Supersymmetric Quantum Fields via Quantum Probability

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Abstract. The super version of imprimitivity theorem is available now to describe global supersymmetry of systems using the representations of super Lie groups (SLG). This result uses the equivalence between super Harish-Chandra pairs and super Lie groups at the categorigal level and is applicable to super Poincaré group and generalizes a smooth SI to super context. We apply the result to build supersymmetric quantum fields. Towards this end, we set up a super fock space of a disjoint union of super Hilbert spaces which is equivalent to super tensoring of boson (even) part symmetrically and that of fermion (odd) part antisymmetrically of the super particle Hilbert space. This leads to a super fock space that is disjoint union of bosonic and fermionic spaces, that is Z_2 graded. We derive covariant Weyl operators for light-like fields, with the massless super spinorial multiplet as an illustrative example. First, we build a representation of a light-like little group in terms of Weyl operators. We then use this construction to induce a representation of Poincaré group to construct the fields via super version of imprimitivity theorem.

1. Introduction

Systems of imprimitivity (SI) is a way to characterize the unitary representations of a Lie group in a comprehensive way. SI is a composite object (G, Ω) of a representation of a group G and its action on a G-space Ω and we say it lives on Ω. Mackey machinery is a set of techniques to induce representations of a group, from that of a subgroup H, that are systems of imprimitivity. The configuration spaces of interest to us in this work are orbits of little groups (space-like, time-like, and light-like) defined on the forward mass hyperboloid and the homogeneous space G/H where H is a closed subgroup of G that consists of left cosets $gH, g \in G$. In the super context this homogeneous space stays the same as we assume the odd part of it as the entire super Lie algebra of G that forms the habitat of super systems of imprimitivity (SSI) with different systems live on various orbits. When SI construction is applied to the Poincare group the projective unitary irreducible representations (PUIR) form the quantum states of fundamental particles with Ω being the configuration spacetime of the particles. From SI characterizations we can derive the canonical commutation relations and infinitesimal forms in terms of differential equations (*Schrödinger*, Heisenberg, and Dirac etc) [13] [15] and [17]. The machinery originally applicable to semidirect products, Poincare group

2010 *Mathematics Subject Classification.* Primary Quantum theory; Secondary Relativity and gravitational theory.

Key words and phrases. supersymmetry; Harish-Chandar pair; superWeyl operators; super-Systems of Imprimitivity; superFock space; Light-like particles.

is a semidirect product of homogeneous Lorentz and $\mathbb{R}^4$, have been generalized in so many ways including a c*-algebraic version and a supersymmetric (SUSY) formulation. In this work, we will apply the SSI techniques to build Weyl operators on super fock spaces and then construct annihilation and creation field operators that are indexed by members of Poincare. In our earlier work we have constructed covariant Quantum Fields via Lorentz Group Representation of Weyl Operators [14]. Here we generalize them to supersymmetry by building the super fock spaces for massless super multiplets (spinorial) and then the covariant filed operators. We glossed over domain consideration and we refer the reader to Varadarajan et.al [2] work for a detailed discussion.

Definition 1.1. A super Hilbert space is a Z_2-graded super vector space $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ over $\mathbb{C}$ with a scalar product $(.,.)$ where the $\mathscr{H}_i (i = 0, 1)$, referred as even and odd, are closed mutually orthogonal subspaces. We set up the parity operator as

$$p(x) = \begin{cases} 0, & \text{if x } \in \mathscr{H}_1, \\ 1, & \text{if x } \in \mathscr{H}_2. \end{cases}$$

We define an even super Hilbert form

$$\langle x, y\rangle = \begin{cases} 0, & \text{if x and y are of opposite parity} \\ (x, y), & \text{if x and y are even} \\ i(x, y), & \text{if x and y are odd} \end{cases} .$$

We have

$$\langle y, x\rangle = (-1)^{p(x)p(y)}\overline{\langle x, y\rangle}.$$

If $T(\mathscr{H} \to \mathscr{H})$ is a bounded linear operator, we denote by T^* its Hilbert space adjoint and by $T^\dagger$ its super adjoint given by $\langle Tx, y\rangle = (-1)^{p(T)p(x)}\langle x, T^\dagger y\rangle$.

Definition 1.2. A super Lie group is (G_0, g) is a a super Harish-Chandra pair if G_0 is a classical Lie group and g is a super Lie algebra with an action of G_0 on it such that (i) $\text{Lie}(G_0)) = g_0 =$ the even part of g. (ii) The action of G_0 on g is the adjoint action of G_0; more precisely, the adjoint action of G_0 on g is the differential of the action of G_0 on g. A representation of a super Lie group is a triple $(\pi, \gamma, \mathscr{H})$ where π is an even representation of G_0 in a super Hilbert space $\mathscr{H}$ and γ is a super representation of g in $\mathscr{H}$.

Definition 1.3. A super Lie algebra is a super vector space g with a bilinear bracket $[,]$ such that g_0 is an ordinary Lie algebra with $[.,.]$ and g_1 is a g_0-module for the action $a \to ad(a) : b \to [a, b], (b \in g_1)$. Further, $a \otimes b \to [a, b]$ is a symmetric g_0-module map from $g_1 \otimes g_1$ to g_0. It also satisfies the nonlinear condition

$$[a, [a, a]] = 0, \forall g \in g_1.$$

One way to ensure this last condition is met is to ensure that the range of the odd bracket g_2 is a subset of g_0 which acts on g_1 trivially as

$$g_2 \subset g_0 \Rightarrow [g_1, g_0] = 0.$$

A super algebra A is an algebra of endomorphisms of linear maps on a super vector space V. The maps that preserve he grading of V are designated as even and that reverse them are called odd. To get a super Lie algebra from A we ca use the bracket

$$[a, b] = ab - (-1)^{p(a)p(b)}ba$$

.

Let us review the notions to describe systems of imprimitivity (SI) and an important result by Mackey that characterizes such systems in terms of induced representations, key notions in Clifford algebras, spinor fields, and Schwartz spaces [1] before discussing our main result in the super context. We provide the SUSY generalizations along with their classical counterparts using the notations and notions from the works of Varadarajan [1, 2, 3].

Definition 1.4. [1] A G-space of a Borel group G is a Borel space X with a Borel automorphism $\forall g \in G, t_g : x \to g.x, x \in X$ such that

$$t_e \text{ is an identity} \tag{1.1}$$

$$t_{g_1,g_2} = t_{g_1} t_{g_2} \tag{1.2}$$

The group G acts on X transitively if $\forall x, y \in X, \exists g \in G \ni x = g.y$.

Definition 1.5. [1] A system of imprimitivity for a group G acting on a Hilbert space $\mathscr{H}$ is a pair (U, P) where $P : E \to P_E$ is a projection valued measure defined on the Borel space X with projections defined on the Hilbert space and U is a representation of G satisfying

$$U_g P_E U_g^{-1} = P_{g.E} \tag{1.3}$$

Systems may be decomposed into SI $(G_0, \Omega = G_0/H_0)$ where H_0 closed subgroup of G_0 and a stabilizer at $\omega_0 \in \Omega$ on orbits by the transitive actions of the group and there exists a functor between the category of unitary representations of H_0 and the category of SI (G_0, Ω). In the case of Poincare group transitive SI is of interest to us we use the specialized version of the Mackey machinery. Let σ be a representation of H_0 on a Hilbert space $\mathscr{K}^\sigma$ then there is a canonical SI (π^σ, P^σ) for G_0 based on Ω with the representation induce by that of H_0 and the natural projection valued measure on $\mathscr{K}^\sigma$. The Hilbert space is the equivalent class of measurable functions $f : G_0 \to \mathscr{K}^\sigma$ satisfying:

$$f(x\eta) = \sigma(\eta)^{-1} f(x), \text{ for almost all } \eta \in H_0. \tag{1.4}$$

$$\int |f(x)|^2_{\mathscr{K}^\sigma} dx < \infty. \tag{1.5}$$

The representation π^σ acts by left translation and the SI relation $\sigma \to (\pi^\sigma, \mathscr{K}^\sigma)$ states that there is functor exists between the category of and the unitary representations of H_0 and the category of SI based on Ω. One can develop an intuition [3] as $\mathscr{K}^\sigma$ as attached to the fixed point ω_0 and for all the non-fixed points $\omega = g[\omega_0]$ a Hilbert space $\mathscr{K}^\sigma_\omega$ via an unitary isomorphism. This results in a fiber bundle $\mathscr{V}^\sigma = \mathscr{K}^\sigma \times G_0/\sim$, where the equivalence relation is defined by $(g, \psi) \sim (g\eta, \sigma(\eta)^{-1}\psi)$. The group G_0 has a a natural right action on the bundle.

Definition 1.6. [3] A super system of imprimitivity is a tuple $(\pi, \rho^\pi, \mathscr{H}, P)$ for a SLG $G = (G_0, \mathfrak{g})$ living on $\Omega = G_0/H_0$ where $H = (H_0, \mathfrak{h})$ is a special subgroup

of G satisfying the following properties:
(1) The tuple $(\pi, \rho^{\pi}, \mathscr{H})$ is a unitary representation of the SLG G.
(2) The tuple $(\pi, \mathscr{H}, P)$ is a classical system of imprimitivity for G_0 in $\mathscr{H}$, based on Ω, with P an even operator.
(3) The projection valued measure P commutes with ρ^{π}; that is, the spectral projections of the odd algebra $\rho^{\pi}(X), X \in g_1$ commute with the projections $P_E, E \subset \Omega$.

The last condition may be unpacked by starting from the assumption that in the super context the configuration space $(\Omega = G/H = G_0)/H_0$ is purely even. This implies $Xf = 0, X \in g_1, f \in C_c^{\infty}(\Omega)$. Now, the commutation follows as

$$[\rho^{\pi}(X), A(f)] = A(X(f)) = 0.$$

This definition of SSI lets us lift our earlier result on SI [14] representation of classical Poincarè to the super context by retaining the even part and making sure that the odd part is compatible.

2. Super fiber bundle representation and super semidirect products

The states of a freely evolving relativistic quantum particles are described by unitary irreducible representations of Poincaré that has a geometric interpretation in terms of fiber bundles.

Definition 2.1. *semidirect product of groups Let A and H be two groups and for each $h \in H$ let $t_h : a \to h[a]$ be an automorphism (defined below) of the group A. Further, we assume that $h \to t_h$ is a homomorphism of H into the group of automorphisms of A so that*

$$h[a] = hah^{-1}, \forall a \in A. \tag{2.1}$$

$$h = e_H, \text{ the identity element of } H. \tag{2.2}$$

$$t_{h_1 h_2} = t_{h_1} t_{h_2}. \tag{2.3}$$

Now, $G = H \rtimes A$ is a group with the multiplication rule of $(h_1, a_1)(h_2, a_2) = (h_1 h_2, a_1 t_{h_1}[a_2])$. The identity element is (e_H, e_A) and the inverse is given by $(h, a)^{-1} = (h^{-1}, h^{-1}[a^{-1}])$.

When H is the homogeneous Lorentz group L_0 and A is the translation group $T = \mathbb{R}^4$ we get the Poincaré group $\mathscr{P} = H \rtimes T$ via this construction. The covering group of inhomogeneous Lorentz is also a semidirect product as $\mathscr{P}^* = H^* \rtimes \mathbb{R}^4$ and as every irreducible projective representation of $\mathscr{P}$ is uniquely induced from a representation of $\mathscr{P}^*$ we will work with the covering group, whose orbits in momentum space are smooth, in the following. We make the assumption that the semidirect product is regular in that for the action of H on the dual H^* there is a Borel cross section. In other words all the $H-$orbits in H^* are locally closed [11].

We will need the following lemma for our discussions on constructing induced representations using characters of an abelian group as in the case of equation (3.1).

Lemma 2.2. *(Lemma 6.12 [1]) Let $h \in H$. Then, $\forall x \in \hat{A}$ where $\hat{A}$ is the set of characters of the group A (which in our case is $\mathbb{R}^4$), there exists one and only*

$y \in \hat{A}$ such that $y(a) = x(h^{-1}[a]), \forall a \in A$. If we write $y = h[x]$, then $h, x \to h[x]$ is continuous from $H \times \hat{A}$ into $\hat{A}$ and $\hat{A}$ becomes a H-space. Here, y can be thought as the adjoint for action of H on $\hat{A}$ and the map $\hat{p}$ in equation(3.1) *is such an example that is of interest to our constructions. In essence, we have Fourier analysis when restricted to the abelian group A of the semidirect product.*

Definition 2.3. A super translation group is a super Lie group (T_0, t) where T_0 is abelian with a trivial action on then even part of the super Lie algebra t_0. As $[t_1, t_1] \subset t_0$ and T_0 acts trivially on t_1 the nonlinear condition of super Lie algebra is satisfied.

Now, let us suppose that t_1 is an L_0-module (Lorentz module) and that the super commutator map $a, b \to [a, b]$ is an L_0-invariant from $t_1 \times t_1$ into t_0. Then, $g = \ell_0 \oplus t$ is a super Lie algebra with $g_0 = \ell_0 \oplus t_0 = Lie(G_0), t_1 = g_1$. Here again the odd bracket has t_0 as range subset of t_0 and acts trivially on t_1 to make it as a super Lie algebra. This way we can construct a whole family of super Lie groups (G_0, g) including super Poincare using homogeneous Lorentz and super translation group. we can generalize the SI example on SO_3 (or the double cover as the spin group) as the closed subgroup $S_0 \subset L_0$ and $H_0 = T_0 S_0$ of G_0. SO, $(H_0, \hbar$ is a special super Lie group of $(G), g)$ with $\hbar = \hbar_0 \oplus t_1$.

3. Super Little groups (stabilizer subgroups)

We consider different SI that live on the orbits of the stabilizer subgroups as concrete examples. It is helpful to have the picture that SI is an irreducible unitary representation of Poincaré group $\mathscr{P}^+$ induced from the representation of a subgroup of homogeneous Lorentz such as SO_3. Using the IRR $U_m, m \in SO_3$ we can induce a representation as

$$U_{(m,g)}\psi(k) = e^{i\{k,g\}}\psi(R_r^{-1}k)$$

where g belongs to the $\mathscr{R}^4$ portion of the Poincaré group, m is a member of the rotation group, and the duality between the configuration space $\mathbb{R}^4$ and the momentum space $\mathbb{P}^4$ is expressed using the character the irreducible representation of the group $\mathbb{R}^4$ as:

$$\{k, g\} = k_0 g_0 - k_1 g_1 - k_2 g_2 - k_3 g_3, p \in \mathbb{P}^4. \tag{3.1}$$

$$\hat{p} : x \to e^{i\{k,g\}}. \tag{3.2}$$

$$\{Lx, Lp\} = \{x, p\}. \tag{3.3}$$

$$\hat{p}(L^{-1}x) = \hat{Lp}(x). \tag{3.4}$$

In the above L is a matrix representation of Lorentz group acting on $\mathbb{R}^4$ as well as $\mathbb{P}^4$ and it is easy to see that $p \to Lp$ is the adjoint of L action on $\mathbb{P}^4$. The $\mathbb{R}^4$ space is called the configuration space and the dual $\mathbb{P}^4$ is the momentum space of a relativistic quantum particle with a real mass.

The stabilizer subgroup (Light-like particles) of the Poincaré group $\mathscr{P}^+$ at $(1, 0, 0, 1)$ [19]: has generators in terms of the Pauli matrix σ_3, and two matrices $N_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$, that are a rotation around the Z axis and the boost

Λ_p in the spatial direction [19]. There is is a Lorentz frame where the momentum is proportional to $(1,0,0,1)$ but there is no rest frame for the particle. The stabilizer subgroup at this point is isomorphic to the two dimensional Euclidean group E_2, and the orbits of this group on $\mathbb{R}^4$ host SI. In the supersymmetric version only admissible orbits host SSI.

Let $L_0^\lambda, \lambda \in T_0^*$ be the stabilizer of λ in L_0 and denote the corresponding super Lie algebra as

$$g^\lambda = t_0 \oplus \ell_0^\lambda \oplus g_1, \ell_0^\lambda = Lie(L_0^\lambda). \tag{3.5}$$

Note that the even part of the super Lie algebra is assumed to be that of the whole group Poincaré. Then, the little super group at λ is $S^\lambda = (T_0 L_0^\lambda, g^\lambda)$ which is a special super subgroup of (G_0, g). A unitary representation (σ, ρ^σ) of super little group S^λ is λ-admissible if $\sigma(t) = e^{i\lambda t}\mathbb{I}, \forall t \in T_0$. The point λ is admissible if there is an irreduicble unitary representation S^λ that is λ-admissible. The set of all λ-admissible points that is also L_0-invariant (L_0-orbit) is denoted by

$$T_0^+ = \{\lambda \in T_0^* | \lambda - admissible\}.$$

We can obtain a spectral measure P on T^* from an unitary representation (π, ρ^π) of a super Lie group S by restricting it to T^* and Fourier transforming it as

$$\pi(t) = \int_{T^*} e^{i\lambda t} dP(\lambda), t \in T^*.$$

Theorem 3.1. [2] *The spectrum of every irreducible representation of the super Lie group $S = (G_0, g)$ is in some orbit of T_0^+. For each orbit in T_0^+ and choice of λ in it, the assignment that takes a λ-admissible unitary representation $\gamma = (\sigma, \rho^\sigma)$ of S^λ into the unitary representation U^γ of (G_0, g) induced by it, is a functor which is an equivalence of categories between the category of the λ-admissible unitary representation of S^λ and the category of unitary representation of (G_0, g) with their spectra in their orbit. Varying λ in that orbit changes into an equivalent one. In particular this functor gives a bijection between the respective sets of equivalence classes of irreducible unitary representations.*

One point worth noting is that if we restrict the UIR of G_0, g to G_0, that from supersymmetric Poincarè to regular Poincarè the URs will be on the same orbit. Moreover, the theorem says the restricted representations will in fact be UIRs. This implies that the super partners will have the same mass until SUSY is broken.

4. Super context: Clifford Algebras and Spinor Fields

As we have to incorporate spinor filds in to the description of super particles whose symmetry is described by super Poincarè let us construct a representation for ρ^σ with nondegenerate bilinear quadratic form. For the unitary representations S^λ of interests to us satisfy

$$\sigma(t) = e^{i\lambda(t)}\mathbb{I}, \lambda \in T^*$$

we can define the derivative

$$-id(Z) = \lambda(Z)\mathbb{I}, (Z \in t_0).$$

As $[X_1, X_2] \in t_0)\forall X_i \in g_1$ holds we have

$$[\rho^\sigma(X_1), \rho^\sigma(X_2)] = 2\Phi_\lambda(X_1, X_2)\mathbb{I}, \forall X_i \in g_1.$$

Now, Φ_λ is the symmetric bilinear form defined as

$$\Phi_\lambda(X_1, X_2) = \frac{1}{2}\lambda([X_1, X_2])$$

on the algebra $\mathcal{C}_\lambda = g_1 \otimes g_1/\sim, X^2 = Q_\lambda(X), Q_\lambda(X) = \Phi_\lambda(X, X), X \in g_1$. We can think of ρ^σ as a representation of the Clifford algebra $\mathcal{C}_\lambda$. As L_0 acts on g_1 with Q_λ invariance we can extend the action $x, a \longmapsto x[a], x \in L_0, a \in g_1$ to $\mathcal{C}_\lambda = g_1 \otimes g_1$. We thus get a representation $S^\lambda = (\sigma, \tau)$ where τ is a self-adjoint representation of g_1 and σ a unitary representation of $T_0 L_0^\sigma$ that satisfies the relation

$$\sigma(t) = e^{i\lambda(t)}\mathbb{I}, (t \in T_0), \tau(x[a]) = \sigma(a)\tau(x)\sigma(x)^{-1}, \forall a \in \mathcal{C}_\lambda, \forall x \in L_0^\sigma.$$

The λ-admissible IRR is equivalent to positive energy condition $Q_\lambda(X) \geq 0, X \in g_1$ [3], Hamiltonians are based on these operators, that exclude imaginary mass tachyons in SUSY context. In this work we are concerned with massless particles, SSI will live on zero mass orbits of the hyperboloid, to keep the set up simple.

Our requirement for super Poincarè group (G_0, t) is that the even part of the super algebra t_0 should have a non degenerate symmetric bilinear form and the odd part t_1 should have a spin structure to incorporate fermionic statistics. In addition, $t = t_0 \oplus t_1$ has to be a real G_0 module. Spin modules are not modules for orthogonal group (Lorentz) we have to consider its double cover $\mathcal{SL}(2, \mathbb{C})$ (the spin group). To have spinors the super Lie algebra part of the Harish-Chandra pair has to be real and we have to set $t_1 = \mathbb{C}^2 \oplus \overline{\mathbb{C}}^2$, which would describe the majorana fermions, and complexify the group with two copies as $\mathcal{SL}(2, \mathbb{C}) \otimes \mathcal{SL}(2, \mathbb{C})$. The number N of spin modules determine the N-generators of the extended supersymmetry. The spin group embeds into the even part of the Clifford algebra we described earlier. This makes the Clifford algebra covariant with respect to Lorentz group which will enable the derivation of Dirac equation [3].

For our result that builds a representation for super Poincarè on a super fock space we need a version of SSI theorem specific for this SLG that we state here without proof.

Theorem 4.1. [2] *The irreducible unitary representations of a super Poincarè group $S = (G_0, g)$ are parameterized by the orbits of p with $p_0 \geq 0, \langle p, p\rangle \geq 0$, and for such p, by irreducible unitary representations of the stabilizer L_0^p at p. Let τ_p be an irreducible SA representation of the Clifford algebra $\mathcal{C}_p$ and let κ_p be the representation of L_0^p in the space of τ_p defined earlier. Then, for any irreducible unitary representation θ of L_0^p the pair (σ, ρ^σ) defined by*

$$\sigma = e^{ip}\theta \otimes \kappa_p, \rho\sigma(x) = \mathbb{I} \otimes \tau_p, (x \in g_1).$$

is an irreducible unitary representation of the little super group

$$S^p = (T_0 L_0^p, \ell_0 \oplus \ell_0^p \oplus g_1)$$

, and all irreducible unitary representations of S^p are obtained in this manner. The unitary representation $\Theta_{\theta p}$ of the super Lie group S induced by it is irreducible

and all irreducible unitary representations of S are obtained in this manner, the correspondence $(p,\theta) \longmapsto \Theta_{\theta p}$ being bijective up to equivalence.

5. Supersymmetric quantum fields

A super Poincare SLG is a semidirect product between homogeneous Lorentz (classical group) and super spacetime translation group. That is, in SUSY for Poincarè the spacetime is augmented with fermionic degrees of freedom and the semdirect product with homogeneous Lorentz taken. We will stick to this setup but use a representation of Lorentz on a super fock space for the semidirect product. To keep things simple in terms of mass of super multiplets our super fock space will have particles of zero mass but with different spins.

Let us construct the single particle Hilbert space before second quantize the system. We need to describe few ingredients to construct the Hilbert space of a Weyl fermion (equivalent of fermionic photino) namely, the fiber bundle, the fiber vector space, an inner product for the fibers, and an invariant measure. The 3+1 spacetime Lorentz group $\hat{O}(3,1)$-orbits of the momentum space $\mathscr{R}^4$, where the systems of imprimitivity established will live, described by the symmetry $\hat{O}(3,1) \rtimes \mathbb{R}^4$. The orbits have an invariant measure α_m^+ whose existence is guaranteed as the groups and the stabilizer groups concerned are unimodular and in fact it is the Lorentz invariant measure $\frac{dp}{p_0}$ for the case of forward mass hyperboloid. The orbits are defined as (we use the standard spin quantum number for the particles differing from Varadarajan who uses twice that number):

$$X_m^{+,1/2} = \{p : p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2, p_0 > 0\}, \text{forward mass hyperboloid}. \quad (5.1)$$

$$X_m^{+,1/2} = \{p : p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2, p_0 < 0\}, \text{backward mass hyperboloid}. \quad (5.2)$$

$$X_{00} = \{0\}, \text{origin}. \quad (5.3)$$

Each of these orbits are invariant with respect to $\hat{O}(3,1)$ and let us consider the stabilizer subgroup of the first orbit at p=(1,1,0,0). Now, assuming that the spin of the particle is 1 and mass $m \to 0$ (massless fermion) let us define the corresponding fiber bundles (vector) for the positive mass hyperboloid that corresponds to the positive-energy states by building the total space as a product of the orbits and the group $SL(2,C)$.

$$\hat{B}_m^{+,1/2} = \{(p,v)\ p \in \hat{X}_m^{+1},\ v \in \mathscr{C}^4, \sum_{r=0}^{3} p_r\gamma_r v = 0\}. \quad (5.4)$$

$$\hat{\pi} : (p,v) \to p. \text{ Projection from the total space } \hat{B}_m^{+,1/2} \text{ to the base } \hat{X}_m^{+,1/2}. \quad (5.5)$$

It is easy to see that if $(p,v) \in B_0^{+,1/2}$ then so is also $(\delta(h)p, S(h^{*-1})v)$. Thus, we have the following Poincaré group symmetric action on the bundle that encodes spinors into the fibers:

$$h,(p,v) \to (p,v)^h = (\delta(h)p, S(h^{*-1})v). \quad (5.6)$$

For $m > 0$ the fiber of $B_M^{+,1/2}$ at $p^(m) = ((a+m^2)^{1/2}, 1, 0, 0)$ is spanned by the vectors

$$v_1^{(m)} = \frac{1}{2}me_1 + \frac{1}{2}(1+(1+m^2)^{1/2})e_3. \tag{5.7}$$

$$v_2^{(m)} = \frac{1}{2}me_4 + \frac{1}{2}(1+(1+m^2)^{1/2})e_2. \tag{5.8}$$

When we take the limit $m \to 0+$ $v_1, V-2$ converge to e_3, e_2 that space the fiber of B_0^+ at (1,1,0,0). The covering group H^* is transitive on X_m^+, X_0^+ implies that the same convergence is true for any point. That is, if $p \in X_0^+$ then there are points $p^{(m)} \in X_m^+$ that converge to p as $m \to 0+$. This has the property that any vector v in the fiber of B_0^+ at p can be expressed as the limit of $v^{(m)}$ which is in fiber of $B_m^{1/1}$ at $p^{(m)}$. The same set of arguments can be applied to $B_{-m}^{+,1/2}$ implying that the bundles $B_{-m}^{+,1/2}$ also converge to B_0^+. The endomorphism (chirality or helicity operator) $\Gamma = i\gamma_0\gamma_1\gamma_2\gamma_3$ transforms $B_m^{+,1/2}(p) \to B_{-m}^{+,1}(p), \forall p \in X_m^+, m > 0$ as it anticommutes with all the gammas. In the limit Γ leaves the fibers of B_0^+ invariant leading to higher degenerecies with $\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This means Γ commutes with all of $S(h)$ implying that $(p, v) \in B_0^+ \Rightarrow (p, \Gamma v) \in B_0^+$. If we impose either of the condition $\Gamma\psi = \pm\psi$ then we can use 2x2 the Pauli matrices for the γs and we get the description for a Weyl fermion.

Γ has eigen values ± 1 at fiber (1, 0, 0, 1) and hence is true of all fibers. The stability group E^* at (1, 0, 0, 1) given by the matrices $\begin{bmatrix} z & a \\ 0 & (z^{-1}) \end{bmatrix}, z, a \in \mathscr{C}, |z| = 1$, induces a representation on the fibers as $m_{z,a} \to z^{\pm 1}$. Next step is to ensure that there is an Hermitian form on the fibers that is positive definite and left invariant with respect to S. The form $v \to p_0^{-1}\langle v, v\rangle$ can be shown to satisfy the condition and by letting $m \to 0+$ the form is still invariant and we have E_2 is the stabiliser group at (1,0,0,1).

Now, we can define the states of the light like particles on the Hilbert space $\hat{\mathscr{H}}_0^{+,\mp 1/2}$, square integrable functions on Borel sections of the bundle $\hat{B}_0^{+,1} = \{(p, v) : (p, v) \in B_0^+, \Gamma v = \mp v\}$.

The states of the particles are defined on the Hilbert space $\hat{\mathscr{H}}_m^{+,1/2}$, square integrable functions on Borel sections of the bundle $\hat{B}_m^{+,2}$ with respect to the invariant measure $\beta_0^{+,1}$, whose norm induced by the inner product is given below:

$$\|\phi\|^2 = \int_{X_m^+} p_0^{-1}\langle \phi p, \phi p\rangle . d\beta_0^{+,1}(p). \tag{5.9}$$

The invariant measure and the induced representation of the Poincaré group from that of the Weyl fermion are given below:

$$d\beta_0^{+,1}(p) = \frac{dp_1 dp_2 dp_3}{2(p_1^2 + p_2^2 + p_3^2)}. \tag{5.10}$$

$$(U_{h,x}\phi)(p) = \exp \mathrm{i}\{x, p\}\phi(\delta(h)^{-1}p)^h. \tag{5.11}$$

Definition 5.1. A super fock space Γ of a super Hilbert space $\mathscr{H}$ is a disjoint union of bosonic fock space of $\Gamma_s(\mathscr{H})$ and fermionic fock space $\Gamma_a(\mathscr{H})$. That is, the even part of the Hilbert space are tensored symmetrically and the odd part of the Hilbert space are tensored antisymmetrically and a disjoint union is formed. The even part of the super fock space supports Weyl operators just as in the classical case. The odd operators of the super Hilbert space are tensored to form the odd operators of the SUSY system.

Let us now state and discuss the main result for the case of massless super multiplets with light-like momentum by constructing a strict cocycle from the representation of a subgroup following the prescription (lemma 5.24) in Varadarajan's text. The SI is a consequence of strict cocycle property and the construction is not canonical.

Lemma 5.24 [1]: Let m be a Borel homomorphism of H_0, which is a subgroup of G into M. Then there exists a Borel map b of G into M such that

$$b(e) = 1. \tag{5.12}$$

$$b(gh) = b(g)m(h), \forall (g,h) \in G \times G_0. \tag{5.13}$$

Corresponding to any such map b, there is a unique strict (G, X)-cocycle f such that

$$f(g, g^1) = b(gg^1)b(g^1)^{-1}. \tag{5.14}$$

$\forall (g, g^{-1}) \in G \times G$. f defines m at x_0. Conversely, when f is a strict (G, X)-cocycle and b is a Borel map such that it satisfies equation 5.12 pair, then the restriction of b to H_0 coincides with the homomorphism m defined of at x_0 and b satisfies equation 5.14.

Theorem 5.2. *Light-like Weyl representation (G_0, g) of super Poincarè group on the super fock space $\Gamma = \Gamma_s(\hat{\mathscr{H}}_0^{+,1}) \oplus \Gamma_a(\hat{\mathscr{H}}_0^{+,1/2})$ is a transitive super system of imprimitivity $(\pi, \rho^\pi, \Gamma, P)$ that lives on $\Omega = G_0/H_0)$. This is a system of 1-extended SUSY.*

Proof. Let us first construct the classical SI (π, Γ, P) on the super fock space with the homomorphism $g : L_0^p \otimes L_0^p \to U_g(\hat{\mathscr{H}}_0^{+,1})$ from the two dimensional Euclidean group $L_0^p = E_2$ to the unitary representation of the group in $\hat{\mathscr{H}}_0^{+,1}$. We note that it is a stabilizer subgroup which is also closed at the momentum $p = (m, 0, 0, m)$ and so H/L_0^p is a transitive space and so the super version is a special subgroup (the odd part is the whole super Lie algevra).

Consider a map, from the light-like particle Hilbert space, $v(g) : L_0^p \to \hat{\mathscr{H}}_0^{+,1/2}$ satisfying the first order cocycle relation $v(gh) = v(g) + U_g v(h), g, h \in L_0^p$. An example of such a map is the following: [7]

$$\mathscr{H} = \oplus_{j=0}^{\infty} \mathscr{H}_j.$$
$$H = 1 \oplus \oplus_{j=1}^{\infty} H_j.$$
$$U_t = e^{-itH}, t \in L_0^p.$$
$$v(t) = tu_0 \oplus \oplus_{j=1}^{\infty} (e^{-itH_j} u_j - u_j).$$

Now, we can define the Weyl operator $V_g = W_g(v(g), U_g)$ where $g \in L_0^p$ for even part of the super fock space $\Gamma_s(\hat{\mathscr{H}}_0^{+,1/2})$.

This is a projective unitary representation satisfying the commutator relation $V_g V_h = e^{iIm\langle v_g, U_g v_h\rangle} V_h V_g$ and let us denote the homomorphism from L_0^p to V_g by m. This guarantees a map (lemma 5.24, [1]) b that satisfies $b(gh) = b(g)m(h), g \in G_0, h \in L_0^p$ and such map can be constructed by considering the map $c(x \to c(x)$ as Borel section of $\mathscr{P}/L_0^p$ (the choice of this section not a canonical one but immaterial to our purpose here) with $c(x_0) = e$. The map β maps $g \in G_0 \to gL_0^p$

$$a(g) = c(\beta(g))^{-1}. \tag{5.15}$$

$$b(g) = m(a(g)). \tag{5.16}$$

Then the strict cocycle ϕ satisfies $\phi(g_1, g_2) = b(g_1 g_2) b(g_2)^{-1}$.

We can now set the SI relation using the above cocycle as follows:

$$U_{h,x}\phi(p) = e^{i\{x,p\}}\phi(g, g^{-1}x) f(g^{-1}x), f \in \mathscr{H}$$

character representation is defined in equation (3.1).

$P_E F = \chi_E f$ Position operator.

We can construct the conjugate pair of field operators for the Fock space $\Gamma_s(\hat{\mathscr{H}}_0^{+,1/2})$ as follows:

Let p_g be the stone generator for the family of operators $P_{gt,p}, g \in \mathscr{P}, t \in \mathbb{R}$ and q(g) = p(ig) and we get the creation and annihilation operators as $a(g)^\dagger = \frac{1}{2}(q(g) - ip(g)$ and $a(g) = \frac{1}{2}(q(g) + ip(g)$.

With the identification π as the Weyl representation on the even part of the super fock space we have the tuple (π, Γ, P). We can set up the representation ρ^π on the super fock space of the super Lie algebra as per the equation (3.5). The third condition on SSI is also satisfied with our selection of purely even super homogeneous subgroup.

We can lift the representation to π to $L_0^p \otimes L_0^p$ and more precisely to its double cover $SL(2, \mathbb{C} \otimes SL(2, \mathbb{C})$. Now, the super Lie algebra t_1 of odd operators of the SLG will be a module of this spin group and we get the spinors that form the fibers of the bundle whose base is the orbit Ω. ■ □

The fermionic creation and annihilation operators $c^\dagger, c$ of the fermionic fock space $\Gamma_a(\mathscr{H}_0^{+,1/2})$ form the generator of a SUSY system with the Hamiltonian $H = \frac{1}{2}\{c, c^\dagger\}$.

Another SUSY system can be described using the configuration space $\mathbb{R}^4$ of the super particles with the Hamiltonian

$$H = \frac{1}{2}\{Q, Q^\dagger\}.$$

$$Q = (p_i - i\partial_i h(\psi^i)), i = 1, ...4.$$

$$Q\dagger = (p_i + i\partial_i h(\psi^{i\dagger})), i = 1, ...4.$$

In the above h is a real function and $\psi^i, \psi^{i\dagger} \in \Gamma_a(\hat{\mathscr{H}}_0^{+,1/2})$.

6. Summary and conclusions

We derived the covariant super field operators in a Minkowsky space using induced representations of groups and expressed them in terms of super systems of imprimitivity. We established the results for the super multiplets (photons and photnios) case by inducing a representation of Poincaré group from the subgroup that is a stabilizer at the momentum (m,0,0,m).This sets the stage for studying SUSY breaking in Minkowskian signature using the tools of SSI.

7. Appendix

8. Quantum Stochastic Fields

We recapitulate some notions in stochastic processes in the quantum context that are Poisson processes and a pair of conjugate Brownian motions that form classes of non commuting Hermitian operators.

Definition 8.1. Poisson process on symmetric Boson Fock space: Fock space is a Hilbert space as defined below as the set of square integrable functions on a space with respect to a measure that assigns equal probabilities to jump times $t_1, t_2, \ldots, t_n$ as would be expected of a poisson process.

$$\begin{aligned}
&\Omega = \cup_n \Omega_n;\\
&\Omega_0 = \emptyset;\\
&\Omega_n = \{t_1, t_2, \ldots, t_n\}; t_1 < t_2 < \cdots < t_n \in [0, T].
\end{aligned}$$

$$P_n(\Omega_n) = \frac{e^{-T} T^n}{n!}.$$

$$\mathscr{H} = L^2\big(\omega, \mathscr{F}, \rho\big).$$

$$\mathscr{W} = B(\mathscr{H}).\text{Bounded linear operators}$$

This space has a continuous tensor product structure as shown below that facilitates defining time-continuous stochastic processes.

$$\begin{aligned}
\Omega_{[s,t]} &= \Omega_{s]} \otimes \Omega_{[s,t]} \otimes \Omega_{[t}, s, t \in [0, T].\\
\mathscr{F}_{[s,t]} &= \mathscr{F}_{s]} \otimes \mathscr{F}_{[s,t]} \otimes \mathscr{F}_{[t}.\\
\mathscr{W}_{[s,t]} &= \mathscr{W}_{s]} \otimes \mathscr{W}_{[s,t]} \otimes \mathscr{W}_{[t}.
\end{aligned}$$

Let us now define special vectors called coherent vectors that also factorize continuously in time and their inner product as:

$$\begin{aligned}
e(f)(\emptyset) &= 1.\\
e(f)(\tau) &= \prod_{t\in\tau}^{\infty} f(t); f \in L^\infty([0, T]).\\
\langle e(f), e(g)\rangle &= e^{\|f\|_2^2 - T}.\\
e(f) &= e(f_{s]}) \otimes e(f_{[s,t]}) \otimes e(f_{[t}).
\end{aligned}$$

Let us denote by D the linear span of the exponential vectors that form the domain of the operators of the stochastic processes. It is enough to define operators with domain as D as it is dense in the Hilbert space the operations are uniquely defined. The Hilbert space we have constructed is on a classical probability space and so we can define Poisson random variables. The random variable $N_t(\tau) = |\tau \cap [0,t]|$ counts the number of jumps up to time t and it is a Poisson process with unit rate under the probability measure P. An operator process can be constructed from this as follows:

$$(\Lambda_t \Psi)(\tau) = N_t(\tau) = |\tau \cap [0,t]|\psi(\tau), \Psi \in \mathscr{F}, \tau \in \Omega, t \in [0,T]. \tag{8.1}$$

We can now define a family of states called coherent states as

$$\mathbb{P}_f(X) = \langle e(f), Xe(f)\rangle^{T-\|f\|_2^2}$$

. The physical intuition is clear as these states describe the coherent states of quantum optics. We will designate $\mathbb{P}_0 = \emptyset$ as the vacuum state and $e(0) = \Phi$ as the vacuum vector. The process $\{\Lambda_t\}$ defined is called the gauge process which is commutative and later after defining the Brownian motions we will construct a non commutative version that will form part of the quantum noises. We have ignored technicalities such as affiliated process that are analogous to adapted processes in classical system and refer the readers to [20] for details.

Definition 8.2. Brownian motions: We continue to work with the Fock space defined on a continuum as above and define a Weyl operator as

$$W(f)e(g) = e^{-\int_0^T (f^\dagger(t)g(t)+\frac{1}{2}f^\dagger(t)f(t))dt}e(f+g) = e^{-\langle f,g\rangle - \|f\|_2^2}e(f+g). \tag{8.2}$$

This is a unitary operator that may be considered as a second quantized on the Fock space and can be shown to posses the continuous tensor property. Let us fix $f \in L^\infty([0,T])$ and construct the group of unitary operators $\{W(tf)\}_{t\in R}$ and by Stone's theorem guarantee's a self adjoint operator B(f) such that $W(kf) = e^{kB(f)}$. This is similar to constructing a unitary out of a Hamiltonian and in this context these operators are called field operators. It is easy to establish that the probability distribution of these random variables is gaussian in the coherent state $\mathbb{P}_g$. Like in the case of the gauge process we can construct another operator process as $\{B_t^\phi = B(e^{i\phi}\chi[0,T]) : t \in [0,T]$ for some fixed real function $\phi \in L^\infty([0,T])$ that has Gaussian probability law at each time epoch. Now, we can define the following pair of conjugate Brownian motions A and $A^\dagger$:

$$\begin{aligned}
Q_t &= B(i\chi[0,T]).\\
P_t &= B(-\chi[0,T).\\
A_t &= \frac{(Q_t + iP_t)}{2}.\\
A_t^\dagger &= \frac{(Q_t - iP_t)}{2}.
\end{aligned}$$

The process Λ_t vanishes in vacuum coherent state and so let us define the process $\Lambda_t(f) = W(f)^\dagger \Lambda_t W(f)$ that has same statistics in state $\mathbb{P}_f$ as Λ_t in vacuum state. The sandwiching with a Weyl operator (defined below) provides a way to transform statistics of a coherent state to vacuum which could be the choice to carry out all the analysis. The three processes $(A_t^\phi, A^{\dagger\phi}, \Lambda_t^\phi)$ are called the quantum noises that can be used to describe the dynamics of an open quantum system. These quantum noises are mathematical objects called white noises that are good approximations to wide band noises encountered in quantum optics [21], [22]. The dynamics of an open system interacting with an environment can be described by a quantum stochastic differential equation (QSDE) of Hudson-Parthasarathy kind [7] as

$$dU_t = \{(S-1)d\Lambda + LdA_t^\dagger - SL^\dagger dA_t + (iH - \frac{1}{2}L^\dagger L)dt\}U_t. \tag{8.3}$$

In the above equation, the unitary operator U is defined on the combined system and the Fock space described by the Equation (8.1). The operator L, it is actually a vector with one element per noise channel, and its conjugate are the Lindbladians corresponding to the channels of decoherence, and the operator S is a similar noise channel that is of discrete in time. When the above equation is traced out with respect to the bath we obtain the quantum master equation with the operator S missing for the obvious reasons. The three parameters S,L, and H of the QSDE characterize the open evolution of a system that are coefficients of Hudson-Parthasarathy quantum stochastic differential equations denoting the internal energy of the system in terms of the Hamiltonian H, couplings to the environment via the Lindbladians, and the scattering by the fields by the S matrix. We refer the readers to the work of Gough and James [22] for the details of the SLH mathematical framework that derives the QSDE for composite systems connected in a network.

Definition 8.3. Weyl operators: The three second quantized operators

$$(A^\phi, A^{\dagger\phi}, \Lambda^\phi)$$

are members of a representation of the Euclidian group over $\mathscr{H}$ [7] whose generic form is:

$$W(u,U)e(v) = \{exp(-\frac{1}{2}\|u\|^2) - \langle u, Uv\rangle\}e(Uv+u), \forall v \in \mathscr{H}.$$

These Weyl operators used to glue together components of quantum optical circuits [6].

Acknowledgment.

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