# Magnetoelectric response of lattice bosonic insulators 

Gautam K. Naik, Michael O. Flynn, Chris R. Laumann ${ }^{1}$<br>${ }^{1}$ Department of Physics, Boston University, 590 Commonwealth Avenue, Boston, Massachusetts 02215, USA

(Dated: March 5, 2024)


#### Abstract

In the absence of parity and time-reversal symmetries, insulators can exhibit magnetoelectric responses, in which applied magnetic fields induce charge polarization and, conversely, applied electric fields induce magnetization. While there is a long history of the study of magnetoelectric response in fermionic insulators, the same for bosonic insulators has been limited. We consider the magnetoelectric response in lattice insulators built out of charged bosonic degrees of freedom and derive a bulk formula for the corresponding linear response tensor. The resulting formulae feature several contributions including a Chern-Simons integral over the bands of the bosonic excitations. We construct several minimal microscopic models that illustrate the ingredients required to obtain a sizable bosonic magnetoelectric response. Our formalism can be applied to bosonic Mott insulators subject to synthetic gauge fields and/or tilted potentials as well as to the spinon sector in the Coulomb phase of a $U(1)$ quantum spin liquid.


## I. INTRODUCTION

The linear magnetoelectric polarizability of a threedimensional insulator is captured by the magnetoelectric tensor,

$$
\begin{equation*}
\alpha_{j}^{i}=\frac{\partial P^{i}}{\partial B^{j}}=\frac{\partial M_{j}}{\partial E_{i}} \tag{1}
\end{equation*}
$$

where $E$ and $P$ are the electric field and polarization and $B$ and $M$ are the magnetic field and magnetization. The second equality is a thermodynamic Maxwell relation, and thus holds at low frequency in quasi-equilibrium. While the study of $\alpha$ has a long history in magnetic materials ${ }^{1 / 4}$, a theoretical framework for computing the orbital contribution to $\alpha$ in band insulators was only developed relatively recently $5+8$. This was motivated by the discovery of fermionic topological insulators, where the orbital contribution is quantized to a non-zero value even when inversion $\mathscr{P}$ and time-reversal $T$ symmetry cause all other contributions to vanish 9 .

In this article, we consider the magnetoelectric response of non-topological bosonic insulators in the absence of inversion and time-reversal symmetry. A motivating example is provided by the bosonic Mott insulator in an optical lattice. Here, the 'insulator' blocks transport of the conserved $U(1)$ charge corresponding to that of the underlying neutral atoms. These do not couple to a true electromagnetic field; nonetheless, the magnetoelectric response $\alpha$ can be probed using local potentials and synthetic gauge fields ${ }^{11} \sqrt{13}$. For example, a tilted optical lattice can play the role of an applied electric field, and the magnetization $M_{j}=\alpha_{j}^{i} E_{i}$ is reflected in lattice scale circulating currents of the bosonic constituents.

A more exotic physical setting is provided by the $U(1)$ Coulomb quantum spin liquid $\sqrt{14} \sqrt[17]{17}$. The spinon sector may be viewed as a bosonic insulator. Unlike the atomic Mott insulator, the charge of the spinons couples to an emergent dynamical electromagnetic field. Below the spinon gap, the magnetoelectric response $\alpha$ couples into the dynamics of the emergent electromagnetism, appearing like a $\theta$-term in the effective theory.

In this article, we derive closed formulae for the magnetoelectric response $\alpha$ of lattice systems of gapped bosonic oscillators with a $U(1)$ charge. Our formalism applies to the quadratic approximation to the excitations around a mean-field insulating state, and we expect it to be quantitatively well-controlled by the gap of the insulator. The derivation of a bulk formula for $\alpha$ is more complicated than one might expect due to the Maxwell relation in Eq. (1). More precisely, the magnetoelectric tensor can


FIG. 1. (a) A bosonic insulator model with monopoles in the two tetrahedra of a bipyramid and a staggered potential is the simplest model to show a magnetoelectric response. The green and red highlights at the sites indicate a positive and negative potential, respectively. Applying an electric field along the 3 -fold rotation axis leads to a current loop and magnetization. (b) The plot on top shows the magnetization response to the applied electric field, and the plot on the bottom shows the polarization response to the applied magnetic field along the 3 -fold rotation axis for the bipyramid model with the Hamiltonian in Eq. (7) having parameter values $\lambda=.34, t=1, v=1$ and $m=0.1$. We use units with $\hbar=1$ and $e=1$. The $x$ and $y$ coordinates of these plots have dimensions that are dependent on the length scales of the system, but the zero field slope, which gives the linear magnetoelectric response coefficient, is dimensionless and, hence, independent of the length scale.


FIG. 2. Band structure of fermionic and bosonic insulators. (a) The single particle excitations of fermionic systems are obtained by adding electrons to unoccupied states (blue) or removing electrons from occupied states (orange). All of these electron and hole modes have positive energy. (b) In bosonic insulators, there are excitation bands corresponding to the positive and negative charged mode creation operators. Stable insulators require a finite positive neutral excitation gap. In such systems, the chemical potential can be adjusted so that all of the charged modes have positive excitation energy.
be decomposed into a pseudo-scalar and a traceless part:

$$
\begin{equation*}
\alpha_{j}^{i}=\tilde{\alpha}_{j}^{i}+\alpha_{\theta} \delta_{j}^{i} \tag{2}
\end{equation*}
$$

The problem is that Faraday's law, $\partial_{t} \boldsymbol{B}+\nabla \times \boldsymbol{E}=0$, in conjunction with the Maxwell relation, ensures that the pseudo-scalar part $\alpha_{\theta}$ cancels out of the bulk current response to applied low-frequency fields (see Fig. 3.). Hence, standard bulk calculations that neglect surface currents fail to compute $\alpha_{\theta}$.

Despite this difficulty, several approaches have been
developed to obtain $\alpha_{\theta}$ for fermionic systems. Ref. 5 utilized a field-theoretic dimensional reduction approach which applies directly only to systems with sufficient symmetry. They found that $\alpha_{j}^{i}=\alpha_{C S} \delta_{j}^{i}$, where $\alpha_{C S}$ is given by a Chern-Simons integral in momentum space. More generally, Essin et al. ${ }^{[7]}$ developed an elegant approach to computing $\alpha$ in general fermionic band insulators by considering an adiabatic protocol in which the bulk Hamiltonian varies in time in the presence of constant magnetic field. They found an additional 'crossgap' contribution $\alpha_{G}$,

$$
\begin{equation*}
\alpha_{j}^{i}=\alpha_{C S} \delta_{j}^{i}+\left(\alpha_{G}\right)_{j}^{i} \tag{3}
\end{equation*}
$$

which, notably, also contributes to the trace $\alpha_{\theta}$. We will review this approach in more detail below, as our bosonic derivation mirrors it. We note that the same result for fermionic insulators was also obtained by considering a constant background electric field in Ref. 8, There have been a few alternate derivations ${ }^{18119}$ as well as generalizations to disordered systems ${ }^{20}$, interacting systems ${ }^{21122}$ and response at finite frequency $\sqrt[23]{25}$, when the Maxwell relation in Eq. (1) does not hold.

Our main result is the following closed formula for the magnetoelectric tensor expressed as a trace over the bosonic excitation modes in momentum space ( $e=\hbar=$ 1),

$$
\begin{align*}
\alpha_{C S} & =\frac{1}{8} \int_{0}^{1} d \beta \int_{\mathrm{BZ}} \frac{d^{3} k}{(2 \pi)^{3}} \epsilon_{\mu \nu \gamma \lambda} \operatorname{Tr} P\left(\partial^{\mu} P \partial^{\nu} P-\partial^{\nu} P \partial^{\mu} P\right) P\left(\partial^{\gamma} P \partial^{\lambda} P-\partial^{\lambda} P \partial^{\gamma} P\right)  \tag{4}\\
\left(\alpha_{G}\right)_{j}^{i} & =\int_{\mathrm{BZ}} \frac{d^{3} k}{(2 \pi)^{3}} \sum_{n, m} \frac{\operatorname{Re} \operatorname{Tr} P_{-n} \partial^{i} P \epsilon_{j \mu \nu} P_{+m}\left\{\partial^{\mu} h_{o}, \partial^{\nu} P\right\}+2 \operatorname{Im} \operatorname{Tr} P_{-n} \partial^{i} P P_{+m}\left(\partial h^{\prime} / \partial B^{j}\right)}{E_{-n}+E_{+m}} \tag{5}
\end{align*}
$$

As promised, $\beta$ is an adiabatic parameter relating our Hamiltonian of interest to a reference Hamiltonian with vanishing ${ }^{26} \alpha$. The projector $P$ picks out the annihilation operators for the negatively charged modes; this plays a role analogous to projection onto occupied states in the fermionic case, see Fig. 2. The 'cross-gap' term depends on the energies $E_{ \pm m}(\boldsymbol{k})$ of the positively and negatively charged bosonic bands and their corresponding mode projectors $P_{ \pm m}(\boldsymbol{k})$. The Hamiltonian appears explicitly in $\alpha_{G}$ through the dynamical matrix $h=h_{o}+h^{\prime}$, which governs the mode dynamics. Precise mathematical definitions can be found in Sec. III

The 'Chern-Simons' contribution $\alpha_{C S}$ in Equation (4) can be rewritten in terms of a second Chern form:

$$
\begin{equation*}
\alpha_{C S}=-\frac{1}{8} \int_{0}^{1} d \beta \int_{\mathrm{BZ}} \frac{d^{3} k}{(2 \pi)^{3}} \epsilon_{\mu \nu \gamma \lambda} \operatorname{Tr} F^{\mu \nu} F^{\gamma \lambda} \tag{6}
\end{equation*}
$$

where, $F^{\mu \nu}=i P\left(\partial^{\mu} P \partial^{\nu} P-\partial^{\nu} P \partial^{\mu} P\right) P$. This 4D second Chern form can further be rewritten as a 3 D mo-
mentum space Chern-Simons form of the corresponding Berry connection along the boundary of the $\beta$ integral. However, this way of rewriting the term and ignoring the adiabatic change from a reference Hamiltonian introduces a gauge dependence to the integral. The gauge freedom also makes this integral harder to compute numerically. Hence, in this article, we only use the second Chern form to compute $\alpha_{C S}$.

The expressions we have in Eqs. (4) and (5) are similar to those for fermionic insulators 7 , with the similarity most evident in the electron-hole picture (see Fig. 2p of the insulator. The distinction is that the bosonic mode operators are obtained by generalized Bogoliubov transformations, while the fermionic modes are obtained by unitary transformations. This has several consequences; for example, the projector $P$ is self-adjoint with respect to a conjugate symmetric sesquilinear form with mixed signature rather than a more familiar positive definite inner product.


FIG. 3. The bulk current within the insulator, $\mathbf{J}_{b}=\partial_{t} \mathbf{P}+\nabla \times$ $\mathbf{M}$, has contributions from the bulk polarization and magnetization. (a) Consider a system with uniform time-independent $\alpha_{\theta}$ and zero $\tilde{\alpha}$. Switching on a magnetic field has a small time period with time-varying magnetic fields accompanied by induced electric fields. The magnetoelectric response to these fields leads to the cancellation of the terms in the bulk current. The current that generates the polarization flows only along the boundary surface. (b) In a system with time-varying $\alpha_{\theta}$ and constant magnetic field, there is no induced electric field, and there is a non-zero bulk current ${ }^{[7]}$.

Note that there are several existing bosonic band formulations and mappings of bosonic systems to fermionic systems ${ }^{[27+32}$. These existing mappings are usually used to simplify the calculation of topological properties of the bosonic bands. The mapping we introduce simply points out the mathematical resemblance between the calculations done in $U(1)$ conserving bosonic systems and $U(1)$ conserving fermionic systems and allows us to write out results that apply to bosonic systems from existing results that apply to fermionic systems.

Our formalism applies to generic quadratic lattice systems of bosonic oscillators with a globally conserved $U(1)$ charge. This charge may be understood as the 'angular momentum' of the 2D harmonic oscillator at each lattice site. We describe these oscillators in terms of a pair of complex scalars, $\Phi_{r}$ and $\Pi_{r}$, at each lattice site $\boldsymbol{r}$, with canonical commutator relations $\left[\Phi_{\boldsymbol{r}}, \Pi_{\boldsymbol{r}^{\prime}}^{\dagger}\right]=i \delta_{\boldsymbol{r} \boldsymbol{r}^{\prime}}$. The general Hamiltonian can be expressed,

$$
\mathscr{H}=\sum_{r \boldsymbol{r}^{\prime}}\left(\begin{array}{ll}
\Pi_{r}^{\dagger} & \Phi_{r}^{\dagger}
\end{array}\right)\left(\begin{array}{cc}
M_{\boldsymbol{r} \boldsymbol{r}^{\prime}}^{-1} & i V_{r \boldsymbol{r}^{\prime}}  \tag{7}\\
-i V_{\boldsymbol{r} \boldsymbol{r}^{\prime}}^{\dagger} & K_{\boldsymbol{r} \boldsymbol{r}^{\prime}}
\end{array}\right)\binom{\Pi_{\boldsymbol{r}^{\prime}}}{\Phi_{\boldsymbol{r}}}
$$

where we view $M$ as a mass matrix and $K$ as a 'spring' coupling matrix. The off-diagonal $V$ matrix may be viewed as a generalized potential, as the diagonal part couples to the local charge, $Q_{r}$.

There are several studies on topological bosonic insulators and fractional topological bosonic insulators ${ }^{33+35}$, which can have non-zero quantized values and certain fractions of the quantized value for $\alpha_{\theta}$. We believe our formalism applies to $\mathscr{P}$ and $T$ breaking Hamiltonians close to trivial bosonic insulators. We leave the question calculating this response for $\mathscr{P}$ and $T$ breaking around bosonic topological insulators as an open question.

There are many experimental realizations of fermionic magnetoelectric response in multiferroics ${ }^{3 / 436}$. However, materials in which the orbital contribution to the response is dominant are primarily, but not limited to, topological insulators ${ }^{10 / 37 / 42}$.

The rest of the article is organized as follows: Section II introduces simple bosonic models that show a magnetoelectric response. Section III introduces the general formalism we use to study bosonic systems with a global $\mathrm{U}(1)$ symmetry. In Section IV, we present numerical results for a lattice bosonic insulator. We conclude and discuss future prospects in Section $V$. The appendix includes the following: Appendix A elucidates the proof for two of the key properties of the correlation matrices introduced in Section III In Appendix B , we go over the derivation of the expressions in Eqs. (4) and (5). Appendix C clarifies the group structure of the diagonalization of Hamiltonians in Eq. (7). Appendix D shows how the formalism we introduce in Section III can be generalised to any $U(1)$ symmetric quadratic bosonic Hamiltonian.

## II. TOY MODEL

In this section, we illustrate the magnetoelectric effect by introducing small models that break all the required symmetries to have a non-zero $\alpha$. These 'toy models' help illustrate that additional microscopic ingredients are required to observe a magnetoelectric response in bosonic systems rather than fermionic systems. We also use the toy models to benchmark our numerical computation of the integrals in Eqs. (4) and (5).

Before turning to bosonic toy models, let us review the simplest fermionic hopping mode ${ }^{77}$ exhibiting a magnetoelectric response. This consists of fermions hopping on a tetrahedron containing a background magnetic monopole. The Hamiltonian is $\mathscr{H}_{f}=-\sum_{\left\langle\boldsymbol{r}, \boldsymbol{r}^{\prime}\right\rangle} c_{\boldsymbol{r}}^{\dagger} t_{\boldsymbol{r} \boldsymbol{r}^{\prime}} c_{\boldsymbol{r}^{\prime}}$, where, $t_{\boldsymbol{r r}^{\prime}}$ is chosen such that the Aharonov-Bohm phase from hopping around any face of the tetrahedron is $\pi / 2$ (as shown in Fig. 4(a)). It is straightforward to diagonalize this model and derive the isotropic magneto-electric response, $\alpha_{j}^{i}=\frac{1}{\sqrt{6}} \frac{e^{2}}{\hbar} \delta_{j}^{i}$, as shown in Ref. 7.

It is instructive to review the symmetries of the fermionic monopole-tetrahedron to see how they permit a non-zero $\alpha$. The system is symmetric under the proper rotational symmetries of the tetrahedron, which leaves both the magnetic monopole field and the tetrahedron invariant. This ensures that the magnetoelectric response $\alpha_{j}^{i}=\alpha_{\theta} \delta_{j}^{i}$ is isotropic. The magnetic monopole breaks the improper reflection symmetries, $M_{i}$, of the tetrahedron as the monopole field reverses under such reflections. However, as time-reversal $T: c \rightarrow c, i \rightarrow-i$ also reverses the monopole field, $H_{f}$ is symmetric under the combined action of $M_{i} T$. Finally, $H_{f}$ is symmetric under (unitary) charge conjugation, $C: c_{r} \rightarrow c_{r}^{\dagger}$. Mathematically, all of the symmetries must be implemented along with appropriate gauge transformations to leave $\mathscr{H}_{f}$ invariant. Crucially, these symmetries are all proper in the $3+1 \mathrm{D}$ sense, and thus, non-zero $\alpha$ is permitted.

Let us now attempt to construct a bosonic system with a magnetoelectric response. The simplest attempt is to consider bosonic modes, $a_{\boldsymbol{r}}$, at every corner of the tetrahedron with hopping the same as that for the fermionic


FIG. 4. Minimal systems for isotropic magnetoelectric response. (a) A quadratic fermionic system with hopping between the four corners of a tetrahedron, such that it encloses a magnetic monopole. The arrows in the image indicate a gauge choice in which every nearest neighbor hop carries phase $e^{i \pi / 2}$. (b) In the case of bosons, a minimal isotropic model is that of a deformed cube with a magnetic monopole and an alternating potential. A particular gauge choice that accomplishes this is indicated by the arrows. The green and red highlights at the sites indicate a positive and negative local potential. (c) The isotropic part of magnetoelectric tensor $\alpha_{\theta}$ (where $e=\hbar=1$ ) as a function of the charge staggering parameter $v$ for the deformed cube model with $\lambda=2.6, t=1, m=0.1$ and the inner vertices pushed in halfway to the center of the cube. The red curve is obtained from the zero field slope of the $P$ vs $B$ curve of a single deformed cube. The black dots are the points obtained by performing the integrals in Eqs. (4) and (5). Note that the insulator gap decreases with increasing $v$.
toy model, i.e. $\mathscr{H}_{b}=-\sum_{\left\langle\boldsymbol{r}, \boldsymbol{r}^{\prime}\right\rangle} a_{\boldsymbol{r}}^{\dagger} t_{\boldsymbol{r} \boldsymbol{r}^{\prime}} a_{\boldsymbol{r}^{\prime}}$. However, this model has bosonic negative energy modes and is unstable. If we add diagonal terms of the form $\lambda \sum_{\boldsymbol{r}} a_{\boldsymbol{r}}^{\dagger} a_{\boldsymbol{r}}$, and make $\lambda$ large enough to ensure a finite positive charge gap in the system, we obtain a rather trivial insulator. The ground state is the Fock vacuum of the $a$-modes, which is unperturbed by any perturbations to $\mathscr{H}_{b}$.

To obtain a more interesting insulator, we must allow both positively and negatively charged excitations. We consider two bosonic modes, $a_{+\boldsymbol{r}}$ and $a_{-\boldsymbol{r}}$, at every site and set the local charge to be $Q_{r}=a_{+\boldsymbol{r}}^{\dagger} a_{+\boldsymbol{r}}-a_{-\boldsymbol{r}}^{\dagger} a_{-\boldsymbol{r}}$. With this setup, we obtain a non-trivial ground state by coupling the charges via terms of the form $\left(a_{+\boldsymbol{r}}^{\dagger} a_{-\boldsymbol{r}^{\prime}}^{\dagger}+\right.$ $a_{+\boldsymbol{r}} a_{-\boldsymbol{r}^{\prime}}$ ).

A perhaps more natural way to describe a pair of bosonic modes at each site is with complex scalars, $\Phi_{r}$ and $\Pi_{r}$, which satisfy the commutation relations $\left[\Phi_{r}, \Pi_{\boldsymbol{r}^{\prime}}^{\dagger}\right]=i \delta_{\boldsymbol{r} \boldsymbol{r}^{\prime}}$. The local charge is the angular momentum of the 2 D oscillator, $Q_{r}=i\left(\Pi_{r}^{\dagger} \Phi_{r}-\Pi_{r} \Phi_{r}^{\dagger}\right)$.

We now consider toy models with a Hamiltonian of the form

$$
\mathscr{H}=\sum_{r \boldsymbol{r}^{\prime}}\left(\begin{array}{ll}
\Pi_{r}^{\dagger} & \Phi_{r}^{\dagger}
\end{array}\right)\left(\begin{array}{cc}
m^{-1} & i v V_{r}  \tag{8}\\
-i V_{r} & \lambda+t K_{r \boldsymbol{r}^{\prime}}
\end{array}\right)\binom{\Pi_{\boldsymbol{r}^{\prime}}}{\Phi_{r}}
$$

where, $m, t, \lambda$ and $v$ are real parameters. Here, $m$ is the uniform local mass of each oscillator, $t$ determines the strength of the off-diagonal nearest neighbor coupling matrix $K, \lambda$ gives the diagonal couplings, and $v$ determines the strength of a staggered site-local charge potential, $V_{r} \in \pm 1$. For any choice of geometry (encoded in $K$ and $V_{r}$ ), this produces a three-dimensional phase space as the overall scale of $\mathscr{H}$ is unimportant for the dimensionless magnetoelectric response $\alpha$.

Let us now consider a bosonic tetrahedron-monopole system $\mathscr{H}_{T M}$ with complex scalars attached to each corner. This is represented by Eq. (8) with the coupling matrix $K_{r r^{\prime}}$ matching that of the fermionic hopping matrix $t_{\boldsymbol{r} \boldsymbol{r}^{\prime}}$. One might expect that this model has all the ingredients required to exhibit non-zero $\alpha$. However, it turns out that the symmetries of this bosonic system aren't the same as that of the corresponding fermionic system. The magnetic monopole still breaks time reversal and mirror symmetries while leaving the proper rotational symmetries of the tetrahedron intact. However, the charge conjugation symmetry $C$ behaves quite differently. For the bosonic system, time reversal is defined by $T: \phi_{r} \rightarrow \phi_{r}, \Pi_{r} \rightarrow-\Pi_{r}, i \rightarrow-i$ and charge conjugation is defined by $C: \phi_{r} \rightarrow \phi_{r}^{\dagger}, \Pi_{r} \rightarrow \Pi_{r}^{\dagger}$. One can check that the $\mathscr{H}_{T M}$ model has $C T$ symmetry. $\alpha$ is odd under the action of $C T$, and this implies that the bosonic model will not show a magnetoelectric response unless $C T$ is broken.

To break $C T$, we can add non-uniform charge potentials that explicitly break all charge conjugation symmetries. One way to do this while still maintaining some rotational symmetries in 3D is to consider a bipyramid with monopoles in both tetrahedrons and opposing potentials at the apexes and the base (see Fig. 11). Although this five-site bipyramid model is the simplest ${ }^{43}$ bosonic model that shows a magnetoelectric response, it is anisotropic. Fig. 1 shows the response of this model along the 3 -fold symmetry axis.

A simple model with isotropic magnetoelectric response has the geometry of a cube deformed such that four of the eight corners are pushed in (as shown in Fig. 4(b)). The Hamiltonian for this system is given by Eq. (8), with the phases in the spring coupling matrix elements $K_{r r^{\prime}}$ chosen so that the deformed cube encloses a negative monopole, and $V_{r}$ chosen to be positive at the outer corners and negative at the inner corners. The only remaining symmetries of this system are the proper tetrahedral rotations, which ensure that the system has an isotropic response.

Tiling copies of this toy model along a cubic lattice gives us a system that has translational symmetry. For such a system of non-interacting deformed cubes, the polarization and hence $\alpha$ is the same as that of the single

| Fermions | Bosons |
| :--- | :--- |
| $H:$ Hamiltonian matrix | $h:$ Dynamical matrix |
| $\rho:$ Density Matrix | $C:$ Correlation matrix |
| $E:$ Energy | $\Lambda:$ Frequency |
| $U:$ Unitary | $R:$ Generalized Bogoliubov |
| diagonalization | diagonalization (Appendix C ) |

TABLE I. Summary of the replacements required to translate the existing fermionic derivation ${ }^{77}$ to one that applies for bosonic systems. For a fermionic system characterized by the Hamiltonian $\mathscr{H}_{f}=c^{\dagger} H c$, all observables can be computed from the single body density matrix $\rho_{i j}=\left\langle c_{i}^{\dagger} c_{j}\right\rangle$. Eqs. 23) and (24) are key examples that show that $C$ and $h$ can be replaced by $\rho$ and $H$, respectively, to translate between expressions that hold for bosonic and fermionic systems.
deformed cube up to a geometrical factor. The polarization of a single deformed cube at small applied magnetic fields can be computed by diagonalizing a small matrix (of size $16 \times 16$ ). Hence, such a system forms a simple toy example to test the validity of the expressions in Eqs. (4) and (5). We use numerical integration of these expressions to compute $\alpha$ and show that this matches with what is obtained from our numerics of a single deformed cube in Fig. 4 (c). Breaking $C T$ by a non-uniform $V_{r}$ is a generic way to tune from a model with $\alpha=0$ to $\alpha \neq 0$. So, we use the strength $v$ of such fields as the adiabatic parameter $\beta$ in Eq. (4).

## III. ANALYZING QUADRATIC BOSONIC SYSTEMS WITH A CONSERVED CHARGE

After identifying the correct mathematical definitions to translate from free fermions to general $U(1)$ conserving quadratic bosons, our derivation of Eqs. (4) and (5) algebraically mirrors that of Essin et a ${ }^{17}$. In this section, we present the appropriate mathematical dictionary (see Table (I) by introducing the formalism to analyze, solve and compute ground state observables within quadratic bosonic systems with a Hamiltonian of the form in Eq. (7). While the properties of the relevant mathematical objects are somewhat different, particularly in that the objects are self-adjoint under different sesquilinear forms, ultimately the derivation goes through. For completeness, we include the bosonic derivation in Appendix B. The key definitions and properties required to push through the derivation as well as make sense of the terms in the resulting Eqs. (4) and (5) are presented below.

In quadratic fermionic systems, unitary diagonalization of the Hamiltonian matrix that accompanies the fermionic modes allows one to obtain all fermionic eigenmodes. In quadratic bosonic systems, unitary diagonalization of the Hamiltonian matrix accompanying the bosonic modes does not give bosonic operators. Instead, a generalized Bogoliubov diagonalization of the dynam-
ical matrix, i.e. the matrix that gives the equations of motion (EOM) of the bosonic modes, is required to obtain the eigenmodes ${ }^{44 \mid 45}$.
a. Spinor Formalism - We consider a generic lattice bosonic system with a $U(1)$ symmetry and $N$ sites, each of which can be excited with both positively and negatively charged excitations. We introduce a Nambu spinor, $\psi=\left[\begin{array}{c}\Pi \\ \Phi\end{array}\right]$, with $\Phi$ and $\Pi$ being a complex scalar field and its conjugate momentum, so that $\left[\Phi, \Pi^{\dagger}\right]=$ $i \mathbb{1}_{N \times N}$. In terms of the spinor $\psi$, the canonical commutation relations can be expressed

$$
\begin{equation*}
\left[\psi, \psi^{\dagger}\right]=\sigma^{y} \tag{9}
\end{equation*}
$$

where, the RHS is understood to be the $2 N \times 2 N$ Pauli matrix $\sigma^{y} \otimes \mathbb{1}_{N \times N}$. The most general quadratic Hamiltonian $\mathscr{H}$ which respects the $U(1)$ symmetry of the complex scalar can be written,

$$
\begin{equation*}
\mathscr{H}=\psi^{\dagger} \sigma^{y} h \psi \quad \text { with EOM } \quad i \partial_{t} \psi=h \psi \tag{10}
\end{equation*}
$$

where $h$ is the $2 N \times 2 N$ dynamical matrix governing the equations of motion for the modes. The conserved charge is,

$$
\begin{equation*}
Q=i\left(\Pi^{\dagger} \Phi-\Pi \Phi^{\dagger}\right)=-\psi^{\dagger} \sigma^{y} \psi-N \tag{11}
\end{equation*}
$$

b. Self-Adjointness and the Sequilinear Form-

Hermiticity of $\mathscr{H}$ requires that $\sigma_{y} h$ is Hermitian as a matrix, while the dynamical matrix $h$ need not be. However, $h$ can be seen as a linear map acting on the space of all charge-increasing mode operators $\left(u=u^{i} \psi_{i}, u^{i} \in \mathbb{C}\right)$. This linear map is self-adjoint with respect to the nondegenerate sesquilinear form:

$$
\begin{equation*}
(u, v)=\left[u^{\dagger}, v\right]=\left(u^{i}\right)^{*}\left[\psi_{i}^{\dagger}, \psi_{j}\right] v^{j}=-\left(u^{i}\right)^{*} \sigma_{i j}^{y} v^{j} \tag{12}
\end{equation*}
$$

This form is conjugate symmetric and can be viewed as a complex inner product with mixed signature $(N, N)$.
c. Bogoliubov Diagonalization - The system is diagonalized by finding the similarity transformation, $R$, that diagonalizes the EOM in Eq. 10 :

$$
B=R \psi=\left[\begin{array}{l}
b_{-}  \tag{13}\\
b_{+}^{\dagger}
\end{array}\right] \quad \text { and } \quad i \partial_{t} B=\Lambda B
$$

where, $\Lambda$ is a diagonal frequency matrix and $B$ is a new set of bosonic operators that contains $N$ annihilation operators, $b_{-}$, of negatively charged modes and $N$ creation operators, $b_{+}^{\dagger}$, of positively charged modes. These operators satisfy the commutation relations

$$
\begin{equation*}
\left[B, B^{\dagger}\right]=\sigma^{z} \tag{14}
\end{equation*}
$$

The positive (negative) charged modes are the modes whose creation operators $b_{+}^{\dagger}\left(b_{-}^{\dagger}\right)$ increase (decrease) the total charge of a state by one. The required similarity transformation, $R$, that diagonalizes the dynamical matrix and gives operators with bosonic commutation relations should satisfy:

$$
\begin{equation*}
h=R^{-1} \Lambda R \quad \text { and } \quad R \sigma^{y} R^{\dagger}=\sigma^{z} \tag{15}
\end{equation*}
$$

where, $\Lambda$ is a diagonal matrix with the frequencies of the bosonic modes. The diagonalized Hamiltonian can be expressed as:

$$
\begin{equation*}
\mathscr{H}=B^{\dagger} \sigma^{z} \Lambda B \tag{16}
\end{equation*}
$$

with the energy spectrum of the bosonic modes given by the diagonal matrix $E=\sigma^{z} \Lambda$.

As $h$ is not hermitian, the right and left eigenvectors of $h$ are not related by a complex conjugate transpose. Nonetheless, we can express the dynamical matrix as

$$
\begin{equation*}
h=\sum_{n} \Lambda_{n n} w_{n} v_{n}^{\dagger} \tag{17}
\end{equation*}
$$

where, $w_{n}\left(v_{n}^{\dagger}\right)$ are column (row) vectors that are right (left) eigenvectors of the matrix $h$. The eigenvectors form a basis and satisfy conditions

$$
\begin{equation*}
v_{m}^{\dagger} \cdot w_{n}=\delta_{m n} \quad \text { and } \quad \sum_{n} w_{n} v_{n}^{\dagger}=\mathbb{1} \tag{18}
\end{equation*}
$$

The diagonalization allows us to construct projectors on to the $n^{\text {th }}$ bosonic mode with positive/negative charge

$$
\begin{align*}
& P_{-n}=w_{n} v_{n}^{\dagger}=R^{-1} \Gamma_{n} R \\
& P_{+n}=w_{N+n} v_{N+n}^{\dagger}=R^{-1} \Gamma_{N+n} R \tag{19}
\end{align*}
$$

where, $\Gamma_{n}$ is a matrix with only a single non-zero element viz. the $n^{\text {th }}$ element along the diagonal being one. The above equation gives the matrix representation of these projectors in the $\psi$ operator basis and these matrices are not hermitian. However, similar to $h$, these projectors viewed as linear maps on the operator space are selfadjoint under the sesquilinear form in Eq. 12 .
d. Ground State Correlations - In the ground state of quadratic systems, all observables follow from bilinear correlators (by bosonic Wick's theorem). We define the correlation matrix $C\left(C_{B}\right)$ of a state in the $\psi(B)$ operator basis to be the following expectation value:

$$
\begin{equation*}
C=\left\langle\psi \psi^{\dagger}\right\rangle \sigma^{y} \quad \text { and } \quad C_{B}=\left\langle B B^{\dagger}\right\rangle \sigma^{z} \tag{20}
\end{equation*}
$$

These are related by the similarity transformation,

$$
\begin{equation*}
C=R^{-1} C_{B} R \tag{21}
\end{equation*}
$$

The ground state of a bosonic insulator with the Hamiltonian in Eq. 10 is the state annihilated by all of the Bogoliubov mode annihilation operators ( $b_{-}$and $\left.b_{+}\right)$. It is straightforward to show that the ground state correlation matrix $C_{B}^{g}$ is a diagonal projector onto the negative mode space, and hence $C^{g}$ is also a projector onto the negative mode space:

$$
\begin{align*}
& C_{B}^{g}=\left\langle B B^{\dagger}\right\rangle \sigma_{z}=\left[\begin{array}{ll}
\left\langle b_{-} b_{-}^{\dagger}\right\rangle & -\left\langle b_{-} b_{+}\right\rangle \\
\left\langle b_{+}^{\dagger} b_{-}\right\rangle & -\left\langle b_{+}^{\dagger} b_{+}\right\rangle
\end{array}\right]=\left[\begin{array}{cc}
\mathbb{1}_{N \times N} & 0 \\
0 & 0
\end{array}\right] \\
& C^{g}=R^{-1} C_{B}^{g} R=\sum_{n} P_{-n} \tag{22}
\end{align*}
$$

The projector $P$ in Eqs. (4) and (5) is the ground state correlator of a lattice insulator in the momentum space (see Eq. $\overline{\text { B17 }}$ ).

While it is evident that the ground state correlation matrix is a projector from the above relations, it can be shown that a more general set of correlation matrices are projectors; correlation matrices $C$ of states with zero charge obey (see section A 1)

$$
\begin{equation*}
C^{2}=C \tag{23}
\end{equation*}
$$

The correlation matrix $C(t)$ of a time-evolved state satisfies the relation (see section A 2)

$$
\begin{equation*}
i \dot{C}(t)=[h, C(t)] \tag{24}
\end{equation*}
$$

The expectation value of any quadratic operator, expressed as $m=\psi^{\dagger} \sigma^{y} M \psi$, in the state whose correlation matrix is $C$ is given by

$$
\begin{equation*}
\langle m\rangle=\operatorname{Tr} M(C-\mathbb{1}) \tag{25}
\end{equation*}
$$

Using the above expression, we can show that the total current density in a lattice is (refer Appendix B 2)

$$
\begin{equation*}
\left\langle\boldsymbol{J}_{T}\right\rangle=\frac{i}{\Omega} \operatorname{Tr} C[\mathscr{R}, h] \tag{26}
\end{equation*}
$$

where, $\Omega$ is the volume of a unit cell and $\mathscr{R}$ is a position operator that specifies the position of the complex scalar in the bosonic lattice (refer Eq. (B11)).

## IV. NUMERICAL DEMONSTRATION FOR A LATTICE MODEL

In this section, we compute $\alpha$ for a non-trivial pyrochlore lattice model using two independent methods: the $k$-space integration of Eqs. (4) and (5) and finite size diagonalization in real space. The model that we present here does not apply to any physical system that we are aware of. Nonetheless, it serves as an example of a lattice bosonic insulator that shows a magnetoelectric response and allows us to verify our expression for $\alpha$.

A simple lattice bosonic insulator that shows a magnetoelectric response can be obtained by considering a pyrochlore lattice, with monopoles in all tetrahedra and a staggered potential in alternating planes. We consider the Hamiltonian in Eq. (7), with the sum on $r$ now going over all the pyrochlore lattice points, and the charge potential $V_{r}$ to be -1 for the points in the Kagome planes perpendicular to the z-direction and +1 for the remaining points in the triangular lattice planes. The hopping matrix $K$ is set so that every face of every tetrahedron has an outward flux of $\pi / 2$, i.e., every tetrahedron has a magnetic monopole in it. The presence of monopoles breaks $\mathscr{P}, T$ and all the mirror symmetries, while the staggered charge potential breaks CT.

For finite-size lattice computations, we consider periodic boundary conditions in the $x y$-plane with $L$ unit


FIG. 5. Magnetoelectric response of the system of monopoles in a pyrochlore lattice with a staggered potential. The rotational symmetries of the model ensures that $\alpha_{i j}$ is diagonal and $\alpha_{11}=\alpha_{22}$. The plots show (a) $\alpha_{11}$, (b) $\alpha_{33}$ and (c) $\theta=4 \pi^{2} \alpha_{\theta}$ as a function of the charge staggering strength $v$ with the other parameters chosen to be $\lambda=4.4642, t=1$ and $m=1$. The grey and black curves are obtained by computing the integrals in Eqs. (4) and (5) using the quadrature rule with a linear discretization size $\delta$. The colored points are obtained by measuring the polarization of different finite-sized pyrochlore lattices periodic in the $x-y$ plane with $L \times L$ unit cells and open in the $z$-direction with $L_{z}$ unit cells, with a magnetic field applied along the $z$-direction. Due to commensurability issues of the lattice vectors and the periodic planes, we restrict our finite-size computations to the response in the $z$-direction. Note that the parameters are chosen such that a large magnetoelectric effect can be seen. The gap decreases as $v$ goes to 1 for these parameters.
cells in each direction and an open boundary with $L_{z}$ unit cells along the $z$-direction. For this computation, one needs to make sure that the edges of the open boundary are both the same kind of planes (we consider Kagome planes), to ensure that the system does not have finite polarization at zero external field. We then diagonalize the Hamiltonian of this system subjected to small applied magnetic fields and obtain the charge distribution in the ground state (from Eq. (B8)). This allows us to compute polarization and estimate the magnetoelectric response along the open direction. The numerical values of $\alpha$ estimated in this system along with the values of $\alpha$ obtained from the numerical $\boldsymbol{k}$-space integration of Eqs. (4) and (5) is shown in Fig. 5.

## V. OUTLOOK

In this article, we discuss the ingredients required to construct quadratic microscopic models that exhibit magnetoelectric response in bosonic insulators. We derive an expression for the magnetoelectric response coefficient and numerically verify the expression by calculating the magnetoelectric tensor for the models we present. When compared to similar simple models of magnetoelectric fermionic systems, the bosonic systems have the added complexity of having to explicitly break $C T$ symmetry.

The derived expression allows for the calculation of the magnetoelectric response of bosonic insulators. The pri-
mary example is the Mott insulating phase of ultracold bosonic atoms with lattice potentials designed to appropriately break inversion and time reversal symmetries. The magnetoelectric response naturally shows up in two different probes. Uniform synthetic magnetic fields 11.13 induce polarization, in which the bosons shift in the direction of the applied field. Alternatively, tilting the lattice potential corresponds to a synthetic electric field which induces magnetization - that is, microscopic loop currents of the underlying bosons.

A more exotic example is provided by the Coulomb quantum spin liquid ${ }^{\sqrt[14]{117}}$ where coexistence with $\mathscr{P}$ and $T$ breaking orders can lead to axion electrodynamics ${ }^{46}$. The methods developed here can be used to calculate the effective axion coupling by working with the effective Hamiltonian of the bosonic spinons. Unlike the ultracold Mott insulator, here, the magnetoelectric effect is coupled to a true dynamical gauge field and thus can be probed through the emergent electrodynamic response. We leave these avenues of study for future work.

## ACKNOWLEDGMENTS

The authors are grateful to S. Pace, M. Bellitti, D. Long, Yuan-Ming Lu, J. Moore, C. Chamon, A. Vishwanath, and D. Arovas for stimulating discussions. C.R.L. acknowledges support from the National Science Foundation through Grant No. PHY-1752727.
${ }^{1}$ J.-P. Rivera, The European Physical Journal B 71, 299 (2009)
${ }^{2}$ J. Íñiguez, Physical Review Letters 101, 117201 (2008)
${ }^{3}$ C. Xu, H. Yu, J. Wang, and H. Xiang, Annual Review of Condensed Matter Physics (2024), 10.1146/annurev-
conmatphys-032922-102353.
${ }^{4}$ N. A. Spaldin and R. Ramesh, Nature Materials 18, 203 (2019)
X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Physical Review B 78, 195424 (2008)
${ }^{6}$ A. M. Essin, J. E. Moore, and D. Vanderbilt, Physical Review Letters 102, 146805 (2009)
${ }^{7}$ A. M. Essin, A. M. Turner, J. E. Moore, and D. Vanderbilt, Physical Review B 81, 205104 (2010).
${ }^{8}$ A. Malashevich, I. Souza, S. Coh, and D. Vanderbilt, New Journal of Physics 12, 053032 (2010)
${ }^{9}$ X.-L. Qi and S.-C. Zhang, Reviews of Modern Physics 83, 1057 (2011)
${ }^{10}$ A. Sekine and K. Nomura, Journal of Applied Physics 129, 141101 (2021)
${ }^{11}$ N. Goldman, G. Juzeliūnas, P. Öhberg, and I. B. Spielman, Reports on Progress in Physics 77, 126401 (2014)
${ }^{12}$ C. Gross and I. Bloch, Science 357, 995 (2017)
${ }^{13}$ F. Schäfer, T. Fukuhara, S. Sugawa, Y. Takasu, and Y. Takahashi, Nature Reviews Physics 2, 411 (2020), arxiv:2006.06120 [cond-mat]
14 M. Udagawa and L. Jaubert, eds., Spin Ice, Springer Series in Solid-State Sciences, Vol. 197 (Springer International Publishing, Cham, 2021).
${ }^{15}$ C. L. Henley, Annual Review of Condensed Matter Physics 1, 179 (2010)
${ }^{16}$ L. Savary and L. Balents, Physical Review Letters 108, 037202 (2012)
${ }^{17}$ L. Savary, X. Wang, H.-Y. Kee, Y. B. Kim, Y. Yu, and G. Chen, Physical Review B 94, 075146 (2016).
${ }^{18}$ K.-T. Chen and P. A. Lee, Physical Review B 84, 205137 (2011)
${ }^{19}$ P. T. Mahon and J. E. Sipe, Physical Review Research 2, 033126 (2020)
${ }^{20}$ B. Leung and E. Prodan, Journal of Physics A: Mathematical and Theoretical 46, 085205 (2013)
${ }^{21}$ K. Everschor-Sitte, M. Sitte, and A. H. MacDonald, Physical Review B 92, 245118 (2015)
${ }^{22}$ D. Schmeltzer and A. Saxena, Physics Letters A 377, 1631 (2013)
${ }^{23}$ P. T. Mahon and J. E. Sipe, Physical Review Research 2, 043110 (2020)
${ }^{24}$ A. H. Duff and J. E. Sipe, Physical Review B 106, 085413 (2022)
${ }^{25}$ K. Shinada, A. Kofuji, and R. Peters, Physical Review B 107, 094106 (2023).
${ }^{20}$ More generally, Eq. (4) can be interpreted as the change in $\alpha_{C S}$ on tuning $\beta$.
${ }^{27}$ R. Shindou, R. Matsumoto, S. Murakami, and J.-i. Ohe, Physical Review B 87, 174427 (2013)
${ }^{28}$ L. Zhang, J. Ren, J.-S. Wang, and B. Li, Physical Review B 87, 144101 (2013).
${ }_{29}$ R. Matsumoto, R. Shindou, and S. Murakami, Physical Review B 89, 054420 (2014).
${ }^{30}$ E. Iacocca and O. Heinonen, Physical Review Applied 8, 034015 (2017)
${ }^{31} \mathrm{~F} . \mathrm{Lu}$ and Y.-M. Lu, "Magnon band topology in spinorbital coupled magnets: Classification and application to $\alpha-\mathrm{RuCl}_{3}, "$ (2018), arxiv:1807.05232 [cond-mat].
${ }^{32}$ P. A. McClarty, Annual Review of Condensed Matter Physics 13, 171 (2022)
${ }^{53}$ A. Vishwanath and T. Senthil, Physical Review X 3, 011016 (2013)
${ }^{34}$ M. Levin, F. J. Burnell, M. Koch-Janusz, and A. Stern, Physical Review B 84, 235145 (2011)
${ }^{35}$ B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Physical Review B 83, 195139 (2011).
${ }_{36} \frac{\text { Z. X. Li, Y. Cao, and P. Yan, Physics Reports Topological }}{}$

Insulators and Semimetals in Classical Magnetic Systems, 915, 1 (2021).
${ }^{3}$ K. N. Okada, Y. Takahashi, M. Mogi, R. Yoshimi, A. Tsukazaki, K. S. Takahashi, N. Ogawa, M. Kawasaki, and Y. Tokura, Nature Communications 7, 12245 (2016)
${ }^{38}$ V. Dziom, A. Shuvaev, A. Pimenov, G. V. Astakhov, C. Ames, K. Bendias, J. Böttcher, G. Tkachov, E. M. Hankiewicz, C. Brüne, H. Buhmann, and L. W. Molenkamp, Nature Communications 8, 15197 (2017)
${ }^{39}$ L. Wu, M. Salehi, N. Koirala, J. Moon, S. Oh, and N. P. Armitage, Science 354, 1124 (2016)
${ }^{40}$ A. Malashevich, S. Coh, I. Souza, and D. Vanderbilt, Physical Review B 86, 094430 (2012)
${ }^{41}$ B. Göbel, A. Mook, J. Henk, and I. Mertig, Physical Review B 99, 060406 (2019)
${ }^{42}$ M. Kammermeier, P. Wenk, and U. Zülicke, Physical Review B 100, 075421 (2019).
${ }^{43}$ More precisely, this is the smallest model of the form in Eq. (8) with nonzero $\alpha$ and sufficient symmetry to rule out a dipole moment and magnetization in the absence of applied fields.
${ }^{44}$ J. H. P. Colpa, Physica A: Statistical Mechanics and its Applications 93, 327 (1978)
${ }^{45}$ K. Kustura, C. C. Rusconi, and O. Romero-Isart, Physical Review A 99, 022130 (2019).
${ }^{46}$ S. D. Pace, C. Castelnovo, and C. R. Laumann, Physical Review Letters 130, 076701 (2023).

## Appendix A: Properties of the correlation matrix

## 1. Proof of $C^{2}=C$ for states with zero charge

For a considered state, we define the matrix

$$
\begin{equation*}
e=\left\langle\psi \psi^{\dagger}\right\rangle \tag{A1}
\end{equation*}
$$

so that the correlation matrix in the same state is given by $C=C \sigma^{y}$. Since the operator $\psi^{\dagger}$ is a charge-lowering operator, for the correlations in a zero-charge state, we get

$$
\begin{align*}
C & =\left\langle\psi \psi^{\dagger}\right\rangle=\left\langle\psi(-Q) \psi^{\dagger}\right\rangle \\
& =\left\langle\psi\left(\psi^{\dagger} \sigma^{y} \psi+1\right) \psi^{\dagger}\right\rangle \\
& =\left\langle\psi_{i} \psi_{k}^{\dagger} \sigma^{y}{ }_{k l} \psi_{l} \psi_{j}^{\dagger}\right\rangle+\left\langle\psi_{i} \psi_{j}^{\dagger}\right\rangle \\
& =\left\langle\psi_{i} \psi_{k}^{\dagger}\right\rangle \sigma^{y}{ }_{k l}\left\langle\psi_{l} \psi_{j}^{\dagger}\right\rangle+\left\langle\psi_{i} \psi_{j}^{\dagger}\right\rangle\left\langle\left(\psi_{k}^{\dagger} \sigma^{y}{ }_{k l} \psi_{l}+1\right)\right\rangle \\
& =C \sigma^{y} C-C\langle Q\rangle \tag{A2}
\end{align*}
$$

We have only specified the Nambu indices in the above equations. We used Wick's theorem for the simplification above. The second term in the last equation is zero if the charge of the state is zero. This leaves us with $C=C \sigma^{y} C$, which leads to $C=C^{2}$.

## 2. Proof of $\dot{C}=i[C, h]$

In a system with Hamiltonian $\mathscr{H}=\psi^{\dagger} \sigma^{y} h \psi$, the equations of motion are:

$$
\begin{equation*}
i \dot{\psi}=h \psi \quad \text { and } \quad i \dot{\psi}^{\dagger}=-\psi^{\dagger} \sigma^{y} h \sigma^{y} \tag{A3}
\end{equation*}
$$

Taking the time derivative of the product and then the state expectation value,

$$
\begin{align*}
i \partial_{t}\left(\psi \psi^{\dagger}\right) & =h \psi \psi^{\dagger}-\psi \psi^{\dagger} \sigma^{y} h \sigma^{y} \\
i \dot{C} & =h C-C \sigma^{y} h \sigma^{y} \tag{A4}
\end{align*}
$$

Multiplying the above equation with $\sigma^{y}$ from the right, we get

$$
i \dot{C}=[h, C] .
$$

## Appendix B: Derivation of the magnetoelectric tensor

We consider a time-dependent Hamiltonian whose magnetoelectric response varies with time. When this system is subjected to a constant magnetic field, it gains polarization over time, and the current leading to the polarization is a bulk current (see Fig. 3). We analytically calculate the current and integrate it in order to obtain the polarization and extract the linear response coefficient $\alpha$. This derivation is a bosonic version of the derivation by Essin et.al. ${ }^{77}$. We start with the mathematical descriptions of magnetic translational symmetry and current in bosonic systems and then go on to the derivation.

## 1. Magnetic Translational Symmetry (MTS)

Consider a translationally symmetric system in a uniform magnetic field. The Hamiltonian in the position basis is given by

$$
\begin{equation*}
\mathcal{H}=\sum_{i, j, \boldsymbol{r}, \boldsymbol{r}^{\prime}, \alpha, \alpha^{\prime}} \psi_{i \boldsymbol{r} \alpha}^{\dagger}\left(\sigma^{y} h\right)_{i \boldsymbol{r} \alpha, j \boldsymbol{r}^{\prime} \alpha^{\prime}} \psi_{j \boldsymbol{r}^{\prime} \alpha^{\prime}} \tag{B1}
\end{equation*}
$$

where, for $\psi_{i_{r} \alpha}, i$ is the Nambu index that specifies if $\Phi$ or $\Pi$ is chosen, $\boldsymbol{r}$ specifies the position of the unit cell, and $\alpha$ is the unit cell index which specifies the position to be $\boldsymbol{r}+\delta \boldsymbol{r}_{\alpha}$. Despite the presence of translational symmetry, the Hamiltonian (and the dynamical matrix $h$ ) will not have the same symmetry, since the vector potential corresponding to a uniform magnetic field cannot be chosen to be translationally symmetric. However, $h$ can be broken down into a translationally symmetric term times a phase, which depends on the choice of gauge for the vector potential.

We choose the symmetric gauge for the vector potential, $\mathcal{A}(\mathbf{r})=\frac{\mathbf{1}}{\mathbf{2}} \mathbf{B} \times \mathbf{r}$. Then, $h$ is said to have magnetic translational symmetry (MTS) if it satisfies the relation

$$
\begin{equation*}
h_{i\left(\boldsymbol{r}+\boldsymbol{r}_{o}\right) \alpha, j\left(\boldsymbol{r}^{\prime}+\boldsymbol{r}_{o}\right) \alpha^{\prime}}=e^{\frac{i}{2} \boldsymbol{B} \cdot\left(\boldsymbol{r}_{o} \times\left(\boldsymbol{r}-\boldsymbol{r}^{\prime}\right)\right)} h_{i \boldsymbol{r} \alpha, i \boldsymbol{r}^{\prime} \alpha^{\prime}} . \tag{B2}
\end{equation*}
$$

If $h$ has MTS, then it can be expressed as

$$
\begin{equation*}
h_{i \boldsymbol{r} \alpha, j \boldsymbol{r}^{\prime} \alpha^{\prime}}=\bar{h}_{i \boldsymbol{r} \alpha, j \boldsymbol{r}^{\prime} \alpha^{\prime}} e^{-\frac{i}{2} \boldsymbol{B} \cdot\left(\boldsymbol{r} \times \boldsymbol{r}^{\prime}\right)} \tag{B3}
\end{equation*}
$$

where, $\bar{h}$ is translationally symmetric. Further,

$$
\begin{equation*}
\bar{h}=h_{o}+h^{\prime}(B) \tag{B4}
\end{equation*}
$$

where, $h_{o}$ is the dynamical matrix of the system when $B=0$ and $h^{\prime}(B)$ is the dependence of the dynamical matrix on $B$ that isn't accounted for by adding minimal coupling to the hopping matrix elements.

If the Hamiltonian (or dynamical matrix) of a system has MTS, then the correlation matrix corresponding to the ground state of the same Hamiltonian also has MTS and

$$
\begin{align*}
C_{i \boldsymbol{r} \alpha, j \boldsymbol{r}^{\prime} \alpha^{\prime}} & =\bar{C}_{i \boldsymbol{r} \alpha, j \boldsymbol{r}^{\prime} \alpha^{\prime}} e^{-\frac{i}{2} \boldsymbol{B} \cdot\left(\boldsymbol{r} \times \boldsymbol{r}^{\prime}\right)}  \tag{B5}\\
\bar{C} & =C_{o}+C^{\prime} \tag{B6}
\end{align*}
$$

where, $\bar{C}$ is translationally symmetric, $C_{o}$ is independent of $\boldsymbol{B}$ and $C^{\prime}$ encodes all the dependence of $\bar{C}$ on $\boldsymbol{B}$.

## 2. Charge and Current operators

The total charge of the system is given by Eq. (11). The local charge at a point $\boldsymbol{r}+\delta \boldsymbol{r}_{\alpha}$ is

$$
\begin{equation*}
Q_{\boldsymbol{r} \alpha}=-\sum_{i j} \psi_{i \boldsymbol{r} \alpha}^{\dagger} \sigma_{i j}^{y} \psi_{j \boldsymbol{r} \alpha}-1 \tag{B7}
\end{equation*}
$$

The expectation of local charge in a state with correlation matrix $C$ (using Eq. 25 ) is given by:

$$
\begin{equation*}
\left\langle Q_{\boldsymbol{r} \alpha}\right\rangle=1-C_{1 \boldsymbol{r} \alpha}-C_{2 \boldsymbol{r} \alpha} \tag{B8}
\end{equation*}
$$

In the equations that follow, we suppress the Nambu indices when they are unnecessary. The local current can be found by using the continuity equation $\left.\operatorname{div}(\mathscr{G})\right|_{\boldsymbol{r}+\delta \boldsymbol{r}_{\alpha}}=\partial_{t} Q_{\boldsymbol{r} \alpha}$ to be

$$
\begin{align*}
\mathscr{I r}_{\boldsymbol{r} \alpha, \boldsymbol{r}^{\prime} \alpha^{\prime}}=i( & \psi_{\boldsymbol{r} \alpha}^{\dagger}\left(\sigma^{y} h\right)_{\boldsymbol{r} \alpha, \boldsymbol{r}^{\prime} \alpha^{\prime}} \psi_{\boldsymbol{r}^{\prime} \alpha^{\prime}} \\
& \left.-\psi_{\boldsymbol{r}^{\prime} \alpha^{\prime}}^{\dagger}\left(\sigma^{y} h\right)_{\boldsymbol{r}^{\prime} \alpha^{\prime}, \boldsymbol{r} \alpha} \psi_{\boldsymbol{r} \alpha}\right) \tag{B9}
\end{align*}
$$

The local current density vector is given by

$$
\begin{equation*}
\boldsymbol{J}_{\boldsymbol{r} \alpha}=\frac{1}{\Omega} \sum_{\boldsymbol{r}^{\prime}, \alpha^{\prime}}\left(\boldsymbol{r}+\delta \boldsymbol{r}_{\alpha}-\boldsymbol{r}^{\prime}-\delta \boldsymbol{r}_{\alpha^{\prime}}\right) \mathscr{I}_{\boldsymbol{r} \alpha \boldsymbol{r}^{\prime} \alpha^{\prime}} \tag{B10}
\end{equation*}
$$

where, $\Omega$ is the volume of a unit cell. The total current density is $\boldsymbol{J}_{T}=\sum_{\boldsymbol{r}, \alpha} \boldsymbol{J}_{\boldsymbol{r} \alpha}$

We now define a position operator in this space as

$$
\begin{equation*}
\mathscr{R}_{i \boldsymbol{r} \alpha, j \boldsymbol{r}^{\prime} \alpha^{\prime}}=\left(\boldsymbol{r}+\delta \boldsymbol{r}_{\alpha}\right) \delta_{\boldsymbol{r} \boldsymbol{r}^{\prime}} \delta_{\alpha, \alpha^{\prime}} \delta_{i j} \tag{B11}
\end{equation*}
$$

With the above definition of $\mathscr{R}$, we can now express the total current density of a state with correlator $C$ as

$$
\begin{equation*}
\boldsymbol{J}=\left\langle\boldsymbol{J}_{T}\right\rangle=\frac{i}{\Omega} \operatorname{Tr} C[h, \mathscr{R}] . \tag{B12}
\end{equation*}
$$

Using Eq. 23) the total current density can be expressed as

$$
\begin{equation*}
\boldsymbol{J}=\frac{i}{\Omega} \operatorname{Tr}[C,[C, \mathscr{R}]][C, h] \tag{B13}
\end{equation*}
$$

For a state being time evolved by the dynamical matrix $h$, using Eq. 24), the total current density can be expressed as

$$
\begin{equation*}
J=\frac{1}{\Omega} \operatorname{Tr}[C,[C, \mathscr{R}]] \dot{C} \tag{B14}
\end{equation*}
$$

## 3. Hamiltonian and state correlators

We consider a time-dependent Hamiltonian whose magnetoelectric response varies with time. Consider a $U(1)$ conserving system of non-interacting bosons in a uniform magnetic field $\boldsymbol{B}$, with the Hamiltonian $\mathscr{H}(\beta, \boldsymbol{B})=\psi^{\dagger} \sigma^{y} h \psi$ containing a time dependant parameter $\beta(t)$, which is such that the magnetoelectric polarizibility $\alpha$ vanishes when $\beta=0$. For example, $\beta$ could be the parameter that accompanies a term that breaks $\mathscr{P}$ (parity) or $T$ (time-reversal) or both symmetries.

Let $C^{g}$ and $C_{o}$ be the correlators corresponding to the ground state of $\mathcal{H}(\beta, \boldsymbol{B})$ and $\mathscr{H}(\beta, \boldsymbol{B}=0)$ respectively. We imagine a situation where the system at time $t=0$ is initialized in the ground state of $\mathcal{H}(\beta=0, \boldsymbol{B})$ and is adiabatically evolved with the Hamiltonian $\mathcal{H}(\beta(t), \boldsymbol{B})$, where $\beta(0)=0$. The correlator corresponding to the state at time $t$ is $C(t)$.

The derivation can be broken down into two steps: first, we find the perturbative corrections to $C(t)$ in $\boldsymbol{B}$ using MTS, and second, we integrate the perturbative expansion of the current. In the rest of the derivation, we suppress the Nambu index and the unit cell index when they are not important.

## 4. Perturbative expansion in B using MTS

Since the system has MTS (see section B 1), we have

$$
\begin{align*}
C_{\boldsymbol{r}_{1} \boldsymbol{r}_{2}}^{g} & =\bar{C}^{g} \boldsymbol{r}_{1} \boldsymbol{r}_{2} e^{-\frac{i}{2} \boldsymbol{B} \cdot\left(\boldsymbol{r}_{1} \times \boldsymbol{r}_{2}\right)} \\
\bar{C}^{g} & =C_{o}+C^{\prime} \tag{B15}
\end{align*}
$$

Using a Fourier transformation

$$
\begin{equation*}
\psi_{\boldsymbol{r} \alpha}=\int_{\mathrm{BZ}} \frac{d^{3} k}{(2 \pi)^{3}} e^{i \boldsymbol{k} \cdot\left(\boldsymbol{r}+\delta \boldsymbol{r}_{\alpha}\right)} \psi_{\boldsymbol{k} \alpha} \tag{B16}
\end{equation*}
$$

the ground state correlator at zero magnetic field $C_{o}$ is simply given by

$$
\begin{aligned}
C_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} & =\int_{\mathrm{BZ}} \frac{d^{3} k}{(2 \pi)^{3}} e^{i \boldsymbol{k} \cdot \boldsymbol{r}_{1}} P_{\boldsymbol{k}} e^{-i \boldsymbol{k} \cdot \boldsymbol{r}_{2}} \\
\text { with } P_{\boldsymbol{k}} & =\sum_{n=1}^{N} w_{\boldsymbol{k} n} v_{\boldsymbol{k} n}^{\dagger}
\end{aligned}
$$

where, $w_{\boldsymbol{k} n}$ and $v_{\boldsymbol{k} \boldsymbol{n}}^{\dagger}$ are the right and left eigenvectors of $h_{o \boldsymbol{k}}$, the momentum space dynamical matrix of the Hamiltonian $\mathcal{H}(\beta, \boldsymbol{B}=0) . C^{\prime}$ is the $\boldsymbol{B}$ dependent component of $\bar{C}^{g}$ and we find $C^{\prime}$ to first order in $\boldsymbol{B}$ by using some simple relations and MTS.

The ground state of the system has zero total charge, and hence we have (see section A 1):

$$
\begin{equation*}
C^{g}=\left(C^{g}\right)^{2} \tag{B18}
\end{equation*}
$$

Expanding the above expression in the position basis and using MTS, we get

$$
\begin{equation*}
\bar{C}^{g} \boldsymbol{r}_{1} \boldsymbol{r}_{3}=\sum_{\boldsymbol{r}_{2}} \bar{C}^{g} \boldsymbol{r}_{1} \boldsymbol{r}_{2} \bar{C}^{g} \boldsymbol{r}_{2} \boldsymbol{r}_{3} e^{-\frac{i}{2} \boldsymbol{B} \cdot\left(\boldsymbol{r}_{1} \times \boldsymbol{r}_{2}+\boldsymbol{r}_{2} \times \boldsymbol{r}_{3}+\boldsymbol{r}_{3} \times \boldsymbol{r}_{1}\right)} \tag{B19}
\end{equation*}
$$

Expanding the above equation to first order in $\boldsymbol{B}$ and using the relation $\left(\boldsymbol{r}_{1} \times \boldsymbol{r}_{2}+\boldsymbol{r}_{2} \times \boldsymbol{r}_{3}+\boldsymbol{r}_{3} \times \boldsymbol{r}_{1}\right)=\left(\boldsymbol{r}_{2}-\right.$ $\left.\boldsymbol{r}_{1}\right) \times\left(\boldsymbol{r}_{3}-\boldsymbol{r}_{2}\right)$, we get

$$
\begin{equation*}
\left(1-C_{o}\right) C^{\prime}\left(1-C_{o}\right)-C_{o} C^{\prime} C_{o}=-\frac{i}{2} \boldsymbol{B} \cdot\left[C_{o}, \mathscr{R}\right] \times\left[C_{o}, \mathscr{R}\right] \tag{B20}
\end{equation*}
$$

Recall that $C_{o}$ is a projector, and hence, the above equation allows us to find all the components of $C^{\prime}$ projected onto the same negative/positive mode.

To find the remaining 'off-diagonal' components of $C^{\prime}$, note that the matrices $h$ and $C^{g}$ have the same eigenstates, and hence we have

$$
\begin{equation*}
\left[h, C^{g}\right]=0 \tag{B21}
\end{equation*}
$$

Rewriting the above equation in the position basis, using MTS (see section B1) and expanding to linear order in $\boldsymbol{B}$ gives:

$$
\begin{align*}
{\left[C^{\prime}, h_{o}\right]=} & \frac{i}{2} \boldsymbol{B} \cdot\left(\left[C_{o}, \mathscr{R}\right] \times\left[h_{o}, \mathscr{R}\right]-\left[h_{o}, \mathscr{R}\right] \times\left[C_{o}, \mathscr{R}\right]\right) \\
& -\left[C_{o}, h^{\prime}\right] \tag{B22}
\end{align*}
$$

Using Eqs. 19) and B17, we get

$$
\begin{align*}
P_{-n} C^{\prime} P_{+m}= & \frac{i B^{j} \epsilon_{j a b} P_{-n}\left\{\partial^{a} h_{o}, \partial^{b} P\right\} P_{+m}}{E_{-n}+E_{+m}} \\
& +\frac{P_{-n} h^{\prime} P_{+m}}{E_{-n}+E_{+m}} \tag{B23}
\end{align*}
$$

In the above equation, momentum labels have been dropped for brevity.

## 5. Expanding Current in B

With the thought experiment considered, the adiabatic time evolution of the state would be given by

$$
\begin{equation*}
i \dot{C}(t)=[h(t), C(t)] \tag{B24}
\end{equation*}
$$

The total current density at time $t$ is given by

$$
\begin{equation*}
J=\frac{1}{\Omega} \operatorname{Tr}[C,[C, \mathscr{R}]] \dot{C} \tag{B25}
\end{equation*}
$$

Using the adiabatic approximation $C \approx C^{g}$ in the above equation and expanding in position space,

$$
\begin{equation*}
\boldsymbol{J}=\frac{1}{\Omega} \sum_{\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, \boldsymbol{r}_{3}}\left(\boldsymbol{r}_{1}-2 \boldsymbol{r}_{2}+\boldsymbol{r}_{3}\right) C_{\boldsymbol{r}_{1} \boldsymbol{r}_{2}}^{g} C_{\boldsymbol{r}_{2} \boldsymbol{r}_{3}}^{g} \dot{C}^{g} \boldsymbol{r}_{3} \boldsymbol{r}_{1} \tag{B26}
\end{equation*}
$$

$$
\begin{align*}
& J=\frac{1}{\Omega} \sum_{\boldsymbol{r}_{1}, \boldsymbol{r}_{2}, \boldsymbol{r}_{3}}\left(\boldsymbol{r}_{1}-2 \boldsymbol{r}_{2}+\boldsymbol{r}_{3}\right)\left(C_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} C_{o \boldsymbol{r}_{2} \boldsymbol{r}_{3}} \dot{C}^{\prime} \boldsymbol{r}_{3} \boldsymbol{r}_{1}+C^{\prime} \boldsymbol{r}_{1} \boldsymbol{r}_{2} C_{o \boldsymbol{r}_{2} \boldsymbol{r}_{3}} \dot{C}_{o \boldsymbol{r}_{3} \boldsymbol{r}_{1}}+C_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} C_{\boldsymbol{r}_{2} \boldsymbol{r}_{3}}^{\prime} \dot{C}_{o \boldsymbol{r}_{3} \boldsymbol{r}_{1}}\right.  \tag{B27}\\
&\left.-\frac{i}{2} \boldsymbol{B} \cdot\left(\boldsymbol{r}_{1} \times \boldsymbol{r}_{2}+\boldsymbol{r}_{2} \times \boldsymbol{r}_{3}+\boldsymbol{r}_{3} \times \boldsymbol{r}_{1}\right) C_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} C_{o \boldsymbol{r}_{2} \boldsymbol{r}_{3}} \dot{C}_{o \boldsymbol{r}_{3} \boldsymbol{r}_{1}}\right)
\end{align*}
$$

The first term above can be expanded in terms of a total derivative

$$
\begin{align*}
& C_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} C_{o \boldsymbol{r}_{2} \boldsymbol{r}_{3}} \dot{C}^{\prime} \boldsymbol{r}_{3} \boldsymbol{r}_{1}=\partial_{t}\left(C_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} C_{o \boldsymbol{r}_{2} \boldsymbol{r}_{3}} C_{\boldsymbol{r}_{3} \boldsymbol{r}_{1}}^{\prime}\right) \\
& \quad-\dot{C}_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} C_{o \boldsymbol{r}_{2} \boldsymbol{r}_{3}} C^{\prime} \boldsymbol{r}_{3} \boldsymbol{r}_{1}-C_{o \boldsymbol{r}_{1} \boldsymbol{r}_{2}} \dot{C}_{o \boldsymbol{r}_{2} \boldsymbol{r}_{3}} C^{\prime} \boldsymbol{r}_{3} \boldsymbol{r}_{1} \tag{B28}
\end{align*}
$$

The total derivative in the above equation can be rewritten as

$$
\begin{equation*}
\boldsymbol{J}_{G}=\frac{1}{\Omega} \partial_{t}\left(\operatorname{Tr}\left[C_{o}, \mathscr{R}\right]\left[C^{\prime}, C_{o}\right]\right) \tag{B29}
\end{equation*}
$$

Using the expression for the off-diagonal elements of $C^{\prime}$ (see Eq. (B23) in the above expression and using $J_{G}^{i}=\partial_{t}\left(\alpha_{G}\right)_{j}^{l} B^{J}$, we get the cross gap contribution to the magnetoelectric polarizability, $\alpha_{G}$, in Eq. (5)

The rest of the terms in the equation for total current density can be rewritten as

$$
\begin{align*}
J_{C S 1} & =-\frac{3}{\Omega} \operatorname{Tr} C^{\prime}\left[\dot{C}_{o},\left[C_{o}, \mathscr{R}\right]\right]  \tag{B30}\\
\boldsymbol{J}_{C S 2} & =-\frac{i}{2} B^{j} \epsilon_{j a b} \operatorname{Tr}\left[C_{o}, \mathscr{R}\right]\left[C_{o}, \mathscr{R}^{a}\right]\left[\mathscr{R}^{b}, \dot{C}_{o}\right]+\text { c.c. } \tag{B31}
\end{align*}
$$

Taking the Fourier transform and simplifying, we get

$$
\begin{align*}
\boldsymbol{J}_{C S} & =\boldsymbol{J}_{C S 1}+\boldsymbol{J}_{C S 2} \\
& =\int_{\mathrm{BZ}} \frac{d^{3} k}{(2 \pi)^{3}} \boldsymbol{B} \operatorname{Tr} P_{\boldsymbol{k}}\left(\left[\dot{P}_{\boldsymbol{k}}, \partial^{x} P_{\boldsymbol{k}}\right]\left[\partial^{y} P_{\boldsymbol{k}}, \partial^{z} P_{\boldsymbol{k}}\right]\right. \\
& \left.+\left[\dot{P}_{\boldsymbol{k}}, \partial^{y} P_{\boldsymbol{k}}\right]\left[\partial^{z} P_{\boldsymbol{k}}, \partial^{x} P_{\boldsymbol{k}}\right]+\left[\dot{P}_{\boldsymbol{k}}, \partial^{z} P_{\boldsymbol{k}}\right]\left[\partial^{x} P_{\boldsymbol{k}}, \partial^{y} P_{\boldsymbol{k}}\right]\right) \tag{B32}
\end{align*}
$$

Using $F^{\mu \nu}=i P\left(\partial^{\mu} P \partial^{\nu} P-\partial^{\nu} P \partial^{\mu} P\right) P$, the above equation can be rewritten as

$$
\begin{equation*}
\boldsymbol{J}_{C S}=-\frac{\boldsymbol{B}}{8} \int_{\mathrm{BZ}} \frac{d^{3} k}{(2 \pi)^{3}} \epsilon_{\mu \nu \gamma \lambda} \operatorname{Tr} F^{\mu \nu} F^{\gamma \lambda} \tag{B33}
\end{equation*}
$$

which, along with $\boldsymbol{J}_{C S}=\partial_{t} \alpha_{C S} \boldsymbol{B}$, gives us the expression for $\alpha_{C S}$ in Eq. (6).

Using Eq. B15 and expanding linear in $\boldsymbol{B}$, we get

## Appendix C: Group structure of the generalized Bogoliubov transformation

The set of generalized Bogoliubov transformations $R$ defined in Eq. (15) that diagonalize dynamical matrices $h$ don't form a group themselves. However, we can define transformations $S$ for every $R$ by

$$
\begin{equation*}
R=S e^{-i \frac{\pi}{4} \sigma^{x}} \tag{C1}
\end{equation*}
$$

With this, for the transformations $S$, we have the condition:

$$
\begin{equation*}
S \sigma^{z} S^{\dagger}=\sigma^{z} \tag{C2}
\end{equation*}
$$

The transformations $S$ are part of the generalized unitary group $U(N, N)$. Note that $U(N, N) \cong G(2 N)$, where $G$ is the conjugate symplectic group.

## Appendix D: Generality of formalism

The formalism introduced in Section III can be further generalized to any bosonic system with a $U(1)$ symmetry. We can consider a system of bosons that have $N$ charge raising creation operators $a_{+i}^{\dagger}$ defined by $\left[Q, a_{+i}^{\dagger}\right]=a_{+i}^{\dagger}$ and $M$ charge lowering operators $a_{-i}^{\dagger}$ defined by $\left[Q, a_{-i}^{\dagger}\right]=-a_{-i}^{\dagger}$. The most general quadratic Hamiltonian conserving the charge $Q$ can be expressed in terms of $A=\left[\begin{array}{c}a_{-}^{-} \\ a_{+}^{\dagger}\end{array}\right]$, an $N+M$ size column vector of operators, as

$$
\begin{align*}
\mathcal{H} & =A^{\dagger} o^{z} h A  \tag{D1}\\
\text { with } o^{z} & =\left[\begin{array}{cc}
\mathbb{1}_{M \times M} & 0 \\
0 & -\mathbb{1}_{N \times N}
\end{array}\right]
\end{align*}
$$

where, $o^{z}=\left[A, A^{\dagger}\right]$ is the matrix that specifies the commutation relations of the bosonic operators. This Hamiltonian can be diagonalized by the similarity transformation $S$, which gives the diagonal Bogoluibons

$$
B=S A=\left[\begin{array}{l}
b_{-}  \tag{D2}\\
b_{+}^{\dagger}
\end{array}\right]
$$

These are operators that satisfy $\left[Q, b_{ \pm i}^{\dagger}\right]= \pm b_{ \pm i}^{\dagger}$ and $\left[H, b_{ \pm i}^{\dagger}\right]=E_{ \pm i} b_{ \pm i}^{\dagger}$. The required transformation can be
found by diagonalizing the matrix $h$ and ensuring that the new set of operators thus found still satisfies the bosonic commutation relations:

$$
\begin{equation*}
h=S^{-1} \Lambda S \quad \text { and } \quad S o^{z} S^{\dagger}=o^{z} \tag{D3}
\end{equation*}
$$

The systems considered in the main text are those that have $N=M$, where every pair of operators $a_{ \pm i}$ can be thought of as charge increasing and decreasing modes of the local 2D oscillator.

