GLOBAL HÖLDER SOLVABILITY OF SECOND ORDER ELLIPTIC EQUATIONS WITH LOCALLY INTEGRABLE LOWER-ORDER COEFFICIENTS

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Abstract. We prove existence of globally Hölder continuous solutions to elliptic partial differential equations with lower-order terms. Our result is applicable to coefficients controlled by a negative power of the distance from the boundary.

1. Introduction

This paper deals with the global Hölder solvability of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(A\nabla u) + \boldsymbol{b}\cdot\nabla u + \mu u = \nu & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $A \in L^\infty(\Omega)^{n\times n}$ is a matrix valued function satisfying the uniformly ellipticity condition

$$|\xi|^2 \leq A(x)\xi\cdot\xi \leq L|\xi|^2 \quad \forall \xi \in \mathbb{R}^n,\ \forall x \in \Omega \tag{1.2}$$

with a fixed constant $1 \leq L < \infty$, g is a Hölder continuous function on the boundary $\partial\Omega$ of Ω. Assumptions on other coefficients to be explained later. We temporarily assume that $\boldsymbol{b} \in L^2_{\mathrm{loc}}(\Omega)^n$, $\mu, \nu \in \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the set of all measures on Ω in the sense of Bourbaki. For Ω, we further assume that

$$\exists \gamma > 0 \quad \frac{\operatorname{cap}(\overline{B(\xi,R)} \setminus \Omega, B(\xi,2R))}{\operatorname{cap}(\overline{B(\xi,R)}, B(\xi,2R))} \geq \gamma \quad \forall R > 0,\ \forall \xi \in \partial\Omega \tag{1.3}$$

where $B(x,r)$ is a ball centered at x with radius $r > 0$, and $\operatorname{cap}(K,U)$ is the relative capacity of an open set $U \subset \mathbb{R}^n$ and compact set $K \subset U$, which is defined by

$$\operatorname{cap}(K,U) := \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|^2\,dx \colon u \in C_c^\infty(U),\ u \geq 1 \text{ on } K \right\}.$$

There are many prior works for existence and regularity results of solutions to (1.1) with various aspects. We treat (1.1) using its divergence structure and refer to [23, 15] for basics of weak solutions. Sharp interior regularity estimates for solutions to (1.1) studied extensively, especially since the 1980s. We refer to [2, 10, 8, 25] for overview. Local Hölder estimates for equations with Morrey coefficients can be found in e.g. [24, 11, 29, 12]. For recent developments in interior regularity theory, see also [28, 30, 16, 17, 27]. There are not a few results on boundary regularity as well, which will be discussed later.

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Let us recall known results for the global Hölder regularity of weak solutions to

$$\begin{cases} -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

It is well known that (1.3) is sufficient for the desired boundary estimate, although there is no explicit reference (see [20, p.130] for related work). The proof of it (see e.g. [26, 14, 15, 20]) consists of three major steps. (i) Prove an interior regularity estimate. (ii) Prove a regularity estimate at each boundary point by using (1.3) and the result of (i). (iii) If the result of (ii) holds at all boundary points, then the desired global regularity follows. As a consequence, if (1.3) holds, then the operator

$$C^{\beta_0}(\partial\Omega) \ni g \mapsto u \in C^{\beta_0}(\overline{\Omega}) \tag{1.5}$$

is bounded for some $\beta_0 \in (0,1)$. Conversely, [1, Theorem 3] (see also [3, Lemma 3]) showed that the boundedness of (1.5) gives (1.3) if Ω has no irregular point.

For global regularity of solutions to (1.1), one approach is to directly repeat the above three steps (e.g. [31, 14, 25, 15, 13, 6, 27]). However, this approach has problems in local estimation at the boundary (ii). Indeed, the fact that the condition (1.3) holds at all boundary points is not exploited. Also, since it requires extending the equation out of the domain, this strategy cannot be applied to locally integrable $\boldsymbol{b}$, μ and ν.

Another popular approach to (1.1) is to construct the Green function of (1.4) and regard the lower-order terms as a perturbation. This approach is often used in the context of potential theory (e.g. [9, 7, 4, 21, 22]). The problems in (ii) above do not occur in this method because the Green function is a global concept. As a results, under smoothness assumptions on $\partial\Omega$, it is possible to deal with coefficients that diverge by negative powers of $\delta(x) := \operatorname{dist}(x, \partial\Omega)$, as in

$$\delta(x)^{1-\beta}|\boldsymbol{b}(x)| \in L^\infty(\Omega), \tag{1.6}$$

$$\mu = c(x)m \quad \text{and} \quad \delta(x)^{2-\beta}c(x) \in L^\infty(\Omega), \tag{1.7}$$

where $\beta \in (0,1)$ and m is the Lebesgue measure. Unfortunately, it is difficult to give explicit estimates of Green functions for domains with complexity boundaries. However, [3] treated such domains without giving explicit formulas for them. This strategy seems good for obtaining Hölder continuous solutions, but there is no known literature dedicated to this direction.

We construct globally Hölder continuous solutions to (1.1) using the Fredholm alternative and a Hölder estimate in [19]. The estimate is a refinement of results in [3, 18]. We control $\boldsymbol{b}$, μ and ν using a Morrey space and apply the the Fredholm alternative to it. For the sake of simplicity, we use slightly different notation from [19].

Definition 1.1. For $q \geq 1$, define

$$\mathsf{M}^q(\Omega) := \left\{ \nu \in \mathcal{M}(\Omega) \colon |||\nu|||_{q,\Omega} < \infty \right\},$$

where

$$|||\nu|||_{q,\Omega} := \operatorname{diam}(\Omega)^{2-n/q} \sup_{\substack{x\in\Omega \\ 0<r<\delta(x)/2}} r^{n/q-n}|\nu|(B(x,r)). \tag{1.8}$$

Also, for a function u on $E \subset \mathbb{R}^n$ and $\beta \in (0,1]$, we define

$$\|u\|_{C^\beta(E)} := \sup_E |u| + \operatorname{diam}(E)^\beta \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\beta}.$$

Our main result is as follows.

Theorem 1.2. *Assume* (1.2) *and* (1.3). *Suppose that*

$$|\boldsymbol{b}|^2 m \in \mathsf{M}^{n/(2-2\beta)}(\Omega), \tag{1.9}$$

$$\mu \in \mathsf{M}^{n/(2-\beta)}(\Omega), \tag{1.10}$$

where $\beta \in (0,1)$ *and that* $\mu \geq 0$. *Then, for each* $\nu \in \mathsf{M}^{n/(2-\beta)}(\Omega)$ *and* $g \in C^\beta(\partial\Omega)$, *there exists a unique weak solution* $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ *to* (1.1). *Moreover, there exists a positive constant* $\beta_\star$ *depending only on* n, L, β *and* γ *such that*

$$\|u\|_{C^{\beta_\star}(\overline{\Omega})} \leq C \left(|||\nu|||_{n/(2-\beta),\Omega} + \|g\|_{C^\beta(\partial\Omega)} \right), \tag{1.11}$$

where C *is a positive constant independent of* ν *and* g.

Remark 1.3. The solution u in Theorem 1.2 may not have finite energy. Note that we do not assume that $\nu \in H^{-1}(\Omega)$ or that g is the trace of an $H^1(\Omega)$ function.

Remark 1.4. We give some remarks to the conditions on $\boldsymbol{b}$ and μ.

(1) Local Hölder estimates under (1.9) and (1.10) are well-known. We prove (1.11) under the same conditions.
(2) The conditions (1.6) and (1.7) imply (1.9) and (1.10), respectively. Similar known result in [3] used weighted Lebesgue spaces, which are more general, but still tighter than our results.
(3) There is no size restriction of $\left|\left|\left| |\boldsymbol{b}|^2 m \right|\right|\right|_{n/(2-2\beta),\Omega}$. This existence result holds even if the problem is not coercive in the sense of bilinear form on $H^1_0(\Omega)$.

We note here limitations of Theorem 1.2. First, we have no information of the optimal value of $\beta_\star$. Second, we do not know sharp conditions for existence of globally continuous solutions. Finally, cancellation of the coefficients or the regularizing effect of the zeroth-order term μu are not used. These are topics for future work.

Organization of the paper. In Section 2, we discuss properties of $\mathsf{M}^q(\Omega)$. In Section 3, we show compactness of lower-order perturbations and prove Theorem 1.2 for the homogeneous boundary data $g = 0$. In Section 4, we complete the proof of Theorem 1.2.

Notation. Throughout below, $\Omega \subsetneq \mathbb{R}^n$ is a bounded open set. We denote by $\delta(x)$ the distance from the boundary Ω.

- $C_c(\Omega) :=$ the set of all continuous functions with compact support in Ω.
- $C_c^\infty(\Omega) := C_c(\Omega) \cap C^\infty(\Omega)$.

We denote by $\mathcal{M}(\Omega)$ the set of all measures on Ω in the sense in [5]. Using the Riesz representation theorem, we identify them with continuous linear functional functionals on $C_c(\Omega)$. When the Lebesgue measure must be indicate clearly, we use the letter m. For a function u on B, we use the notation $\fint_B u\,dx := m(B)^{-1}\int_B u\,dx$.

For a function u, we define $\operatorname{osc} u := \sup u - \inf u$. The letter C denotes various constants.

2. Morrey spaces and elliptic regularity

We first consider properties of the Morrey space $\mathsf{M}^q(\Omega)$. Since Ω is bounded, for any $1 \le q_1 \le q_2$, we have $|||\nu|||_{q_1,\Omega} \le |||\nu|||_{q_2,\Omega}$. In particular, for $q > n/2$, we have

$$|||\nu|||_{n/2,\Omega} \le |||\nu|||_{q,\Omega}. \tag{2.1}$$

If $f \in L^q(\Omega)$, then $fm \in \mathsf{M}^q(\Omega)$. On the other hand, this space is significantly larger than $L^q(\Omega)$ in the sense of boundary behavior. In fact, for a locally integrable function c in (1.7), we have $cm \in \mathsf{M}^{n/(2-\beta)}(\Omega)$ (see [19, Proposition 6.1]).

Theorem 2.1. *The space* $\left(\mathsf{M}^q(\Omega), |||\cdot|||_{q,\Omega}\right)$ *is a Banach space.*

Proof. We can check that $|||\cdot|||_{q,\Omega}$ is a norm on $\mathsf{M}^q(\Omega)$. Let us prove the completeness of it. For the sake of simplicity, we assume that $\operatorname{diam}(\Omega) = 1$ without loss of generality. Let $\{\mu_j\}$ be a Cauchy sequence in $\mathsf{M}^q(\Omega)$. Then, for any $\epsilon > 0$, there exists j_ϵ such that

$$|\mu_j - \mu_i|(B) \le \epsilon \operatorname{diam}(B)^{n-n/q}$$

whenever $j, i \ge j_\epsilon$ and $2B \subset \Omega$. Then, we have

$$\left| \int_\Omega \varphi \, d(\mu_j - \mu_i) \right| \le \epsilon \operatorname{diam}(B)^{n-n/q}$$

for all

$$\varphi \in C_c(B), \quad \|\varphi\|_{L^\infty(\Omega)} \le 1. \tag{2.2}$$

If $K \subset \Omega$ is compact, then, we can choose finitely many balls $\{B_k\}$ such that $2B_k \subset \Omega$ and $K \subset \bigcup_k B_k$. Using (2.2) and a partition of unity, we find that $\{\mu_j\}$ is bounded in the sense of the dual of $C_c(\Omega)$. Therefore, there exists a subsequence $\{\mu_{j_k}\}$ of $\{\mu_j\}$ and $\mu \in \mathcal{M}(\Omega)$ such that μ_{j_k} converges to μ vaguely.

Fix a ball B and φ satisfying (2.2) again. Taking the limit $i \to \infty$ along the above subsequence, we obtain

$$\left| \int_\Omega \varphi \, d(\mu_j - \mu) \right| \le \epsilon \operatorname{diam}(B)^{n-n/q} \tag{2.3}$$

and

$$\left| \int_\Omega \varphi \, d\mu \right| \le (|||\mu_j|||_{q,\Omega} + \epsilon) \operatorname{diam}(B)^{n-n/q}.$$

It follows from assumption on φ that

$$|\mu|(B) \le (|||\mu_j|||_{q,\Omega} + \epsilon) \operatorname{diam}(B)^{n-n/q}.$$

Therefore, $\mu \in \mathsf{M}^q(\Omega)$. Using (2.3) again, we obtain

$$|||\mu_j - \mu|||_{q,\Omega} \le \epsilon.$$

Consequently, $\mu_j \to \mu$ in $\mathsf{M}^q(\Omega)$. The uniqueness of μ and the convergence of the whole sequence follows from the usual manner. $\square$

We understand (1.1) in the sense of distributions.

Definition 2.2. Let $\boldsymbol{b} \in L^2_{\mathrm{loc}}(\Omega)$, and let $\mu, \nu \in \mathcal{M}(\Omega)$. We say that a function $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is a weak solution to (1.1) if

$$\int_\Omega A\nabla u \cdot \nabla \varphi + \boldsymbol{b} \cdot \nabla u \varphi \, dx + \int_\Omega u\varphi \, d\mu = \int_\Omega \varphi \, d\nu$$

for all $\varphi \in C_c^\infty(\Omega)$.

The following weak Harnack inequality can be found in e.g. [25, Theorem 3.13].

Lemma 2.3. *Suppose that* (1.2), (1.9) *and* (1.10) *hold for some* $\beta \in (0,1)$. *Let* u *be a nonnegative weak supersolution to* $-\operatorname{div}(A\nabla u) + \boldsymbol{b} \cdot \nabla u + \mu u = 0$ *in* Ω. *Then, we have*

$$\fint_{B(x,r)} u \, dx \le C \inf_{B(x,r)} u$$

whenever $B(x,2r) \subset \Omega$, *where* C *is a constant depending only on* n, L, β, $\left|\left|\left| |\boldsymbol{b}|^2 m \right|\right|\right|_{n/(2-2\beta),\Omega}$ *and* $|||\mu|||_{n/(2-\beta),\Omega}$.

Proposition 2.4. *Suppose that* (1.2), (1.9) *and* (1.10) *hold for some* $\beta \in (0,1)$. *Assume further that* $\mu \ge 0$. *Let* $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ *be a weak solution to*

$$\begin{cases} -\operatorname{div}(A\nabla u) + \boldsymbol{b} \cdot \nabla u + \mu u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then, $u = 0$.

Proof. This follows from the strong maximum principle. Let $M = \sup_\Omega u \ge 0$. We note that

$$-\operatorname{div}(A\nabla(M-u)) + \boldsymbol{b} \cdot \nabla(M-u) + \mu(M-u) = M\mu \ge 0 \quad \text{in } \Omega.$$

Assume that $M > 0$, and consider the set $E := \{x \in \Omega \colon u(x) = M\}$. Take $x \in E$ such that $\delta(x) = \operatorname{dist}(E, \partial\Omega) > 0$. By Lemma 2.3, we have

$$\fint_{B(x,\delta(x/2))} (M-u) \, dx \le C \inf_{B(x,\delta(x)/2)} (M-u) = 0.$$

Since $B(x,\delta(x)/2) \subset E$, it follows from an elementary geometrical consideration that $\operatorname{dist}(E,\partial\Omega) \le \delta(x)/2$. This contradicts to the definition of x. Therefore, $M = 0$. By the same way, $\inf_\Omega u = 0$. $\square$

For $\boldsymbol{b} = \boldsymbol{0}$, $\mu = 0$ and $g = 0$, we have the following existence theorem.

Lemma 2.5 ([19, Theorem 1.3])**.** *Assume that* (1.2) *and* (1.3) *hold. Suppose that* $\nu \in \mathsf{M}^q(\Omega)$ *for some* $q > n/2$. *Then, there exists a unique weak solution* $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ *to*

$$\begin{cases} -\operatorname{div}(A\nabla u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Moreover, there exist positive constants C_1 *and* β_1 *depending only on* n, L, q *and* γ *such that*

$$\|u\|_{C^{\beta_1}(\Omega)} \le C_1 |||\nu|||_{q,\Omega}. \tag{2.6}$$

We use the following notation.

Definition 2.6. Let $\nu \in \mathsf{M}^q(\Omega)$ with $q > n/2$. We denote by $\mathbf{G}_0\nu$ the weak solution $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ to (2.5).

3. Lower-order terms

Let us recall the Fredholm alternative.

Lemma 3.1 ([15, Theorem 5.3])**.** *Let X be a normed space, and let T be a compact linear operator from X into itself. Then, either (i) the homogeneous equation*

$$x - Tx = 0$$

has a nontrivial solution $x \in X$, or (ii) for each $y \in X$, the equation

$$x - Tx = y$$

has a unique solution $x \in X$. Moreover, in case (ii), the operator $(I-T)^{-1}$ exists and is bounded.

We apply the above theorem to the operator

$$T\colon \mathsf{M}^q(\Omega) \ni \nu \mapsto T\nu := -\left(\boldsymbol{b}\cdot\nabla + \mu\right)\mathbf{G}_0\nu \in \mathsf{M}^q(\Omega). \tag{3.1}$$

Lemma 3.2. *Assume that (1.9) and (1.10) hold. Let $q = n/(2-\beta)$. Then, the operator T in (3.1) is a compact operator from $\mathsf{M}^q(\Omega)$ into itself. Moreover, we have*

$$|||T\nu|||_{q,\Omega} \le C_2 \left(|||\,|\boldsymbol{b}|^2 m|||_{n/(2-2\beta),\Omega}^{1/2} + |||\mu|||_{n/(2-\beta),\Omega} \right) |||\nu|||_{q,\Omega} \tag{3.2}$$

for all $\nu \in \mathsf{M}^q(\Omega)$.

Proof. Let $u = \mathbf{G}_0\nu$. By (2.5), we have

$$\|u\|_{L^\infty(\Omega)} \le C_1 |||\nu|||_{q,\Omega}. \tag{3.3}$$

Let $B(x,r)$ be a ball such that $B(x,4r) \subset \Omega$. Take $\eta \in C_c^\infty(B(x,2r))$ such that $\eta = 1$ on $B(x,r)$ and $|\nabla\eta| \le C/r$. Testing (2.5) with $u\eta^2$, we obtain

$$\int_{B(x,r)} |\nabla u|^2 \, dx \le C \left(\frac{1}{r^2} \int_{B(x,2r)} |u|^2 \, dx + \int_{B(x,2r)} |u| \, d|\nu| \right).$$

By (2.1), we also get

$$\int_{B(x,r)} |\nabla u|^2 \, dx \le C \left(\|u\|_{L^\infty(\Omega)}^2 + \|u\|_{L^\infty(\Omega)} |||\nu|||_q \right) r^{n-2}. \tag{3.4}$$

The right-hand side is estimated by (3.3). Meanwhile, by Hölder's inequality, we have

$$\int_{B(x,r)} |\boldsymbol{b}\cdot\nabla u| \; dx \le \left(\int_{B(x,r)} |\boldsymbol{b}|^2 \, dx \right)^{1/2} \left(\int_{B(x,r)} |\nabla u|^2 \, dx \right)^{1/2}.$$

Combining these inequalities with (1.9), we obtain

$$\int_{B(x,r)} |\boldsymbol{b}\cdot\nabla u| \, dx \le C |||\,|\boldsymbol{b}|^2 m|||_{n/(2-2\beta),\Omega}^{1/2} |||\nu|||_{q,\Omega} r^{n-2+\beta}. \tag{3.5}$$

Meanwhile, by (1.10) and (3.3), we have

$$\int_{B(x,r)} |u| \, d|\mu| \le C |||\mu|||_{n/(2-\beta),\Omega} |||\nu|||_{q,\Omega} r^{n-2+\beta}.$$

By a simple covering argument, we find that (3.2) holds.

Let us prove the compactness of T. Let $\{\nu_j\}$ be a bounded sequence in $\mathsf{M}^q(\Omega)$ and assume that $|||\nu_j|||_{q,\Omega} \le M < \infty$. Set $u_j = \mathbf{G}_0\nu_j$. Since $\{u_j\}$ is bounded in $C^{\beta_0}(\overline{\Omega})$, by the Ascoli-Arzelà theorem, we can take a subsequence of $\{u_j\}$ and $u \in C(\overline{\Omega})$ such that $u_j \to u$ uniformly in Ω. Meanwhile, by (3.4), we have

$$\int_{B(x,r)} |\nabla(u_j - u_i)|^2 \, dx \\ \le C \left(\|u_j - u_i\|_{L^\infty(\Omega)}^2 + 2\|u_j - u_i\|_{L^\infty(\Omega)} M \right) r^{n-2}$$

for all $i, j \ge 1$. It follows from (1.9) that $\{(\boldsymbol{b} \cdot \nabla u_j) m\}$ is a Cauchy sequence in $\mathsf{M}^q(\Omega)$. By the same way, $\{\mu u_j\}$ is a Cauchy sequence in $\mathsf{M}^q(\Omega)$. By Theorem 2.1, T is compact. □

Corollary 3.3. *Assume that* (1.9) *and* (1.10) *hold. Let* $q = n/(2-\beta)$. *Then, either (i) the homogeneous equation* (2.4) *has a nontrivial solution* $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, *or (ii) for each* $\nu \in \mathsf{M}^q(\Omega)$, *the equation*

$$\begin{cases} -\operatorname{div}(A\nabla u) + \boldsymbol{b} \cdot \nabla u + \mu u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

has a unique solution $u \in H^1_{\mathrm{loc}}(\Omega) \cap C^{\beta_1}(\overline{\Omega})$. *Moreover, in case (ii), the operator*

$$\mathbf{G}_T \colon \mathsf{M}^q(\Omega) \ni \nu \mapsto \mathbf{G}_T \nu := u \in H^1_{\mathrm{loc}}(\Omega) \cap C^{\beta_1}(\overline{\Omega}) \tag{3.7}$$

exists and is bounded.

Proof. If there is a non-trivial solution $\sigma \in \mathsf{M}^q(\Omega)$ to

$$\sigma - T\sigma = 0, \tag{3.8}$$

then, $u := \mathbf{G}_0 \sigma \in H^1_{\mathrm{loc}}(\Omega) \cap C^{\beta_1}(\overline{\Omega})$ is a non-trivial solution to (2.4). We prove the converse statement. Assume the existence of a non-trivial solution $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ to (2.4). Take a ball $B(x, 4r) \subset \Omega$ and $\eta \in C_c^\infty(B(x, 2r))$ such that $\eta = 1$ on $B(x, r)$ and $|\nabla \eta| \le C/r$. Testing (2.4) with $u\eta^2$, we get

$$\int_{B(x,2r)} |\nabla u|^2 \eta^2 \, dx \le \frac{C}{r^2} \int_{B(x,2r)} u^2 \, dx \\ + \left| \int_{B(x,2r)} \boldsymbol{b} \cdot \nabla u u \eta^2 \, dx + \int_{B(x,2r)} u^2 \eta^2 \, d\mu \right|.$$

By the Young inequality $ab \le (\epsilon/2)a^2 + (2\epsilon)^{-1} b^2$ $(a, b, \epsilon \ge 0)$, we have

$$\left| \int_{B(x,2r)} \boldsymbol{b} \cdot \nabla u u \eta^2 \, dx \right| \le \frac{\epsilon}{2} \int_{B(x,2r)} |\nabla u|^2 \eta^2 \, dx + \frac{1}{2\epsilon} \int_{B(x,2r)} |\boldsymbol{b}|^2 u^2 \eta^2 \, dx.$$

Combining these inequalities with (1.9), (1.10) and (2.1), we obtain

$$\int_{B(x,r)} |\nabla u|^2 \, dx \le C \|u\|_{L^\infty(\Omega)}^2 r^{n-2}.$$

It follows from (1.9) that $(\boldsymbol{b} \cdot \nabla u) m \in \mathsf{M}^q(\Omega)$. Meanwhile, $\mu u \in \mathsf{M}^q(\Omega)$ because u is bounded and (1.10) holds. Therefore, $\sigma := -\operatorname{div}(A\nabla u)$ belongs to $\mathsf{M}^q(\Omega)$. It is also a non-trivial solution to (3.8).

Assume that there is no non-trivial solution to (3.8). By Lemmas 3.1 and 3.2, for each $\nu \in \mathsf{M}^q(\Omega)$, there exists a unique solution $\sigma \in \mathsf{M}^q(\Omega)$ to $\sigma - T\sigma = \nu$. Then, $u := \mathbf{G}_0 \sigma \in H^1_{\mathrm{loc}}(\Omega) \cap C^{\beta_1}(\overline{\Omega})$ satisfies

$$-\operatorname{div}(A\nabla u) = \nu + T\sigma = \nu - (\boldsymbol{b} \cdot \nabla + \mu) u.$$

Let us prove the uniqueness of u. If there are two different solutions u_1 and u_2 to (3.6), then $v = u_1 - u_2$ is a non-trivial solution to (2.4). Since $\sigma := -\operatorname{div}(A\nabla v)$ belongs to $\mathsf{M}^q(\Omega)$, this contradicts to assumption. □

Remark 3.4. Assume further that $\||\,|\boldsymbol{b}|^2 m\,|\|_{n/(2-2\beta),\Omega}^{1/2} + \||\mu|\|_{n/(2-\beta),\Omega} \le (2C_1C_2)^{-1}$, where C_1 and C_2 are constants in Lemmas 2.5 and 3.2, respectively. Then, we can get an explicit bound of (3.7) by the contractive mapping theorem.

4. Inhomogeneous boundary data

Lemma 4.1. *Let $g \in C^\beta(\partial\Omega)$. Then, there exists a unique weak solution $w \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ to* (1.4). *Moreover, there exists a positive constants C and $0 < \beta_0 \le \beta$ such that*

$$\|w\|_{C^{\beta_0}(\Omega)} \le C\|g\|_{C^\beta(\partial\Omega)}.$$

Assume further that (1.9) *and* (1.10) *hold. Then, $\boldsymbol{b}\cdot\nabla w + \mu w \in \mathsf{M}^{n/(2-\beta)}(\Omega)$ and*

$$\||\boldsymbol{b}\cdot\nabla w + \mu w|\|_{n/(2-\beta),\Omega} \le C\|w\|_{L^\infty(\Omega)}. \tag{4.1}$$

Proof. As mentioned in Section 1, the existence of w and its Hölder estimate are well-known (see e.g. [20, Theorem 6.44]). By the comparison principle, we have

$$\operatorname*{osc}_{\Omega} w \le \operatorname*{osc}_{\partial\Omega} g.$$

As the proof of (3.4), we have

$$\int_{B(x,r)} |\nabla w|^2\, dx \le C\left(\operatorname*{osc}_{\Omega} w\right)^2 r^{n-2}$$

whenever $B(x,4r) \subset \Omega$. By (1.9) and (1.10), we obtain (4.1). □

Theorem 4.2. *Suppose that* (1.2), (1.3), (1.9) *and* (1.10) *hold. Assume further that there is no non-trivial solution to* (2.4). *Then, for each $\nu \in \mathsf{M}^{n/(2-\beta)}(\Omega)$ and $g \in C^\beta(\partial\Omega)$, there exists a unique weak solution $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ to* (1.1). *Moreover, there exists a positive constant $\beta_\star$ depending only on n, L, β and γ satisfying* (1.11), *where C is a positive constant independent of ν and g.*

Proof. Let w be a weak solution in Lemma 4.1. Consider the problem

$$\begin{cases} -\operatorname{div}(A\nabla v) + \boldsymbol{b}\cdot\nabla v + \mu v = \nu - \boldsymbol{b}\cdot\nabla w - \mu w & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Since the right-hand side is in $\mathsf{M}^q(\Omega)$, this equation has a unique solution $v \in H^1_{\mathrm{loc}}(\Omega) \cap C^{\beta_1}(\overline{\Omega})$. Then, $u = v + w \in C^{\beta_\star}(\overline{\Omega})$ satisfies (1.11), where $\beta_\star = \min\{\beta_1, \beta_0\}$. □

Proof of Theorem 1.2. Combine Theorem 4.2 and Proposition 2.4. □

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