THE INTERMEDIATE EXTENSION, VANISHING CYCLES, AND PERVERSE EIGENSPACE OF ONE

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Abstract. We prove a number of results involving the kernel of the identity minus the monodromy on the vanishing cycles.

1. Introduction

Suppose that $\mathcal{U}$ is a non-empty open subset of $\mathbb{C}^{n+1}$ and $f : \mathcal{U} \to \mathbb{C}$ is a nowhere locally constant, reduced, complex analytic function such that $f^{-1}(0)$ is non-empty. Then $V(f) = f^{-1}(0)$ is an affine hypersurface inside $\mathcal{U}$.

In Theorem 2.5 of [7], we proved that we have a short exact sequence in the abelian category of perverse sheaves on $V(f)$ (using $\mathbb{Z}$ as our base ring):

$$(\dagger) \qquad 0 \to \ker\{\mathrm{id} - \widetilde{T}_f\} \to \mathbb{Z}^\bullet_{V(f)}[n] \to \mathbf{I}^\bullet_{V(f)} \to 0,$$

where $\mathbf{I}^\bullet_{V(f)}$ is the intersection cohomology complex on $V(f)$ (with constant $\mathbb{Z}$ coefficients) and $\widetilde{T}_f$ is the vanishing cycle monodromy operator.

In Theorem 2.6 of [7], we note that the dual statement is also true. That dual result is, letting $j : V(f) \to \mathcal{U}$ denote the inclusion, there exists a short exact sequence

$$(\ddagger) \qquad 0 \to \mathbf{I}^\bullet_{V(f)} \to j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to \mathrm{coker}\{\mathrm{id} - \widetilde{T}_f\} \to 0,$$

in $\mathrm{Perv}(V(f))$.

In this paper, we will prove a number of other results involving $\ker\{\mathrm{id} - \widetilde{T}_f\}$, which is the perverse eigenspace of one for the vanishing cycle monodromy; in particular, in Theorem 5.5, we prove generalizations of (†) and (‡).

2. Preliminary notation, definitions and results

In this section, we give notations and well-known results on the derived category and perverse sheaves. Basic references are [1], [4], [2], [5], [6], [10], [3], and [8]. We should remark that we adopt a convention which is now standard; because we will work solely in the derived category, we will **not** write an R in front of derived functors since all of our functors are derived.

Let X be a complex analytic space. We use $D^b_c(X)$ to denote the derived category of bounded, constructible complexes of sheaves of $\mathbb{Z}$-modules on X. We let ${}^{\mu}\mathbf{D}^{\leqslant 0}(X)$ (respectively, ${}^{\mu}\mathbf{D}^{\geqslant 0}(X)$) denote the full subcategory of $D^b_c(X)$ of those complexes which satisfy the support (respectively, cosupport) condition. We note the abelian category of perverse sheaves on X by $\mathrm{Perv}(X)$; thus $\mathrm{Perv}(X) = {}^{\mu}\mathbf{D}^{\leqslant 0}(X) \cap {}^{\mu}\mathbf{D}^{\geqslant 0}(X)$. We let ${}^{\mu}H^k$ denote the degree k cohomology with respect to the perverse t-structure.

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Let $f : X \to \mathbb{C}$ be a nowhere locally constant complex analytic function on X. We let $j : V(f) \hookrightarrow X$ be the (closed) inclusion and $i : X\backslash V(f) \hookrightarrow X$ be the (open) inclusion. Let $\psi_f[-1]$ and $\phi_f[-1]$ denote the shifted nearby cycle and vanishing cycle functors, respectively, from $D_c^b(X)$ to $D_c^b(V(f))$; we denote the respective Milnor monodromy (natural) automorphisms of these functors by T_f and $\widetilde{T}_f$.

The functors $\psi_f[-1]$ and $\phi_f[-1]$ are t-exact with respect to the perverse t-structure. This means that each of these functors takes ${}^{\mu}\mathbf{D}^{\leqslant 0}(X)$ (respectively, ${}^{\mu}\mathbf{D}^{\geqslant 0}(X)$) to ${}^{\mu}\mathbf{D}^{\leqslant 0}(V(f))$ (respectively, ${}^{\mu}\mathbf{D}^{\geqslant 0}(V(f))$). In particular, this means that $\psi_f[-1]$ and $\phi_f[-1]$ take perverse sheaves to perverse sheaves.

For $\mathbf{A}^\bullet \in D_c^b(X)$, there are natural distinguished triangles:

The vanishing triangle:

$$\to j^*[-1]\mathbf{A}^\bullet \xrightarrow{\text{comp}} \psi_f[-1]\mathbf{A}^\bullet \xrightarrow{\text{can}} \phi_f[-1]\mathbf{A}^\bullet \xrightarrow{[1]};$$

and

The dual vanishing triangle:

$$\to \phi_f[-1]\mathbf{A}^\bullet \xrightarrow{\text{var}} \psi_f[-1]\mathbf{A}^\bullet \xrightarrow{\text{pmoc}} j^![1]\mathbf{A}^\bullet \xrightarrow{[1]} .$$

Automorphisms of these triangles are given respectively by $(\mathrm{id}, T_f, \widetilde{T}_f)$ and $(\widetilde{T}_f, T_f, \mathrm{id})$. Furthermore, $\mathrm{var}\circ\mathrm{can} = \mathrm{id} - T_f$ and $\mathrm{can}\circ\mathrm{var} = \mathrm{id} - \widetilde{T}_f$.

It is very helpful to give a name to one more map:

Definition 2.1. *We let* $\omega_f := \mathrm{pmoc}\circ\mathrm{comp}$ *be the natural transformation from* $j^*[-1]$ *to* $j^![1]$ *and refer to this as the* **Wang transformation** *(or the* **Wang morphism** *for a given* $\mathbf{A}^\bullet$*).*

Let d be an integer, and let $f : Y \to X$ be a morphism of complex spaces such that, for all $\mathbf{x} \in X$, $\dim f^{-1}(\mathbf{x}) \leqslant d$. Then,

1) f^* sends ${}^{\mu}\mathbf{D}^{\leqslant 0}(X)$ to ${}^{\mu}\mathbf{D}^{\leqslant d}(Y)$;

2) $f^!$ sends ${}^{\mu}\mathbf{D}^{\geqslant 0}(X)$ to ${}^{\mu}\mathbf{D}^{\geqslant -d}(Y)$;

3) if $\mathbf{F}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant 0}(Y)$ and $f_!\mathbf{F}^\bullet \in \mathbf{D}_c^b(X)$, then $f_!\mathbf{F}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant d}(X)$;

4) if $\mathbf{F}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 0}(Y)$ and $f_*\mathbf{F}^\bullet \in \mathbf{D}_c^b(X)$, then $f_*\mathbf{F}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant -d}(X)$.

For closed embeddings, we also have the following:

Let $g_1, \dots, g_e$ be complex analytic functions on X. Let m denote the inclusion of $Y := V(g_1, \dots, g_e)$ into X. Then,

i) m^* sends ${}^{\mu}\mathbf{D}^{\geqslant 0}(X)$ to ${}^{\mu}\mathbf{D}^{\geqslant -e}(Y)$;

ii) $m^!$ sends ${}^{\mu}\mathbf{D}^{\leqslant 0}(X)$ to ${}^{\mu}\mathbf{D}^{\leqslant e}(Y)$.

3. A Fundamental Exact Sequence

Suppose that $\mathbf{P}^\bullet \in \mathrm{Perv}(X)$. Then, applying perverse cohomology to the vanishing and dual vanishing triangles, we immediately conclude:

Proposition 3.1. *${}^\mu H^k(j^![1]\mathbf{P}^\bullet)$ is possibly non-zero only for $k = -1, 0$, ${}^\mu H^k(j^*[-1]\mathbf{P}^\bullet)$ is possibly non-zero only for $k = 0, 1$, and there are exact sequences*

$$0 \to {}^\mu H^0(j^*[-1]\mathbf{P}^\bullet) \xrightarrow{{}^\mu H^0(\mathrm{comp})} \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{\mathrm{can}} \phi_f[-1]\mathbf{P}^\bullet \to {}^\mu H^1(j^*[-1]\mathbf{P}^\bullet) \to 0;$$

and

$$0 \to {}^\mu H^{-1}(j^![1]\mathbf{P}^\bullet) \to \phi_f[-1]\mathbf{P}^\bullet \xrightarrow{\mathrm{var}} \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{{}^\mu H^0(\mathrm{pmoc})} {}^\mu H^0(j^![1]\mathbf{P}^\bullet) \to 0.$$

Thus, ${}^\mu H^0(j^*[-1]\mathbf{P}^\bullet) \cong \ker\{\mathrm{can}\}$, ${}^\mu H^1(j^*[-1]\mathbf{P}^\bullet) \cong \mathrm{coker}\{\mathrm{can}\}$, ${}^\mu H^{-1}(j^![1]\mathbf{P}^\bullet) \cong \ker\{\mathrm{var}\}$, and ${}^\mu H^0(j^![1]\mathbf{P}^\bullet) \cong \mathrm{coker}\{\mathrm{var}\}$.

Theorem 3.2. *Suppose that $\mathbf{P}^\bullet$ is a perverse sheaf on X. Then, there is an exact sequence in $\mathrm{Perv}(V(f))$:*

$$0 \to \frac{\ker\{\mathrm{id} - \widetilde{T}_f\}}{\ker\{\mathrm{var}\}} \longrightarrow {}^\mu H^0(j^*[-1]\mathbf{P}^\bullet) \xrightarrow{{}^\mu H^0(\omega_f)} {}^\mu H^0(j^![1]\mathbf{P}^\bullet) \longrightarrow \frac{\mathrm{im}\{\mathrm{can}\}}{\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\}} \to 0,$$

where we view $\ker\{\mathrm{var}\}$ as a sub-perverse sheaf of $\ker\{\mathrm{id} - \widetilde{T}_f\}$ by the canonical injection $\ker\{\mathrm{var}\} \hookrightarrow \ker\{\mathrm{can} \circ \mathrm{var}\} = \ker\{\mathrm{id} - \widetilde{T}_f\}$, and we view $\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\}$ as a sub-perverse sheaf of $\mathrm{im}\{\mathrm{can}\}$ by the canonical injection $\mathrm{im}\{\mathrm{id}\text{-}\widetilde{\mathrm{T}}_\mathrm{f}\} = \mathrm{im}\{\mathrm{can} \circ \mathrm{var}\} \hookrightarrow \mathrm{im}\{\mathrm{can}\}$.

Proof. One easily verifies that there is an exact sequence

$$0 \to \ker\{\mathrm{var}\} \to \ker\{\mathrm{can} \circ \mathrm{var}\} \to \ker\{\mathrm{can}\} \to \mathrm{coker}\{\mathrm{var}\} \to \mathrm{coker}\{\mathrm{can} \circ \mathrm{var}\} \to \mathrm{coker}\{\mathrm{can}\} \to 0,$$

where

- the second arrow from the left is the canonical injection,
- the third arrow is induced by var,
- the fourth arrow is the composition of the canonical injection of $\ker\{\mathrm{can}\}$ into $\psi_f[-1]\mathbf{P}^\bullet$ with the canonical surjection from $\psi_f[-1]\mathbf{P}^\bullet$ onto $\mathrm{coker}\{\mathrm{var}\}$,
- the fifth arrow is induced by can, and
- the sixth arrow is the canonical surjection.

Now the exact sequence in the statement of the theorem follows immediately from Proposition 3.1. □

4. Splitting

We have the following easy result which tells us when the vanishing cycles are a direct summand of the nearby cycles.

Theorem 4.1. *Let $\mathbf{P}^\bullet$ be a perverse sheaf on X. Then, the following are equivalent:*

(1) $\mathrm{id} - \widetilde{T}_f$ *is an isomorphism;*

(2) $\ker\{\mathrm{id} - \widetilde{T}_f\} = 0$ *and* $\mathrm{coker}\{\mathrm{id} - \widetilde{T}_f\} = 0$*;*

(3) ${}^\mu H^{-1}(j^![1]\mathbf{P}^\bullet) = 0$*,* ${}^\mu H^1(j^*[-1]\mathbf{P}^\bullet) = 0$*, and* ${}^\mu H^0(\omega_f)$ *is an isomorphism;*

(4) $j^![1]\mathbf{P}^\bullet$ *and* $j^*[-1]\mathbf{P}^\bullet$ *are perverse, and* $\omega_f : j^*[-1]\mathbf{P}^\bullet \to j^![1]\mathbf{P}^\bullet$ *is an isomorphism;*

(5) $\omega_f : j^*[-1]\mathbf{P}^\bullet \to j^![1]\mathbf{P}^\bullet$ *is an isomorphism, and* $\mathbf{P}^\bullet$ *is the intermediate extension of* $i^*\mathbf{P}^\bullet = i^!\mathbf{P}^\bullet$ *to* $V(f)$*.*

Furthermore, when these equivalent conditions hold, the vanishing and dual vanishing triangles are short exact sequences in $\mathrm{Perv}(V(f))$ *which split in a manner compatible with the monodromy automorphisms; thus we have isomorphisms*

$$\psi_f[-1]\mathbf{P}^\bullet \cong \phi_f[-1]\mathbf{P}^\bullet \oplus j^*[-1]\mathbf{P}^\bullet \cong \phi_f[-1]\mathbf{P}^\bullet \oplus j^![1]\mathbf{P}^\bullet$$

and, via these isomorphisms, T_f *is identified with* $(\widetilde{T}_f, \mathrm{id})$ *in each case.*

Proof. Given Proposition 3.1 and Theorem 3.2, the equivalences of (1), (2), and (3) are utterly trivial. Clearly (4) implies (3), and (3) together with Theorem 3.2 implies (4), since having zero perverse cohomology outside of degree 0 is equivalent to being perverse (see 1.3.7 of [1] or Proposition 5.1.7 of [3]).

We need to show the equivalence of (4) and (5). Recall the standard results (i) and (ii) from Section 2, which tell us that $j^*[-1]$ sends ${}^\mu\mathbf{D}^{\geqslant 0}(X)$ to ${}^\mu\mathbf{D}^{\geqslant 0}(V(f))$ and $j^![1]$ sends ${}^\mu\mathbf{D}^{\leqslant 0}(X)$ to ${}^\mu\mathbf{D}^{\leqslant 0}(V(f))$; therefore, $j^*[-1]\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\geqslant 0}(V(f))$ and $j^![1]\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\leqslant 0}(V(f))$. But one of the equivalent definitions/characterizations of $\mathbf{P}^\bullet$ being the intermediate extension of $i^*\mathbf{P}^\bullet = i^!\mathbf{P}^\bullet$ is that $j^*[-1]\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\leqslant 0}(V(f))$ and $j^![1]\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\geqslant 0}(V(f))$ (see, for instance, [3] Definition 5.2.6). Thus (4) and (5) are equivalent.

We need to show that (1)-(5) imply the splittings exist. Assuming (1)-(5), we have short exact sequences in $\mathrm{Perv}(X)$:

$$0 \to j^*[-1]\mathbf{P}^\bullet \xrightarrow{\text{comp}} \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{\text{can}} \phi_f[-1]\mathbf{P}^\bullet \to 0$$

and

$$0 \to \phi_f[-1]\mathbf{P}^\bullet \xrightarrow{\text{var}} \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{\text{pmoc}} j^![1]\mathbf{P}^\bullet \to 0,$$

where $\mathrm{can} \circ \mathrm{var} = \mathrm{id} - \widetilde{T}_f$.

Consider $\mathrm{var} \circ (\mathrm{id} - \widetilde{T}_f)^{-1}$ from $\phi_f[-1]\mathbf{P}^\bullet$ to $\psi_f[-1]\mathbf{P}^\bullet$, and $(\mathrm{id} - \widetilde{T}_f)^{-1} \circ \mathrm{can}$ from $\psi_f[-1]\mathbf{P}^\bullet$ to $\phi_f[-1]\mathbf{P}^\bullet$. Then $\mathrm{can} \circ [\mathrm{var} \circ (\mathrm{id} - \widetilde{T}_f)^{-1}] = \mathrm{id}$ and $[(\mathrm{id} - \widetilde{T}_f)^{-1} \circ \mathrm{can}] \circ \mathrm{var} = \mathrm{id}$. The first equality shows that the first short exact sequence splits, and the second equality shows that the second short exact sequence splits. □

Remark 4.2. Throughout this paper, we use $\mathbb{Z}$ as our base ring because we care about torsion. However, we could use any base ring, R, which is a commutative, regular, Noetherian ring, with finite Krull dimension (e.g., $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$). In particular, if we use a base ring which is a field in Theorem 4.1, then $\ker\{\mathrm{id} - \widetilde{T}_f\} = 0$ if an only if $\mathrm{coker}\{\mathrm{id} - \widetilde{T}_f\} = 0$; so, in the field case, $\ker\{\mathrm{id} - \widetilde{T}_f\} = 0$ if and only if $\mathrm{id} - \widetilde{T}_f$ is an isomorphism.

5. The intermediate extension to $V(f)$

In this section, we isolate the properties that allowed us to prove (†) and (‡) from the introduction; this allows us to obtain analogous results in a much more general setting.

We continue with X f, j and i as before, and assume throughout the remainder of the paper that $\mathbf{P}^\bullet$ is a perverse sheaf on X. We let $\Sigma_f := \operatorname{supp}\phi_f[-1]\mathbf{P}^\bullet$, $m : \Sigma_f \hookrightarrow V(f)$ denote the (closed) inclusion and let $\ell : V(f)\backslash\Sigma_f \hookrightarrow V(f)$ denote the (open) inclusion. Finally, we define the (closed) inclusion $\hat{m} := j \circ m : \Sigma_f \to X$.

First, we have the easy:

Proposition 5.1. *The following are equivalent:*

(1) ${}^{\mu}H^{-1}(\hat{m}^*\mathbf{P}^\bullet) = {}^{\mu}H^0(\hat{m}^*\mathbf{P}^\bullet) = {}^{\mu}H^0(\hat{m}^!\mathbf{P}^\bullet) = {}^{\mu}H^1(\hat{m}^!\mathbf{P}^\bullet) = 0$*;*

(2) $\hat{m}^*\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant -2}(\Sigma_f)$ *and* $\hat{m}^!\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 2}(\Sigma_f)$*;*

(3) $\hat{m}^*[-2]\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant 0}(\Sigma_f)$ *and* $\hat{m}^![2]\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 0}(\Sigma_f)$*;*

(4) for all integers k,

$$\dim \operatorname{supp}^{-k}(\hat{m}^*[-2]\mathbf{P}^\bullet) = \dim \overline{\left\{x \in \Sigma_f \mid H^{-k-2}(\mathbf{P}^\bullet)_x \neq 0\right\}} \leq k$$

and

$$\dim \operatorname{cosupp}^{k}(\hat{m}^![2]\mathbf{P}^\bullet) = \dim \overline{\left\{x \in \Sigma_f \mid \mathbb{H}^{k+2}(B^\circ_\epsilon(x) \cap X,\, B^\circ_\epsilon(x) \cap X\backslash\{x\};\, \mathbf{P}^\bullet) \neq 0\right\}} \leq k$$

where $B^\circ_\epsilon(x)$ denotes a open ball of (small) radius $\epsilon > 0$, centered at x and, by convention, the empty set has dimension $-\infty$

Proof. Since $\hat{m}$ is inclusion, $\hat{m}^*$ sends ${}^{\mu}\mathbf{D}^{\leqslant 0}(X)$ to ${}^{\mu}\mathbf{D}^{\leqslant 0}(\Sigma_f)$, and $\hat{m}^!$ sends ${}^{\mu}\mathbf{D}^{\geqslant 0}(X)$ to ${}^{\mu}\mathbf{D}^{\geqslant 0}(\Sigma_f)$. Thus Items (1) and (2) are equivalent, and clearly Item (3) is equivalent to Item (2).

Item (4) is nothing more than a direct translation of the conditions (the support and cosupport conditions) in Item (3). To see this, for all $x \in \Sigma_f$, let $w_x : \{x\} \hookrightarrow \Sigma f$ so that $\breve{w}_x := \hat{m} \circ w_x$ is the inclusion of $\{x\}$ into X. Then use that $H^{-k-2}(\breve{w}_x^*\mathbf{P}^\bullet) \cong H^{-k-2}(\mathbf{P}^\bullet)_x$ and $H^{k+2}(\breve{w}_x^!\mathbf{P}^\bullet) \cong \mathbb{H}^{k+2}(B^\circ_\epsilon(x) \cap X,\, B^\circ_\epsilon(x) \cap X\backslash\{x\};\, \mathbf{P}^\bullet)$. □

Remark 5.2. The conditions $\hat{m}^*\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant -2}(\Sigma_f)$ and $\hat{m}^!\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 2}(\Sigma_f)$ are dual to each other, provided that we use a field for our base ring. To be precise, if our base ring is a field, and $\mathbf{P}^\bullet$ is self-dual (i.e., $\mathcal{D}\mathbf{P}^\bullet \cong \mathbf{P}^\bullet$), then $\hat{m}^*\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant -2}(\Sigma_f)$ if and only if $\hat{m}^!\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 2}(\Sigma_f)$.

Definition 5.3. *When the equivalent conditions of Proposition 5.1 are satisfied, we say that f is* **$\mathbf{P}^\bullet$-reduced**.

The reason for this terminology wll become clear in the next section.

In order to prove a generalization of (†) and (‡), we need something that replaces intersection cohomology of $V(f)$ with constant coefficients. That "something" is:

Definition 5.4. *Let $\mathbf{I}^\bullet_{V(f)}$ be the intermediate extension from $V(f)\backslash\Sigma_f$ to $V(f)$ of the perverse sheaf*

$$\ell^*j^*[-1]\mathbf{P}^\bullet \cong \ell^*\psi_f[-1]\mathbf{P}^\bullet \cong \ell^*j^![1]\mathbf{P}^\bullet \cong \ell^!j^![1]\mathbf{P}^\bullet \cong \ell^!\psi_f[-1]\mathbf{P}^\bullet \cong \ell^!j^*[-1]\mathbf{P}^\bullet.$$

(Note that these isomorphisms follow at once from the vanishing and dual vanishing triangles, since we are restricting to the complement on the support of the vanishing cycles.)

Now we have:

Theorem 5.5. *Suppose that f is $\mathbf{P}^\bullet$-reduced.*

Then,

(1) *$j^*[-1]\mathbf{P}^\bullet$ and $j^![1]\mathbf{P}^\bullet$ are perverse,*

(2) *$\mathbf{P}^\bullet$ is isomorphic to the intermediate extension of $i^*\mathbf{P}^\bullet = i^!\mathbf{P}^\bullet$ from $X\backslash V(f)$ to X,*

(3) *$V(f)$ contains no irreducible component of* $\operatorname{supp}\mathbf{P}^\bullet$*,*

(4) *Σ_f contains no irreducible component of* $\operatorname{supp} j^*[-1]\mathbf{P}^\bullet$ *or* $\operatorname{supp} j^![1]\mathbf{P}^\bullet$*, and*

(5) *there are short exact sequences in* $\operatorname{Perv}(V(f))$*:*

$$0 \to \ker\{\mathrm{id} - \widetilde{T}_f\} \to j^*[-1]\mathbf{P}^\bullet \to \mathbf{I}^\bullet_{V(f)} \to 0$$

and

$$0 \to \mathbf{I}^\bullet_{V(f)} \to j^![1]\mathbf{P}^\bullet \to \operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\} \to 0$$

Proof.

Proof of (1): Recall that we showed in Theorem 3.2 that ${}^\mu H^k(j^![1]\mathbf{P}^\bullet)$ is possibly non-zero only for $k = -1, 0$, and ${}^\mu H^k(j^*[-1]\mathbf{P}^\bullet)$ is possibly non-zero only for $k = 0, 1$. Thus, to show (1), we need to show that ${}^\mu H^{-1}(j^![1]\mathbf{P}^\bullet) = 0$ and ${}^\mu H^1(j^*[-1]\mathbf{P}^\bullet) = 0$. We will show that ${}^\mu H^1(j^*[-1]\mathbf{P}^\bullet) = 0$ and leave the dual argument for ${}^\mu H^{-1}(j^![1]\mathbf{P}^\bullet)$ to the reader.

Note that $\ell^! j^*[-1]\mathbf{P}^\bullet$ is perverse and so $\ell_!\ell^! j^*[-1]\mathbf{P}^\bullet$ satisfies the support condition, i.e., ${}^\mu H^k(\ell_!\ell^! j^*[-1]\mathbf{P}^\bullet) = 0$ for $k \geq 1$. Also note that our hypothesis that ${}^\mu H^0(\hat{m}^*\mathbf{P}^\bullet) = 0$ implies that

$${}^\mu H^1(m_* m^* j^*[-1]\mathbf{P}^\bullet) = {}^\mu H^0(m_*\hat{m}^*\mathbf{P}^\bullet) = 0.$$

Now apply perverse cohomology to the distinguished triangle

$$\to \ell_!\ell^! j^*[-1]\mathbf{P}^\bullet \to j^*[-1]\mathbf{P}^\bullet \to m_* m^* j^*[-1]\mathbf{P}^\bullet \xrightarrow{[1]}$$

to reach the desired conclusion.

Proof of (2): That $\mathbf{P}^\bullet$ is isomorphic to the intermediate extension of $i^*\mathbf{P}^\bullet = i^!\mathbf{P}^\bullet$ from $X\backslash V(f)$ to X follows immediately from (1), since one of the equivalent characterizations of the intermediate extension is that $j^*[-1]\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\leqslant 0}(V(f))$ and $j^![1]\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\geqslant 0}(V(f))$. See, for instance, [3], Definition 5.2.6.

Proof of (3): This follows immediately from (2).

Proof of (4): Suppose that Σ_f contains an irreducible component C of $\operatorname{supp} j^*[-1]\mathbf{P}^\bullet$ or $\operatorname{supp} j^![1]\mathbf{P}^\bullet$, where we know that $\operatorname{supp} j^*[-1]\mathbf{P}^\bullet$ and $\operatorname{supp} j^![1]\mathbf{P}^\bullet$ are perverse by (1). Then, restricting to an open dense subset of C either $m^* j^*[-1]\mathbf{P}^\bullet$ or $m^! j^![1]\mathbf{P}^\bullet$ would be perverse and non-zero. But this would contradict either ${}^\mu H^{-1}(\hat{m}^*\mathbf{P}^\bullet) = 0$ or ${}^\mu H^1(\hat{m}^!\mathbf{P}^\bullet) = 0$.

Proof of (5): By (1), Proposition 3.1, and Theorem 3.2, we have an exact sequence in $\operatorname{Perv}(V(f))$:

$$0 \to \ker\{\mathrm{id} - \widetilde{T}_f\} \to j^*[-1]\mathbf{P}^\bullet \xrightarrow{\omega_f} j^![1]\mathbf{P}^\bullet \to \operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\} \to 0.$$

Let $\mathbf{J}^\bullet := \operatorname{im}\{\omega_f\}$. Then we have two short exact sequences in $\operatorname{Perv}(V(f))$:

$$0 \to \ker\{\mathrm{id} - \widetilde{T}_f\} \to j^*[-1]\mathbf{P}^\bullet \to \mathbf{J}^\bullet \to 0.$$

and

$$0 \to \mathbf{J}^\bullet \to j^![1]\mathbf{P}^\bullet \to \operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\} \to 0.$$

We claim that $\mathbf{J}^\bullet \cong \mathbf{I}^\bullet_{V(f)}$.

First, by applying $\ell^* = \ell^!$ to the short exact sequences, we see that $\mathbf{J}^\bullet$ is an extension of $\ell^* j^*[-1]\mathbf{P}^\bullet \cong \ell^! j^![1]\mathbf{P}^\bullet$. We need to show that $m^*\mathbf{J}^\bullet \in {}^\mu\mathbf{D}^{\leqslant -1}(\Sigma_f)$ and $m^!\mathbf{J}^\bullet \in {}^\mu\mathbf{D}^{\geqslant 1}(\Sigma_f)$, i.e., that ${}^\mu H^0(\hat{m}^*\mathbf{J}^\bullet) = 0$ and ${}^\mu H^0(\hat{m}^!\mathbf{J}^\bullet) = 0$.

Apply m^* to the first short exact sequence/distinguished triangle above, apply $m!$ to the second short exact sequence/distinguished triangle, and take the long exact sequences on perverse cohomology. We obtain exact sequences

$$\to {}^\mu H^0(m^* j^*[-1]\mathbf{P}^\bullet) \to {}^\mu H^0(m^*\mathbf{J}^\bullet) \to {}^\mu H^1(m^* \ker\{\mathrm{id} - \widetilde{T}_f\}) \to$$

and

$$\to {}^\mu H^{-1}(m^!\operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\}) \to {}^\mu H^0(m^!\mathbf{J}^\bullet) \to {}^\mu H^0(m^! j^![1]\mathbf{P}^\bullet) \to .$$

Now ${}^\mu H^1(m^* \ker\{\mathrm{id} - \widetilde{T}_f\}) = 0$ and ${}^\mu H^{-1}(m^!\operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\}) = 0$ because $m^* \ker\{\mathrm{id} - \widetilde{T}_f\}$ and $m^!\operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\}$ are perverse since they are restrictions/upper-shrieks of perverse sheaves to sets containing the supports of the initial perverse sheaves. Furthermore, ${}^\mu H^0(m^* j^*[-1]\mathbf{P}^\bullet) = {}^\mu H^{-1}(\hat{m}^*\mathbf{P}^\bullet) = 0$ and ${}^\mu H^0(m^! j^![1]\mathbf{P}^\bullet) = {}^\mu H^1(\hat{m}^!\mathbf{P}^\bullet) = 0$ by hypothesis. And so, we are finished. □

6. The constant sheaf on affine space

Let us consider the classical case from the introduction, where we replace X with $\mathcal{U}$, a non-empty open subset of $\mathbb{C}^{n+1}$, and replace $\mathbf{P}^\bullet$ with $\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$.

We begin with an easy, but fundamental, lemma. This lemma is well-known, but the proof is short, so we include it.

Lemma 6.1. *Let Y be a closed complex analytic subspace of $\mathcal{U}$, which we do not assume is pure-dimensional. Denote by c the codimension of Y in $\mathcal{U}$, i.e., let $c := n + 1 - \dim Y$. Let $r : Y \hookrightarrow \mathcal{U}$ be the inclusion.*

Then,

$$r^*[-c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\leqslant 0}(Y) \qquad \text{and} \qquad r^![c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\geqslant 0}(Y).$$

Proof. First, we will show that

$$r^*[-c]\mathbb{Z}^\bullet_Y[n+1] \cong \mathbb{Z}^\bullet_Y[\dim Y] \in {}^\mu\mathbf{D}^{\leqslant 0}(\Sigma),$$

which says that $r^*[-c]\mathbb{Z}^\bullet_Y[n+1]$ satisfies the support condition. This argument is simple.

Let p be an integer. We need to show that $\dim \operatorname{supp}^{-p}\left(\mathbb{Z}^\bullet_Y[\dim Y]\right) \leq p$. We have

$$\operatorname{supp}^{-p}\left(\mathbb{Z}^\bullet_Y[\dim Y]\right) = \overline{\{x \in Y \mid H^{\dim Y - p}(\mathbb{Z}^\bullet_Y)_x \neq 0\}}.$$

Now $H^{\dim Y - p}(\mathbb{Z}^\bullet_Y)_x = 0$ unless $p = \dim Y$. Thus the support condition is satisfied.

Now we will show that

$$r^![c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\geqslant 0}(Y),$$

which says that $r^![c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ satisfies the cosupport condition. Let p be an integer. We need to show that $\dim \operatorname{cosupp}^p\left(r^![c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\right) \leq p$.

For all $x \in Y$, let $v_x : \{x\} \hookrightarrow Y$ denote the inclusion, and let $\hat{v}_x := r \circ v_x$; hence, $\hat{v}_x$ is the inclusion of $\{x\}$ into $\mathcal{U}$. Then, we have

$$\operatorname{cosupp}^p\left(r^![c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\right) = \overline{\{x \in Y \mid H^p(v_x^! r^![c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) \neq 0\}} =$$

$$\overline{\{x \in Y \mid H^{n+1+c+p}(\hat{v}_x^! \mathbb{Z}^\bullet_{\mathcal{U}}) \neq 0\}} = \overline{\{x \in Y \mid H^{n+1+c+p}(B^\circ_\epsilon(x), B^\circ_\epsilon(x)\backslash\{x\}; \mathbb{Z}) \neq 0\}} =$$

$$\overline{\{x \in Y \mid \widetilde{H}^{n+c+p}(S^{2n+1}; \mathbb{Z}) \neq 0\}}.$$

where $B^\circ_\epsilon(x)$ again denotes a open ball of (small) radius $\epsilon > 0$, centered at x, S^{2n+1} denotes a sphere in $\mathcal{U}$ of real dimension $2n+1$ (its center and radius are irrelevant), and $\widetilde{H}$ denotes reduced cohomology. Now $\widetilde{H}^{n+c+p}(S^{2n+1}; \mathbb{Z}) = 0$ unless $p = n+1-c = \dim Y$. Thus the cosupport condition is satisfied. $\square$

Now, as before, let $f : \mathcal{U} \to \mathbb{C}$ be a nowhere locally constant complex analytic function such that $V(f)$ is non-empty. Note that we have **not** assumed that f is reduced. Let $\Sigma_f := \operatorname{supp} \phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$.

There is a notion of the singular set of the analytic set $V(f)$; it is the set of points at which $V(f)$ fails to be an analytic submanifold of $\mathcal{U}$ (which, using results about Milnor fibrations, is also the set of points where $V(f)$ fails to be even a topological submanifold of $\mathcal{U}$). We denote this singular set by $\Sigma V(f)$, and note that it always has dimension at most $n-1$.

If f is not reduced, then Σ_f will contain an irreducible component of $V(f)$, while $\Sigma V(f)$ will not. However, if f is reduced, then $\Sigma_f = \Sigma V(f)$, and so the intermediate extension to $V(f)$ of the shifted constant sheaf on $V(f)\backslash\Sigma V(f)$ – the intersection cohomology on $V(f)$ – is the same as the intermediate extension to $V(f)$ of the shifted constant sheaf on $V(f)\backslash\Sigma_f$, which is how we defined $\mathbf{I}^\bullet_{V(f)}$ in Definition 5.4.

As in the general case, we let $j : V(f) \hookrightarrow \mathcal{U}$ and $\hat{m} : \Sigma_f \hookrightarrow \mathcal{U}$ denote the inclusions.

The following lemma explains our terminology in Definition 5.3.

Lemma 6.2. *The function f is reduced (in the algebraic sense) if and only if f is $\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$-reduced, i.e., if and only if*

$$\hat{m}^*\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\leqslant -2}(\Sigma_f) \qquad \text{and} \qquad \hat{m}^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\geqslant 2}(\Sigma_f).$$

Proof. Suppose that

$$\hat{m}^*\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\leqslant -2}(\Sigma_f) \qquad \text{and} \qquad \hat{m}^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\geqslant 2}(\Sigma_f).$$

Then, Item (4) of Theorem 5.5 tells us that Σ_f does not contain an irreducible component of $V(f)$, i.e., f is reduced.

Now we must prove the converse. Let c be the codimension of Σf in $\mathcal{U}$, i.e., $c = n+1-\dim\Sigma_f$. Assume that f is reduced, so that $\dim \Sigma_f \leq n-1$ and so $c \geq 2$.

By Lemma 6.1, we have that

$$\hat{m}^*[-c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\leqslant 0}(\Sigma_f) \qquad \text{and} \qquad \hat{m}^![c]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\geqslant 0}(\Sigma_f),$$

that is

$$\hat{m}^*\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\leqslant -c}(\Sigma_f) \qquad \text{and} \qquad \hat{m}^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \in {}^\mu\mathbf{D}^{\geqslant c}(\Sigma_f).$$

As $c \geq 2$, we are finished.

$\square$

From this lemma and Theorem 5.5, we immediately conclude a new proof of (†) and (‡) from the introduction, which we state here as:

Theorem 6.3. *Suppose that f is reduced. Then, there are short exact sequences in the abelian category of perverse sheaves on $V(f)$:*

$$0 \to \ker\{\mathrm{id} - \widetilde{T}_f\} \to \mathbb{Z}^\bullet_{V(f)}[n] \to \mathbf{I}^\bullet_{V(f)} \to 0,$$

and

$$0 \to \mathbf{I}^\bullet_{V(f)} \to j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to \operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\} \to 0,$$

where $\mathbf{I}^\bullet_{V(f)}$ is the intersection cohomology complex on $V(f)$.

As our final result, we will prove a theorem about integral cohomology (homology) manifolds. First, we need a lemma.

Lemma 6.4. *For all $x \in V(f)$, let $E_{f,x}$ be the total space of the Milnor fibration of f at x. Then the following are equivalent:*

(1) $j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ has stalk cohomology isomorphic to that of $j^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \cong \mathbb{Z}^\bullet_{V(f)}[n]$, and

(2) for all $x \in V(f)$,

$$H^k(E_{f,x};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 1, 0; \\ 0, & \text{if } k \neq 1, 0. \end{cases}$$

Proof. Item (1) means that

$$H^p(j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_x \cong \begin{cases} \mathbb{Z}, & \text{if } p = -n; \\ 0, & \text{if } p \neq -n. \end{cases}$$

Now,

$$H^p(j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_x \cong \mathbb{H}^{n+p+2}(B^\circ_\epsilon(x),\, B^\circ_\epsilon(x)\backslash V(f);\mathbb{Z}) \cong \widetilde{H}^{n+p+1}(B^\circ_\epsilon(x)\backslash V(f);\mathbb{Z}),$$

where $B^\circ_\epsilon(x)$ is a small open ball, centered at x, and $B^\circ_\epsilon(x)\backslash V(f)$ is homotopy-equivalent to $E_{f,x}$, the total space of the Milnor fibration of f at x.

Thus, Item (1) is equivalent to

$$\widetilde{H}^k(E_{f,x};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 1; \\ 0, & \text{if } k \neq 1, \end{cases}$$

or, equivalently,

$$H^k(E_{f,x};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 1, 0; \\ 0, & \text{if } k \neq 1, 0. \end{cases}$$

□

Theorem 6.5. *Suppose that f is reduced, and that $j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ has stalk cohomology isomorphic to that of $j^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$. Then,*

(1) $\mathrm{id} - \widetilde{T}_f$ is an isomorphism,

(2) $j^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ and $j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ are perverse sheaves,

(3) $\omega_f : j^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ *is an isomorphism, and both of these complexes are isomorphic to* $\mathbf{I}^\bullet_{V(f)}$*,*

(4) $V(f)$ *is an integral cohomology/homology manifold, and*

(5) there is an isomorphism

$$\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \cong \phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \oplus \mathbf{I}^\bullet_{V(f)}.$$

Proof. The non-zero part of the cohomological version of the Wang sequence from Lemma 8.4 in [9] begins as follows:

$$0 \to H^0(E_{f,x};\,\mathbb{Z}) \to H^0(F_{f,x};\,\mathbb{Z}) \xrightarrow{(\mathrm{id}-T_f)_x^{-n}} H^0(F_{f,x};\,\mathbb{Z}) \to$$

$$H^1(E_{f,x};\,\mathbb{Z}) \to H^1(F_{f,x};\,\mathbb{Z}) \xrightarrow{(\mathrm{id}-T_f)_x^{-n+1}} H^1(F_{f,x};\,\mathbb{Z}) \to$$

$$H^2(E_{f,x};\,\mathbb{Z}) \to H^2(F_{f,x};\,\mathbb{Z}) \xrightarrow{(\mathrm{id}-T_f)_x^{-n+2}} H^2(F_{f,x};\,\mathbb{Z}) \to,$$

where $F_{f,x}$ is the Milnor fiber of f at x , T_f is the monodromy automorphism on the **nearby** cycles, and the subscript k in $(\mathrm{id}-T_f)_x^k$ denotes the degree (not exponentiation).

By Lemma 6.4, we know that

$$H^k(E_{f,x};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 1, 0; \\ 0, & \text{if } p \neq 1, 0. \end{cases}$$

Since f is reduced, $F_{f,x}$ and $E_{f,x}$ are path-connected. We also know that $H^1(F_{f,x};\,\mathbb{Z})$ is torsion-free by the Universal Coefficient Theorem. Therefore we obtain the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{(\mathrm{id}-T_f)_x^{-n}} \mathbb{Z} \to \mathbb{Z} \to H^1(F_{f,x};\,\mathbb{Z}) \xrightarrow{(\mathrm{id}-T_f)_x^{-n+1}} H^1(F_{f,x};\,\mathbb{Z}) \to$$

$$0 \to H^2(F_{f,x};\,\mathbb{Z}) \xrightarrow{(\mathrm{id}-T_f)_x^{-n+2}} H^2(F_{f,x};\,\mathbb{Z}) \to 0 \dots,$$

where the first map from $\mathbb{Z}$ to $\mathbb{Z}$ is an isomorphism, $(\mathrm{id}-T_f)_x^{-n} = 0$, the third map from $\mathbb{Z}$ to $\mathbb{Z}$ is an isomorphism, and $(\mathrm{id}-T_f)_x^k$ is an isomorphism for $k \neq -n$. However, $T_{f,x}^k \cong \widetilde{T}_{f,x}^k$ outside of degree $-n$. Therefore, for all $x \in V(f)$, $(\mathrm{id}-\widetilde{T}_f)_x^*$ is an isomorphism and thus so is $\mathrm{id}-\widetilde{T}_f$.

Now Items (2), (3), and (5) are immediate from Theorem 4.1 and Theorem 6.3. It remains for us to demonstrate Item (4).

For all $x \in V(f)$, let $v_x : \{x\} \hookrightarrow V(f)$ denote the inclusion. Let $\hat{v}_x := j \circ v_x$ so that $\hat{v}_x$ is the inclusion of $\{x\}$ into $\mathcal{U}$. Applying $v_x^!$ to the isomorphism in Item (3), for all $x \in V(f)$, we have an isomorphism

$$(\star) \qquad v_x^! j^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \cong v_x^! j^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \cong \hat{v}_x^![1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1].$$

Let $B_\epsilon^\circ(x)$ denote an open ball in $\mathcal{U}$ of (small) radius $\epsilon > 0$, centered at x. Then $(\star)$ tells us that, for all p,

$$\mathbb{H}^{p+n}(B_\epsilon^\circ(x) \cap V(f),\, B_\epsilon^\circ(x) \cap V(f)\backslash\{x\};\,\mathbb{Z}) \;\cong\; \mathbb{H}^{p+n+2}(B_\epsilon^\circ(x),\, B_\epsilon^\circ(x)\backslash\{x\};\,\mathbb{Z}),$$

i.e., for all k,

$$\mathbb{H}^k(B_\epsilon^\circ(x) \cap V(f),\, B_\epsilon^\circ(x) \cap V(f)\backslash\{x\};\,\mathbb{Z}) \;\cong\; \widetilde{\mathbb{H}}^{k+1}(S^{2n+1};\,\mathbb{Z}).$$

As the real dimension of $V(f)$ is $2n$, we conclude that $V(f)$ is an integral cohomology/homology manifold.

$\square$

7. A question

We continue with all of our previous notation.

Theorem 3.2 and the definition of the intermediate extension motivate the following definition.

Definition 7.1. *We define the* **intermediate Wang restriction** *of* $\mathbf{P}^\bullet$ *to be the (perverse) image*

$$\mathbf{J}^\bullet_f := \operatorname{im}\left\{{}^\mu H^0(j^*[-1]\mathbf{P}^\bullet) \xrightarrow{{}^\mu H^0(\omega_f)} {}^\mu H^0(j^![1]\mathbf{P}^\bullet)\right\}.$$

The point is that now Theorem 3.2 and Theorem 5.5 tell us immediately that we have:

Proposition 7.2. *There are exact sequences in* $\operatorname{Perv}(V(f))$*:*

$$0 \to \frac{\ker\{\mathrm{id} - \widetilde{T}_f\}}{\ker\{\mathrm{var}\}} \longrightarrow {}^\mu H^0(j^*[-1]\mathbf{P}^\bullet) \xrightarrow{\alpha} \mathbf{J}^\bullet_f \to 0$$

and

$$0 \to \mathbf{J}^\bullet_f \xrightarrow{\beta} {}^\mu H^0(j^![1]\mathbf{P}^\bullet) \longrightarrow \frac{\operatorname{im}\{\mathrm{can}\}}{\operatorname{im}\{\mathrm{id} - \widetilde{T}_f\}} \to 0,$$

where $\beta \circ \alpha = {}^\mu H^0(\omega_f)$*.*

Furthermore, if $\hat{m}^*\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\leqslant -2}(\Sigma)$ *and* $\hat{m}^!\mathbf{P}^\bullet \in {}^\mu\mathbf{D}^{\geqslant 2}(\Sigma)$*, then these two short exact sequences collapse to those of Theorem 5.5; in particular, the intermediate Wang restriction* $\mathbf{J}^\bullet_f$ *is isomorphic to the intermediate extension* $\mathbf{I}^\bullet_{V(f)}$*.*

Of course, the big question is:

Question 7.3. Does $\mathbf{J}^\bullet_f$ have any interesting properties in general, even when it is not isomorphic to the intermediate extension $\mathbf{I}^\bullet_{V(f)}$?

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