^{-1} \circ S_{\tau'}(0)| \geq 1$ or it is the same as $|S_{\sigma^{(k)}}^{-1} \circ S_{\tau^{(k)}}(0)|$ for some $k \leq n$. Assuming $\cap J_n$ is non-empty, taking $a \in \cap J_n$ gives $\mathcal{S}$ satisfies the WSP.

As we choose between the two options in a non-periodic way, we see that we choose second option infinitely often. As such, we have that $\cap J_n = \cap \text{clos}(J_n)$ is non-empty, and a singleton. This is our value a.

2.4. **The IFS satisfies OSC$_{\text{co}}$ but not OSC.**

It is easy to see that this does not satisfy the convex open set condition.

To see that it satisfies the OSC, we will construct an open set V such that $\cup S_i(V) \subset S$ and $S_i(V) \cap S_j(V) = \emptyset$ for $i \neq j$.

Let $I_n = \cup_{|\sigma|=n} S_\sigma([0,1])$. We see that $K = \cap I_n$. Let $V_n = \cup_{|\sigma|=n} S_{\sigma((3/7,4/7))}$. Set $V = \cup V_n$.

By construction we see that $S_1(V), S_2(V), S_3(V) \subset V$. It is easy to see that $S_1(V) \cap S_3(V) = S_2(V) \cap S_3(V) = \emptyset$. So it remains to show that $S_1(V) \cap S_2(V) = emptyset$.

We note that $S_1(V_0) \cap S_2(V_0) = \emptyset$ for trivial reasons. By noting that $V_n \subset I_n$ and $V_n \cap I_{n+1} = \emptyset$ for all n, we see that if $n_1 \neq n_2$ then $S_1(V_{n_1}) \cap S_2(V_{n_2}) = \emptyset$ Hence we need only check that $S_1(V_n) \cap S_2(V_n) = \emptyset$ for all n. Note $\sigma^{(1)} = 1$ and $\tau^{(1)} = 2$. By our construction of a we can see that $S_{\sigma^{(1)}}(V_n) \cap S_{\tau^{(1)}}(V_n) = S_{\sigma^{(2)}}(V_{n-1}) \cap S_{\tau^{(2)}}(V_{n-1})$. Continuing in this way we get $S_{\sigma^{(1)}}(V_n) \cap S_{\tau^{(1)}}(V_n) = S_{\sigma^{(n+1)}}(V_0) \cap S_{\tau^{(n+1)}}(V_0)$. The last intersection is empty.

This proves that $\mathcal{S}$ satisfies OSC.

2.5. **The IFS does not satisfy FTC$_{\text{co}}$.**

As the IFS satisfies OSC, it satisfies FTC with the same V. In this case it would have only one neighbourhood type, namely $\{\text{Id}\}$.

As we choose between the two options in a non-periodic way, see that $N(S_{\sigma^{(n)}})$ will all be distinct neighbourhood sets. (Recall this neighbourhood type is taken with respect to the open set $(0,1) = \text{int}(\text{hull}(K))$.) To see this we note that $\sigma^{(n)}$ and $\tau^{(n)}$ are completely determined by a. Let $\sigma^{(n)}$ be the length n word determined by a, and similarly for $\tau^{(n)}$. We see that if $7^n(S_{\sigma^{(n)}}(0) - S_{\tau^{(n)}}(0)) = 7^m(S_{\sigma^{(m)}}0) - S_{\tau^{(m)}}(0))$, then we get then we have $7^{n+k}(S_{\sigma^{(n+k)}}(0) - S_{\tau^{(n+k)}}(0)) = 7^{m+k}(S_{\sigma^{(m+k)}}0) - S_{\tau^{(m+k)}}(0))$ for all k. As such, this would imply the choice between two options would be eventually periodic.

This proves that $\mathcal{S}$ does not satisfy FTC$_{\text{co}}$.

2.6. **Example.**

For the a given in equation (1), we choose the first option or second option depending on if the nth term of the Thue-Morse sequence was 0 or 1. (See for example [1].) That is, the first five choices are first option, second option, second option, first option, and second option. Any non-periodic sequence would have worked.

In Figure 2.1 we present the overlap at level n for each new type.

The first graph is the level 1 maps, $S_0([0,1]), S_1([0,1])$ and $S_2([0,1])$. We note that $S_0([0,1])$ and $S_1([0,1])$ overlap, as is shown by the vertical lines. This is the first non-trivial neighbourhood type. Here $\sigma^{(1)} = 0$ and $\tau^{(1)} = 1$.

For the second graph, we expand to level 2 cylinders the non-trivial neighbourhood type from the first graph. The upper three intervals are those coming from the right neighbour of the neighbourhood type from the first graph, and the lower three intervals from the left most neighbour. We see that the $S_{02}([0,1])$ overlaps $S_{10}([0,1])$, as indicated by the vertical lines. This is the second non-trivial neighbourhood type. It is worth noting that we also have overlaps at $S_{00}([0,1])$ with

$S_{01}([0,1])$ and $S_{10}([0,1])$ with $S_{11}([0,1])$. Neither of these are new neighbourhood types, as they are equivalent to those found at the first level. Here $\sigma^{(2)} = 02$ and $\tau^{(2)} = 10$, as we choose the first option.

We continue in this manner, expanding the new neighbourhood type found at level $n-1$ to the level n cylinders, and observing that there is a new neighbourhood type at level n.

3. Construction of second example

This is very similar to the first example. We have two additional maps to force an exact overlap of level 2 cylinders. This exact overlap allows us to show that it does not satisfy OSC.

The constructed IFS will be of the form

$$\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$$

where

$$\begin{aligned}
S_1(x) &= x/16 \\
S_2(x) &= x/16 + a \\
S_3(x) &= x/16 + 15/16 - 16a \\
S_4(x) &= x/16 + 11/16 \\
S_5(x) &= x/16 + 15/16
\end{aligned}$$

Here a is chosen such that the IFS satisfies the weak separation property, but does not satisfy the convex finite type condition. Further, a is chosen so that the resulting IFS does not satisfy OSC, as there is an exact overlap given by $S_{15} = S_{23}$.

One particular value of a (again based on the Thue-Morse sequence) is given by

$$a \approx \frac{0.7493705552}{16} \approx .04683565970 \tag{2}$$

$$15/16 - 16a \approx \frac{3.010071117}{16} \approx 0.1881294448$$

3.1. **Construction of IFS.**

Let $\sigma^{(1)} = 1$ and $\tau^{(1)} = 2$. Let $J_1 = (0, 1/16)$. We see $a \in J_1$ if and only if $S_{\sigma^{(1)}}(0) < S_{\tau^{(1)}}(0) < S_{\sigma^{(1)}}(1)$.

We set $\sigma^{(2)} = \sigma^{(1)}4$ and $\tau^{(2)} = \tau^{(1)}1$. We choose $J_2 \subset J_1$ such that $a \in J_2$ if and only if $S_{\sigma^{(2)}}(0) < S_{\tau^{(2)}}(0) < S_{\sigma^{(2)}}(1)$.

We next proceed as in Section 2,

The first option is to choose J_{n+1} such that $a \in J_{n+1}$ if and only if $S_{\sigma^{(n)}5}(0) < S_{\tau^{(n)}1}(0) < S_{\sigma^{(n)}5}(1)$. In this case we would set $\sigma^{(n+1)} = \sigma^{(n)}5$ and $\tau^{(n+1)} = \tau^{(n)}1$.

The second option is to choose J_{n+1} such that $a \in J_{n+1}$ if and only if $S_{\tau^{(n)}2}(0) < S_{\sigma^{(n)}5}(0) < S_{\tau^{(n)}2}(1)$. In this case we would set $\sigma^{(n+1)} = \tau^{(n)}2$ and $\tau^{(n+1)} = \sigma^{(n)}5$.

The proof is the same as before.

3.2. **The IFS satisfies WSP and not $\mathrm{FTC}_{\mathrm{co}}$, OSC, nor $\mathrm{OSC}_{\mathrm{co}}$.**

As before, it is easy to see that the IFS satisfies WSP and not $\mathrm{FTC}_{\mathrm{co}}$.. As there are exact overlaps, the IFS clearly does not satisfy OSC, nor $\mathrm{OSC}_{\mathrm{co}}$.

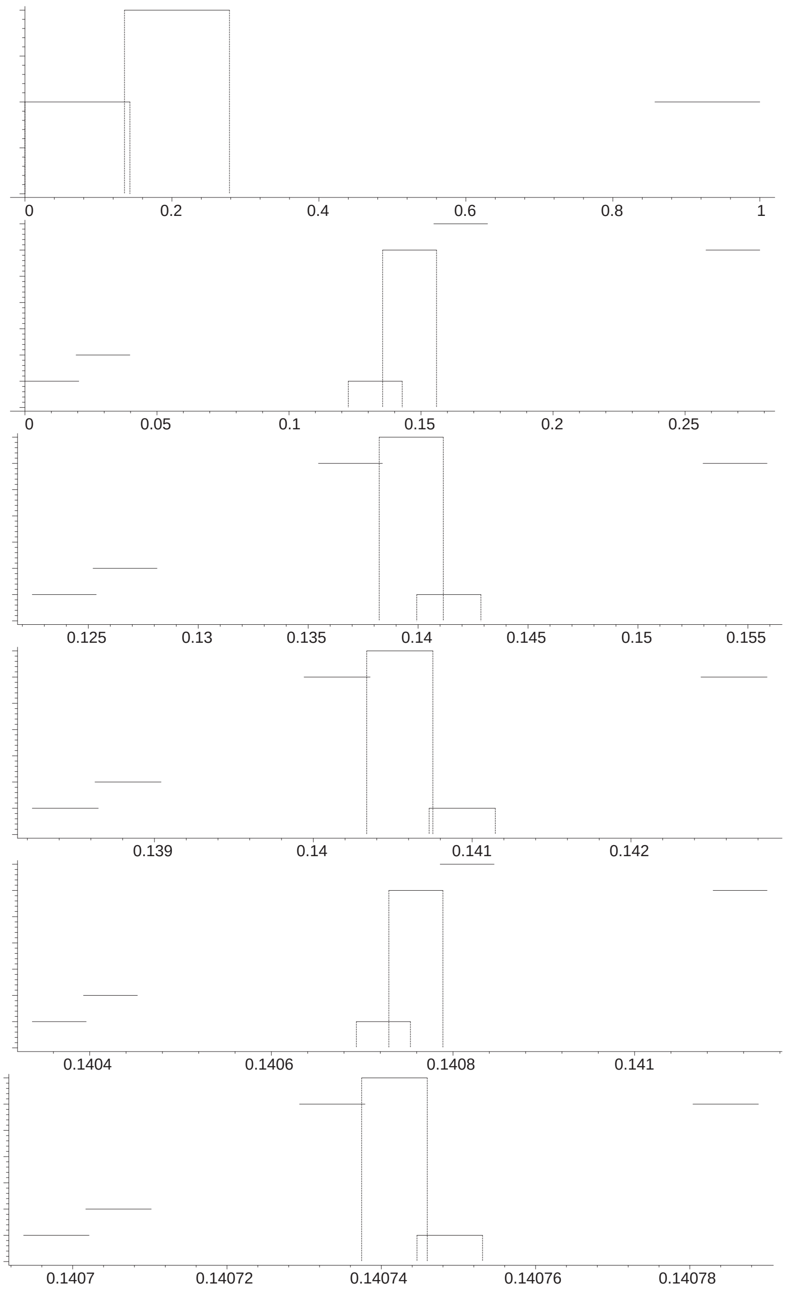

Figure 2.1: Level n cylinders for Example 2.6

3.3. **The IFS satisfies FTC.** Although the IFS does not satisfy FTC$_{co}$, nor OSC, it does satisfy FTC.

To see that it satisfies the FTC, it suffices to construct a open set V and demonstrate that there are only finitely many neighbourhood types with this open set.

Let $I_n = \cup_{|\sigma|=n} S_\sigma([0,1])$. We see that $K = \cap I_n$. Let $V_n = \cup_{|\sigma|=n} S_{\sigma((7/16,8/16))}$.

Let $V = \cup V_n$. We see by construction that $S_1(V), S_2(V), \ldots, S_5(V) \subset V$.

We see that $S_{23}(V) = S_{14}(V) \subset S_2(V) \cap S_1(V)$. Hence $N(S_2) = \{\mathrm{Id}, S_2^{-1} \circ S_1\}$ and $N(S_1) = \{\mathrm{Id}, S_1^{-1} \circ S_2\}$. There are no other overlaps at level 1, and hence $N(S_3) = N(S_4) = N(S_5) = \{\mathrm{Id}\}$.

For the children under S_1 we have $N(S_{11}) = N(S_1)$ and $N(S_{12}) = N(S_2)$. All other children have neighbourhood type $\{\mathrm{Id}\}$.

Similarly the children under S_2 have neighbourhood types $N(S_1), N(S_2)$ or $\{\mathrm{Id}\}$.

This gives that there are three neighbourhood types, namely

$$\{\mathrm{Id}\}, \qquad \{\mathrm{Id}, S_2^{-1} \circ S_1\}, \qquad \{\mathrm{Id}, S_1^{-1} \circ S_2\}.$$

which proves that this IFS satisfies FTC.

3.4. **Example.**

In Figure 3.2 we present the overlap at level n for each new type. We choose, in order, option 1, option 2, option 2, option 1, option 2, ... We choose this based on the Thue-Morse sequence, which is a well known non-periodic sequence, although any non-periodic sequence would have worked.

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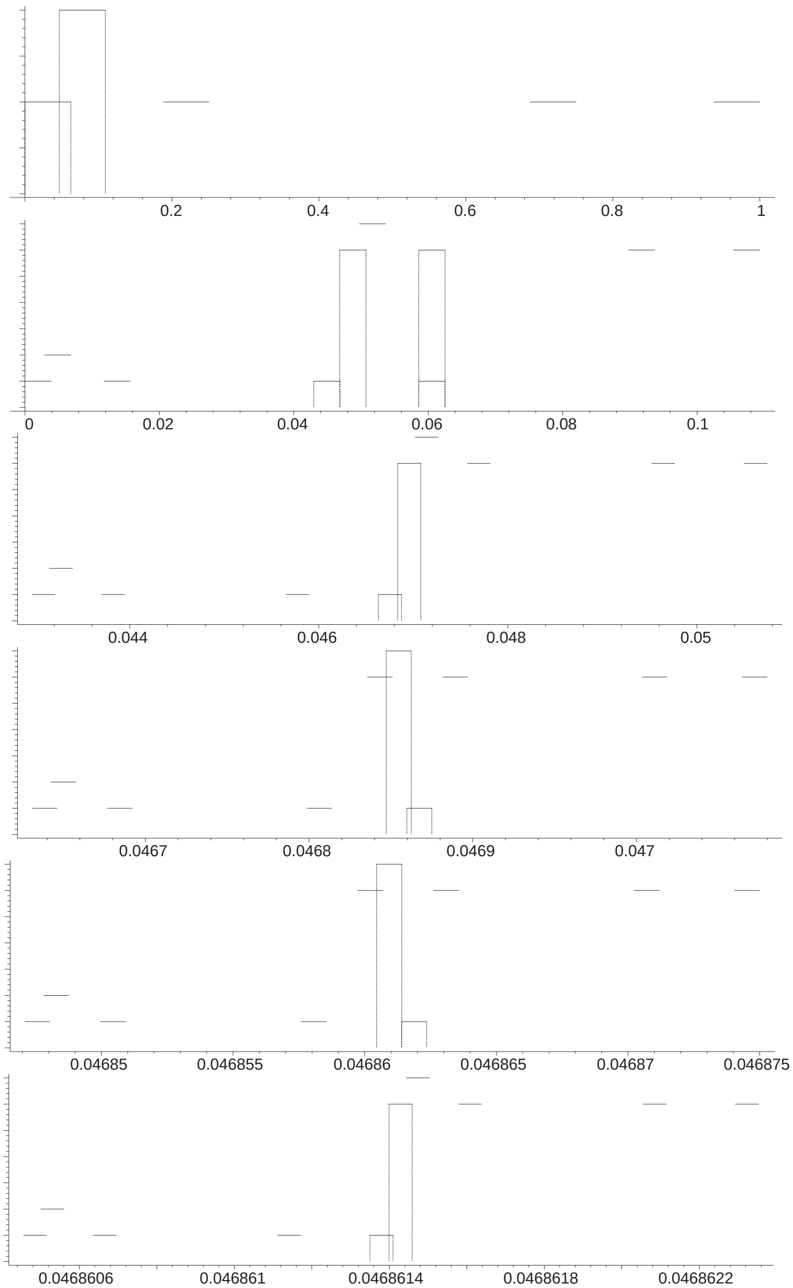

Figure 3.2: Level n cylinders for Example 3.4