Asymptotic expansion of the drift estimator for the fractional Ornstein-Uhlenbeck process *

Ciprian A. Tudor[1] and Nakahiro Yoshida[2,3,4]

[1]Université de Lille 1 †
[2]Graduate School of Mathematical Sciences, University of Tokyo ‡
[3]CREST, Japan Science and Technology Agency
[4]The Institute of Statistical Mathematics

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Abstract

We present an asymptotic expansion formula of an estimator for the drift coefficient of the fractional Ornstein-Uhlenbeck process. As the machinery, we apply the general expansion scheme for Wiener functionals recently developed by the authors [27]. The central limit theorem in the principal part of the expansion has the classical scaling $T^{1/2}$. However, the asymptotic expansion formula is a complex in that the order of the correction term becomes the classical $T^{-1/2}$ for $H \in (1/2, 5/8)$, but T^{4H-3} for $H \in [5/8, 3/4)$.

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1 Asymptotic expansion of an estimator for a fractional Ornstein-Uhlenbeck process

We consider the Langevin equation

$$\begin{cases} dX_t = -\theta X_t dt + \sigma dB_t, & t \geq 0, \\ X_0 = x_0, \end{cases} \tag{1.1}$$

where x_0 is a constant and $(B_t, t \geq 0)$ is a fractional Brownian motion with Hurst index $H \in (1/2, 1)$. Suppose that the parameter space Θ is a bounded open set in $\mathbb{R}$ satisfying $\overline{\Theta} \subset (0, \infty)$, and that the true value of θ is in Θ. In what follows, the true value is also denoted by θ for notational simplicity.

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†Université de Lille 1: 59655 Villeneuve d'Ascq, France
‡Graduate School of Mathematical Sciences, University of Tokyo: 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan. e-mail: nakahiro@ms.u-tokyo.ac.jp

From (1.1),

$$X_t = e^{-\theta t}x_0 + \int_0^t e^{-\theta(t-s)}\sigma dB_s, \tag{1.2}$$

where the stochastic integral is regarded as a Wiener integral, i.e., an divergence integral with respect to the fractional Brownian motion B.

Hu and Nualart [8] investigated the estimator $\widetilde{\theta}_T$ defined by

$$\widetilde{\theta}_T = \left(\frac{1}{\sigma^2 H\Gamma(2H)T}\int_0^T X_t^2\right)^{-\frac{1}{2H}}. \tag{1.3}$$

In the inferential theory, the estimator $\widetilde{\theta}_T$ is regarded as an M-estimator for the estimating equation

$$\int_0^T X_t^2 dt - \widetilde{\nu}_T(\vartheta) = 0 \tag{1.4}$$

for

$$\widetilde{\nu}_T(\vartheta) = \mu(\vartheta)T \quad \text{with} \quad \mu(\vartheta) = \sigma^2 H\Gamma(2H)\vartheta^{-2H}. \tag{1.5}$$

We remark that $\widetilde{\theta}_T$ is an approximately moment estimator but not the exact moment estimator since $\overline{\nu}_T(\theta) := E\big[\int_0^T X_t^2 dt\big]$ is decomposed as $\overline{\nu}_T(\theta) = \widetilde{\nu}_T(\theta) + \overline{b}_T(\theta)$ and $\overline{b}_T(\theta)$ does not vanish though it is of order of $O(1)$ as $T \to \infty$, according to Lemma 5.3. Since it is common to use a bias-corrected estimator in the higher-order inference, we will consider the estimator

$$\widehat{\theta}_T^o = \widetilde{\theta}_T - T^{-\frac{1}{2}-\mathsf{q}(H)}\beta\big(\widetilde{\theta}_T\big),$$

where $\beta = \beta_H \in C_B^\infty(\Theta)$, i.e., β is smooth on Θ and all its derivatives are bounded on Θ, and $\mathsf{q} = \mathsf{q}(H)$ is a number define by (1.7). The value of $\widehat{\theta}_T^o$ can exceed the boundary of Θ, not necessarily due to the β term, therefore the estimator $\widehat{\theta}_T$ we will consider is more precisely defined as

$$\widehat{\theta}_T = \begin{cases} \widehat{\theta}_T^o & \text{if } \widetilde{\theta}_T \in \Theta \text{ and } \widehat{\theta}_T^o \in \Theta, \\ \theta_* & \text{otherwise,} \end{cases} \tag{1.6}$$

where θ_* is a prescribed value in Θ. The choice of the value θ_* will not affect asymptotically in any order of expansion.

Hu and Nualart [8] proved that, for $H \in \big(\frac{1}{2}, \frac{3}{4}\big)$,

$$\sqrt{T}\big(\widetilde{\theta}_T - \theta\big) \to^d N(0, c_0)$$

as $T \to \infty$, with c_0 defined as (4.2). On the other hand, Hu et al. showed in [5] that the estimator (1.6) converges to a non-Gaussian distribution (a Rosenblatt random variable), when $H \in (3/4, 1)$. Other estimators for the drift parameter of the fractional Ornstein-Uhlenbeck process have been analyzed, among others, in Brouste and Kleptsyna [3], Chen and Li [5], Cheng and Zhou [6], and El Onsy, Es-Sebaiy and Viens [7].

In this paper, we will give an asymptotic expansion for the distribution of $\sqrt{T}\big(\widehat{\theta}_T - \theta\big)$. The order q of the expansion is defined as

$$\mathsf{q} = \mathsf{q}(H) = \begin{cases} \frac{1}{2} & \big(H \in \big(\frac{1}{2}, \frac{5}{8}\big]\big) \\ -4H + 3 & \big(H \in \big(\frac{5}{8}, \frac{3}{4}\big)\big) \end{cases} \tag{1.7}$$

The k-th Hermite polynomial $H_k(x;0,c_0)$ is defined by

$$H_k(x;0,c_0) \quad = \quad e^{2^{-1}c_0^{-1}x^2}(-\partial_x)^k e^{-2^{-1}c_0^{-1}x^2} \qquad (x\in\mathbb{R}).$$

We consider the approximate density function

$$\begin{aligned} p_{H,T}(x) \quad = \quad \phi(x;0,c_0)\bigg(1 &+ 1_{\{H\in[\frac{5}{8},\frac{3}{4})\}}2^{-1}c_2H_2(x;0,c_0)T^{4H-3} \\ &+1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}3^{-1}c_3H_3(x;0,c_0)T^{-\frac{1}{2}} \\ &+c_1H_1(x;0,c_0)T^{-\mathfrak{q}(H)}\bigg), \end{aligned} \tag{1.8}$$

where the constants $c_0,...,c_3$ depending on H and θ will be specified later at (4.2) and (6.4). For $a,b>0$, we denote by $\mathcal{E}(a,b)$ the set of measurable functions $g:\mathbb{R}\to\mathbb{R}$ such that $|g(x)|\le a(1+|x|^b)$ for all $x\in\mathbb{R}$. The main theorem of this paper is here.

Theorem 1.1. *Suppose that $H\in(1/2,3/4)$. Then*

$$\sup_{g\in\mathcal{E}(a,b)}\left|E\big[g\big(T^{1/2}(\widehat{\theta}_T-\theta)\big)\big]-\int_{\mathbb{R}}g(x)p_{H,T}(x)dx\right| \quad = \quad o(T^{-\mathfrak{q}(H)}) \tag{1.9}$$

as $T\to\infty$, for every $a,b>0$.

The function β set so as to satisfy $c_1=0$ corrects the second-order bias. In Section 7, the real performance of the formula $p_{H,T}$ will be investigated in several cases by simulations.

We will treat mainly the asymptotic expansion formula (1.8) with the threshold $5/8$ changing the shape of the formula by the indicator functions. The expansion formula is still valid even if we remove the indicator functions and keep all terms because the exponents of T automatically count the order of terms and the smaller terms, even if they remain in the formula, do not affect the error bound for a given value of H. More precisely,

Theorem 1.2. *Suppose that $H\in(1/2,3/4)$. Then there exist constants $c_0,c_{1,1}^+,c_{1,2}^+,c_2,c_3$ such that for*

$$\begin{aligned} p^+_{H,T}(x) \quad = \quad \phi(x;0,c_0)\bigg(1 &+ 2^{-1}c_2H_2(x;0,c_0)T^{4H-3}+3^{-1}c_3H_3(x;0,c_0)T^{-\frac{1}{2}} \\ &+c^+_{1,1}H_1(x;0,c_0)T^{-\frac{1}{2}}+c^+_{1,2}H_1(x;0,c_0)T^{-\mathfrak{q}(H)}\bigg), \end{aligned} \tag{1.10}$$

it holds that

$$\sup_{g\in\mathcal{E}(a,b)}\left|E\big[g\big(T^{1/2}(\widehat{\theta}_T-\theta)\big)\big]-\int_{\mathbb{R}}g(x)p^+_{H,T}(x)dx\right| \quad = \quad o(T^{-\mathfrak{q}(H)}) \tag{1.11}$$

as $T\to\infty$, for every $a,b>0$.

The constants c_0, c_2, c_3 are the same as those of $p_{H,T}$. The constants $c^+_{1,1}$ and $c^+_{1,2}$ are given in (6.5).

In asymptotic expansions in general, such a "redundant" formula may give in practice a better approximation to the distribution though there is no theoretical explanation except for an intuition that such a primitive formula has more information than the slimmed formulas obtained by further neglecting smaller terms.

Concluding this section, here are several comments. Hu, Nualart and Zhou [9] presented limit theorems for general Hurst parameter. The Berry-Esseen bound for the parameter estimation is discussed, among others, in Kim and Park [13], Chen, Kuang and Li [4], and Chen and Li [5].

For estimation of the Hurst coefficient, we refer the reader to Istas and Lang [12], Kubilius and Mishura [14], Kubilius, Mishura and Ralchenko [15] and Berzin, Latour and León [1]. Asymptotic expansions are discussed in Mishura, Yamagishi and Yoshida [18]. A related expansion for the quadratic form for a stochastic differential equation driven by a fractional Brownian motion (in particular for the estimator for a constant volatility parameter) is in Yamagishi and Yoshida [28, 29]. Tudor and Yoshida [26] gave asymptotic expansion of the quadratic variation of a mixed fractional Brownian motion.

In this article, we consider an asymptotic expansion for a fractional process, while this problem has been studied well for diffusion processes: Mykland [19], Yoshida [30, 31], Kusuoka and Yoshida [16], Sakamoto and Yoshida [22, 23, 24] and Kutoyants and Yoshida [17], just to mention a few.

The general expansion formula by Tudor and Yoshida [27] was applied in this article. A different formulation using a limit theorem to specify the correction term is in Tudor and Yoshida [25].

The following sections are devoted to the proof of Theorem 1.1. The asymptotic expansion formula is specified with the Gamma factors defined in Section 2. Since the stochastic expansion of the error of the estimator will be expressed with certain basic variables, we derive expansions for their Gamma factors in Section 3. From these expansions, Section 4 gives an asymptotic expansion of the sum $\mathbb{S}_T$ of the basic variables (Proposition 4.4). In Section 5, we obtain a stochastic expansion of the error of the estimator by using $\mathbb{S}_T$ (Equation (5.21)), and in Section 6, it will be used to prove Theorem 1.1, with the aid of the perturbation method. Theorem 1.2 is proved by a minor change of that of Theorem 1.1.

2 Gamma factors and their representations

To get the asymptotic expansion (1.8) of the estimator $\widehat{\theta}_T$ of (1.6), we will use the method developed in Tudor and Yoshida [27], which is based on the analysis of its gamma factors. Therefore, we introduce below these random variables and then we study their asymptotic behavior in the later sections for the functionals associated with the stochastic expansion of $\widehat{\theta}_T$.

To accommodate a fractional Brownian motion, prepare the set $\mathcal{E}$ of step functions on $\mathbb{R}_+ = [0, \infty)$, and introduce an inner product on $\mathcal{E}$ such that

$$\langle 1_{[0,t]}, 1_{[0.s]} \rangle_{\mathcal{H}} \;=\; R_H(t,s) := \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big)$$

for $t, s \in \mathbb{R}_+$. Define the Hilbert space $\mathcal{H}$ as the closure of $\mathcal{E}$ with respect to $\|\cdot\|_{\mathcal{H}} = \langle \cdot, \cdot \rangle_{\mathcal{H}}^{1/2}$. In the case $H \in (1/2, 1)$, the space $\mathcal{H}$ has a subspace $|\mathcal{H}|$ of all measurable functions $\mathsf{h} : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\int_0^\infty \int_0^\infty |\mathsf{h}_t||\mathsf{h}_s||t-s|^{2H-2} ds dt \;<\; \infty.$$

For elements $\mathsf{h}, \mathsf{g} \in |\mathcal{H}|$,

$$\langle \mathsf{h}, \mathsf{g} \rangle_{\mathcal{H}} \;=\; \alpha_H \int_0^\infty \int_0^\infty \mathsf{h}_t \mathsf{g}_s |t-s|^{2H-2} ds dt, \quad \alpha_H \;=\; H(2H-1).$$

We consider an isonormal Gaussian process $\mathbb{W} = \big(\mathbb{W}(\mathsf{h})\big)_{\mathsf{h} \in \mathcal{H}}$ on the Hilbert space $\mathcal{H}$. Then, $B_t = \mathbb{W}(1_{[0,t]})$ $(t \in \mathbb{R}_+)$ form a fractional Brownian motion with the Hurst coefficient H. We will apply the Malliavin calculus associated with $\mathbb{W}$. We denote the Malliavin derivative by D, and the

Malliavin operator by L. See Nualart [21], Nourdin and Peccati [20] and Ikeda and Watanabe [11] for the concepts of the Malliavin calculus.

For $\mathsf{F} = (\mathsf{F}_i)_{i=1,...,d} \in \mathbb{D}_{1,2}{}^d$, the gamma factors $\Gamma^{(m)}(\mathsf{F}_{i_1},...,\mathsf{F}_{i_m})$ for $(i_1,...,i_m) \in \{1,...,d\}^m$ are defined as

$$\begin{aligned} \Gamma^{(1)}(\mathsf{F}_{i_1}) &= \Gamma^{(1)}_{i_1}(\mathsf{F}) \;=\; \mathsf{F}_i - E[\mathsf{F}_{i_1}], \\ \Gamma^{(m)}(\mathsf{F}_{i_1},...,\mathsf{F}_{i_m}) &= \Gamma^{(m)}_{i_1,...,i_m}(\mathsf{F}) \;=\; \big\langle D(-L)^{-1}\Gamma^{(m-1)}(\mathsf{F}_{i_1},...,\mathsf{F}_{i_{m-1}}), D\mathsf{F}_{i_m}\big\rangle_{\mathcal{H}} \quad (m \geq 2). \end{aligned}$$

The map $(\mathsf{F}_{i_1},...,\mathsf{F}_{i_m}) \mapsto \Gamma^{(m)}(\mathsf{F}_{i_1},...,\mathsf{F}_{i_m})$ is multi-linear. Tudor and Yoshida [27] used the notation $\Gamma^{(m)}_{i_1,...,i_m}(\mathsf{F})$ for $\Gamma^{(m)}(\mathsf{F}_{i_1},...,\mathsf{F}_{i_m})$. The second gamma factor $\Gamma^{(2)}(\mathsf{F}_{i_1},\mathsf{F}_{i_2})$ is in general different from the carré du champ $\Gamma(\mathsf{F}_{i_1},\mathsf{F}_{i_2}) = \langle \mathsf{F}_{i_1},\mathsf{F}_{i_2}\rangle_{\mathcal{H}}$.

Suppose that a d-dimensional random variable $\mathsf{F} = (\mathsf{F}_i)_{i=1,...,d}$ has the representation

$$\mathsf{F}_i = I_2(\mathsf{f}_i) + c_i \tag{2.1}$$

for some $\mathsf{f}_i \in \mathcal{H}^{\odot 2}$ and $c_i \in \mathbb{R}$. In this special case, the gamma factors have the following expressions:

$$\begin{aligned} \Gamma^{(1)}(\mathsf{F}_{i_1}) &= \mathsf{F}_{i_1} - c_{i_1}, \\ \Gamma^{(2)}(\mathsf{F}_{i_1},\mathsf{F}_{i_2}) &= 2\langle I_1(\mathsf{f}_{i_1}), I_1(\mathsf{f}_{i_2})\rangle_{\mathcal{H}} \;=\; 2I_2(\mathsf{f}_{i_1} \otimes_1 \mathsf{f}_{i_2}) + 2\langle \mathsf{f}_{i_1},\mathsf{f}_{i_2}\rangle_{\mathcal{H}^{\otimes 2}} \\ \Gamma^{(3)}(\mathsf{F}_{i_1},\mathsf{F}_{i_2},\mathsf{F}_{i_3}) &= 2^2\langle I_1(\mathsf{f}_{i_1} \otimes_1 \mathsf{f}_{i_2}), I_1(\mathsf{f}_{i_3})\rangle_{\mathcal{H}} \\ &= 2^2 I_2\big(\mathsf{f}_{i_1} \otimes_1 \mathsf{f}_{i_2} \otimes_1 \mathsf{f}_{i_3}\big) + 2^2\big\langle \mathsf{f}_{i_1} \otimes_1 \mathsf{f}_{i_2}, \mathsf{f}_{i_3}\big\rangle_{\mathcal{H}^{\otimes 2}}. \end{aligned}$$

Generally,

$$\Gamma^{(m)}(\mathsf{F}_{i_1},...,\mathsf{F}_{i_m}) \;=\; 2^{m-1}I_2\big(\mathsf{f}_{i_1} \otimes_1 \cdots \otimes_1 \mathsf{f}_{i_m}\big) + 2^{m-1}\big\langle \mathsf{f}_{i_1} \otimes_1 \widetilde{\cdots \otimes_1} \mathsf{f}_{i_{m-1}}, \mathsf{f}_{i_m}\big\rangle_{\mathcal{H}^{\otimes 2}} \tag{2.2}$$

for $(i_1,...,i_m) \in \{1,...,d\}^m$ and F_i of (2.1), where $\tilde{}$ means the symmetrization.

3 Estimates of the gamma factors of the basic variables

3.1 Basic variables

Let

$$\begin{aligned} u_T(s,t) &= K_U T^{-1/2} e^{-\theta|s-t|} 1_{[0,T]^2}(s,t) \text{ with } K_U \;=\; -\frac{\theta^{2H}}{4H^2\Gamma(2H)}, \\ v_T(s,t) &= K_V T^{-1/2} e^{-\theta(T-s)-\theta(T-t)} 1_{[0,T]^2}(s,t) \text{ with } K_V \;=\; \frac{\theta^{2H}}{4H^2\Gamma(2H)}, \\ w_T(t) &= K_W T^{-1/2} (e^{-\theta t} - e^{-2\theta T+\theta t}) 1_{[0,T]}(t) \text{ with } K_W \;=\; -\frac{x_0\theta^{2H}}{2\sigma H^2\Gamma(2H)}. \end{aligned}$$

We will treat the multiple integrals

$$U_T = I_2(u_T), \quad V_T = I_2(v_T) \;\text{ and }\; W_T = I_1(w_T). \tag{3.1}$$

These variables will play an important role in this article to derive the asymptotic expansion. In fact, the estimator $\widehat{\theta}_T$ will be related with the sum of them in (5.3).

3.2 Gamma factors of U_T and V_T

Since U_T and V_T have the form of (3.1), the formula (2.2) gives

$$\Gamma^{(m)}(\mathsf{F}_T, ..., \mathsf{F}_T) \;=\; 2^{m-1} I_2\big(\underbrace{\mathsf{f}_T \otimes_1 \cdots \otimes_1 \mathsf{f}_T}_{m}\big) + 2^{m-1}\big\langle \underbrace{\mathsf{f}_T \otimes_1 \cdots \otimes_1 \mathsf{f}_T}_{m-1}, \mathsf{f}_T\big\rangle_{\mathcal{H}^{\otimes 2}} \tag{3.2}$$

for $m \geq 2$ and $\mathsf{F}_T = I_2(\mathsf{f}_T) = U_T$ and V_T with $\mathsf{f}_T = u_T$ and v_T, respectively.

3.3 Estimates for $E[\Gamma^{(m)}(U_T, ..., U_T)]$

Let

$$a(x_1, x_2, x_3) \;=\; e^{-\theta|x_1 - x_2|}|x_2 - x_3|^{2H-2} \tag{3.3}$$

for $x_1, x_2, x_3 \in \mathbb{R}$, and

$$\overline{a}(x) \;=\; \int_{\mathbb{R}} e^{-\theta|z|}|z - x|^{2H-2} dz$$

for $x \in \mathbb{R}$. Then

$$\overline{a}(x) = \overline{a}(|x|) \quad \text{and} \quad \overline{a}(x - y) \;=\; \int_{\mathbb{R}} a(x, z, y)dz \;\geq\; \int_A a(x, z, y)dz \tag{3.4}$$

for any $x, y \in \mathbb{R}$ and any one-dimensional Borel set A. The functions $a(x_1, x_2, x_3)$ and $\overline{a}(x)$ depend on θ and H.

Lemma 3.1. *There exists a positive constant C depending on (θ, H), such that*

$$\overline{a}(r) \leq C(1 \wedge r^{2H-2}) \quad (\forall r \geq 0). \tag{3.5}$$

Proof. Notice that $2|z| \geq 1$ for $|z - 1| \leq 1/2$. For $r > 0$, we have

$$\begin{aligned}
\overline{a}(r) \;&=\; r^{2H-2} \int_{\mathbb{R}} re^{-\theta r|z|}|z - 1|^{2H-2} dz \\
&\leq\; r^{2H-2}\bigg(2^{2-2H} \int_{\{z:|z-1|>1/2\}} re^{-\theta r|z|}dz + \int_{\{z:|z-1|\leq 1/2\}} \sup_{z' \in \mathbb{R}} \big(2|z'|\, re^{-\theta r|z'|}\big)|z - 1|^{2H-2} dz\bigg) \\
&\leq\; 2^{3-2H}\theta^{-1}\big(1 + (2H - 1)^{-1}e^{-1}\big)r^{2H-2} \quad \text{since } H > 1/2,
\end{aligned}$$

besides, $\overline{a}(r) \leq \int_{\{z:|z-r|\geq 1\}} e^{-\theta|z|}dz + \int_{\{z:|z-r|<1\}} |z - r|^{2H-2}dz < 2\theta^{-1} + 2(2H - 1)^{-1} < \infty.$ □

Here is a common estimate for a multiple integral.

Lemma 3.2. *Let $m \geq 2$ and $H \in \big(\frac{1}{2}, \frac{m+1}{2m}\big)$. Suppose that functions $\alpha_i : \mathbb{R} \to \mathbb{R}_+$ $(i = 1, ..., m)$ satisfy*

$$\alpha_i(x) \;\leq\; C(1 \wedge |x|^{2H-2}) \quad (x \in \mathbb{R}) \tag{3.6}$$

for some positive constant C. Then

$$\int_{\mathbb{R}^{m-1}} \alpha_1(x_1)\alpha_2(x_1 - x_2) \cdots \alpha_{m-1}(x_{m-2} - x_{m-1})\alpha_m(x_{m-1})dx_1 \cdots dx_{m-1} \;<\; \infty.$$

Proof. By Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{m-1}} \alpha_1(x_1)\alpha_2(x_1-x_2)\cdots\alpha_{m-1}(x_{m-2}-x_{m-1})\alpha_m(x_{m-1})dx_1\cdots dx_{m-1} \\
= \;& \big\|(\alpha_1 * \cdots * \alpha_{m-1}) \times \alpha_m\big\|_{L^1(\mathbb{R})} \;\le\; \prod_{i=1}^{m} \|\alpha_i\|_{L^{\frac{m}{m-1}}}.
\end{aligned} \tag{3.7}$$

Since $H < \frac{m+1}{2m}$, we have $(2H-2)\frac{m}{m-1} < -1$, and hence $\|\alpha_i\|_{L^{\frac{m}{m-1}}} < \infty$ from the inequality (3.6). $\square$

Let

$$\begin{aligned}
C_U(m,H,\theta) \;=\; & 2^m m K_U^m \alpha_H^m \int_{(0,\infty)^{2m-1}} a(0,x_2,x_3)a(x_3,x_4,x_5)\cdots \\
& \cdots a(x_{2m-3},x_{2m-2},x_{2m-1})a(x_{2m-1},x_{2m},0)\, dx_2\cdots dx_{2m}.
\end{aligned}$$

According to Hu and Nualart [8],

$$\int_{(0,\infty)^3} a(0,x_2,x_3)a(x_3,x_4,0)dx_2dx_3dx_4 \;=\; \theta^{-4H+1}d_H$$

for

$$d_H \;=\; (4H-1)\Big\{\frac{\Gamma(2H-1)^2}{2} + \frac{\Gamma(2H-1)\Gamma(3-4H)\Gamma(4H-2)}{\Gamma(2-2H)}\Big\}.$$

Therefore,

$$C_U(2,H,\theta) \;=\; \frac{\theta(4H-1)}{(2H)^2}\Big\{1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2H)\Gamma(2-2H)}\Big\}. \tag{3.8}$$

Lemma 3.3. *Let $m \ge 2$. Assume $H \in \big(\frac{1}{2}, \frac{m+1}{2m}\big)$. Then $C_U(m,H,\theta)$ is finite and*

$$\begin{aligned}
E\big[\Gamma^{(m)}(U_T,...,U_T)\big] \;&=\; 2^{m-1}\big\langle \underbrace{u_T \otimes_1 \cdots \otimes_1 u_T}_{m-1}, u_T\big\rangle_{\mathcal{H}^{\otimes 2}} \\
&=\; T^{-\frac{1}{2}(m-2)}C_U(m,H,\theta) + o\big(T^{-\frac{1}{2}(m-2)}\big)
\end{aligned} \tag{3.9}$$

as $T \to \infty$.

Proof. Let

$$I_T \;=\; \int_{[0,T]^{2m}} a(x_1,x_2,x_3)a(x_3,x_4,x_5)\ldots a(x_{2m-1},x_{2m},x_1)dx_1\cdots dx_{2m} \tag{3.10}$$

and

$$I'_\infty \;=\; 2m\int_{(0,\infty)^{2m-1}} a(0,x_2,x_3)a(x_3,x_4,x_5)\cdots a(x_{2m-3},x_{2m-2},x_{2m-1})a(x_{2m-1},x_{2m},0)\, dx_2\cdots dx_{2m}.$$

From (3.4), we obtain

$$(2m)^{-1}I'_\infty \;\le\; \int_{\mathbb{R}^{m-1}} \overline{a}(x_1)\overline{a}(x_1-x_2)\cdots\overline{a}(x_{m-2}-x_{m-1})\overline{a}(x_{m-1})dx_1\cdots dx_{m-1}, \tag{3.11}$$

and $I'_\infty < \infty$ by using the estimate (3.5) of Lemma 3.1, and Lemma 3.2.

By L'Hôpital's rule,

$$\begin{aligned}
\lim_{T\to\infty} \frac{I_T}{T} &= \lim_{T\to\infty} \frac{dI_T}{dT} \\
&= 2m \lim_{T\to\infty} \int_{[0,T]^{2m-1}} a(T,x_2,x_3)a(x_3,x_4,x_5)\cdots a(x_{2m-1},x_{2m},T)dx_2...dx_{2m} \\
&= 2m \lim_{T\to\infty} \int_{[0,T]^{2m-1}} a(0,x_2,x_3)a(x_3,x_4,x_5)\cdots a(x_{2m-1},x_{2m},0)dx_2...dx_{2m} \\
&= I'_\infty, \qquad (3.12)
\end{aligned}$$

where we changed variables as $\tilde{x}_i = T - x_i$ for $i = 2, ..., m$.

From (3.2) and the expression of the scalar product in $\mathcal{H}^{\otimes 2}$,

$$\begin{aligned}
E\big[\Gamma^{(m)}(U_T,...,U_T)\big] &= 2^{m-1}\big\langle \underbrace{u_T \otimes_1 \cdots \otimes_1 u_T}_{m-1}, u_T\big\rangle_{\mathcal{H}^{\otimes 2}} \\
&= 2^{m-1}K_U^m T^{-m/2}\alpha_H^m I_T \qquad (3.13)
\end{aligned}$$

for $m \geq 2$. Now we obtain (3.9) from (3.12) and (3.13) since $C_U(m,H,\theta) = 2^{m-1}K_U^m\alpha_H^m I'_\infty$. □

Lemma 3.4. *Let $m \geq 2$. Suppose that $H = \frac{m+1}{2m}$. Then, for any $\epsilon > 0$,*

$$E\big[\Gamma^{(m)}(U_T,...,U_T)\big] = 2^{m-1}\big\langle \underbrace{u_T \otimes_1 \cdots \otimes_1 u_T}_{m-1}, u_T\big\rangle_{\mathcal{H}^{\otimes 2}} = o\big(T^{-\frac{1}{2}(m-2)+\epsilon}\big) \qquad (3.14)$$

as $T \to \infty$.

Proof. Recall that the functions $a_T(x,z,y)$, $\overline{a}(x)$ are associated with $H = \frac{m+1}{2m}$. By (3.10) and (3.4),

$$I_T \leq \int_{[0,T]^m} \overline{a}(x_1-x_2)\overline{a}(x_2-x_3)\cdots\overline{a}(x_{m-1}-x_1)dx_1\cdots dx_m. \qquad (3.15)$$

For any $\epsilon_1 > 0$, Lemma 3.1 yields

$$\begin{aligned}
\overline{a}(r) &\leq C(1\wedge r^{2H-2}) = C\big(1\wedge r^{2H-2-\epsilon_1}(r/T)^{\epsilon_1}T^{\epsilon_1}\big) \\
&\leq \widetilde{a}(r)T^{\epsilon_1} \quad (\forall r \in (0,T);\ T \geq 1), \qquad (3.16)
\end{aligned}$$

where $\widetilde{a}(x) = C\big(1\wedge |x|^{2H-2-\epsilon_1}\big)$ for $x \in \mathbb{R}$. Let

$$\widetilde{I}_T = \int_{[0,T]^m} \widetilde{a}(x_1-x_2)\widetilde{a}(x_2-x_3)\cdots\widetilde{a}(x_{m-2}-x_{m-1})\widetilde{a}(x_{m-1}-x_1)dx_1\cdots dx_m. \qquad (3.17)$$

Then

$$\begin{aligned}
\lim_{T\to\infty} \frac{d\widetilde{I}_T}{dT} &= m \lim_{T\to\infty} \int_{[0,T]^{m-1}} \widetilde{a}(T-x_2)\widetilde{a}(x_2-x_3)\cdots\widetilde{a}(x_{m-2}-x_{m-1})\widetilde{a}(x_{m-1}-T)dx_2\cdots dx_m \\
&= m \lim_{T\to\infty} \int_{[0,T]^{m-1}} \widetilde{a}(x_2)\widetilde{a}(x_2-x_3)\cdots\widetilde{a}(x_{m-2}-x_{m-1})\widetilde{a}(x_{m-1})dx_2\cdots dx_m \quad (x_i \leftarrow T-x_i) \\
&= m \int_{[0,\infty)^{m-1}} \widetilde{a}(x_2)\widetilde{a}(x_2-x_3)\cdots\widetilde{a}(x_{m-2}-x_{m-1})\widetilde{a}(x_{m-1})dx_2\cdots dx_m =: \widetilde{I}'_\infty \qquad (3.18)
\end{aligned}$$

The limit $\widetilde{I}'_\infty$ is finite by Lemma 3.2 applied to $\alpha_i(x) = \widetilde{a}(x) = C\big(1 \wedge |x|^{2H-2-\epsilon_1}\big)$.

Set $\epsilon_1 = \epsilon/m$ for a given $\epsilon > 0$. Now, (3.15) and (3.16) give $I_T \le T^{m\epsilon_1}\widetilde{I}_T$. Therefore, from (3.13),

$$\begin{aligned} 0 \ &\le\ T^{\frac{1}{2}(m-2)-\epsilon} E\big[\Gamma^{(m)}(U_T,...,U_T)\big] \ =\ 2^{m-1}K_U^m T^{-m/2}\alpha_H^m\, T^{-1-\epsilon} I_T \\ &\le\ 2^{m-1}K_U^m T^{-m/2}\alpha_H^m\, T^{-1}\widetilde{I}_T \ \to_{T\to\infty}\ 2^{m-1}K_U^m T^{-m/2}\alpha_H^m \widetilde{I}'_\infty \ <\ \infty \end{aligned}$$

by L'Hôpital's rule. This completes the proof. $\square$

For $p_1,...,p_m \in \mathbb{R}$, define $\mathsf{B}_m(p_1,p_2,...,p_m)$ by

$$\mathsf{B}_m(p_1,p_2,...,p_m) \ =\ \int_{[0,1]^m} |x_1-x_2|^{p_1}|x_2-x_3|^{p_2}\cdots|x_{m-1}-x_m|^{p_{m-1}}|x_m-x_1|^{p_m}\,dx_1...dx_m \in [0,\infty].$$

Define $a_T(x,y)$ by

$$a_T(x,y) \ =\ \int_0^T a(x,z,y)dz \ =\ \int_0^T e^{-\theta|x-z|}|z-y|^{2H-2}dz \quad (x,y\in\mathbb{R}). \tag{3.19}$$

Lemma 3.5. *Let $m \ge 2$. Suppose that $H \in (\frac{m+1}{2m},1)$. Then $\mathsf{B}_m(2H-2,...,2H-2) < \infty$ and*

$$\begin{aligned} &\lim_{T\to\infty} T^{-(2H-1)m}\int_{[0,T]^m} a_T(x_1,x_2)a_T(x_2,x_3)\dots a_T(x_m,x_1)dx_1...dx_m \\ &\quad = 2^m\theta^{-m}\mathsf{B}_m(2H-2,...,2H-2). \end{aligned}$$

Proof. We have

$$a_T(x,y) \ =\ 2T^{2H-2}A_T(T^{-1}x,T^{-1}y), \tag{3.20}$$

where

$$A_T(x,y) \ =\ \frac{T}{2}\int_0^1 e^{-T\theta|x-z|}|z-y|^{2H-2}dz.$$

By (3.4) and Lemma 3.1, for some constant C, $a_T(x,y) \le C|x-y|^{2H-2}$ for $x,y\in\mathbb{R}$, in particular,

$$A_T(x,y) \ =\ 2^{-1}T^{-2H+2}a_T(Tx,Ty) \ \le\ 2^{-1}T^{-2H+2}\overline{a}(|Tx-Ty|) \ \le\ 2^{-1}C|x-y|^{2H-2} \tag{3.21}$$

for $x,y\in\mathbb{R}$. Furthermore, by using the convergence of the Laplace distribution to the delta-measure, it is not difficult to show

$$A_T(x,y) \ \to\ \theta^{-1}|x-y|^{2H-2} \quad (T\to\infty) \tag{3.22}$$

for $(x,y)\in(0,1)^2$, $x\ne y$. Lebesgue's theorem with (3.21) and (3.22) ensures

$$\begin{aligned} &T^{-(2H-1)m}\int_{[0,T]^m} a_T(x_1,x_2)a_T(x_2,x_3)\dots a_T(x_m,x_1)dx_1...dx_m \\ &=\ 2^m\int_{[0,1]^m} A_T(x_1,x_2)A_T(x_2,x_3)\dots A_T(x_m,x_1)dx_1...dx_m \\ &\to\ 2^m\theta^{-m}\mathsf{B}_m(2H-2,...,2H-2) \quad (T\to\infty) \end{aligned}$$

if $\mathsf{B}_m(2H-2,...,2H-2) < \infty$. However, we know $\mathsf{B}_m(2H-2,...,2H-2) < \infty$ when $H > \frac{m+1}{2m}$. See Lemma 3.6 below. $\square$

Lemma 3.6. *Let $m \in \mathbb{Z}_{\geq 2}$. Suppose that the numbers $p_1, ..., p_m > -1$ satisfy $\sum_{i=1}^m p_i + m - 1 > 0$. Then $\mathsf{B}_m(p_1, p_2, ..., p_m) < \infty$.*

Proof. The variance gamma distribution $\mathrm{VG}(\lambda, \alpha, \beta, \mu)$ is a probability distribution on $\mathbb{R}$ with the density function

$$p(x) = \frac{1}{\sqrt{\pi}\Gamma(\lambda)}(\alpha^2 - \beta^2)^\lambda \left(\frac{|x-\mu|}{2\alpha}\right)^{\lambda - \frac{1}{2}} K_{\lambda - \frac{1}{2}}(\alpha|x-\mu|)\exp(\beta(x-\mu)) \quad (x \in \mathbb{R}),$$

where $\lambda, \alpha \in (0, \infty)$, $\beta \in \mathbb{R}$ ($\alpha > |\beta|$) and $\mu \in \mathbb{R}$ are parameters, and K_ν is the Bessel function of the third kind with index ν. See e.g. Iacus and Yoshida [10] for the variance gamma distribution and the related variance gamma process. Here we will use the variance gamma distribution $\mathrm{VG}(\lambda, 1, 0, 0)$ for $\lambda > 0$. Denote the density of $\mathrm{VG}(\lambda, 1, 0, 0)$ by $p(x; \lambda)$.

The following facts are known:

(i) $K_{-\nu}(z) = K_\nu(z)$

(ii) $K_\nu(z) \sim 2^{-1}\Gamma(\nu)(z/2)^{-\nu}$ as $z \to 0$ when $\mathrm{Re}(\nu) > 0$, and $K_0(z) \sim -\log z$.

(iii) As $z \to \infty$ under $|\arg z| \leq 3\pi/2 - \epsilon$ with $\epsilon > 0$,

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^\infty \frac{a_k(\nu)}{z^k},$$

where

$$a_k(\nu) = \frac{(4\nu^2 - 1)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2k-1)^2)}{8^k k!}.$$

Around $x = 0$, the density function $p(x; \lambda)$ has the singularity $|x|^{2\lambda - 1}$ when $2\lambda - 1 < 0$, $-\log|x|$ when $2\lambda - 1 = 0$, and no singularity when $2\lambda - 1 > 0$. Moreover, the function $p(x; \lambda)$ rapidly decays when $|x| \to \infty$. Thus, we have the estimate

$$|x|^{2\lambda - 1} 1_{\{|x| \leq 1\}} \leq C_\lambda \, p(x; \lambda) \qquad (x \in \mathbb{R}) \tag{3.23}$$

for some constant C_λ depending on $\lambda > 0$.

The family of variance gamma distributions is closed under convolution. In fact, in our case, the characteristic function of $\mathrm{VG}(\lambda, 1, 0, 0)$ is

$$\varphi_{\mathrm{VG}(\lambda,1,0,0)}(u) = \left(1 + u^2\right)^{-\lambda} \quad (u \in \mathbb{R})$$

and hence

$$\mathrm{VG}(\lambda_1, 1, 0, 0) * \mathrm{VG}(\lambda_2, 1, 0, 0) = \mathrm{VG}(\lambda_1 + \lambda_2, 1, 0, 0) \tag{3.24}$$

for $\lambda_1, \lambda_1 > 0$.

Suppose that $p_i > -1$ for $i = 1, ..., m$. Let $\lambda_i = (p_i + 1)/2 > 0$ for $i = 1, ..., m$. Then

$$\begin{aligned}
&\int_{[0,1]^m} |x_1 - x_2|^{p_1}|x_2 - x_3|^{p_m} \cdots |x_{m-1} - x_m|^{p_{m-1}}|x_m - x_1|^{p_m} dx_1 ... dx_m \\
\lesssim\ & \int_{\mathbb{R}^m} 1_{[0,1]}(x_1)p(x_1 - x_2; \lambda_1)p(x_2 - x_3; \lambda_2)\cdots p(x_{m-1} - x_m; \lambda_{m-1})p(x_m - x_1; \lambda_m)dx_1...dx_m \quad ((3.23)) \\
=\ & \int_{\mathbb{R}^2} 1_{[0,1]}(x_1)p(x_1 - x_m; \lambda_1 + \cdots + \lambda_{m-1})p(x_m - x_1; \lambda_m)dx_m dx_1 \quad ((3.24)) \\
=\ & \int_{\mathbb{R}} 1_{[0,1]}(x_1)p(0; \lambda_1 + \cdots + \lambda_m)dx_1 \quad ((3.24)) \\
=\ & p(0; \lambda_1 + \cdots + \lambda_m).
\end{aligned}$$

On the other hand, $p(0;\lambda_1+\cdots+\lambda_m)<\infty$ since the density function $p(x;\lambda_1+\cdots+\lambda_m)$ has no singularity at the origin due to

$$2(\lambda_1+\cdots+\lambda_m)-1 \;=\; \sum_{i=1}^{m} p_i + m - 1 \;>\; 0$$

by assumption. $\square$

Under the assumption of Lemma 3.5, obviously Lemma 3.6 ensures $\mathtt{B}_m(2H-2,...,2H-2)<\infty$ since $2H-2>-1$ by $H>1/2$, and $m(2H-2)+m-1=2mH-m-1>0$.

Lemma 3.7. *Let $m\geq 2$ and $C'_U(m,H,\theta)=2^{2m-1}K_U^m\alpha_H^m\theta^{-m}\mathtt{B}_m(2H-2,...,2H-2)$. Suppose that $H\in(\frac{m+1}{2m},1)$. Then $C'_U(m,H,\theta)<\infty$ and*

$$\begin{aligned} T^{(\frac{3}{2}-2H)m}E\big[\Gamma^{(m)}(U_T,...,U_T)\big] &= 2^{m-1}\big\langle \underbrace{u_T\otimes_1\cdots\otimes_1 u_T}_{m-1}, u_T\big\rangle_{\mathcal{H}^{\otimes 2}}T^{(\frac{3}{2}-2H)m} \\ &\to C'_U(m,H,\theta) \end{aligned} \tag{3.25}$$

as $T\to\infty$.

Proof. From (2.2) and (3.20), we obtain

$$\begin{aligned} &E\big[\Gamma^{(m)}(U_T,...,U_T)\big] \\ =\;& 2^{m-1}\big\langle u_T\otimes_1\cdots\otimes_1 u_T, u_T\big\rangle_{\mathcal{H}^{\otimes 2}} \\ =\;& 2^{m-1}K_U^m\alpha_H^m T^{-m/2}\int_{[0,T]^m} a_T(x_1,x_2)a_T(x_2,x_3)\ldots a_T(x_m,x_1)dx_1...dx_m. \end{aligned} \tag{3.26}$$

Now the convergence (3.25) follows from Lemma 3.5. $\square$

3.4 Expansion of $E\big[\Gamma^{(2)}(U_T,U_T)\big]$

Let

$$C''_U(2,H,\theta) \;=\; -\frac{(2H-1)\theta^{4H-2}}{2H^2(3-4H)\Gamma(2H)^2}. \tag{3.27}$$

Lemma 3.8. *Suppose that $H\in(1/2,3/4)$. Then*

$$\begin{aligned} E\big[\Gamma^{(2)}(U_T,U_T)\big] &= 2\big\langle u_T,u_T\big\rangle_{\mathcal{H}^{\otimes 2}} \\ &= C_U(2,H,\theta)+C''_U(2,H,\theta)T^{4H-3}+o(T^{4H-3}) \end{aligned}$$

as $T\to\infty$.

Proof. From (3.13),

$$E\big[\Gamma^{(2)}(U_T,U_T)\big] \;=\; 2\big\langle u_T,u_T\big\rangle_{\mathcal{H}^{\otimes 2}} \;=\; 2K_U^2\alpha_H^2T^{-1}I_T^{(2)}, \tag{3.28}$$

where

$$I_T^{(2)} \;=\; \int_{[0,T]^4} a(x_1,x_2,x_3)a(x_3,x_4,x_1)dx_1\cdots dx_4.$$

In Lemma 3.3 and its proof, we already know

$$\frac{dI_T^{(2)}}{dT} = 4\int_{[0,T]^3} a(0,x_2,x_3)a(x_3,x_4,0)dx_2dx_3dx_4$$

and

$$I_\infty^{(2)\prime} := \lim_{T\to\infty}\frac{I_T^{(2)}}{T} = \lim_{T\to\infty}\frac{dI_T^{(2)}}{dT} = \left(2K_U^2\alpha_H^2\right)^{-1}C_U(2,H,\theta). \tag{3.29}$$

In the following equalities of (3.30), $=^{***}$ is obvious, and $=^{**}$ is verified by L'Hôpital's rule with the aid of $\frac{dI_T^{(2)}}{dT} - I_\infty^{(2)\prime} \to 0$ as $T\to\infty$. As will be seen, the limit on the right-hand side of $=^{***}$ is non-zero. Therefore, $I_T^{(2)} - TI_\infty^{(2)\prime} = \int_0^T \big(\frac{dI_t^{(2)}}{dt} - I_\infty^{(2)\prime}\big)dt \to \infty$ since $\int_1^\infty t^{4H-3}dt = \infty$. With this fact, L'Hôpital's rule applies to the equalities $=^*$. In this way, we obtain

$$\begin{aligned}
\lim_{T\to\infty}\big(T^{-1}I_T^{(2)} - I_\infty^{(2)\prime}\big)/T^{4H-3} &= \lim_{T\to\infty}\big(I_T^{(2)} - TI_\infty^{(2)\prime}\big)/T^{4H-2} \\
&=^* \lim_{T\to\infty}\big(\frac{dI_T^{(2)}}{dT} - I_\infty^{(2)\prime}\big)/\big((4H-2)T^{4H-3}\big) \\
&=^{**} \lim_{T\to\infty}\frac{d^2I_T^{(2)}}{dT^2}/\big((4H-2)(4H-3)T^{4H-4}\big) \\
&=^{***} \lim_{T\to\infty} 4(4H-2)^{-1}(4H-3)^{-1}T^{4-4H}\big(I_T^{(2,1)} + I_T^{(2,2)} + I_T^{(2,3)}\big),
\end{aligned} \tag{3.30}$$

where

$$I_T^{(2,1)} = \int_{[0,T]^2} a(0,T,x_3)a(x_3,x_4,0)dx_3dx_4,$$

$$I_T^{(2,2)} = \int_{[0,T]^2} a(0,x_2,T)a(T,x_4,0)dx_2dx_4$$

and

$$I_T^{(2,3)} = \int_{[0,T]^2} a(0,x_2,x_3)a(x_3,T,0)dx_2dx_3.$$

For $I_T^{(2,i)}$ $(i=1,2,3)$, we have the following estimates:

$$I_T^{(2,1)} = \int_{[0,T]^2} e^{-\theta T}|T-x_3|^{2H-2}e^{-\theta|x_3-x_4|}|x_4|^{2H-2}dx_3dx_4 \lesssim e^{-\theta T/2}, \tag{3.31}$$

$$\begin{aligned}
I_T^{(2,2)} &= \int_{[0,T]^2} e^{-\theta|x_2|}|x_2-T|^{2H-2}e^{-\theta|T-x_4|}|x_4|^{2H-2}dx_2dx_4 \\
&= T^2\int_{[0,1]^2} e^{-\theta Tx_2}|Tx_2-T|^{2H-2}e^{-\theta|T-Tx_4|}|Tx_4|^{2H-2}dx_2dx_4 \\
&= T^{4H-2}\int_{[0,1]^2} e^{-\theta Tx_2}|x_2-1|^{2H-2}e^{-\theta T|1-x_4|}|x_4|^{2H-2}dx_2dx_4 \\
&= T^{4H-4}\theta^{-2}\int_{[0,1]^2} \theta Te^{-\theta Tx_2}|x_2-1|^{2H-2}\,\theta Te^{-\theta T|1-x_4|}|x_4|^{2H-2}dx_2dx_4 \\
&\sim T^{4H-4}\theta^{-2}
\end{aligned} \tag{3.32}$$

and

$$
\begin{aligned}
I_T^{(2,3)} &= \int_{[0,T]^2} e^{-\theta x_2}|x_2-x_3|^{2H-2}e^{-\theta|x_3-T|}T^{2H-2}dx_2dx_3 \\
&= T^{2H}\int_{[0,1]^2} e^{-\theta T x_2}|Tx_2-Tx_3|^{2H-2}e^{-\theta|Tx_3-T|}dx_2dx_3 \\
&= T^{4H-2}\int_{[0,1]^2} e^{-\theta T x_2}|x_2-x_3|^{2H-2}e^{-\theta T|1-x_3|}dx_2dx_3 \\
&= T^{4H-4}\theta^{-2}\int_{[0,1]^2} \theta T e^{-\theta T x_2}|x_2-x_3|^{2H-2}\ \theta T e^{-\theta T|1-x_3|}dx_2dx_3 \\
&\sim T^{4H-4}\theta^{-2} \qquad (3.33)
\end{aligned}
$$

as $T\to\infty$.

Thus, we obtain

$$
\begin{aligned}
\lim_{T\to\infty}\big(T^{-1}I_T^{(2)}-I_\infty^{(2)\prime}\big)/T^{4H-3} &= \lim_{T\to\infty} 4(4H-2)^{-1}(4H-3)^{-1}T^{4-4H}\big(I_T^{(2,1)}+I_T^{(2,2)}+I_T^{(2,3)}\big) \\
&= 8(4H-2)^{-1}(4H-3)^{-1}\theta^{-2} \qquad (3.34)
\end{aligned}
$$

as $T\to\infty$, from (3.30), (3.31), (3.32) and (3.33).

From (3.28), (3.29) and (3.34),

$$
\begin{aligned}
E\big[\Gamma^{(2)}(U_T,U_T)\big] &= 2K_U^2\alpha_H^2T^{-1}I_T^{(2)} \\
&= C_U(2,H,\theta)+C_U''(2,H,\theta)T^{4H-3}+o(T^{4H-3})
\end{aligned}
$$

as $T\to\infty$. This completes the proof. □

3.5 Estimate of U_T, V_T and W_T

The (s,p)-Sobolev norm of functional F is defined as $\|F\|_{s,p}=\|(1-L)^{s/2}F\|_p$ for $s\in\mathbb{R}$ and $p>1$. Let $D_\infty=\cap_{s\in\mathbb{R},p>1}D_{s,p}$.

Lemma 3.9. *$U_T=O_{D_\infty}(1)$, i.e., $\|U_T\|_{s,p}=O(1)$ as $T\to\infty$ for every $s\in\mathbb{R}$ and $p>1$.*

Proof. $E[U_T^2]=2\langle u_T,u_T\rangle_{\mathfrak{H}^{\otimes 2}}=E\big[\Gamma^{(2)}(U_T,U_T)\big]=O(1)$ thanks to Lemma 3.8. Hypercontractivity and a fix chaos give the result. □

Lemma 3.10. *$V_T=O_{D_\infty}(T^{-1/2})$.*

Proof. We have

$$
\begin{aligned}
E[V_T^2] &= 2\langle v_T,v_T\rangle_{\mathfrak{H}^{\otimes 2}} \\
&= 2\alpha_H^2K_V^2T^{-1}\int_{[0,T]^4} e^{-\theta(T-t_1)-\theta(T-t_2)}|t_2-t_3|^{2H-2}e^{-\theta(T-t_3)-\theta(T-t_4)}|t_4-t_1|^{2H-2}dt_1dt_2dt_3dt_4 \\
&\lesssim T^{-1}\int_{[0,T]^2} e^{-\theta(T-t_1)}|T-t_3|^{2H-2}e^{-\theta(T-t_3)}|T-t_1|^{2H-2}dt_1dt_3 \\
&\qquad \text{(Use (3.4) and (3.5) for the integrals with respect to } t_2 \text{ and } t_4) \\
&\le T^{-1}\bigg(\int_{[0,\infty)} e^{-\theta t}t^{2H-2}dt\bigg)^2 = \big(T^{-1/2}\theta^{1-2H}\Gamma(2H-1)\big)^2
\end{aligned}
$$

for all $T > 0$. Then we obtain the results by hypercontractivity. $\square$

Lemma 3.11. $W_T = O_{D_\infty}(T^{-1/2})$.

Proof. It is sufficient to observe that

$$\begin{aligned}
E[W_T^2] &= \langle w_T, w_T\rangle_{\mathfrak{H}} \\
&= T^{-1}K_W^2\alpha_H \int_{[0,T]^2} (e^{-\theta t} - e^{-2\theta T+\theta t})|t-s|^{2H-2}(e^{-\theta s} - e^{-2\theta T+\theta s})dtds \\
&\leq T^{-1}K_W^2\alpha_H \int_{[0,T]^2} e^{-\theta t}|t-s|^{2H-2}e^{-\theta s}dtds \\
&\qquad (\because 0 \leq e^{-\theta t} - e^{-2\theta T+\theta t} = e^{-\theta t}(1-e^{-2\theta(T-t)}) \leq e^{-\theta t}) \\
&\lesssim T^{-1}\int_{[0,T]} e^{-\theta t}t^{2H-2}dt \quad ((3.4) \text{ and } (3.5)) \\
&\leq T^{-1}\theta^{1-2H}\Gamma(2H-1)
\end{aligned}$$

for all $T > 0$. $\square$

3.6 Cross-gamma factors

Lemma 3.12. $E\big[\Gamma^{(2)}(U_T,V_T)\big] = E\big[\Gamma^{(2)}(V_T,U_T)\big] = O(T^{-1})$ *as* $T \to \infty$.

Proof. We have

$$\begin{aligned}
E\big[\Gamma^{(2)}(U_T,V_T)\big] &= E\big[\Gamma^{(2)}(V_T,U_T)\big] \;=\; 2^{-1}E\big[\langle DU_T, DV_T\rangle\big] \\
&= 2\langle u_T, v_T\rangle_{\mathfrak{H}^{\otimes 2}} \;=\; C(\theta,H)T^{-1}J_T,
\end{aligned}$$

where $C(\theta,H)$ is a constant and

$$J_T = \int_{[0,T]^4} e^{-\theta|t_1-s_1|}|s_1-s_2|^{2H-2}e^{-\theta|T-s_2|-\theta|T-t_2|}|t_2-t_1|^{2H-2}ds_1ds_2dt_1dt_2.$$

Then we have

$$J_T = O(1) \tag{3.35}$$

as $T \to \infty$. Indeed, by using (3.4), and (3.5) of Lemma 3.1, we obtain

$$\begin{aligned}
J_T &\lesssim \int_{[0,T]^2} \big(1\wedge|t_1-s_2|^{2H-2}\big)e^{-\theta(T-s_2)}\big(1\wedge|T-t_1|^{2H-2}\big)ds_2dt_1 \\
&\lesssim \int_{[0,T]} \big(1\wedge|T-t_1|^{2H-2}\big)\big(1\wedge|T-t_1|^{2H-2}\big)dt_1 \\
&\leq \int_{[0,T]} \big(1\wedge|T-t_1|^{4H-4}\big)dt_1 < \int_{[0,\infty)} \big(1\wedge t^{4H-4}\big)dt \;<\; \infty
\end{aligned}$$

due to $4H-4 < -1$ when $H < 3/4$. $\square$

Lemma 3.13. *Let $m \geq 3$. Then*

$$\begin{aligned} E\big[\Gamma^{(m)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] &= O(T^{-\frac{m}{2}})1_{\{H\in(\frac{1}{2},\frac{m+1}{2m})\}} + O(T^{-\frac{m}{2}+})1_{\{H=\frac{m+1}{2m}\}} \\ &\quad + O(T^{-\frac{m}{2}(3-4H)})1_{\{H\in(\frac{m+1}{2m},1)\}} \end{aligned}$$

as $T \to \infty$, for any $(\mathsf{F}_1, ..., \mathsf{F}_m) \in \{U_T, V_T\}^m$, if $\#\{i \in \{1, ..., m\};\ \mathsf{F}_i = V_T\} = 1$.

Proof. Suppose that $m \geq 3$ and $\#\{i \in \{1, ..., m\};\ \mathsf{F}_i = V_T\} = 1$. Then we have

$$E\big[\Gamma^{(m)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] = 2^{m-1}\langle u_T \otimes_1 \cdots \otimes_1 u_T, v_T\rangle_{\mathfrak{H}^{\otimes 2}} = C(m,\theta,H)T^{-m/2}J_T^*, \tag{3.36}$$

where $C(m, \theta, H)$ is a constant and

$$\begin{aligned} J_T^* &= \int_{[0,T]^{2m}} e^{-\theta|t_1-s_1|}|s_1-t_2|^{2H-2}e^{-\theta|t_2-s_2|}|s_2-t_3|^{2H-2} \\ &\quad \cdots e^{-\theta|t_{m-1}-s_{m-1}|}|s_{m-1}-t_m|^{2H-2}e^{-\theta|T-t_m|-\theta|T-s_m|}|s_m-t_1|^{2H-2}ds_1dt_2\cdots ds_mdt_1. \end{aligned}$$

1) Case $H \in (\frac{1}{2}, \frac{m+1}{2m})$. By using (3.4), and (3.5) of Lemma 3.1, we obtain

$$\begin{aligned} J_T^* &\lesssim \int_{[0,T]^{m+1}} \big(1\wedge|t_1-t_2|^{2H-2}\big)\big(1\wedge|t_2-t_3|^{2H-2}\big)\cdots\big(1\wedge|t_{m-2}-t_{m-1}|^{2H-2}\big) \\ &\qquad \times\big(1\wedge|t_{m-1}-t_m|^{2H-2}\big)e^{-\theta|T-t_m|-\theta|T-s_m|}|s_m-t_1|^{2H-2}dt_1\cdots dt_mds_m \\ &\lesssim \int_{[0,T]^{m-1}} \big(1\wedge|t_1-t_2|^{2H-2}\big)\big(1\wedge|t_2-t_3|^{2H-2}\big)\cdots\big(1\wedge|t_{m-2}-t_{m-1}|^{2H-2}\big) \\ &\qquad \times\big(1\wedge|t_{m-1}-T|^{2H-2}\big)\big(1\wedge|T-t_1|^{2H-2}\big)dt_1\cdots dt_{m-1} \\ &= \int_{[0,T]^{m-1}} \big(1\wedge|t_1-t_2|^{2H-2}\big)\big(1\wedge|t_2-t_3|^{2H-2}\big)\cdots\big(1\wedge|t_{m-2}-t_{m-1}|^{2H-2}\big) \\ &\qquad \times\big(1\wedge t_{m-1}^{2H-2}\big)\big(1\wedge t_1^{2H-2}\big)dt_1\cdots dt_{m-1}. \end{aligned} \tag{3.37}$$

We will estimate the right-hand side of (3.37). By the same reasoning as the proof of $I'_\infty < \infty$ around (3.11) by Young's inequality and Hölder's inequality. we see $J_T^* = O(1)$. Hence $E\big[\Gamma^{(m)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] = O(T^{-m/2})$.

2) Case $H = \frac{m+1}{2m}$. For an estimation of the right-hand side of (3.37), we can follow the proof of $\widetilde{I}'_\infty < \infty$ around (3.18), with a discounted function $\widetilde{a}$. Therefore we obtain $E\big[\Gamma^{(m)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] = O(T^{-\frac{m}{2}+})$.

3) Case $H \in (\frac{m+1}{2m}, 1)$. Since $|T - t_m| + |T - s_m| \geq |t_m - s_m|$, we have

$$\begin{aligned} J_T^* &\leq \int_{[0,T]^{2m}} e^{-\theta|t_1-s_1|}|s_1-t_2|^{2H-2}e^{-\theta|t_2-s_2|}|s_2-t_3|^{2H-2} \\ &\quad \cdots e^{-\theta|t_{m-1}-s_{m-1}|}|s_{m-1}-t_m|^{2H-2}e^{-\theta|t_m-s_m|}|s_m-t_1|^{2H-2}ds_1dt_2\cdots ds_mdt_1 \\ &= \int_{[0,T]^m} a_T(t_1,t_2)a_T(t_2,t_3)\cdots a_T(t_m,t_1)dt_1\cdots dt_m, \end{aligned}$$

where the function a_T is defined in (3.19). Now Lemma 3.5 gives the estimate $J_T^* = O(T^{m(2H-1)})$, and hence $E\big[\Gamma^{(m)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] = O(T^{m(2H-3/2)})$ from (3.36). This completes the proof of Lemma 3.13 □

Lemma 3.14. *Let $H \in (1/2, 3/4)$. Suppose that $m \geq 2$ and $1 \leq k \leq m$. Then*

$$E\big[\Gamma^{(m)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] \;=\; O(T^{-\frac{k}{2}})$$

as $T \to \infty$, for any $(\mathsf{F}_1, ..., \mathsf{F}_m) \in \{U_T, V_T\}^m$, if $\#\{i \in \{1, ..., m\};\ \mathsf{F}_i = V_T\} = k$.

Proof. We obtain these estimates from Lemmas 3.9 and 3.10, if hypercontractivity and Lemma 3.1 of Tudor and Yoshida [27]. $\square$

Lemma 3.15. **(a)** *$\|W_T\|_{s,p} = O(T^{-1/2})$ as $T \to \infty$ for $s \in \mathbb{R}$ and $p > 1$.*

(b) *Let $m \geq 2$. Then $E\big[\Gamma^{(k)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] = 0$ for any $(\mathsf{F}_1, ..., \mathsf{F}_m) \in \{U_T, V_T, W_T\}^m$, if $\#\{i \in \{1, ..., m\};\ \mathsf{F}_i = W_T\} = 1$.*

(c) *Let $m \geq 2$ and $k \leq m$. Then $E\big[\Gamma^{(m)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big] = O(T^{-\frac{k}{2}})$ as $T \to \infty$, for any $(\mathsf{F}_1, ..., \mathsf{F}_m) \in \{U_T, V_T, W_T\}^m$, if $\#\{i \in \{1, ..., m\};\ \mathsf{F}_i = W_T\} = k$.*

Proof. (a) is nothing but Lemma 3.11. (b) follows from the fact that $E\big[\Gamma^{(k)}(\mathsf{F}_1, ..., \mathsf{F}_m)\big]$ is the expectation of an element of the first chaos. (a) implies (c). $\square$

4 Gamma factors and asymptotic expansion of the sum of the basic variables

Define $\mathbb{S}_T$ by

$$\mathbb{S}_T \;=\; U_T + V_T + W_T, \tag{4.1}$$

and c_0 and c_2 by

$$c_0 \;=\; C_U(2, H, \theta) \quad \text{and} \quad c_2 \;=\; C''_U(2, H, \theta), \tag{4.2}$$

respectively. See (3.8) and (3.27) for these constants.

Lemma 4.1. *Let $H \in (\frac{1}{2}, \frac{3}{4})$. Then*

$$E\big[\Gamma^{(2)}(\mathbb{S}_T, \mathbb{S}_T)\big] \;=\; c_0 + c_2 T^{4H-3} + o(T^{4H-3})$$

as $T \to \infty$.

Proof. From (4.1) and Lemmas 3.12, 3.14 and 3.15, we see

$$E\big[\Gamma^{(2)}(\mathbb{S}_T, \mathbb{S}_T)\big] \;=\; E\big[\Gamma^{(2)}(U_T, U_T)\big] + O(T^{-1})$$

as $T \to \infty$. We obtain the result from Lemma 3.8. $\square$

Let

$$c'_3 \;=\; C_U(3, H, \theta). \tag{4.3}$$

Lemma 4.2. **(a)** *For $H \in (\frac{1}{2}, \frac{2}{3})$, $E\big[\Gamma^{(3)}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T)\big] = c'_3 T^{-\frac{1}{2}} + o\big(T^{-\frac{1}{2}}\big)$.*

(b) *For $H = \frac{2}{3}$, $E\big[\Gamma^{(3)}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T)\big] = O(T^{-\frac{1}{2}+})$.*

(c) *For* $H \in (\frac{2}{3}, 1)$, $E\big[\Gamma^{(3)}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T)\big] \;=\; O(T^{-\frac{3}{2}(3-4H)})$.

Proof. By using Lemmas 3.13, 3.14 and 3.15, we obtain

$$
\begin{aligned}
E\big[\Gamma^{(3)}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T)\big] \;&=\; E\big[\Gamma^{(3)}(U_T, U_T, U_T)\big] \\
&\quad + O(T^{-\frac{3}{2}}) 1_{\{H \in (\frac{1}{2}, \frac{2}{3}\}} + O(T^{-\frac{3}{2}+}) 1_{\{H = \frac{2}{3}\}} \\
&\quad + O(T^{-\frac{3}{2}(3-4H)}) 1_{\{H \in (\frac{2}{3}, 1)\}} + O(T^{-1})
\end{aligned}
$$

as $T \to \infty$. Then the desired estimates follow from Lemmas 3.3, 3.4 and 3.7. $\square$

The centered $\Gamma^{(p)}$ is denoted by $\widetilde{\Gamma^{(p)}}$. Let

$$
\mathbb{I}_T \;=\; \widetilde{\Gamma^{(3)}}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T) \;=\; \Gamma^{(3)}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T) - E\big[\Gamma^{(3)}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T)\big].
$$

Lemma 4.3. *As* $T \to \infty$,

$$
\begin{aligned}
\mathbb{I}_T \;&=\; 1_{\{H \in (\frac{1}{2}, \frac{7}{12})\}} O_{D_\infty}(T^{-1}) + 1_{\{H = \frac{7}{12}\}} O_{D_\infty}(T^{-1+}) \\
&\quad + 1_{\{H \in (\frac{7}{12}, 1)\}} O_{D_\infty}(T^{\frac{3}{2}(4H-3)}).
\end{aligned}
$$

Proof. (I) Estimation of the centered third-order gamma factors involving U_T and V_T. It holds that

$$
\begin{aligned}
E\big[\big(\widetilde{\Gamma^{(3)}}(U_T, U_T, U_T)\big)^2\big] \;&=\; 2^4 E\big[I_2(u_T \otimes_1 u_T \otimes_1 u_T)^2\big] \;=\; 2^5 \langle \underbrace{u_T \otimes_1 \cdots \otimes_1 u_T}_{5}, u_T \rangle_{\mathfrak{H}^{\otimes 2}} \\
&=\; E\big[\Gamma^{(6)}(U_T, ..., U_T)\big] \\
&=\; 1_{\{H \in (\frac{1}{2}, \frac{7}{12})\}} O(T^{-2}) + 1_{\{H = \frac{7}{12}\}} O(T^{-2+}) + 1_{\{H \in (\frac{7}{12}, 1)\}} O(T^{3(4H-3)})
\end{aligned}
\tag{4.4}
$$

from Lemmas 3.3, 3.4 and 3.7. These estimates are enhanced to D_∞, that is,

$$
\begin{aligned}
\widetilde{\Gamma^{(3)}}(U_T, U_T, U_T) \;&=\; 1_{\{H \in (\frac{1}{2}, \frac{7}{12})\}} O_{D_\infty}(T^{-1}) + 1_{\{H = \frac{7}{12}\}} O_{D_\infty}(T^{-1+}) \\
&\quad + 1_{\{H \in (\frac{7}{12}, 1)\}} O_{D_\infty}(T^{\frac{3}{2}(4H-3)}).
\end{aligned}
\tag{4.5}
$$

For a mixed centered third-order Gamma factor of U_T and V_T, we have

$$
\begin{aligned}
& E\big[\big(\widetilde{\Gamma^{(3)}}(U_T, U_T, V_T)\big)^2\big] \\
=\;& 2^4 E\big[I_2(u_T \otimes_1 u_T \otimes_1 v_T)^2\big] \quad \text{(tensor symmtrized)} \\
\sim\;& \langle \underbrace{v_T \otimes_1 u_T \otimes_1 \cdots \otimes_1 u_T}_{5}, v_T \rangle_{\mathfrak{H}^{\otimes 2}} + \cdots + \langle \underbrace{u_T \otimes_1 u_T \otimes_1 \cdots \otimes_1 v_T}_{5}, v_T \rangle_{\mathfrak{H}^{\otimes 2}} \\
\lesssim\;& T^{-3} \int_{[0,T]^{12}} a(t_1, s_1, t_2) a(t_2, s_2, t_3) \cdots a(t_5, s_5, t_6) a(t_6, s_6, t_1) dt_1 \cdots dt_6 ds_1 \cdots ds_6.
\end{aligned}
$$

Here we used $|T - x| + |T - y| \geq |x - y|$ for one v_T to alter it into the function a. Since

$$
E\big[\big(\widetilde{\Gamma^{(3)}}(U_T, U_T, V_T)\big)^2\big] \quad \lesssim \quad E\big[\widetilde{\Gamma^{(6)}}(U_T, ..., U_T)\big]
$$

by (3.10) and (3.13), $\widetilde{\Gamma^{(3)}}(U_T, U_T, V_T)$ admits the same estimate as (4.4), and hence the estimate (4.5). On the other hand, Lemmas 3.9 and 3.10 give $\widetilde{\Gamma^{(3)}}(V_T, V_T, V_T) = O_{D_\infty}(T^{-3/2})$ and $\widetilde{\Gamma^{(3)}}(V_T, V_T, U_T) = \widetilde{\Gamma^{(3)}}(V_T, U_T, V_T) = \widetilde{\Gamma^{(3)}}(U_T, V_T, V_T) = O_{D_\infty}(T^{-1})$. In conclusion,

$$\begin{aligned}\widetilde{\Gamma^{(3)}}(U'_T, U''_T, U'''_T) &= 1_{\{H\in(\frac{1}{2},\frac{7}{12})\}}O_{D_\infty}(T^{-1}) + 1_{\{H=\frac{7}{12}\}}O_{D_\infty}(T^{-1+}) \\ &\quad +1_{\{H\in(\frac{7}{12},1)\}}O_{D_\infty}(T^{\frac{3}{2}(4H-3)}).\end{aligned} \tag{4.6}$$

for $U'_T, U''_T, U'''_T \in \{U_T, V_T\}$.

(II) Estimation of the centered third-order gamma factors involving at least one W_T. We consider $\widetilde{\Gamma^{(3)}}(U'_T, U''_T, W_T)$ for $U'_T, U''_T \in \{U_T, V_T\}$. In order to estimate $E\big[\big(\widetilde{\Gamma^{(3)}}(U'_T, U''_T, W_T)\big)^2\big]$, it suffices to show

$$\langle \underbrace{w_T \otimes_1 u_T \otimes_1 \cdots \otimes_1 u_T}_{k}, \underbrace{w_T \otimes_1 u_T \otimes_1 \cdots \otimes_1 u_T}_{6-k} \rangle_{\mathfrak{H}} = O(T^{-3}) \tag{4.7}$$

for $k = 0, 1, ..., 5$. Here we used the domination of the kernel of v_T by that of u_T, once again. We also notice that $e^{-2\theta T+\theta t} \leq e^{-\theta t}$ for $t \in [0, T]$. Therefore, it is sufficient to use the following estimates:

$$\begin{aligned}J^{**}_T &:= T^{-3}\int_{[0,T]^9} a(t, s_1, t_1)a(t_1, s_2, t_2)\cdots a(t_{k-1}, s_k, t_k)e^{-\theta t_k} \\ &\qquad \times a(t, s_{k+1}, t_{k+1})a(t_{k+1}, s_{k+2}, t_{k+2})\cdots a(t_3, s_4, t_4)e^{-\theta t_4} \\ &\qquad \times dt_1\cdots dt_4 ds_1\cdots ds_4 dt \\ &\lesssim T^{-3}\int_{[0,T]^3}(1\wedge|r_1|^{2H-2})(1\wedge|r_1-r_2|^{2H-2})(1\wedge|r_2-r_3|^{2H-2})(1\wedge|r_3|^{2H-2}) \\ &\qquad \times dr_1dr_2dr_3 \qquad\qquad (4.8) \\ &\lesssim T^{-(2-\epsilon)}\int_{[0,T]^3}(1\wedge|r_1|^{2H_1-2})(1\wedge|r_1-r_2|^{2H_1-2})(1\wedge|r_2-r_3|^{2H_1-2})(1\wedge|r_3|^{2H_1-2}) \\ &\qquad \times dr_1dr_2dr_3, \qquad\qquad (4.9)\end{aligned}$$

where $H_1 = H - (1+\epsilon)/8$, $\epsilon \geq -1$, and $T \geq 1$. The last inequality of (4.9) is verified by the estimate

$$T^{-\frac{1+\epsilon}{4}}(1\wedge|r|^{2H-2}) \leq 1\wedge\big(|r|^{-\frac{1+\epsilon}{4}}|r|^{2H-2}\big) = 1\wedge|r|^{2H_1-2}$$

for $r \in [-T, T] \setminus \{0\}$ and $T \geq 1$.

When $H \in (\frac{5}{8}, \frac{3}{4})$, take $\epsilon =$ to have $H_1 \in (\frac{1}{2}, \frac{5}{8})$. We apply Lemma 3.2 to $\alpha_i(x) = 1 \wedge |x|^{2H_1-2}$ in the case $m = 4$ and H_1 for H under $\epsilon = 0$, to verify the integral on the right-hand side of (4.9) is finite. Hence $J^{**}_T = O(T^{-2})$.

When $H = \frac{5}{8}$, it is possible to show that the integral on the right-hand side of (4.9) is finite for any $\epsilon \in (-1, \infty)$. Therefore, $J^{**}_T = O(T^{-3+})$.

When $H \in (\frac{1}{2}, \frac{5}{8})$, we directly apply Lemma 3.2 to $\alpha_i(x) = 1 \wedge |x|^{2H-2}$ in the case $m = 4$ and H, and see integral on the right-hand side of (4.8) is finite, therefore, $J^{**}_T = O(T^{-3})$.

Consequently, for any $H \in (\frac{1}{2}, \frac{3}{4})$, $J^{**}_T = O(T^{-2})$, which implies $\widetilde{\Gamma^{(3)}}(U'_T, U''_T, W_T) = O_{D_\infty}(T^{-1})$ as $T \to \infty$, for $U'_T, U''_T \in \{U_T, V_T\}$. In the same fashion, it is possible to show $\widetilde{\Gamma^{(3)}}(U'_T, W_T, U''_T) = O_{D_\infty}(T^{-1})$ and $\widetilde{\Gamma^{(3)}}(W_T, U'_T, U''_T) = O_{D_\infty}(T^{-1})$ for $U'_T, U''_T \in \{U_T, V_T\}$.

Moreover, Lemmas 3.9-3.11 show $\widetilde{\Gamma^{(3)}}(W_T, W_T, U'_T)$, $\widetilde{\Gamma^{(3)}}(W_T, U'_T, W_T)$ and $\widetilde{\Gamma^{(3)}}(U'_T, W_T, W_T)$ are of order $O_{D_\infty}(T^{-1})$ for $U'_T \in \{U_T, V_T\}$. Similarly, $\widetilde{\Gamma^{(3)}}(W_T, W_T, W_T) = O_{D_\infty}(T^{-3/2})$.

After all that,

$$\widetilde{\Gamma^{(3)}}(U'_T, U''_T, U'''_T) \;=\; O_{D_\infty}(T^{-1}) \tag{4.10}$$

for $U'_T, U''_T \in \{U_T, V_T, W_T\}$ if $1_{\{U'_T=W_T\}} + 1_{\{U''_T=W_T\}} + 1_{\{U'''_T=W_T\}} \geq 1$.
(III) The proof of Lemma 4.3 is completed by merging (4.6) and (4.10). $\square$

The estimated exponents of T and the ranks of the terms appearing in the asymptotic expansion are summarized in Table 1, together with the estimates for the centered third-order gamma factors. It should be remarked that the change of the second dominant terms is seamless at $H = 5/8$. In the asymptotic expansion, the classical order $-1/2$ becomes the exponent of the first-order correction term for $H \in (1/2, 5/8)$, while $4H - 3$ does for $H \in (5/8, 3/4)$, and both do at $H = 5/8$.

Table 1: Estimated exponents of T and [Rank]s

sequence\interval	$(\frac{1}{2}, \frac{5}{8})$	$(\frac{5}{8}, \frac{7}{12})$	$(\frac{7}{12}, \frac{2}{3})$	$(\frac{2}{3}, \frac{3}{4})$
0th-order term of $E[\Gamma^{(2)}(U_T, U_T)]$	0 [1]	0 [1]	0 [1]	0 [1]
1st-order term of $E[\Gamma^{(2)}(U_T, U_T)]$	$4H-3$ [3]	$4H-3$ [2]	$4H-3$ [2]	$4H-3$ [2]
$E[\Gamma^{(3)}(\mathbb{S}_T, \mathbb{S}_T, \mathbb{S}_T)]$	$-\frac{1}{2}$ [2]	$-\frac{1}{2}$ [3]	$-\frac{1}{2}$ [3]	$\frac{3}{2}(4H-3)$ [3]
$E[\widetilde{\Gamma}^{(3)}(U_T, U_T, U_T)]$	-1	-1	$\frac{3}{2}(4H-3)$	$\frac{3}{2}(4H-3)$
$E[\widetilde{\Gamma}^{(3)}(U_T, U_T, V_T)]$	-1	-1	$\frac{3}{2}(4H-3)$	$\frac{3}{2}(4H-3)$
$E[\widetilde{\Gamma}^{(3)}(U'_T, U''_T, W_T)]$	-1	-1	-1	-1

We shall derive an asymptotic expansion of $\mathbb{S}_T$. Define the density function $p^*_{H,T}(x)$ as

$$\begin{aligned} p^*_{H,T}(x) \;&=\; \phi(x; 0, c_0)\bigg(1 + 1_{\{H\in[\frac{5}{8},\frac{3}{4})\}} 2^{-1} c_2 H_2(x; 0, c_0) T^{4H-3} \\ &\qquad\qquad + 1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}} 3^{-1} c'_3 H_3(x; 0, c_0) T^{-\frac{1}{2}}\bigg). \end{aligned} \tag{4.11}$$

The exponent $\mathsf{q} = \mathsf{q}(H)$ is given in (1.7).

Proposition 4.4. *Suppose that $H \in (1/2, 3/4)$. Then*

$$\sup_{g\in\mathcal{E}(a,b)} \left| E[g(\mathbb{S}_T) - \int_{\mathbb{R}} g(x) p^*_{H,T}(x) dx \right| \;=\; o(T^{-\mathsf{q}(H)}) \tag{4.12}$$

as $T \to \infty$.

Proof. Prepare the following parameters:

$$d = 1, \quad p = 2, \quad \mathsf{k} = 1, \quad \mathsf{q}_0(H) = \frac{2}{3}\mathsf{q}(H), \quad \xi(H) = \frac{1}{9}\mathsf{q}(H), \quad \ell = 11, \quad \ell_1 = 5.$$

Then

$$\begin{aligned} &\mathsf{q}_0(H)(\mathsf{k}+1) > \mathsf{q}(H), \quad \xi(H)(\ell - d) > \mathsf{q}(H), \\ &\ell \geq \ell_1 > p + 1 + d, \quad \mathsf{q}_0(H) \leq \mathsf{q}(H) - 3\xi(H). \end{aligned}$$

Therefore, Condition $[B]$ of Tudor and Yoshida [27] is satisfied for each $H \in (1/2, 3/4)$, thanks to Lemmas 4.1 and 4.2.

From (3.2), the formula (2.2) gives

$$\Gamma^{(2)}(U_T, U_T) \quad = \quad 2I_2\big(u_T \otimes_1 u_T\big) + 2\big\langle u_T, u_T\big\rangle_{\mathcal{H}^{\otimes 2}}. \tag{4.13}$$

Lemma 3.8 shows

$$2\big\langle u_T, u_T\big\rangle_{\mathcal{H}^{\otimes 2}} \quad = \quad C_U(2, H, \theta) + O(T^{4H-3}). \tag{4.14}$$

From (4.13) and (4.14),

$$\Gamma^{(2)}(U_T, U_T) - c_0 \quad = \quad 2I_2\big(u_T \otimes_1 u_T\big) + O(T^{4H-3})$$

Furthermore,

$$\begin{aligned} E\big[I_2\big(u_T \otimes_1 u_T\big)^2\big] &= 2\langle u_T \otimes_1 u_T, u_T \otimes_1 u_T\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= 2\langle u_T \otimes_1 u_T \otimes_1 u_T, u_T\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= 1_{\{H\in(\frac{1}{2},\frac{5}{8}\}}O(T^{-1}) + 1_{\{H=\frac{5}{8}\}}O(T^{-1+}) + 1_{\{H\in(\frac{5}{8},\frac{3}{4})\}}O(T^{2(4H-3)}). \end{aligned}$$

by Lemmas 3.3, 3.4 and 3.7. Therefore, in any case of $H \in (1/2, 3/4)$, we can find a positive constant $\mathsf{a}(H)$ such that

$$\Gamma^{(2)}(U_T, U_T) - c_0 \quad = \quad O_{D_\infty}(T^{-\mathsf{a}(H)})$$

as $T \to \infty$. With the help of Lemmas 3.10 and 3.11, this verifies $[A1]$ (ii) of Tudor and Yoshida [27] for $\Gamma^{(2)}(\mathbb{S}_T, \mathbb{S}_T)$. Lemmas 3.9-3.11 imply $\mathbb{S}_T = O_{D_\infty}(1)$, and $[A1]$ (i) is checked. Thus, $[A1]$ of Tudor and Yoshida [27] holds. Besides, Condition $[A2^\sharp]$ of Tudor and Yoshida [27] has been ensured by Lemma 4.3. We apply Theorem 5.2 of Tudor and Yoshida [27] to conclude (4.12). □

5 Smooth stochastic expansion of the estimator

Let $Q_T = \int_0^T X_t^2 dt$. Define $\mathbf{G}(\vartheta)$ by

$$\mathbf{G}(\vartheta) \quad = \quad \int_0^1 \partial_\theta \mu\big(\theta + u(\vartheta - \theta)\big) du \quad (\vartheta \in (0, \infty)). \tag{5.1}$$

In particular,

$$\mathbf{G}(\theta) \;=\; \partial_\theta \mu(\theta) \;=\; -2\sigma^2 H^2 \Gamma(2H)\theta^{-2H-1}. \tag{5.2}$$

Lemma 5.1.

$$Q_T \quad = \quad T^{1/2}\mathbf{G}(\theta)\big(U_T + V_T + W_T\big) + \overline{\nu}_T(\theta). \tag{5.3}$$

Proof. By the representation

$$X_t \quad = \quad e^{-\theta t}x_0 + I_1\big(\sigma e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot)\big),$$

we have

$$\begin{aligned} X_t^2 \quad &= \quad e^{-2\theta t}x_0^2 + 2e^{-\theta t}x_0 I_1\big(\sigma e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot)\big) \\ &\quad + I_2\bigg(\sigma^2 e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot) \otimes e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot)\bigg) + \sigma^2\big\langle e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot), e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot)\big\rangle_{\mathfrak{H}} \\ &= \quad e^{-2\theta t}x_0^2 + 2e^{-\theta t}x_0 I_1\big(\sigma e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot)\big) + I_2\bigg(\sigma^2 e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot) \otimes e^{-\theta(t-\cdot)}1_{[0,t]}(\cdot)\bigg) \\ &\quad + \sigma^2 \alpha_H \int_{[0,t]^2} e^{-\theta(t-s_1)}e^{-\theta(t-s_2)}|s_1 - s_2|^{2H-2} ds_1 ds_2. \end{aligned} \tag{5.4}$$

Moreover,

$$\begin{aligned}
&\int_0^T \sigma^2 e^{-\theta(t-s_1)}1_{[0,t]}(s_1)e^{-\theta(t-s_2)}1_{[0,t]}(s_2)dt \\
=& \int_{s_1\vee s_2}^T \sigma^2 e^{-2\theta t+\theta(s_1+s_2)}dt 1_{\{s_1,s_2\in[0,T]\}} \\
=& \sigma^2(2\theta)^{-1}\big(e^{-\theta|s_1-s_2|}-e^{\theta(-2T+s_1+s_2)}\big)1_{\{s_1,s_2\in[0,T]\}} \\
=& T^{1/2}\sigma^2(2\theta K_U)^{-1}u_T(s_1,s_2)-T^{1/2}\sigma^2(2\theta K_V)^{-1}v_T(s_1,s_2), \qquad (5.5)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T 2x_0\sigma e^{-2\theta t+\theta s}1_{\{s<t\le T\}}dt &= x_0\sigma\theta^{-1}\big(e^{-\theta s}-e^{-2\theta T+\theta s}\big)1_{\{s\in[0,T]\}} \\
&= T^{1/2}x_0\sigma\theta^{-1}K_W^{-1}w_T(s). \qquad (5.6)
\end{aligned}$$

Therefore, (5.4), (5.5) and (5.6) gives (5.3). □

Lemma 5.2. *For every $\epsilon>0$ and $L>0$, $P\big[|\widehat{\theta}_T-\theta|>\epsilon\big]=O(T^{-L})$ as $T\to\infty$.*

Proof. Take a sufficiently small positive number r such that $U(\theta,\mathsf{r})\equiv\{\theta'\in\mathbb{R};|\theta'-\theta|<\mathsf{r}\}\subset\Theta$. Suppose that $0<2\epsilon<\mathsf{r}$. By definition of $\widehat{\theta}_T$, we have

$$\begin{aligned}
\big\{|\widehat{\theta}_T-\theta|>2\epsilon\big\} &\subset \big\{\big|\widetilde{\theta}_T-\theta\big|>\epsilon\big\}\cup\big\{T^{-1}\|\beta\|_\infty>\epsilon\big\} \\
&\subset \bigg\{|T^{-1}Q_T-\mu(\theta)|\ge\inf_{\theta':|\theta'-\theta|>\epsilon}|\mu(\theta')-\mu(\theta)|\bigg\}\cup\big\{T^{-1}\|\beta\|_\infty>\epsilon\big\} \qquad (5.7)
\end{aligned}$$

since $T^{-1}Q_T=\mu(\widetilde{\theta}_T)$. Then

$$P\big[|\widehat{\theta}_T-\theta|>\epsilon\big] \quad\lesssim\quad E\big[\big|T^{-1}Q_T-T^{-1}\overline{\nu}_T(\theta)\big|^{2L}\big]\;=\;O(T^{-L})$$

as $T\to\infty$ (recall $\overline{\nu}_T(\theta)=E\big[\int_0^T X_t^2dt\big]$) since $T^{-1/2}(Q_T-\overline{\nu}_T(\theta))=O_{L^{\infty-}}(1)$ as $T\to\infty$, i.e., all L^p-norms are bounded, from the representation (5.3) of Q_T and Lemmas 3.9-3.11. □

Let

$$\begin{aligned}
b_\infty(\theta) &= -\sigma^2\alpha_H\Gamma(2H)\theta^{-2H-1}-\frac{1}{2}\sigma^2\alpha_H\Gamma(2H-1)\theta^{-2H-1}+\frac{1}{2\theta}x_0^2 \\
&= -\frac{1}{2}\sigma^2\alpha_H(4H-1)\Gamma(2H-1)\theta^{-2H-1}+\frac{1}{2\theta}x_0^2 \qquad (5.8)
\end{aligned}$$

Lemma 5.3. *$\overline{\nu}_T(\theta)=\widetilde{\nu}_T(\theta)+\overline{b}_T(\theta)$ and $\overline{b}_T(\theta)\to b_\infty(\theta)$ as $T\to\infty$.*

Proof. We see

$$\begin{aligned}
\overline{\nu}_T(\theta) &= E\bigg[\int_0^T X_t^2dt\bigg] \\
&= 2\sigma^2\alpha_H(2\theta)^{-1}\int_0^T e^{-\theta t}t^{2H-2}dt\,T-2\sigma^2\alpha_H(2\theta)^{-1}\int_0^T te^{-\theta t}t^{2H-2}dt \\
&\quad -\sigma^2\alpha_H\int_{[0,T]^2}(2\theta)^{-1}e^{-\theta(s_1+s_2)}|s_1-s_2|^{2H-2}ds_1ds_2+\frac{1-e^{-2\theta T}}{2\theta}x_0^2 \\
&= \widetilde{\nu}_T(\theta)+\overline{b}_T(\theta).
\end{aligned}$$

Remark that

$$2\alpha_H(2\theta)^{-1}\int_0^T e^{-\theta t}t^{2H-2}dt \;=\; H(2H-1)\Gamma(2H-1)\theta^{-2H}+O(e^{-\theta T/2})$$
$$\;=\; H\Gamma(2H)\theta^{-2H}+O(e^{-\theta T/2})$$

as $T\to\infty$. Therefore,

$$\begin{aligned}\lim_{T\to\infty}\bar{b}_T(\theta) &= -2\sigma^2\alpha_H(2\theta)^{-1}\int_0^\infty te^{-\theta t}t^{2H-2}dt\\ &\quad -\sigma^2\alpha_H\int_{[0,\infty)^2}(2\theta)^{-1}e^{-\theta(s_1+s_2)}|s_1-s_2|^{2H-2}ds_1ds_2+\frac{1}{2\theta}x_0^2\\ &= -\sigma^2\alpha_H\Gamma(2H)\theta^{-2H-1}-\frac{1}{2}\sigma^2\alpha_H\Gamma(2H-1)\theta^{-2H-1}+\frac{1}{2\theta}x_0^2.\end{aligned}$$

The proof is completed. □

The effect of the initial value x_0 may appear in the asymptotic expansion possibly in the leading correction term. In this sense, we can say the moment estimator is fairly skewed.

When $\widetilde{\theta}_T\in U(\theta,\mathsf{r})$ and $\widehat{\theta}^o_T\in U(\theta,\mathsf{r})$,

$$\begin{aligned}S_T &:= T^{-1/2}\big(Q_T-\bar{\nu}_T(\theta)\big)\\ &= T^{-1/2}\big(\widetilde{\nu}_T(\widetilde{\theta}_T)-\bar{\nu}_T(\theta)\big)\\ &= \mathbf{G}(\widetilde{\theta}_T)\,T^{1/2}(\widetilde{\theta}_T-\theta)-T^{-1/2}\bar{b}_T(\theta)\end{aligned}\tag{5.9}$$

and

$$S_T \;=\; \mathbf{G}(\theta)\,T^{1/2}(\widetilde{\theta}_T-\theta)+T^{-1/2}\mathbf{C}(\widetilde{\theta}_T)\,T(\widetilde{\theta}_T-\theta)^2-T^{-1/2}\bar{b}_T(\theta),\tag{5.10}$$

where $\mathbf{G}(\vartheta)$ is defined by (5.1) and

$$\mathbf{C}(\vartheta) \;=\; \int_0^1(1-u)\partial_\theta^2\mu\big(\theta+u(\vartheta-\theta)\big)du.$$

By definition, $\mathbf{G}(\theta)\;=\;-2\sigma^2H^2\Gamma(2H)\theta^{-2H-1}$ (see (5.2)) and

$$\mathbf{C}(\theta)\;=\;\sigma^2H^2(2H+1)\Gamma(2H)\theta^{-2H-2}\;=\;2^{-1}\sigma^2H\Gamma(2H+2)\theta^{-2H-2}.$$

Since $\inf_{\vartheta\in\overline{\Theta}}|\mathbf{G}(\vartheta)|>0$, we have

$$T^{1/2}(\widetilde{\theta}_T-\theta) \;=\; \mathbf{G}(\widetilde{\theta}_T)^{-1}S_T\;+T^{-1/2}\mathbf{G}(\widetilde{\theta}_T)^{-1}\bar{b}_T(\theta)\tag{5.11}$$

from (5.9), besides

$$\begin{aligned}T^{1/2}(\widetilde{\theta}_T-\theta) &= \mathbf{G}(\theta)^{-1}S_T-T^{-1/2}\mathbf{G}(\theta)^{-1}\mathbf{C}(\widetilde{\theta}_T)\,T(\widetilde{\theta}_T-\theta)^2\\ &\quad +T^{-1/2}\mathbf{G}(\theta)^{-1}\bar{b}_T(\theta)\end{aligned}\tag{5.12}$$

from (5.10). Substitute the expression of (5.11) for $T(\widetilde{\theta}_T-\theta)^2$ of (5.12) to obtain

$$\begin{aligned}T^{1/2}(\widetilde{\theta}_T-\theta) &= \mathbf{G}(\theta)^{-1}S_T-T^{-1/2}\mathbf{G}(\theta)^{-3}\mathbf{C}(\theta)S_T^2\\ &\quad +T^{-1/2}\mathbf{G}(\theta)^{-1}\bar{b}_T(\theta)+\mathbf{R}^\dagger_T,\end{aligned}\tag{5.13}$$

where

$$\mathbf{R}^{\dagger}_T \;=\; -T^{-1/2}\mathbf{G}(\theta)^{-3}\big(\mathbf{C}(\widetilde{\theta}_T)-\mathbf{C}(\theta)\big)S_T^2 \\ -T^{-1/2}\mathbf{G}(\theta)^{-1}\mathbf{C}(\widetilde{\theta}_T)\big\{2S_T\mathbf{R}^*_T+(\mathbf{R}^*_T)^2\big\} \tag{5.14}$$

with $\mathbf{R}^*_T$ given by

$$\mathbf{R}^*_T(\theta) \;=\; \big(\mathbf{G}(\widetilde{\theta}_T)^{-1}-\mathbf{G}(\theta)^{-1}\big)S_T \;+T^{-1/2}\mathbf{G}(\widetilde{\theta}_T)^{-1}\overline{b}_T(\theta). \tag{5.15}$$

Finally, from (5.13),

$$T^{1/2}(\widehat{\theta}_T-\theta) \;=\; \mathbf{G}(\theta)^{-1}S_T - T^{-1/2}\mathbf{G}(\theta)^{-3}\mathbf{C}(\theta)S_T^2 \\ +T^{-1/2}\mathbf{G}(\theta)^{-1}\overline{b}_T(\theta) - T^{-\frac{1}{2}-\mathfrak{q}(H)}\beta(\theta)+\mathbf{R}^{\ddagger}_T, \tag{5.16}$$

where

$$\mathbf{R}^{\ddagger}_T \;=\; \mathbf{R}^{\dagger}_T - T^{-1/2}\big(\beta(\widetilde{\theta}_T)-\beta(\theta)\big). \tag{5.17}$$

Recall $\mathbb{S}_T=\mathbf{G}(\theta)^{-1}S_T$ has the representation

$$\mathbb{S}_T \;=\; U_T+V_T+W_T.$$

From (5.16).

$$T^{1/2}(\widehat{\theta}_T-\theta) \;=\; \mathbb{S}_T + T^{-1/2}\lambda\mathbb{S}_T^2 + T^{-\mathfrak{q}(H)}\mathbb{d}_T + \mathbf{R}^{\ddagger}_T, \tag{5.18}$$

where

$$\mathbb{d}_T = T^{-\frac{1}{2}+\mathfrak{q}(H)}\mathbf{G}(\theta)^{-1}\overline{b}_T(\theta)-\beta(\theta)$$

and

$$\lambda \;=\; -\mathbf{G}(\theta)^{-1}\mathbf{C}(\theta) \;=\; 2^{-1}(2H+1)\theta^{-1}. \tag{5.19}$$

Take a smooth function $\psi:\mathbb{R}\to[0,1]$ such that $\psi(x)=1$ when $|x|<1/2$ and $\psi(x)=0$ when $|x|>1$. Let

$$\psi_T^{C_1} = \psi\big(C_1\big|T^{-1}Q_T-T^{-1}\overline{\nu}_T(\theta)\big|^2\big). \tag{5.20}$$

In view of (5.7), we can say there exist numbers T_1 and C_1 such that $\widetilde{\theta}_T\in U(\theta,\mathsf{r})$ and $\widehat{\theta}_T\in U(\theta,\mathsf{r})$ whenever $\psi_T>0$ and $T>T_1$. In what follows, we will only consider T such that $T>T_1$. Then the functional $\widetilde{F}_T^{C_1}:=\psi_T^{C_1}T^{1/2}(\widetilde{\theta}_T-\theta)$ is well defined on the whole probability space and it is possible to show $\widetilde{F}_T^{C_1}=O_{D_\infty}(1)$. In this way, we have reached the stochastic expansion

$$F_T^{2C_1} \;:=\; \psi_T^{2C_1}T^{1/2}(\widehat{\theta}_T-\theta) \;=\; \mathbb{S}_T+T^{-1/2}\kappa\mathbb{S}_T^2+T^{-\mathfrak{q}(H)}\mathbb{d}_T+\mathbf{R}_T, \tag{5.21}$$

where

$$\mathbf{R}_T \;=\; \psi_T^{2C_1}\mathbf{R}^{\ddagger}_T-(1-\psi_T^{2C_1})\big(\mathbb{S}_T+T^{-1/2}\kappa\mathbb{S}_T^2+T^{-\mathfrak{q}(H)}\mathbb{d}_T\big). \tag{5.22}$$

Lemma 5.4. *$\mathbf{R}_T\in D_\infty$ and $\mathbf{R}_T=O_{D_\infty}(T^{-1})$ as $T\to\infty$.*

Proof. It is easy to show that $\psi_T^{2C_1}\in D_\infty$ and $\psi_T^{2C_1}-1=O_{D_\infty}(T^{-L})$ for every $L>0$. As for the term $\psi_T^{2C_1}\mathbf{R}^{\ddagger}_T$ in (5.22), it is observed that, on the event $\{\psi_T^{2C_1}>0\}$, the terms appearing in the representation of $\mathbf{R}^{\ddagger}_T$ consist of some functionals of the form $f(\widetilde{\theta}_T)$ for a $f\in C_B^\infty(U(\theta,\mathsf{r}))$. Since $\psi_T^{2C_1}\mathbf{R}^{\ddagger}_T$ has the factor $\psi_T^{2C_1}$, we can replace $f(\widetilde{\theta}_T)$ by $f(\theta+T^{-1/2}\widetilde{F}_T^{C_1})$. The latter is well defined on the whole probability space and indeed it is in D_∞. Along (5.17), (5.14) and (5.15), we can verify that $\mathbf{R}_T\in D_\infty$ and $\mathbf{R}_T=O_{D_\infty}(T^{-1})$ as $T\to\infty$. □

6 Proof of Theorems 1.1 and 1.2

6.1 Proof of Theorem 1.1

The asymptotic expansion $p^*_{H,T}$ for $\mathbb{S}_T$ has already been obtained in Proposition 4.4. We will deal with the last three terms on the right-hand side of (5.21) by the perturbation method of Sakamoto and Yoshida [22]. The stochastic expansion (5.21) of $F_T^{2C_1}$ reads $F_T^{2C_1} = \mathbb{S}_T + T^{-\mathsf{q}(H)}\mathbb{Y}_T$ with the perturbation term $\mathbb{Y}_T = T^{\mathsf{q}(H)-\frac{1}{2}}\kappa\mathbb{S}_T^2 + \mathbb{d}_T + T^{\mathsf{q}(H)}\mathbf{R}_T$. From Proposition 4.4, in particular,

$$(\mathbb{S}_T, \mathbb{Y}_T) \quad \to^d \quad \big(\mathbb{S}_\infty, 1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}\kappa\mathbb{S}_\infty^2 + 1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}\mathbf{G}(\theta)^{-1}b_\infty(\theta) - \beta(\theta)\big)$$

as $T\to\infty$, where $\mathbb{S}_\infty$ is a random variable distributed as $\mathbb{S}_\infty \sim N(0, c_0)$ and $b_\infty(\theta)$ is given in (5.8). We can apply Theorem 2.1 of Sakamoto and Yoshida [22] because asymptotic non-degeneracy of $\mathbb{S}_T$ is obvious. The asymptotic expansion for $F_T^{2C_1}$ is now given by the density function

$$p_{H,T}(x) \quad = \quad p^*_{H,T}(x) + T^{-\mathsf{q}(H)}g(x), \tag{6.1}$$

where

$$g(x) \quad = \quad -\partial_x\big\{(\kappa x^2 + \tau)\phi(x; 0, c_0)\big\}$$

with

$$\kappa \ = \ \kappa(H,\theta) \ = \ 1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}\lambda \quad \text{and} \quad \tau \ = \ \tau(H,\theta) \ = \ 1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}\mathbf{G}(\theta)^{-1}b_\infty(\theta) - \beta(\theta). \tag{6.2}$$

Recall that the constant λ is defined in (5.19). More precisely,

$$\begin{aligned} g(x) &= \phi(x;0,c_0)\big\{-2\kappa x + (\kappa x^2 + \tau)H_1(x, c_0)\big\} \\ &= \phi(x;0,c_0)\big\{(\tau - 2\kappa c_0)H_1(x,c_0) + \kappa x^2 H_1(x,c_0)\big\} \\ &= \phi(x;0,c_0)\big\{(\tau - 2\kappa c_0)H_1(x,c_0) + \kappa c_0^2 H_3(x,c_0) + 3\kappa c_0 H_1(x,c_0)\big\} \\ &= \phi(x;0,c_0)\big\{(\tau + \kappa c_0)H_1(x,c_0) + \kappa c_0^2 H_3(x,c_0)\big\} \end{aligned} \tag{6.3}$$

Remark that $H_3(x,c_0) = c_0^{-3}x^3 - 3c_0^{-2}x$ and

$$x^2 H_1(x,c_0) \quad = \quad c_0^2 H_3(x;0,c_0) + 3c_0 H_1(x;0,c_0).$$

With τ and κ of (6.2) and c_3' of (4.3), set

$$c_1 \ = \ \tau + \kappa c_0 \quad \text{and} \quad c_3 \ = \ c_3' + 3\lambda c_0^2. \tag{6.4}$$

Remark that $1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}c_3 \ = \ 1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}(c_3' + 3\kappa c_0^2)$. Then the resulting asymptotic expansion formula for $F_T^{2C_1}$ is given by $p_{H,T}$ of (1.8).

Since the estimator $\widehat{\theta}_T$ takes values in the bounded set Θ and as already mentioned $\psi_T^{2C_1} - 1 = O_{D_\infty}(T^{-L})$ for every $L > 0$, it is easy to show

$$\sup_{g\in\mathcal{E}(a,b)} \big|E\big[g\big(T^{1/2}(\widehat{\theta}_T - \theta)\big)\big] - E\big[g\big(F_T^{2C_1}\big)\big]\big| \quad = \quad O(T^{-L}) \quad (T\to\infty)$$

for every $L > 0$. Thus, we obtain the asymptotic expansion and its error bound for $T^{1/2}(\widehat{\theta}_T - \theta)$. $\square$

6.2 Proof of Theorem 1.2

Define $c_{1,1}^{+}$ and $c_{1,2}^{+}$ as

$$c_{1,1}^{+} \;=\; \mathbf{G}(\theta)^{-1}b_\infty(\theta)+\lambda c_0 \quad\text{and}\quad c_{1,2}^{+} \;=\; -\beta(\theta). \tag{6.5}$$

Then, by the definition (1.10) of $I\!P_{H,T}^{+}$ and the argument in Section 6.1, we see

$$\sup_{g\in\mathcal{E}(a,b)}\left|\int_{\mathbb{R}} g(x)\big(p_{H,T}(x)-p_{H,T}^{+}(x)\big)dx\right| \;=\; o(T^{-\mathfrak{q}(H)})$$

as $T\to\infty$, for every $a,b>0$. Therefore, (1.11) follows from (1.9) of Theorem 1.1. □

7 Simulation study

The performance of the asymptotic expansion formula $p_{H,T}$ of (1.8) will be investigated by simulations. We consider the parameter values $\theta=2$ and $H\in\{0.55,0.625,0.7\}$. The number of replications in each Monte Carlo simulation is 10^5. The YUIMA package (cf. [2, 10]) is used for the study.

Figure 1 shows the asymptotic expansion formula $p_{0.55,50}$ captures the skewness of the distribution of the estimation error in the time horizon $T=50$. On the other hand, the normal approximation improves for $T=100$ as in Figure 2.

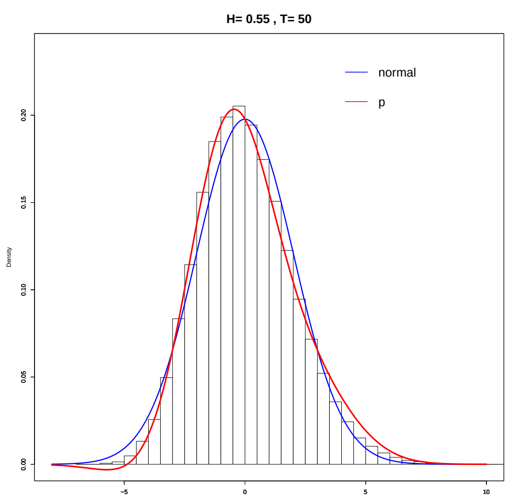


Figure 1: $N(0,c_0)$ and $p_{0.55,50}$

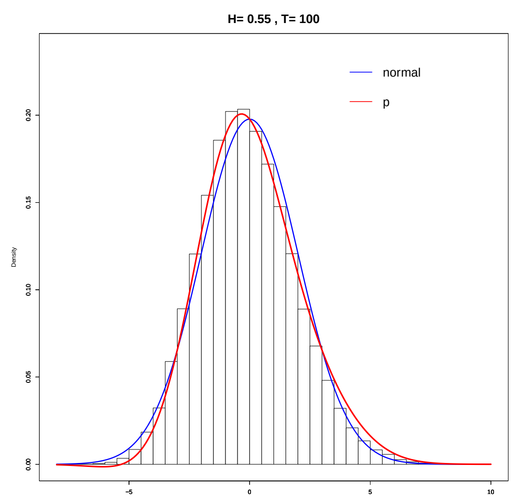
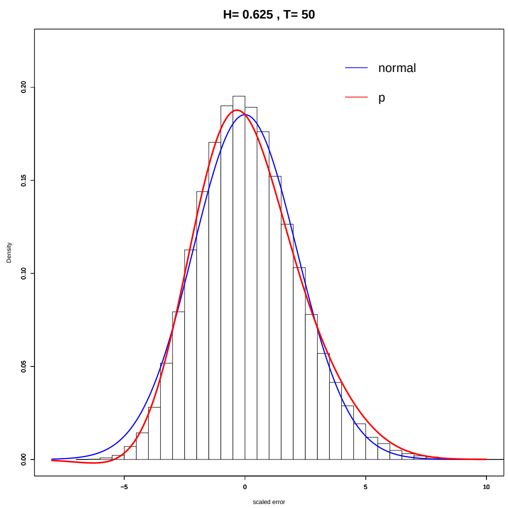


Figure 2: $N(0,c_0)$ and $p_{0.55,100}$

The value $H=5/8=0.625$ is the threshold of T's exponents $-1/2$ and $4H-3$ of the first-order correction term of the asymptotic expansion. Figures 3 and 4 show that the asymptotic expansion formulas have caught the skewness of the distribution. The correction becomes smaller for the larger T. Since the first-order correction by the asymptotic expansion consists of the two terms, it is a bit unexpected that the difference between the histogram and the normal distribution is rather small. However, it is natural in a sense because the relative effect of the skewness decreases down toward 5/8 on $(1/2,5/8]$, and the relative effect of the gap between the real variance and c_0 goes down toward 5/8 on $[5/8,3/4)$.



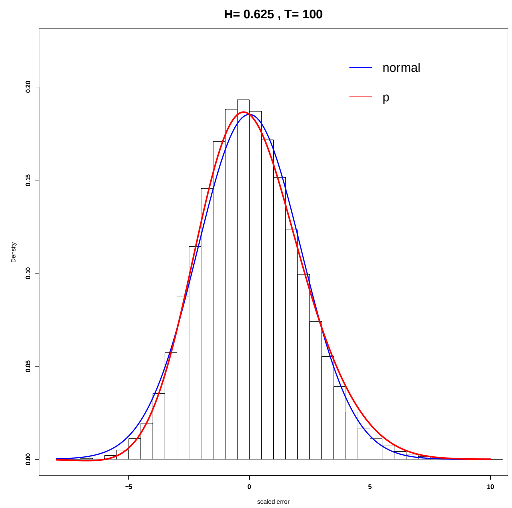
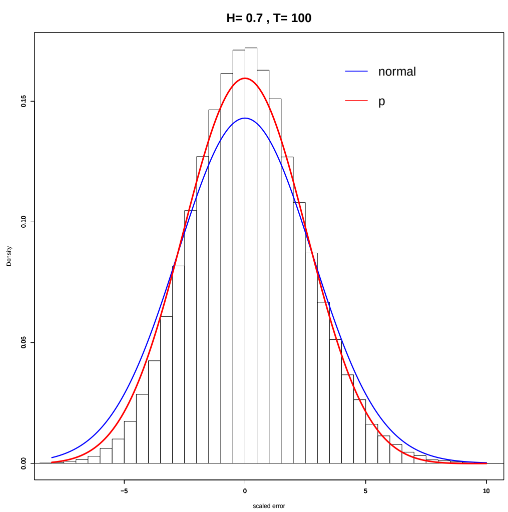


Figure 3: $N(0, c_0)$ and $p_{0.625,50}$

Figure 4: $N(0, c_0)$ and $p_{0.625,100}$

In the case $H = 0.7$, Figure 5 shows the asymptotic expansion fairly improves the normal approximation. However, some discrepancy remains yet between the asymptotic expansion and the histogram, even for $T = 100$, for which the normal approximations performed better when $H = 0.55$ and 0.625, as observed above. The value $H = 0.7$ is near to the upper bound of the interval $(1/2, 3/4)$ (more generally (0,3/4)) of H for the valid normal approximation with the scaling $T^{1/2}$. Hu et al. [9] showed that the limit becomes a normal distribution for $H = 3/4$ with the rate of convergence $T^{1/2}/\sqrt{\log T}$, and a Rosenblatt distribution if H exceeds 3/4 with the rate T^{2-2H}. This fact explains the relatively large discrepancy between the histogram and the normal approximation under rate $T^{1/2}$. The asymptotic expansion is trying to approximate the histogram, while it still has a gap since the first-order asymptotic expansion $p_{0.7,100}$ does not incorporate the effect of the kurtosis nor the higher-order moments of the variable. The approximations by the asymptotic expansion and normal distribution are improved when $T = 400$ as Figure 6 though the error of the normal approximation is not small yet.



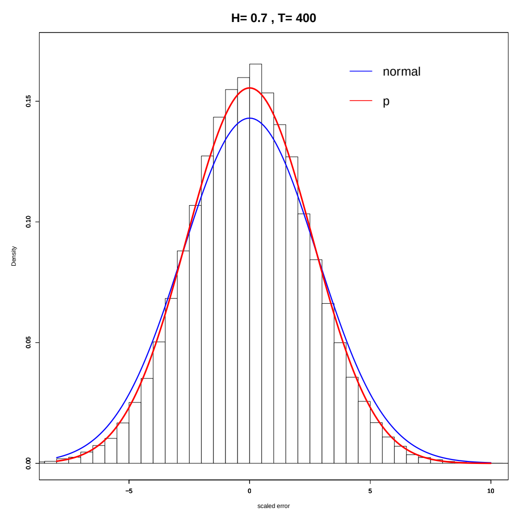


Figure 5: $N(0, c_0)$ and $p_{0.7,100}$

Figure 6: $N(0, c_0)$ and $p_{0.7,400}$

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