

# Integrable and superintegrable quantum mechanical systems with position dependent masses invariant with respect to one parametric Lie groups. 1. Systems with cylindric symmetry.

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## Abstract

Cylindrically symmetric quantum mechanical systems with position dependent masses (PDM) admitting at least one second order integral of motion are classified. It is proved that there exist 68 such systems which are inequivalent. Among them there are twenty seven superintegrable and twelve maximally superintegrable. The arbitrary elements of the corresponding Hamiltonians (i.e., masses and potentials) are presented explicitly.

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# 1 Introduction

Integrals of motion belong to very important part of classical and quantum mechanics. Just the existence of the sufficient number of integrals of motion for a Hamiltonian system makes it integrable or exactly solvable, and we are not supposed to calculate its approximate solutions to describe its behavior. Moreover, sometimes it is possible to describe the main physical properties of the mentioned system using its integrals of motion and ignoring the motion equations. A classical example of such situation was presented by Pauli who had calculated the energy spectrum of the Hydrogen atom using its integrals of motion forming the Laplace-Runge-Lenz vector [1]. And it was done before the discovery of Schrödinger equation!

The systematic search for integrals of motion admitted by the Schrödinger equation equations started with papers [2], [3] were all inequivalent second order integrals of motion for the 2d one particle quantum systems had been classified. And it needed 24 years to extend this result to the 3d case [4], [5].

Papers [2] and [3] were indeed seminal. They were followed by a great number of research works, see, e.g. survey [6]. In particular, integrable and superintegrable system with matrix potentials have been classified for both spin-orbit [7, 8] and Pauli type interactions [9, 10]. One of the modern trends is to search for the third and even arbitrary order integrals of motion [11], see also [12] where the determining equations for such integrals were presented.

The higher (in particular, second) order integrals of motion are requested for description of systems admitting solutions in separated variables [13] just such integrals of motion characterize integrable and superintegrable systems [6]. Let us mention also the nice conjecture of Marquette and Winternitz [14] which predicts a surprising connection of higher order superintegrability with the soliton theory.

In any case the integrable and superintegrable systems of the standard quantum mechanics belong to the well developed research field which, however, still have some white spots. Among them is the classification of arbitrary order symmetry operators of generic form, which still are described only at the level of determining equations [12].

One more important research field which is closely related to the mentioned one is the classification of second order integrals of motion for quantum mechanical systems with position dependent mass. Such systems are used in many branches of modern theoretical physics, whose list can be found, e.g., in [15, 16]. Their symmetries and integrals of motion were studied much less than those ones for the standard SE. However, the classification of Lie symmetries of the PDM Schrödinger equations with scalar potentials have been obtained already [17, 18, 19].

Second order integrals of motion for 2d PDM SEs are perfectly classified [20, 21, 22, 23]. The majority of such systems admits also at least one continuous Lie symmetry. The two dimensional second-order (maximally) superintegrable systems for Euclidean 2-space had been classified even algebraic geometrically [24].

The situation with the 3d systems is still indefinite. At the best of my knowledge the completed classification results were presented only for the maximally superintegrable (i.e., admitting the maximal possible number of integrals of motion) systems [25, 26], and (or) for the system whose integrals of motion are supposed to satisfy some special conditions like the functionally linearly dependence [27]. The nondegenerate systems, i.e, those ones which have 5 linearly independent, contained in 6 linearly independent (but functionally dependent) 2nd order integrals of motion are known [28], see also [29] for the contemporary trends in this field. In addition, a certain progress can be recognized in the classification of the so called

semidegenerate systems which admit five linearly independent integrals of motion and whose potentials are linear combinations of three functionally independent terms [30].

Surely, the maximally superintegrable systems are very important and interesting. In particular, they admit solutions in multi separated coordinates [31, 32, 33, 34]. On the other hand, there are no reasons to ignore the PDM systems which admit second order integrals of motion but are not necessary maximally superintegrable. And just such systems are the subject of our study.

In view of the complexity of the total classification of integrals of motion for 3d PDM systems it is reasonable to separate this generic problem to well defined subproblems which can have their own values. The set of such subproblems can be treated as optimal one if solving them step by step we can obtain the complete classification of PDM systems admitting integrals of motion. We choose the optimal set of subproblems in the following way.

The first subproblem consists in the classification of the PDM systems admitting the first order integrals of motion. This problem is already solved, refer to [17].

The first order integrals of motion are nothing but generators of Lie symmetries. The important aspect of the results presented in [17] is the complete description of possible Lie symmetry groups which can be admitted by the stationary PDM Schrödinger equation. And this property, i.e., the presence of Lie symmetry, can be effectively used to separate the the problem of the classification of the PDM systems admitting second order integrals of motion for PDM systems to a well defined subproblems corresponding to the fixed symmetries.

It was shown in [17] the PDM Schrödinger equation can admit six, four, three, two or one parametric Lie symmetry groups. In addition, there are also such equations which have no Lie symmetry. In other words, there are six well defined classes of such equations which admit  $n$ -parametric Lie groups with  $n = 6, 4, 3, 2, 1$  or do not have any Lie symmetry. And it is a natural idea to search for second order integrals of motion consequently for all these classes.

The systems admitting six- or four-parametric Lie groups are not too interesting since the related Hamiltonians cannot include non-trivial potentials. That is why we started our research with the case of three-parametric groups. The classification of the corresponding PDM systems admitting second order integrals of motion was obtained in [35]. There were specified 38 inequivalent PDM systems together with their integrals of motion. The majority of them are new systems which are not not maximally superintegrable.

Notice that the superintegrable 3d PDM systems invariant with respect to the 3d rotations have been classified a bit earlier in paper [36] where their supersymmetric aspects were discussed also. For relativistic aspects of superintegrability see [37], [38].

The systems admitting two-parametric Lie groups and second order integrals of motion have been classified in [39]. We again find a number of new systems in addition to the known maximally superintegrable ones.

The natural next step is to classify the systems which admit one parametric Lie groups and second order integrals of motion. As it is shown in [17], up to equivalence these groups are reduced to dilatations, shifts along the fixed coordinate axis, rotations around this axis and some specific combinations of the mentioned transformations. We will conventionally call them the natural and combined symmetries respectively.

In the present paper we start the systematic search for integrable and superintegrable PDM systems admitting one parametric Lie symmetry groups. Namely, we present the classification of the mentioned systems which are invariant w.r.t. the rotations around the fixed coordinate axis. The number of such systems appears to be rather extended, and to keep the reasonable

size of paper we restrict ourselves to this particular symmetry, while the systems with the other Lie symmetries will be presented in the following papers. Notice that the number of inequivalent systems which admit the other one parametric symmetry groups is much less extended than in the case of the cylindric symmetry.

In spite of the fact that the usual strategy in studying of superintegrable systems with PDM is to start with the classical Hamiltonian systems and then quantize them if necessary, we deal directly with quantum mechanical systems. This way is more complicated but it guaranties obtaining of all integrals of motion including those ones which can disappear in the classical limit [40].

The main result of the present paper consist in the complete classification of integrable, superintegrable and maximally superintegrable PDM systems with cylindric symmetry. In addition, we optimise the algorithm of solution of the related determining equations which can be used for a classification of other PDM systems.

## 2 Formulation of the problem

We are studying the stationary Schrödinger equations with position dependent mass of the following generic form:

$$H\psi = E\psi, \quad (1)$$

where

$$H = p_a f(\mathbf{x}) p_a + V(\mathbf{x}), \quad (2)$$

$\mathbf{x} = (x^1, x^2, x^3)$ ,  $p_a = -i\partial_a$ ,  $V(\mathbf{x})$  and  $f(\mathbf{x}) = \frac{1}{2m(\mathbf{x})}$  are functions associated with the potential and inverse PDM, and summation from 1 to 3 is imposed over the repeating index  $a$ .

The particular form (2) of the hamiltonian is convenient for study of its symmetries and integrals of motion. Moreover, more generic formulations including the arbitrary ambiguity parameters (refer, e.g. to [16]) are mathematically equivalent to (2).

In paper [17] all equations (1) admitting at least one first order integral of motion was found. Such integrals of motion generate Lie groups which leave the equations invariant. In accordance with [17] there are six inequivalent Lie symmetry groups which can be accepted by the PDM Schrödinger equations. They include three "natural" groups, rotation around the third coordinate axis, shift along this axis and dilatation groups. In addition, we can fix three combined symmetries which are superpositions of rotations and shifts, rotations and dilatations, and shifts, rotations and conformal transformations.

The generic form of the corresponding inverse masses  $f$  and potentials  $V$  can be represented by the following formulae [17]:

$$f = F(\tilde{r}, x_3), \quad V = G(\tilde{r}, x_3), \quad (3)$$

$$f = F(x_1, x_2), \quad V = V(x_1, x_2), \quad (4)$$

$$f = r^2 F(\varphi, \theta), \quad V = V(\varphi, \theta) \quad (5)$$

where  $F(\cdot)$  and  $V(\cdot)$  are arbitrary functions whose arguments are fixed in the brackets,

$$r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \quad \tilde{r} = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad \varphi = \arctan\left(\frac{x_2}{x_1}\right), \quad \theta = \arctan\left(\frac{\tilde{r}}{x_3}\right).$$

In the present paper we classify the PDM systems which admit one of the mentioned natural symmetries, namely the rotations around the third coordinate axis, and, in addition, have at least one second order integral of motion. The generic form of the corresponding inverse masses  $f$  and potentials  $V$  are represented in (3).

Equations (1), (2) with arbitrary parameters presented in (3) admit the following first order integrals of motion

$$L_3 = x_1 p_2 - x_2 p_1 \quad (6)$$

which is nothing but the third component of the orbital momentum.

Our goal is to fix such systems (1), (3) which, in addition to their Lie symmetries, admit second order integrals of motion whose generic form is:

$$Q = \partial_a \mu^{ab} \partial_b + \eta \quad (7)$$

where  $\mu^{ab} = \mu^{ba}$  and  $\eta$  are unknown functions of  $\mathbf{x}$  and summation from 1 to 3 is imposed over all repeating indices.

Operators (7) are formally hermitian. In addition, just representation (7) leads to the most compact and simple systems of determining equations for unknown parameters  $\mu^{ab}$  and  $\eta$ .

By definition, operators  $Q$  should commute with  $H$ :

$$[H, Q] \equiv HQ - QH = 0. \quad (8)$$

Evaluating the commutator in (8) and equating to zero the coefficients for the linearly independent differential operators  $\partial_a \partial_b \partial_c$  and  $\partial_a$  we come to the following determining equations

$$5(\mu_c^{ab} + \mu_b^{ac} + \mu_a^{bc}) = \delta^{ab}(\mu_c^{nn} + 2\mu_n^{cn}) + \delta^{bc}(\mu_a^{nn} + 2\mu_n^{an}) + \delta^{ac}(\mu_b^{nn} + 2\mu_n^{bn}), \quad (9)$$

$$(\mu_a^{nn} + 2\mu_n^{na})f - 5\mu^{an}f_n = 0, \quad (10)$$

$$\mu^{ab}V_b - f\eta_a = 0 \quad (11)$$

where  $\delta^{bc}$  is the Kronecker delta,  $f_n = \frac{\partial f}{\partial x_n}$ ,  $\mu_n^{an} = \frac{\partial \mu^{an}}{\partial x_n}$ , etc., and summation is imposed over the repeating indices  $n$  over the values  $n = 1, 2, 3$ .

Equations (9), (10) and (11) give the necessary and sufficient conditions for commutativity of operators  $H$  (2) and  $Q$  (7) [39].

### 3 Evolution of the determining equations

A particular solution of equations (9) is  $\mu^{ab} = \mu_0^{ab}$  where

$$\mu_0^{ab} = \delta^{ab}g(\mathbf{r}) \quad (12)$$

with arbitrary function  $g(\mathbf{r})$ .

Whenever tensor  $\mu_0^{ab}$  is nontrivial, the determining equations (10) represent the coupled system of three *nonlinear* partial differential equation equations for two unknowns  $g(x)$  and  $f(\mathbf{x})$ . Fortunately, this system can be linearizing by introduction of the new dependent variables

$$M = \frac{1}{f}, \quad N = \frac{g}{f} \quad (13)$$

which reduces (10) to the following form:

$$(\mu_a^{nn} + 2\mu_n^{na})M + 5\mu^{an}M_n + N_a = 0. \quad (14)$$

We see that for variables (13) the determining equation (10) is linear. Equation (11) in its turn can be effectively linearised by introducing the following new dependent variables  $\tilde{M}$  and  $R$ :

$$\tilde{M} = MV, \quad R = \frac{N\tilde{M}}{M} - \eta \quad (15)$$

which reduce (11) to the following equation:

$$(\mu_a^{nn} + 2\mu_n^{na})\tilde{M} + 5\mu^{an}\tilde{M}_n + R_a = 0 \quad (16)$$

which simple coincides with (14). Surely, it does not mean that  $M$  and  $N$  coincide with  $\tilde{M}$  and  $\tilde{N}$  respectively, since these functions can include different arbitrary elements, say, different integration constants. In accordance with (13) and (15) the related inverse mass potential have the form:

$$f = \frac{2}{M}, \quad V = \frac{\tilde{M}}{M} \quad (17)$$

where  $\tilde{M}$  and  $M$  are different solutions of the same equation, while the corresponding functions  $g$  and  $\eta$  are expressed via  $M$  and  $\tilde{M}$  in the following manner:

$$g = \frac{N}{M}, \quad \eta = -\frac{N\tilde{M}}{M} - R. \quad (18)$$

We see that to find the admissible inverse mass and potential it is sufficient to solve the only linear equation (14) and then find the desired functions  $f, V, g$  and  $\eta$  using definitions (17) and (18).

Just linearised determining equation (13) together with the mentioned definitions will be used in the following to solve our classification problem.

Let us represent generic integral of motion (7) in terms of new dependent variables  $M, N, \tilde{M}$  and  $\tilde{N}$  (refer to (13) and (15))

$$Q = P_a\mu^{ab}P_b + (N \cdot H) - R \quad (19)$$

where we denote

$$(N \cdot H) = P_a(Nf)P_a + NV. \quad (20)$$

The latter definition includes a Hermitized product of function  $N$  with Hamiltonian (2).

It is necessary to note than whenever function  $g(\mathbf{r})$  is equal to zero, i.e., tensor  $\mu_0^{ab}$  is trivial, the above presented speculations are forbidden. We still can deal with the determining equations (14), but we are supposed to deal with the initial determining equations (11) instead of (16).

## 4 Equivalence relations

An important step of our classification problem is the definition of equivalence relations which will be presented in this section.

Non degenerated changes of dependent and independent variables of equations (1), (2) are called equivalence transformations provided they keep their generic form up to the explicit expressions for the arbitrary elements  $f$  and  $V$ . They have the structure of a continuous group which however can be extended by some discrete elements. Let us remind that a particular subset of the equivalence transformations are invariance transformation which by definition keep the mentioned arbitrary elements uncharged.

It was shown in [17] that the maximal continuous equivalence group of equation (1) is the group of conformal transformations of the 3d Euclidean space which we denote as  $C(3)$ . The corresponding Lie algebra is a linear span of the following first order differential operators [17]:

$$\begin{aligned} P^a &= p^a = -i \frac{\partial}{\partial x_a}, & L^a &= \varepsilon^{abc} x^b p^c, \\ D &= x_n p^n - \frac{3i}{2}, & K^a &= r^2 p^a - 2x^a D, \end{aligned} \quad (21)$$

where  $r^2 = x_1^2 + x_2^2 + x_3^2$  and  $p_a = -i \frac{\partial}{\partial x_a}$ . Operators  $P^a$ ,  $L^a$ ,  $D$  and  $K^a$  generate shifts, rotations, dilatations and pure conformal transformations respectively.

In addition, equation (1) is form invariant with respect to the following discrete transformations:

$$x_a \rightarrow \tilde{x}_a = \frac{x_a}{r^2}, \quad \psi(\mathbf{x}) \rightarrow \tilde{x}^3 \psi(\tilde{\mathbf{x}}) \quad (22)$$

where  $\tilde{x} = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2}$ .

Notice that the related Lie algebra  $c(3)$  is isomorphic to the algebra  $so(1,4)$  whose basic elements  $S_{\mu\nu}$  can be expressed via generators (21) as:

$$S_{ab} = \varepsilon_{abc} L_c, \quad S_{4a} = \frac{1}{2}(K_a - P_a), \quad S_{0a} = \frac{1}{2}(K_a + P_a), \quad S_{04} = D \quad (23)$$

where  $a, b = 1, 2, 3$ . The related Lie group  $SO(1,4)$  is the Lorentz group in (1+4)-dimensional space. The discrete transformation (22) anticommutes with  $S_{4a}$  and  $S_{40}$  but commutes with the remaining generators (23). Thus its action on operators (23) can be represented as follows:

$$S_{4a} \rightarrow -S_{4a}, \quad S_{04} \rightarrow -S_{04}, \quad S_{0a} \rightarrow S_{0a}, \quad S_{ab} \rightarrow S_{ab}. \quad (24)$$

The presented speculations are valid for an abstract system (1) which is free of any additional constrains. However, for the systems whose arbitrary elements satisfy condition (3) the equivalence group is reduced since it is supposed that it does not change the invariance groups of these equations. It means that the set of generators (23) should be reduced to such ones which commute with  $L_3$ . There are four the generators satisfying this condition, namely:

$$P_3, L_3, K_3, D. \quad (25)$$

They generate the reduced equivalence algebra  $so(2,1) \oplus e(1)$  where  $e(1)$  includes the only basis element  $L_3$ .

Thus, in comparison with the generic case, for special arbitrary elements presented in (3) the admissible continuous equivalence transformations are reduced to group SO(2,1) extended by rotations w.r.t. the third coordinate axis. However, the admissible discrete transformations are extended by the reflection of one out of two the independent variables  $x_1$  and  $x_2$ , i.e.,

$$x_1 \rightarrow -x_1, x_2 \rightarrow x_2, x_3 \rightarrow x_3 \quad (26)$$

or

$$x_1 \rightarrow x_1, x_2 \rightarrow -x_2, x_3 \rightarrow x_3 \quad (27)$$

which keep the related equation (1) invariant. These discrete transformations can be added to the universal discrete transformation (22).

The presented equivalence relations will be used in the following to simplify calculations and to optimize the representation of the classification results.

## 5 Identities in the extended enveloping algebra of $\mathfrak{c}(3)$

It was noted in [39] that integrals of motion (7) where  $\mu^{ab}$  Killing tensors, i.e., solutions of equations can be represented as bilinear combinations of the basic elements of algebra  $\mathfrak{c}(3)$  (21) added by the special term with  $\mu^{ab} = \delta_{ab}g(\mathbf{x})$  and potential term  $\eta$ . In other words they admit the following representation:

$$Q = c^{\mu\nu, \lambda\sigma} \{S_{\mu\nu}, S_{\lambda\sigma}\} + P_a g(\mathbf{x}) P_a + \eta \quad (28)$$

where  $S_{\mu\nu}$  are generators (23) and  $c^{\mu\nu, \lambda\sigma}$  are numeric parameters.

In accordance with (28) the second order integrals of motion of the PDM Schrödinger equation belong to the enveloping algebra of their equivalence algebra, i.e.,  $\mathfrak{c}(3)$ , extended by special terms  $p_a g(\mathbf{x}) p_a$ . Notice that the same is true for the any order integrals of motion with the appropriate generalization of the extending term.

Representation (28) is very important. Being combined with the equivalence relations discussed in the previous section it enables essentially simplify both the calculations and the representation of their results. In addition to the equivalence relations we will use numerous identities in the enveloping algebra of algebra  $\mathfrak{c}(3)$  which take place for its particular realization (21). These identities are presented in the following formulae:

$$\{P_a, D\} + \varepsilon_{abc} \{P_b, L_c\} = 2P_c x_a P_c, \quad (29)$$

$$L_1^2 + L_2^2 + L_3^2 + D^2 = P_a r^2 P_a, \quad (30)$$

$$\{L_a, L_b\} + \{P_a, K_b\} - \delta^{ab} (L_1^2 + L_2^2 + L_3^2) = 2Q^{ab} \quad (31)$$

$$P_1^2 + P_2^2 + P_3^2 = P_a P_a,$$

$$\{P_a, K_b\} - \{P_b, K_a\} = 2\varepsilon_{abc} L_c D, \quad (32)$$

$$P_1 L_1 + P_2 L_2 + P_3 L_3 = 0$$

where  $Q^{ab} = -P_c x_a x_b P_c$ .

The message given by relations (29)-(32) is that the terms in the l.h.s. can be treated as linearly dependent whenever they are included into second order integral of motion (28) since

the latter one includes a yet indefinite term of the kind presented in the r.h.s. of equations (29)-(32).

We will use relations (29)-(32) to produce maximally compact presentations for the integrals of motion.

## 6 Solution of determining equations

The autonomous subsystem (9) defines the conformal Killing tensor which is the fourth order polynomial in variables  $x_a$  and includes an arbitrary function which multiplies the Kronecker delta. The explicit expression for this polynomial are presented in (12) and in the following formulae (see, e.g., [41]) :

$$\mu_1^{ab} = \lambda_1^{ab}, \quad (33)$$

$$\mu_2^{ab} = \lambda_2^a x^b + \lambda_2^b x^a - 2\delta^{ab} \lambda_2^c x^c, \quad (34)$$

$$\mu_3^{ab} = (\varepsilon^{acd} \lambda_3^{cb} + \varepsilon^{bcd} \lambda_3^{ca}) x^d, \quad (35)$$

$$\mu_4^{ab} = (x^a \varepsilon^{bcd} + x^b \varepsilon^{acd}) x^c \lambda_4^d, \quad (36)$$

$$\mu_5^{ab} = \delta^{ab} r^2 + k(x^a x^b - \delta^{ab} r^2), \quad (37)$$

$$\mu_6^{ab} = \lambda_6^{ab} r^2 - (x^a \lambda_6^{bc} + x^b \lambda_6^{ac}) x^c - \delta^{ab} \lambda_6^{cd} x^c x^d, \quad (38)$$

$$\mu_7^{ab} = (x^a \lambda_7^b + x^b \lambda_7^a) r^2 - 4x^a x^b \lambda_7^c x^c + \delta^{ab} \lambda_7^c x^c r^2, \quad (39)$$

$$\mu_8^{ab} = 2(x^a \varepsilon^{bcd} + x^b \varepsilon^{acd}) \lambda_8^{dn} x^c x^n - (\varepsilon^{ack} \lambda_8^{bk} + \varepsilon^{bck} \lambda_8^{ak}) x^c r^2, \quad (40)$$

$$\mu_9^{ab} = \lambda_9^{ab} r^4 - 2(x^a \lambda_9^{bc} + x^b \lambda_9^{ac}) x^c r^2 + (4x^a x^b + \delta^{ab} r^2) \lambda_9^{cd} x^c x^d + \delta^{ab} \lambda_9^{cd} x^c x^d r^2 \quad (41)$$

where  $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ ,  $\lambda_n^a$  and  $\lambda_n^{ab}$  are arbitrary parameters, satisfying the conditions  $\lambda_n^{ab} = \lambda_n^{ba}$ ,  $\lambda_n^{bb} = 0$ .

Thus we have to search for solutions of the determining equations (10) where  $\mu^{ab}$  are linear combinations of tensors (12), (33)-(41).

Formulae (12) and (33)-(41) include an arbitrary function  $g(\mathbf{x})$  and 35 arbitrary parameters  $\lambda_n^a$  and  $\lambda_n^{ab}$ ,  $a, b = 1, 2, \dots, 9$ . In addition, we have eight coefficients which appear in an arbitrary linear combination of tensors (12), and three more unknown function, i.e.,  $f$ ,  $V$  and  $\eta$ . Thus the problem of the complete classification of the 3d PDM systems admitting second order integrals of motion looks to be huge. However, our strategy is to solve it step by step for the systems admitting three, two and one parametric Lie groups, and, finally, for the systems which do not admit any Lie symmetry. The first two steps have been already done in papers [35] and [39]. The third step is the subject of the current paper.

We are studying the PDM systems which are invariant with respect to rotations around the fixed axis (say, the third one). In accordance with (3) the corresponding Hamiltonian (2) is reduced to the following form:

$$H = p_a f(\tilde{r}, x_3) p_a + V(\tilde{r}, x_3) \quad (42)$$

where  $\tilde{r} = \sqrt{x_1^2 + x_2^2}$ .

As it was noted in Section 3 the equivalence group of equation (1) is reduced to  $\text{SO}(2,1) \otimes \text{E}(1)$  provided the related Hamiltonian (2) has the reduced form (42). The corresponding infinitesimal operators are presented in (25). In addition, there is the discrete equivalence transformations (22), (26) and (27).

However, the symmetry fixed above makes it possible to decouple the second order integrals of motion to the following three subclasses: scalars, vectors and second rank tensors with respect to rotations around the third coordinate axis. We will consider them consequently.

## 6.1 Scalar integrals of motion

Since the searched integrals of motion include bilinear combinations of the generators  $S_{\mu\nu}$  of the extended Lorentz group  $SO(1,4)$ , it is relatively easy to select such of them which commute with  $S_{12} = L_3$  and so are scalars with respect to the rotations. First, all  $S_{\mu\nu}$  with  $\mu, \nu \neq 1$  and 2 commute with  $S_{12}$  and so their bilinear combinations are scalars. There exactly three such  $S_{\mu\nu}$ , namely,  $S_{04}, S_{03}$  and  $S_{43}$ . Surely, this list can be added by  $S_{12}$ . Thus we have indicated the following scalars:

$$\begin{aligned} & S_{04}^2, S_{03}^2, S_{34}^2, S_{12}S_{04}, S_{12}S_{03}, S_{12}S_{34}, \\ & S_{34}S_{04} + S_{04}S_{34}, S_{03}S_{04} + S_{04}S_{03}, S_{43}S_{03} + S_{03}S_{43}. \end{aligned} \quad (43)$$

The above presented list does not include the squared generator  $S_{12}$  which commutes with all the considered Hamiltonians by definition.

One more collection of scalars which includes the sums w.r.t. the repeating indices  $n = 1, 2$  and  $m = 1, 2$  is presented in the following formulae:

$$S_{na}S_{nb} + S_{nb}S_{na} \quad (44)$$

and

$$\varepsilon_{3nm}(S_{na}S_{mb} + S_{mb}S_{na}) \quad (45)$$

where  $a, b = 0, 3, 4$ .

Equation (1) including arbitrary parameters (3) is transparently invariant with respect the inversion of variables  $x_1$  or  $x_2$ . The consequence of this observation is that its scalar integrals of motion can be decoupled to two linearly independent parts including proper scalars and pseudo scalars. The proper scalars are collected in (43) and (44) while the pseudo scalars are present in (45).

The generic scalar integral of motion is a linear combination of operators (43), (44), and the term  $P_a g(\vec{r}, x_3) P_a$  (refer to (28)). The pseudo scalar ones are the linear spans of operators (45) added by the same term. Moreover, many terms of this linear combination should be omitted in view of relations (23) and (29)-(32). As a result we will obtain the reduced set of scalars which we write in terms of operators (21):

$$P_3 L_3, D L_3, \{L_3, K_3\}, \quad (46)$$

$$P_3^2, \{P_3, K_3\}, D^2, K_3^2, \{P_3, D\}, \{K_3, D\}. \quad (47)$$

Notice that equation (44) collects the pseudo scalars while the proper scalars are presented in (47).

One more way to specify the scalar integrals of motion is to use the formal expressions for Killing tensors given by relations (12)-(41). Such integrals are generated by the mentioned tensors in two cases: when the only nonzero parameters  $\lambda_n^a$  and  $\lambda_n^{ab}$  correspond to  $a = 3$  and  $a = b = 3$ . One more possibility is  $\lambda^{11} = \lambda^{22} \neq 0$ , but it is reduced to the previous one in

view to the traceless condition for constant tensors  $\lambda_n^{ab}$ . Going over the mentioned values of arbitrary parameters we recover relations (46) and (47).

Let us specify the inequivalent linear combinations of pseudo scalars (46). First we note that all of them are products of operator  $L_3$  and linear combinations of the following operators:

$$S_{04} = D, \quad S_{03} = \frac{1}{2}(K_3 + P_3), \quad S_{43} = \frac{1}{2}(K_3 - P_3) \quad (48)$$

which satisfy the following relations

$$[S_{04}, S_{03}] = -iS_{43}, \quad [S_{04}, S_{43}] = -iS_{03}, \quad [S_{04}, S_{43}] = -iS_{04} \quad (49)$$

and so form a basis of Lie algebra  $so(1,2)$ . Moreover, operators (48) commute with  $L_3$ . It means that the number of inequivalent pseudoscalars which are nothing but the products of linear combinations of operators (48) with operator  $L_3$  is equal to the number of inequivalent subalgebras of algebra  $so(1,2)$ , since just these subalgebras generate our integrals of motion.

The inequivalent subalgebras of algebra  $so(1,2)$  are one dimensional and include the following basis elements:

$$S_{03}, \quad S_{43}, \quad S_{03} \pm S_{43}. \quad (50)$$

Moreover, up to discrete transformation (22) which changes the sign of  $S_{03}$  we can restrict ourselves to the positive sign in the last term in (50). As a result we obtain the following inequivalent symmetries (28) which we present together with the related Killing tensors (33)-(41):

$$\hat{Q}_1 = P_3 L_3 + P_a \tilde{q}_1(\tilde{r}, x_3) P_a, \quad \mu^{ab} = \mu_3^{ab} \quad (51)$$

and

$$\hat{Q}_2 = (K_3 + P_3) L_3 + P_a \tilde{q}_2(\tilde{r}, x_3) P_a, \quad \mu^{ab} = \mu_3^{ab} + \mu_8^{ab}, \quad (52)$$

$$\hat{Q}_3 = (K_3 - P_3) L_3 + P_a \tilde{q}_3(\tilde{r}, x_3) P_a, \quad \mu^{ab} = \mu_3^{ab} - \mu_8^{ab} \quad (53)$$

we remind that the only nonzero parameters in tensors  $\mu_3^{ab}$  and  $\mu_8^{ab}$  are  $\lambda_3^{33}$  and  $\lambda_8^{33}$ .

Let us specify one more pseudo scalar operator

$$\hat{Q}_4 = \{D, L_3\} + P_a \tilde{q}_4(\tilde{r}, x_3) P_a, \quad \mu^{ab} = \mu_4^{ab}, \quad \lambda_4^1 \neq 0. \quad (54)$$

Operator (54) is equivalent to (52) and can be ignored in the analysis of integrable PDM systems. However, we cannot ignore it in the case of superintegrable systems.

The specific arguments of functions  $q(\cdot)$  in (70) are caused by the requested symmetry of (70) with respect to rotations around the third coordinate axis.

The next task is to specify all inequivalent proper scalars. In accordance with (46) and (49) they belong to the enveloping algebra of algebra  $so(1,2)$ . Moreover, it follows from the first of equations (30) that the related Casimir operator of algebra  $so(1,2)$  takes the following form:

$$S_{03}^2 + S_{04}^2 - S_{43}^2 = P_a \tilde{r}^2 P_a - L_3^2. \quad (55)$$

Thus the considered integrals of motion are linear combination of basic elements of the mentioned enveloping algebra:

$$Q = \sum c_\alpha Q_\alpha \quad (56)$$

where  $\alpha = 1, \dots, 6$ ,

$$\begin{aligned} Q_1 &= (S_{03}^2 - S_{43}^2), \quad Q_2 = (S_{03}^2 + S_{43}^2), \quad Q_3 = \{S_{03}, S_{43}\}, \\ Q_4 &= \{S_{03}, S_{04}\}, \quad Q_5 = \{S_{04}, S_{43}\}, \quad Q_6 = S_{04}^2 - S_{03}^2 \end{aligned} \quad (57)$$

and  $c_1, \dots, c_6$  are real constants.

Notice that relation (55) can be rewritten in terms of operators  $Q_1, Q_2$  and  $Q_6$  in the following way:

$$Q_1 + 3Q_2 + 2Q_6 = 2(P_a \tilde{r}^2 P_a - L_3^2). \quad (58)$$

The l.h.s. of relation (58) includes the symmetry operator  $L_3^2$  which commutes with Hamiltonian (2) by definition and the term  $P_a \tilde{r}^2 P_a$  which can be included to the last term of the generic integral of motion (28). It means that operators  $Q_1, Q_2$  and  $Q_6$  can be treated as linearly dependent and so one of the coefficients  $c_1, c_2$  and  $c_3$  can be nullified without loss of generality.

The next step in the simplification of expression (56) can be made using the equivalence transformations generated by the Lie group whose Lie algebra is a linear span of base elements (48). The transformations generated by  $S_{04}$  are the Lorentz transformations which look as:

$$Q_2 \rightarrow Q_2 \cosh(2\lambda) + Q_3 \sinh(2\lambda), \quad (59)$$

$$Q_3 \rightarrow Q_3 \cosh(2\lambda) + Q_2 \sinh(2\lambda),$$

$$Q_4 \rightarrow Q_4 \cosh(\lambda) + Q_5 \sinh(\lambda) \quad (60)$$

$$Q_5 \rightarrow Q_4 \cosh(\lambda) + Q_5 \sinh(\lambda).$$

In addition,  $Q_6$  and  $Q_2$  are transformed to their linear combinations, but such transformations can be ignored in view of relation (58).

Let us use transformations (59) to specify the inequivalent versions of coefficients  $c_2$  and  $c_3$  present in formula (56). For  $c_2^2 > c_3^2$  or  $c_2^2 < c_3^2$  we can nullify  $c_3$  or  $c_2$  correspondingly, while for the special case  $c_4 = \pm c_5$  we can restrict ourselves to the case  $c_4 = -c_5$  up to the discrete equivalence transformations (22). So the inequivalent versions of the pairs  $(c_4, c_5)$  are:

$$c_2 \neq 0, c_3 = 0; \quad c_2 = 0, c_3 \neq 0; \quad c_2 = -c_3. \quad (61)$$

Whenever the last version presented in (61) is true we can use transformations (60) to specify the inequivalent versions of constants  $c_4$  and  $c_5$  which are analogous to the ones presented in (61):

$$c_4 \neq 0, c_5 = 0; \quad c_4 = 0, c_5 \neq 0; \quad c_4 = -c_5 \quad (62)$$

provided  $c_2 = -c_3$ .

The next step is to use the rotation like transformation generated by  $S_{43}$  which look as follows:

$$Q_3 \rightarrow Q_3 \cos(\omega) - Q_5 \sin(\omega), \quad (63)$$

$$Q_5 \rightarrow Q_5 \cos(\omega) + Q_3 \sin(\omega),$$

$$Q_6 \rightarrow Q_6 \cos(2\omega) - Q_4 \sin(2\omega), \quad (64)$$

$$Q_4 \rightarrow Q_4 \cos(2\omega) + Q_6 \sin(2\omega)$$

Let at least one of the constants  $c_3, c_3$  is nontrivial than transformations (63) can be used to transform them to the following form:

$$c_5 = 0 \quad c_3 \neq 0. \quad (65)$$

Alternatively, both  $c_5$  and  $c_3$  can be equal to zero.

Summarising the results presented in (61), (62) and (65) and taking into account that in view of relation (58) coefficients  $c_1, c_2, c_3$  can be treated as linearly dependent we come to the following inequivalent versions of operators (56) which we present together with the related Killing tensors:

$$Q_1 = P_3^2 + P_a q_1(\tilde{r}, x_3) P_a + \eta^1, \quad \mu^{ab} = \mu_1^{ab}, \quad (66)$$

$$Q_2 = \{P_3, D\} + P_a q_2(\tilde{r}, x_3) P_a + \eta^2, \quad \mu^{ab} = \mu_2^{ab}, \quad (67)$$

$$Q_3 = \{P_3, K_3\} + P_a q_3(\tilde{r}, x_3) P_a, \quad \mu^{ab} = \mu_6^{ab} + \eta^3, \quad (68)$$

$$Q_4 = P_3^2 \pm K_3^2 + P_a q_4(\tilde{r}, x_3) P_a + \eta^4, \quad \mu^{ab} = \mu_1^{ab} \pm \mu_9^{ab}, \quad (69)$$

$$Q_5 = (K_3 \pm P_3)^2 + P_a q_5(\tilde{r}, x_3) P_a + \eta^5, \quad \mu^{ab} = \mu_1^{ab} \pm 2\mu_6^{ab} + \mu_9^{ab}, \quad (70)$$

$$Q_6 = \{P_3, K_3 \pm P_3\} + P_a q_6(\tilde{r}, x_3) P_a + \eta^6, \quad \mu^{ab} = \mu_6^{ab} \pm \mu_1^{ab}, \quad (71)$$

$$Q_7 = K_3^2 \pm P_3^2 + 2n\{K_3, P_3\} + P_a q_7(\tilde{r}, x_3) P_a + \eta^7, \quad \mu^{ab} = \mu_9^{ab} - \mu_1^{ab} + 2n\mu_6^{ab}, \quad (72)$$

$$Q_8 = \{D, (K_3 \pm P_3)\} + P_a q_8(\tilde{r}, x_3) P_a + \eta^8, \quad \mu^{ab} = \mu_7^{ab} \pm \mu_1^{ab} \quad (73)$$

and nonzero parameters in tensors  $\mu_1^{ab}, \mu_2^{ab}, \mu_6^{ab}, \mu_7^{ab}$  and  $\mu_9^{ab}$  are  $\lambda_1^{33}, \lambda_2^{33}, \lambda_6^{33}, \lambda_7^3$  and  $\lambda_9^{33}$ .

Thus to classify the scalar integrals of motion it is sufficient to solve the determining equations (10) and (11) where  $\mu^{ab}$  are the Killing tensors fixed in (66) - (73).

## 6.2 Pseudo scalar integrals of motion

Let us search for solutions of the above defined determining equations for pseudo scalar integrals of motion.

Let us start with the pseudo scalar integrals of motion fixed in (51). The related determining equations (14) are reduced to the following form:

$$x_2 \partial_3 M = \partial_1 N, \quad x_1 \partial_3 M = -\partial_2 N, \quad \partial_3 N = 0 \quad (74)$$

and are solved by the following functions:

$$M = \frac{\nu x_3 + F(\tilde{r})}{\tilde{r}^2}, \quad N = \frac{\nu}{2} \varphi \quad (75)$$

where  $\varphi = \arctan\left(\frac{x_2}{x_1}\right)$  is the Euler angle,  $\nu$  is the integration constant, and  $F(\tilde{r})$  is an arbitrary function of  $\tilde{r} = \sqrt{x_1^2 + x_2^2}$ . The corresponding functions  $\tilde{M}$  and  $\tilde{N}$  which generate potential  $V$  and function  $\eta$  are solutions of the same equation and so have the same generic forms as given in (75), but in general with different parameter  $\nu$  and different function  $F(\tilde{r})$ . Substituting the obtained results into equations (18) we come to functions  $f$  and  $V$  presented in Item 1 of Table 1.

The next pseudoscalar operator which we consider is fixed by relations (52) and (53). The corresponding determining equations (14) for  $M = M(\tilde{r}^2, x_3)$  and  $N = N(\tilde{r}^2, x_3)$  are reduced to the form:

$$\partial_3 N = 0, \quad \partial_4 N = 0, \quad (76)$$

$$\partial_\varphi N = \tilde{r}^2(2x_3\tilde{r}^2\partial_4 M + 2x_3 M) - \frac{1}{2}(\tilde{r}^2 - x_3^2 \pm 1)\partial_3 M \quad (77)$$

where  $\partial_3 = \frac{\partial}{\partial x_3}$ ,  $\partial_4 = \frac{\partial}{\partial x_4}$ ,  $x_4 = \tilde{r}^2$ .

In accordance with (76) function  $N$  depends on  $\varphi$  only. Since  $M$  by definition does not depend on this variable, to solve equation (77)  $N$  should be a linear function, i.e.,  $N = c\varphi$ . Then equation (77) is reduced to the following one:

$$2x_3\tilde{r}^2\partial_4 M + 2x_3 M - \frac{1}{2}((\tilde{r}^2 - x_3^2 \pm 1)\partial_3 M) = c. \quad (78)$$

Equation (78) is easily integrable. Its solutions are presented in Items 2 and 3 of Table 1. Notice that these solutions are qualitative different for different signs before 1 in the formulae presented above. And these signs are the same as signs for  $P_3$  in equations (52) and (53).

Let us fix also the PDM system admitting symmetry  $DL_3 + \dots$  which is generated by the Killing tensor (37) with the only nonzero parameter  $\lambda_4^3$ . This symmetry is equivalent to the symmetry  $L_3(P_3 + K_3) + \dots$  considered above, see equation (51). However, its presentation will be useful in searching for the systems admitting more than one second order integrals of motion. We will not present the calculation details but give the corresponding equations (14):

$$\begin{aligned} x_4 x_3 \partial_3 M + 2x_4 \partial_4 M + 2M + \partial_\varphi N &= 0, \\ \partial_3 N = 0, \quad \partial_4 N = 0, \quad \partial_{\varphi\varphi} N &= 0 \end{aligned} \quad (79)$$

and their solutions:

$$M = \frac{F(\theta) - 2\nu \ln(\tilde{r})}{\tilde{r}^2} \quad (80)$$

where  $F(\theta)$  and  $G(\theta)$  are arbitrary functions,  $\varphi$  and  $\theta$  are the Euler angle,  $\mu$  and  $\nu$  are arbitrary parameters.

Just these functions together with the corresponding integrals of motion are presented in Item 4 of Table 1.

Thus we have found all inequivalent systems with position dependent mass which admit pseudo scalar integrals of motion. They are defined up to pairs of arbitrary functions. The related Hamiltonians commute also with the third component of angular momentum and so the found systems are integrable. We will see that for some particular arbitrary functions these systems are superintegrable.

### 6.3 True scalar integrals of motion

Consider now the integrals of motion which are invariant with respect to the space reflections. Their generic form is given by formulae (66)–(72).

Let us represent the corresponding determining equations. They include functions  $M$  and  $N$  which depend on  $\tilde{r}$  and  $x_3$ , but in some cases it is reasonable to treat them as functions of  $r$  and  $x_3$ . To unify the representation we will use the notations  $\tilde{r}^2 = x_4$  and  $r^2 = x_5$ .

Substituting the Killing tensors fixed in (66)-(72) into (14) we come to the following equations:

$$\text{For } Q_1 : \partial_{\tilde{r}}N = 0, \partial_3(M + N) = 0; \quad (81)$$

$$\text{For } Q_2 : \begin{aligned} \partial_3M + 2\partial_5N &= 0, \\ 2\partial_5(x_5M + x_3N) + \partial_3(x_3M + N) - 2M &= 0; \end{aligned} \quad (82)$$

$$\text{For } Q_3 : \begin{aligned} 2\partial_4(x_3^2M + N) - x_3p_3M &= 0, \\ \partial_3(\tilde{r}^2M + N) - 2x_3\tilde{r}^2\partial_4M &= 0; \end{aligned} \quad (83)$$

$$\text{For } Q_4 : \begin{aligned} \partial_4(N + 4x_3^2\tilde{r}^2M) - W_{\pm}\partial_3M &= 0, \\ \partial_3(N + W_{\pm}^2M) - 4x_3\tilde{r}^2W_{\pm}\partial_4M &= 0, \\ W_{\pm} = \tilde{r}^2 \pm 1 - x_3^2 = x_4 - x_3^2 \pm 1; \end{aligned} \quad (84)$$

$$\text{For } Q_5 : \begin{aligned} \partial_4(N + 4(x_3^2 \pm 1)\tilde{r}^2M) - x_3(W_{\pm})\partial_3M &= 0, \\ \partial_3N + ((W_{\pm})^2 + 4\tilde{r}^2)\partial_3M - 4\partial_4(x_3\tilde{r}^2M(W \mp 2)); \end{aligned} \quad (85)$$

$$\text{For } Q_6 : \begin{aligned} (2x_3^2\partial_4M - x_3\partial_3)M &= \partial_4N, \\ (x_4^2 - 1)(\partial_3 - 2x_3x_4\partial_4)M &= \partial_3N; \end{aligned} \quad (86)$$

$$\text{For } Q_7 : \begin{aligned} x_3(x_4 - x_3^2 + n)\partial_3M + 2(N + 2 - 2x_3^2)\partial_4(x_4M) &= \partial_4N, \\ (x_3^4 - 2(x_4 + n)x_3^2 + x_4^2 - 4x_4 \pm 1)\partial_3M \\ - 4x_3(x_4 - x_3^2 + n)\partial_4x_4M &= -\partial_3N, \end{aligned} \quad (87)$$

$$\text{For } Q_8 : \begin{aligned} (3x_3^2 - x_4 \pm 1)\partial_3M + 8x_3\partial_4(x_4M) &= 2\partial_4N, \\ 2(x_4 - 3x_3^2 \pm 1)(x_4\partial_4M + \partial_3(x_3M)) &= -\partial_3N \end{aligned} \quad (88)$$

The systems (81) and (83) are easy solvable. They are solved by the following functions:

$$M = F(\tilde{r}) + G(x_3), \quad N = -G(x_3)$$

and

$$M = \frac{F(\theta) + G(R)}{r^2}, \quad N = -F(\theta)$$

where  $F(\cdot)$  and  $\theta$  are arbitrary functions,  $\theta = \arctan\left(\frac{\tilde{r}}{x_3}\right)$  is the Euler angle. These solutions generate the inverse masses and potentials represented in Items 5 and 6 of Table 1

Equations (84) are a bit more complicated. Excluding unknown variable  $N$  we obtain the following second order equation for  $M$ :

$$\partial_{33}M - 4\partial_{55}(x_5M) + 2\partial_5M = 0. \quad (89)$$

The obtained partial differential equation with variable coefficients appears to be exactly solvable. To discover its exact solutions it is reasonable to represent them as

$$M = \frac{P}{\sqrt{x_5}} = \frac{P}{r} \quad (90)$$

where  $P$  are polynomials in  $r$  and  $x_3$ .

The only first order polynomial satisfying (89), (89) is  $c_1x_3$ , the second order polynomial is  $c_2rx_3 + c_3(r^2 + x_3^2)$ , the third order polynomial is  $c_4(x_3^3 + 3x_3r^2) + c_5(r^3 + 3rx_3^2)$ , etc. In this way we can find the infinite (but countable) set of exact solutions for  $M$ . A more difficult step

is to recognize the fact that such polynomials are only particular case of the generic solution  $P = F(r + x_3) + G(r - x_3)$  were  $F(\cdot)$  and  $G(\cdot)$  are arbitrary functions of the arguments fixed in the brackets. It means that rather complicated second order equation with variable coefficients presented in (89) can be reduced to the D'Alambert equation if we represent the dependent variable in form (90) and change the independent variables  $(x_3, x_5)$  to  $(x_3, \sqrt{x_5})$ .

The corresponding inverse masses and potentials are presented in Item 7 of Table 1.

The remaining tasks, i.e., the constructions of exact solutions for the determining equations (84)-(88) appear to be much more difficult problems which, however, are solvable.

Consider equations (84). Excluding variable  $N$  we obtain the following compatibility conditions for this system:

$$\begin{aligned} & -(W_{\pm}^2 \mp \tilde{r}^3 x_3^2) \partial_{34} M + x_3 W_{\pm} (4x_4 \partial_4 M - \partial_{33} M) \\ & + (20x_4 - 8x_3(x_3 \mp 1)) \partial_4 M + (3W_{\pm} + 6x_3^2) \partial_3 M + 12x_3 M. \end{aligned} \quad (91)$$

This rather complicated partial differential equation of second order can be solved by the following trick. Let us choose new independent variables. To find the first of them we solve equations (84) for  $N = 0$  and obtain the following subclass of solutions for our problem:

$$M = \frac{G\left(\frac{r^2 \pm 1}{\tilde{r}}\right)}{(r^2 \pm 1)^2 \mp 4\tilde{r}^2}. \quad (92)$$

This solution is valid for any arbitrary function  $G\left(\frac{r^2 \pm 1}{\tilde{r}}\right)$ , in particular, for  $G\left(\frac{r^2 \pm 1}{\tilde{r}}\right) = 1$ , when

$$M = \frac{1}{(r^2 \pm 1)^2 \mp 4\tilde{r}^2}. \quad (93)$$

Thus it is reasonable to search for solutions with non-trivial  $N$  in the form:

$$M = \frac{F(r, x_3)}{(r^2 \pm 1)^2 \mp 4\tilde{r}^2}. \quad (94)$$

Substituting this form into (84) we immediately recognize that the latter equations turn to the identities provided  $N = -F(r, x_3)$  and

$$F = F\left(\frac{r^2 \mp 1}{x_3}\right). \quad (95)$$

In this way we come to the solutions presented in Item 8 of Table 1.

In complete analogy with the above one can solve equations (85). The inverse masses and potentials generated by solutions of these equations are presented in Item 9 of Table 1.

Let us consider equations (86). Excluding  $N$  we obtain the following compatibility condition for this system:

$$((x_4 - x_3^2 - 1) \partial_3 \partial_4 + 4x_3 x_4 \partial_4 \partial_4 - x_3 \partial_3 \partial_3 - 3\partial_3 + 8x_3 \partial_4) M(x_4, x_3) = 0. \quad (96)$$

Like (89) equation (96) can be reduced to the D'Alambert equation if we chose the following new independent and dependent variables:

$$x = r^2 \pm 1, \quad y = x_3^2, \quad \tilde{M}(x, y) = \sqrt{x^2 \mp 4y} M(\tilde{r}, x_3). \quad (97)$$

As a result we come to the following generic solution for (96):

$$M = \frac{F(x_-) + G(\tilde{x}_-)}{\sqrt{(r^2 \pm 1)^2 \mp 4x_3^2}} \quad (98)$$

where  $x_- = \sqrt{x^2 \mp 4y} + x$ ,  $\tilde{x}_- = \sqrt{x^2 \pm 4y} - x$ ,  $F(\cdot)$  and  $G(\cdot)$  are arbitrary functions. The corresponding potential  $V$  (17) looks as:

$$V = \frac{\tilde{F}(x_-) + \tilde{G}(\tilde{x}_-)}{F(x_-) + G(\tilde{x}_-)}. \quad (99)$$

For any fixed  $F(x_-)$  and  $G(\tilde{x}_-)$  we can solve equations (86) and find functions  $N = N_{F,G}$  corresponding to  $M$  defined in (98). Unfortunately, it is seemed be impossible to represent functions  $N_{F,G}$  in closed form for  $F$  and  $G(\tilde{x}_-)$  arbitrary. However, we can do it at least for some rather extended classes of  $F(\cdot)$  and  $G$ . In particular it is the case if  $F(x_-) + G(\tilde{x}_-)$  is a homogeneous function of  $x_3^2$  and  $y$ , say, for

$$F(x_-) + G(\tilde{x}_-) = x_-^n + (-1)^{n+1} \tilde{x}_-^n = \Phi_n. \quad (100)$$

Substituting (100) into (97) and integrating the corresponding equations (14) we find the related functions  $M$  and  $N$  in closed form for  $n$  arbitrary:

$$M = M_n = \frac{\Phi_n}{\sqrt{(r^2 - 1)^2 + 4x_3^2}}, \quad (101)$$

$$N = N_n = 2^n x_3^2 M_{n-1}.$$

Thus we found the infinite (but countable) set of solutions of the determining equations (14) which generate integrals of motion (71). Let us represent explicitly some of them:

$$M_1 = r^2 - 1, \quad M_{-1} = \frac{1}{x_3^2}$$

$$M_2 = (r^2 - 1)^2 + x_3^2,$$

$$M_3 = (r^2 - 1)((r^2 - 1)^2 + 2x_3^2)$$

$$M_4 = (r^2 - 1)^4 + 3(r^2 - 1)x_3^4 + x_3^6,$$

$$M_5 = (r^2 - 1)^5 + 4(r^2 - 1)^3 x_3^2 + (r^2 - 1)x_3^4,$$

...

$$M_n = \sum_{m \leq n} \frac{(n-m)! x^{n-2m} x_3^{2m}}{(n-2m)! 2^m m!}.$$

Two more particular solutions are:

$$M = \frac{1}{\tilde{r}^2} \quad \text{and} \quad M = \frac{x_3^2 + 1}{\tilde{r}^4}. \quad (103)$$

Notice that linear combinations of solutions (101), (102) and (103) also solve equations (14), see Item 10 of Table 1.

The last class of symmetries which we are supposed to study is represented in equation (72). The corresponding determining equations (87) are the most complicated, but in many aspects they are analogous to ones requested for the systems admitting integrals of motion of type (71).

Excluding  $N$  we come to the following compatibility condition for the system (87):

$$\begin{aligned} & ((x_4 - x_3^2 + n)^2 - n^2 \pm 1 - 4x_3^2x_4)\partial_3\partial_4 - (n + x_4 - x_3^2)(x_3\partial_3\partial_3 - 4x_3x_4\partial_4\partial_4) \\ & + 3(n + x_4 - 3x_3^2r)\partial_3 - (8n - 8x_3^2 + 20x_4)x_3\partial_4 - 12x_3M = 0. \end{aligned} \quad (104)$$

This frighteningly looking equation appears to be equivalent to the d'Alembert one if we choose the new independent and dependent variables

$$\begin{aligned} y &= \frac{1}{x_4}(\sqrt{z_{\pm}^2 + 2x_3^2x_4} + z_{\pm}), \quad \tilde{y} = \frac{1}{x_4}(\sqrt{z_{\pm}^2 + 2x_3^2x_4} - z_{\pm}), \\ z_{\pm} &= \frac{x_4^2 \pm 1 + 2n(x_4 - x_3^2)}{2\sqrt{2(n^2 \mp 1)}}, \quad 1 - n^2 \leq 0, \quad \hat{M}(y, \tilde{y}) = \sqrt{z_{\pm}^2 + 2x_3^2x_4}M(\tilde{r}, x_3). \end{aligned}$$

As a result equation (104) takes the canonical form

$$\partial_y\partial_{\tilde{y}}\hat{M} = 0 \quad (105)$$

and so the related mass function has the following form:

$$M = \frac{F(y) + G(\tilde{y})}{x_4\sqrt{z_{\pm}^2 + 2x_3^2x_4}}$$

with arbitrary functions  $F(\cdot)$  and  $G(\cdot)$ .

For any particular functions  $F(\cdot)$  and  $F_2(\cdot)$  we can find the corresponding functions  $N$  by direct integration of equations (87). Moreover, for some rather extended classes of the arbitrary functions it is possible to present the related functions  $N$  in closed form like it was done in equations (100)-(103). In particular we can set:

$$M = M_m = \frac{1}{x_4\sqrt{z_{\pm}^2 + 2x_3^2x_4}}(y^m + (-1)^{m+1}\tilde{y}^m) \quad (106)$$

where  $n$  are natural numbers. Then the corresponding functions  $N = N_m$  can be found in the closed form for arbitrary  $m$  by direct integration of equations (87):

$$N_m = 2^{m-1}x_3^2M_{m-1}. \quad (107)$$

We again have a countable set of exact solutions whose linear combinations  $\sum_m c_m M_m$  are solutions also. The corresponding potentials looks as follows:

$$V = \frac{\sum_m \tilde{c}_m (y^m + (-1)^{m+1}\tilde{y}^m)}{\sum_m c_m (y^m + (-1)^{m+1}\tilde{y}^m)}. \quad (108)$$

The obtained in this way solutions are represented in Items 11 and 12 of Table 1. Solutions corresponding to  $n^2 - 1 < 0$  which are represented in Items 13 and 14 of the same table can be obtained in analogous way.

Thus we find all inequivalent PDM systems which, in addition to the cylindric symmetry, admit at least one second order scalar integral of motion and so are integrable. The inverse masses and potentials of these systems are defined up to two arbitrary functions for pseudo scalar integrals of motion and up to four arbitrary functions if integrals of motion are true scalars. For some particular classes of the mentioned arbitrary functions the number of the admitted scalar integrals of motion can be extended as it is indicated in Tables 3-5.

Table 1. Inverse masses, potentials and scalar integrals of motion for integrable systems

No	$f$	$V$	Integrals of motion
1	$\frac{\tilde{r}^2}{c_1 x_3 + F(\tilde{r})}$	$\frac{c_2 x_3 + G(\tilde{r})}{c_1 x_3 + F(\tilde{r})}$	$P_3 L_3 + \frac{c_1}{2}(\varphi \cdot H) - \frac{c_2}{2}\varphi$
2	$\frac{\tilde{r}^2}{F\left(\frac{r^2-1}{\tilde{r}}\right) + c_1 \operatorname{arctanh}\left(\frac{r^2+1}{2x_3}\right)}$	$\frac{G\left(\frac{r^2-1}{\tilde{r}}\right) + c_2 \operatorname{arctanh}\left(\frac{r^2+1}{2x_3}\right)}{F\left(\frac{r^2-1}{\tilde{r}}\right) + c_1 \operatorname{arctanh}\left(\frac{r^2+1}{2x_3}\right)}$	$\{L_3, (K_3 + P_3)\}$ $-2c_1(\varphi \cdot H) + 2c_2\varphi$
3	$\frac{\tilde{r}^2}{c_1 \operatorname{arctan}\left(\frac{r^2-1}{2x_3}\right) + F\left(\frac{r^2+1}{\tilde{r}}\right)}$	$\frac{c_2 \operatorname{arctan}\left(\frac{r^2-1}{2x_3}\right) + G\left(\frac{r^2+1}{\tilde{r}}\right)}{c_1 \operatorname{arctan}\left(\frac{r^2-1}{2x_3}\right) + F\left(\frac{r^2+1}{\tilde{r}}\right)}$	$\{L_3, (K_3 - P_3)\}$ $-2c_1(\varphi \cdot H) + 2c_2\varphi$
4	$\frac{\tilde{r}^2}{F(\theta) - 2c_1 \ln(\tilde{r})}$	$\frac{G(\theta) - 2c_2 \ln(\tilde{r})}{F(\theta) - 2c_1 \ln(\tilde{r})}$	$DL_3 + c_1(\varphi \cdot H) - c_2\varphi$
5	$\frac{1}{F(\tilde{r}) + G(x_3)}$	$\frac{\tilde{F}(\tilde{r}) + \tilde{G}(x_3)}{F(\tilde{r}) + G(x_3)}$	$P_3^2 - (G(x_3) \cdot H) + \tilde{G}(x_3)$
6	$\frac{r^2}{F(\theta) + G(r)}$	$\frac{\tilde{F}(\theta) + \tilde{G}(r)}{F(\theta) + G(r)}$	$L_1^2 + L_2^2 - (F(\theta) \cdot H) + \tilde{F}(\theta)$
7	$\frac{r^2}{F(r+x_3) + G(r-x_3)}$	$\frac{\tilde{F}(r+x_3) + \tilde{G}(r-x_3)}{F(r+x_3) + G(r-x_3)}$	$\{P_3, D\} - P_n x_3 P_n$ $-((F(r+x_3) - G(r-x_3)) \cdot H)$ $+ \tilde{F}(r+x_3) - \tilde{G}(r-x_3)$
8	$\frac{F_{\pm}}{F\left(\frac{r^2 \pm 1}{\tilde{r}}\right) - G\left(\frac{r^2 \mp 1}{x_3}\right)}$	$\frac{\tilde{F}\left(\frac{r^2 \pm 1}{\tilde{r}}\right) + \tilde{G}\left(\frac{r^2 \mp 1}{x_3}\right)}{F\left(\frac{r^2 \pm 1}{\tilde{r}}\right) - G\left(\frac{r^2 \mp 1}{x_3}\right)}$	$(K_3 \mp P_3)^2 + \tilde{G}\left(\frac{r^2 \mp 1}{x_3}\right)$ $+ \left(G\left(\frac{r^2 \mp 1}{x_3}\right) \cdot H\right) - \hat{\eta}$
9	$\frac{\sqrt{F_{\pm}}}{F(x_{\pm}) + G(\tilde{x}_{\pm})}$	$\frac{\tilde{F}(x_{\pm}) + \tilde{G}(\tilde{x}_{\pm})}{F_1(x_{\pm}) + F_2(\tilde{x}_{\pm})}$	$\{P_3, (K_3 \pm P_3)\} + P_a x_3^2 P_a$ $+ ((g(F, G) \cdot H) - \eta(\tilde{F}, \tilde{G}))$
10	$\sum_n c_n F_n, F_n = \frac{x_{\pm}^n + (-1)^{n+1} \tilde{x}_{\pm}^n}{F_{\pm}}$	$\frac{\sum_n \tilde{c}_n \tilde{F}_n}{\sum_n c_n F_n}$	$\{P_3, (K_3 \pm P_3)\} + P_a x_3^2 P_a$ $- (x_3^2 \sum_n c_n 2^n F_{n-1} \cdot H)$ $+ x_3^2 \sum_n \tilde{c}_n 2^n F_{n-1}$
11	$\frac{\sqrt{z_{\pm}^2 + 2x_3^2 \tilde{r}^2}}{F(y_{\pm}) + G(\tilde{y}_{\pm})}$	$\frac{\tilde{F}(y_{\pm}) + \tilde{G}(\tilde{y}_{\pm})}{F(y_{\pm}) + G(\tilde{y}_{\pm})}$	$K_3^2 \pm P_3^2 + 2n\{K_3, P_3\} + 4P_a \tilde{r}^2 P_a$ $+ (g(F, G) \cdot H) - \tilde{\eta}(\tilde{F}, \tilde{G}) - \hat{\eta},$ $1 - n^2 \leq 0$
12	$\sum_m c_m \tilde{F}_m, \tilde{F}_m = \frac{y_{\pm}^m + (-1)^{m+1} \tilde{y}_{\pm}^m}{z_{\pm}}$	$\frac{\sum_m \tilde{c}_m \tilde{F}_m}{\sum_m c_m \tilde{F}_m}$	$K_3^2 \pm P_3^2 + 2n\{K_3, P_3\} + 4P_a \tilde{r}^2 P_a$ $- \left(x_3^2 \sum_m c_m 2^{m-1} \tilde{F}_{m-1} \cdot H\right)$ $+ x_3^2 \sum_m \tilde{c}_m 2^{m-1} \tilde{F}_{m-1} - \hat{\eta},$ $1 - n^2 \leq 0$
13	$\frac{\sqrt{\tilde{z}_{\pm}^2 \mp 2x_3^2 \tilde{r}^2}}{F(s_{\pm}) + G(\tilde{s}_{\pm})}$	$\frac{\tilde{F}(s_{\pm}) + \tilde{G}(\tilde{s}_{\pm})}{F(s_{\pm}) + G(\tilde{s}_{\pm})}$	$K_3^2 \pm P_3^2 + 2n\{K_3, P_3\} + 4P_a \tilde{r}^2 P_a$ $+ (g(F, G) \cdot H) - \tilde{\eta}(\tilde{F}, \tilde{G}) - \hat{\eta},$ $n^2 - 1 < 0$
14	$\sum_m c_m \Phi_m, \Phi_m = \frac{s_{\pm}^m + (-1)^{m+1} \tilde{s}_{\pm}^m}{z_{\pm}}$	$\frac{\sum_m \tilde{c}_m \Phi_m}{\sum_m c_m \Phi_m}$	$K_3^2 \pm P_3^2 + 2n\{K_3, P_3\} + 4P_a \tilde{r}^2 P_a$ $- \left(x_3^2 \sum_m c_m 2^{m-1} \Phi_{m-1} \cdot H\right)$ $+ x_3^2 \sum_m \tilde{c}_m 2^{m-1} \Phi_{m-1} - \hat{\eta},$ $n^2 - 1 < 0$

In Table 1 and the following Tables 2, 3, 4  $F(\cdot), G(\cdot), \tilde{F}(\cdot)$  and  $\tilde{G}(\cdot)$  are arbitrary functions,  $g(F, G)$  and  $\eta(\tilde{F}, \tilde{G})$  are solutions of equation (86), (87) with given  $F_1(x), F_2(\tilde{x})$  and  $G_1, G_2$ ,

$c_1, c_2, ..$  are arbitrary real parameters. In addition we use the following notations which also will be used in all tables:

$$\begin{aligned}
F_{\pm} &= (r^2 \pm 1)^2 \mp 4x_3^2, \quad \tilde{F}_{\pm} = (r^2 \pm 1)^2 \pm 4x_3^2, \\
\hat{\eta} &= 3(r^2 - 5x_3^2), \quad G_{\pm} = (\tilde{r}^2 - 2x_3^2)(r^2 \pm 1)^2 + 4x_3^2\tilde{r}^2, \\
z_{\pm} &= \frac{r^4 \pm 1 + 2n(\tilde{r}^2 - x_3^2)}{2\sqrt{2(n^2 \mp 1)}}, \quad -n^2 \leq 1, \\
y_{\pm} &= \frac{1}{\tilde{r}^2} \left( \sqrt{z_{\pm}^2 + 2x_3^2\tilde{r}^2} + z_{\pm} \right), \quad \tilde{y}_{\pm} = \frac{1}{\tilde{r}^2} \left( \sqrt{z_{\pm}^2 + 2x_3^2\tilde{r}^2} - z_{\pm} \right), \\
\tilde{z}_{\pm} &= \frac{r^4 \pm 1 + 2n(\tilde{r}^2 - x_3^2)}{2\sqrt{2(1 \mp n^2)}}, \quad n^2 < 1 \\
s_{\pm} &= \frac{1}{\tilde{r}^2} \left( \sqrt{\tilde{z}_{\pm}^2 \mp 2x_3^2\tilde{r}^2} + \tilde{z}_{\pm} \right), \quad \tilde{s}_{\pm} = \frac{1}{\tilde{r}^2} \left( \sqrt{\tilde{z}_{\pm}^2 \mp 2x_3^2\tilde{r}^2} - \tilde{z}_{\pm} \right), \\
x_{\pm} &= \sqrt{(r^2 \pm 1)^2 \mp x_3^2} + r^2 \pm 1, \quad \tilde{x}_{\pm} = \sqrt{(r^2 \pm 1)^2 \mp x_3^2} - r^2 \mp 1.
\end{aligned} \tag{109}$$

The mysterious term  $\hat{\eta}$  appears as a result of the following identity:

$$K_3^2 = p_a \mu_8^{ab} p_b + \hat{\eta}$$

where  $\mu_9^{ab}$  is the Killing tensor (41) with  $\lambda_9^3 \neq 0$ , and it is the cost paid for our desire to represent the integrals of motion via the anticommutators of operators (21).

## 7 Vector integrals of motion

The systems admitting vector integrals of motion can be classified in the way analogous to one used for the case of scalar integrals. First, using our knowledge of the requested Killing tensors we will find the generic form of second order integrals of motion which transform as a vectors under rotations with respect to the third coordinate axis. Then, using the discrete equivalence transformations we will select the subsets of such integrals with fixed parities. Then, using the continuous equivalence transformations we will specify all inequivalent linear combinations of the Killing tensors which should be considered. Finally we will solve the determining equations generated by such inequivalent Killing tensors.

### 7.1 Generic vector integrals of motion and their parity properties

Vector integrals of motion are generated by linear combinations of Killing tensors (33)-(41) with nonzero parameters  $\lambda_n^\alpha$  and  $\lambda_n^{3\alpha}$ , where  $\alpha = 1, 2, n = 1, \dots, 9$ . The corresponding integrals of motion (28) are linear combinations of operator  $Q^{(0)}$  generated by  $\mu_0^{ab}$ , i.e.,

$$Q^{(0)} = P_a g(\mathbf{x}) P_a \tag{110}$$

and the following operators:

$$\{S_{12}, S_{3\alpha}\}, \{S_{12}, S_{4\alpha}\}, \{S_{12}, S_{0\alpha}\}, \tag{111}$$

$$\{S_{3\alpha}, S_{03}\}, \{S_{3\alpha}, S_{43}\}, \{S_{3\alpha}, S_{04}\}, \{S_{0\alpha}, S_{43}\}, \{S_{4\alpha}, S_{03}\}. \tag{112}$$

Notice that operators (111) are invariant with respect to the reflections  $x_\alpha \rightarrow -x_\alpha$  while operators (112) change their size under this operation. Thus operators (111) and (112) are true and pseudo vectors respectively.

The generic vector integral of motion includes a linear combination of operators (111) and (112). Since Hamiltonians (42) are invariant with respect to the space reflection, such combinations can include either operators (111) or (112), but never both of them. Thus it is necessary to consider the true vector and pseudo vector integrals of motion separately, while their linear combinations can be rearranged in accordance with the following comment.

Let us note that pseudo vectors (111) have the following property: whenever we change the sign of  $x_1$ , the first components of these vectors are not changed but the second components change their sign. Analogously, if we change the sign of  $x_2$ : the second components are kept unchanged while the first components change their signs also. In other words, if  $V_\alpha$  is one of pseudo vectors (111), then  $\tilde{V}_\alpha = \varepsilon_{\alpha\nu} V_\nu$  (were  $\varepsilon_{\alpha\nu}$  is the antisymmetric unit tensor,  $\alpha$  and  $\nu$  take the values 1 or 2) transforms as a true vector under the space reflections.

That is why the true vectors and pseudo vectors the generic forms of the related operator  $Q^{(0)}$  (110) are

$$Q_\alpha^{(0)} = P_a x_\alpha g(\tilde{r}, x_3) P_a \quad (113)$$

and

$$\tilde{Q}_\alpha^{(0)} = P_a \varepsilon_{\alpha\nu} x_\nu g(\tilde{r}, x_3) P_a \quad (114)$$

correspondingly, were  $\varepsilon_{\alpha\nu}$  is the antisymmetric unit tensor,  $\alpha$  and  $\nu$  take the values 1 or 2.

Like in Section 5, we can use the equivalence transformations from group SO(1,2) whose generators are  $S_{03}, S_{04}$  and  $S_{43}$  to simplify the integrals of motion. Let us discuss these possible simplifications.

In accordance with (33)-(41) it is possible to specify five classes of vector integrals of motion, whose coefficients are zero, first, second, third or four order polynomials in  $x_a$  which are fixed in the following equations (115), (116), (117), (118) and (119) correspondingly:

$$Q_\alpha^{(1)} = P_3 P_\alpha + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_1, \quad \mu^{ab} = \mu_1^{ab}, \quad (115)$$

$$Q_\alpha^{(2)} = \{P_\alpha, D\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_2, \quad \mu^{ab} = \mu_2^{ab}, \quad (116)$$

$$Q_\alpha^{(3)} = \varepsilon_{\alpha\nu} \{P_3, L_\nu\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_3, \quad \mu^{ab} = \mu_2^{ab} + \mu_3^{ab}, \quad (117)$$

$$Q_\alpha^{(4)} = \{K_3, P_\alpha\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_4, \quad \mu^{ab} = \mu_4^{ab}, \quad (117)$$

$$Q_\alpha^{(5)} = \varepsilon_{\alpha\nu} \{D, L_\nu\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_5, \quad \mu^{ab} = \mu_4^{ab} + \mu_6^{ab}, \quad (117)$$

$$Q_\alpha^{(6)} = \{D, K_\alpha\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_6, \quad \mu^{ab} = \mu_7^{ab} \quad (118)$$

$$Q_\alpha^{(7)} = \varepsilon_{\alpha\nu} \{K_3, L_\nu\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_7, \quad \mu^{ab} = \mu_7^{ab} + \mu_8^{ab}, \quad (118)$$

$$Q_\alpha^{(8)} = \{K_3, K_\alpha\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}_8, \quad \mu^{ab} = \mu_9^{ab} \quad (119)$$

where  $\tilde{\eta}_n = \tilde{\eta}_n(\tilde{r}, x_3)$  are an unknown functions,  $\mu_1^{ab}, \dots, \mu_8^{ab}$  are tensors (33)-(41) with the following nonzero coefficients:

$$\lambda_n^\alpha, \lambda_m^{3\alpha}, \quad \alpha = 1, 2, \quad n = 2, 4, 6, \quad m = 1, 3, 7, 8, 9. \quad (120)$$

In accordance with (115)- (119), to generate vector integrals of motion we choose the special Killing tensors whose nonzero parameters are fixed in (120). In addition, functions  $\eta$  and  $g$  in (7) and (110) and functions  $N, K$  in (19) should be changed in the following manner:

$$\eta \rightarrow x_\alpha \tilde{\eta}(\tilde{r}, x_3), \quad N \rightarrow x_\alpha \tilde{N}(\tilde{r}, x_3), \quad K \rightarrow x_\alpha \tilde{K}(\tilde{r}, x_3). \quad (121)$$

Let us note that operators  $Q^{(8)}, Q^{(7)}$  and  $Q^{(6)}$  are equivalent to  $Q^{(1)}, Q^{(3)}$  and  $Q^{(2)}$  respectively up to transformation (24) and so it is sufficient to consider only the potential integrals of motion represented in (115)-(117). The generic vector integral of motion is a linear combination of operators (115)-(119). However, such linear combination can be a priori simplified.

Operators (115)-(119) have different invariance properties with respect to transformation  $x_3 \rightarrow -x_3$ . Half part of them including  $Q^{(1)}, Q^{(4)}, Q^{(5)}$  and  $Q^{(8)}$  change their signs together with  $x_3$  while the remaining operators keep their form.

Let us start with operators  $Q^{(1)}, Q^{(4)}, Q^{(5)}, Q^{(8)}$  and consider their linear combination:

$$Q = c_1 Q^{(1)} + c_4 Q^{(4)} + c_5 Q^{(5)} + c_8 Q^{(8)} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}. \quad (122)$$

Let the coefficients  $c_1$  and  $c_8$  are nontrivial, then up to the dilatation transformation  $c_1 = \pm c_8$ . Setting  $c_8 = 2$  we come to the following version of operator (122):

$$Q = \{K_3, K_\alpha\} \pm 2P_3 P_\alpha + c_4 \{K_3, P_\alpha\} + c_5 \varepsilon_{\alpha\nu} D L_\nu + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}. \quad (123)$$

If only one of the coefficients  $c_1, c_8$  is trivial and  $c_4$  is nontrivial then up to transformation (24) and dilatation we can reduce (122) to the following form:

$$Q = \{(K_3 \pm P_3), P_\alpha\} + c_5 \varepsilon_{\alpha\nu} D L_\nu + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}, \quad (124)$$

while the remaining inequivalent versions of operator (122) are

$$Q = P_3 P_\alpha + c_5 \varepsilon_{\alpha\nu} D L_\nu + Q_\alpha^{(0)} + x_\alpha \tilde{\eta} \quad (125)$$

and

$$c_4 \{K_3, P_\alpha\} + c_5 \varepsilon_{\alpha\nu} D L_\nu + Q_\alpha^{(0)} + x_\alpha \tilde{\eta} \quad (126)$$

Analogous speculations with the linear combinations of operators  $Q_2, Q_3, Q_6$  and  $Q_7$  make it possible to specify the following their inequivalent versions:

$$\begin{aligned} Q &= c_1 \{D, (K_\alpha \pm P_\alpha)\} + c_2 \varepsilon_{\alpha\nu} \{L_\nu, (K_3 \pm P_3)\} + Q_\alpha^{(0)} + c_3 \varepsilon_{\alpha\nu} \{L_\nu, P_3\} + x_\alpha \tilde{\eta}, \\ Q &= c_1 \{D, P_\alpha\} + c_2 \varepsilon_{\alpha\nu} \{L_\nu, (K_3 \pm P_3)\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}, \\ Q &= c_1 \{D, (K_\alpha \pm P_\alpha)\} + c_2 \varepsilon_{\alpha\nu} \{L_\nu, P_3\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}, \\ Q &= c_1 \{D, P_\alpha\} + c_2 \varepsilon_{\alpha\nu} \{L_\nu, P_\alpha\} + Q_\alpha^{(0)} + x_\alpha \tilde{\eta}. \end{aligned} \quad (127)$$

Concerning the linear combinations of operators (115) - (119) with different parities with respect to the reflection of  $x_3$  we can note that such combinations of operator  $Q_\alpha^{(1)}$  with  $Q_\alpha^{(2)}$  or  $Q_\alpha^{(3)}$  are equivalent to  $Q_\alpha^{(2)}$  or  $Q_\alpha^{(3)}$  up to the shift transformations. The same is true for the bilinear combinations of  $Q_\alpha^{(2)}$  with  $Q_\alpha^{(5)}$  and  $Q_\alpha^{(4)}$  with  $Q_\alpha^{(7)}$ : they can be reduced to  $Q_\alpha^{(5)}$  and  $Q_\alpha^{(7)}$  respectively.

We will not discuss the remaining linear combinations since the computing experiments show that they do not correspond to nontrivial integrals of motion.

Thus to classify the PDM systems with cylindric symmetry which admit vector integrals of motion we are supposed to solve the determining equations (14), (16) which correspond to the symmetries represented in equations (115) - (117) and (123) - (127).

## 7.2 Selected calculations

The next step is to solve the the determining equations indicated in the above.

For the symmetry specified in (115) the determining equations (14) have especially simple form. Namely, substituting the Killing tensor  $\mu_1^{3\alpha}$  for  $\alpha = 1$  into (14) we come to the following system:

$$\begin{aligned} \partial_3 M(\tilde{r}, x_3) &= \partial_1(x_1 \tilde{N}(\tilde{r}, x_3)), \\ \partial_2 \tilde{N}(\tilde{r}, x_3) &= 0, \quad \partial_1 M(\tilde{r}, x_3) = \partial_3(x_1 \tilde{N}(\tilde{r}, x_3)) \end{aligned} \quad (128)$$

which is easy solvable. Its generic solution is:

$$M = c_1 r^2 + c_2 x_3 + c_3, \quad \tilde{N} = -c_1 x_3 - c_2. \quad (129)$$

In contrast with the case of scalar integrals of motion (refer to Table 1) the obtained solution do not include arbitrary functions but only arbitrary parameters  $c_1, c_2$  and  $c_3$ . Moreover, parameter  $c_2$  can be reduced to zero via the shift of  $x_3$  provided parameter  $c_1$  is nontrivial. Thus taking into account that at least one of nonzero parameters in parameter in (129) can be reduced to the unity by simultaneous scaling of all independent variables  $x_1, x_2$  and  $x_3$ , without loss of generality we can rewrite the latter equation in the following form:

$$M = \cos(\lambda) r^2 + \sin(\lambda) x_3 + c_3, \quad \tilde{N} = -\cos(\lambda) x_3 - \sin(\lambda) \quad (130)$$

where parameter  $\lambda$  takes two values:  $\lambda = 0$  or  $\lambda = \frac{\pi}{2}$

The reason of the reduced freedom in the arbitrary elements  $M$  lies in the fact that the considered systems by definition admit as minimum two integrals of motion which are the components of the bivector (115). We have found the first component, i.e. choose  $\alpha = 1$  in (115) and (121) but the second one can be obtained simple by the changes  $P_1 \rightarrow P_2, x_1 \rightarrow x_2$ .

Moreover, it happens that our system admits two more integrals of motion and is maximally superintegrable. Its integrals of motion together with the admissible potential are represented in Item 4 of Table 5.

Consider now integral of motion  $Q_\alpha^{(2)}$  presented in (116) where  $Q_\alpha^{(0)} = -P_\alpha x_\alpha P_\alpha$  and  $\alpha = 1$ . The related determining equations (14) have the following form:

$$\begin{aligned} 2\tilde{r}\partial_4 M(\tilde{r}^2, x_3) + x_3\partial_3 M(\tilde{r}^2, x_3) + M(\tilde{r}^2, x_3) + N(\tilde{r}^2, x_3) &= 0, \\ 2x_3\partial_4 M(\tilde{r}^2, x_3) - \partial_3 M(\tilde{r}^2, x_3) + \partial_3 N(\tilde{r}^2, x_3) &= 0, \quad \partial_4 N(\tilde{r}^2, x_3) = 0 \end{aligned} \quad (131)$$

and are easy solvable also. Their generic solution is:

$$M = c_1 + \frac{c_2}{x_3^2} + \frac{c_3}{\tilde{r}}, \quad N = -c_1 + \frac{c_2}{x_3^2}. \quad (132)$$

The corresponding inverse mass and potential are presented in Item 5 of Table 4 where one more integral of motion is indicated also. Thus the considered system is superintegrable.

The next (and the last) example we consider is the system admitting the symmetry presented in (126), i.e.,  $c_1 Q_\alpha^{(4)} + c_2 Q_\alpha^{(5)}$ . Setting  $\alpha = 1$  we come to the following determining equations (14):

$$(c_2 - c_1)(x_4 - x_3^2)\partial_3 M + (2c_1 - c_2)\partial_4 M + 2(2c_1 - c_2)x_3 M - 2x_4\partial_4 N - N = 0,$$

$$2(c_2 - 2c_1)x_3x_4\partial_4M + ((c_2 - c_1)x_3^2 + c_1x_4)\partial_3M + 2(c_2 - 2c_1)x_3\partial_3M + N = 0,$$

$$(c_1 - c_2)(x_3^2 - x_4) - \frac{1}{2}\partial_3Nc_2x_3\partial_3M + c_2M = 0.$$

Excluding  $N$  we come to the following compatibility conditions of the system presented above:

$$(c_2 - 2c_1)x_3x_4\partial_{34}M + \frac{1}{2}(c_2 - c_1)(x_3^2 + c_1x_4)\partial_{34}M + \frac{3}{2}(c_2 - 2c_1)\partial_4M + \frac{1}{4}(c_2 + 2c_1)\partial_3M = 0,$$

$$(c_1 - c_2)(x_4 - x_3^2)\partial_{44}M + \frac{3}{2}\left(c_2 - \frac{2}{3}\right)x_3\partial_{34}M + \left(\frac{5}{2}c_2 - 2c_1\right)\partial_4M - \frac{1}{4}\partial_{33}M = 0.$$

This system has two special and one regular solutions. Namely, for  $c_1 = 2c_2$  and  $c_1 = c_2$  we obtain:

$$M = c_3r \tag{133}$$

and

$$M = \frac{c_3x_3}{r^2} + F(r) \tag{134}$$

respectively, where  $F(r)$  is an arbitrary function. If parameters  $c_1$  and  $c_2$  do not satisfy the conditions presented above, the solution is

$$M = \frac{c_3}{r}. \tag{135}$$

The system with whose mass is given in (134) is maximally superintegrable, see Item 1 of Table 5. The solutions (133) and (135) correspond to the rotationally invariant systems which admit as minimum the three parametric Lie group. Such systems are completely classified in paper [9] and we will not discuss them here.

Solving step by step the determining equations corresponding to the remaining symmetries (117), (118), (119), (123) - (127) we find all inequivalent PDM systems with cylindric symmetry, which admit vector integrals of motion. The obtained results are presented in Tables 2-5.

## 8 Tensor integrals of motion

The last class of symmetries which is supposed to be considered are the tensor integrals of motion. The corresponding Killing tensors are given by equations (33), (35), (38), (40) and (41) where where the indices  $a, b$  of the nonzero parameters  $\lambda_k^{ab}$  independently take the values 1 and 2. In other words, we have to consider exactly five linearly independent tensors and their linear combinations. The corresponding integrals of motion include operators (110) and the following bilinear combinations:

$$P_\alpha P_\nu, \{K_\alpha K_\nu\}, \{P_\alpha K_\nu\} \tag{136}$$

and

$$\{P_\alpha, L_\nu\}, \{K_\alpha L_\nu\} \tag{137}$$

were  $\alpha$  and  $\nu$  take the values 1 and 2.

Notice that operators (136) are true tensors while operators (137) change their signs under the reflection and so are pseudo tensors. It means that we have to consider linear combinations of symmetries (136) and (137) separately.

One more note is that the traceless symmetric tensor  $Q^{\alpha\nu}$  in two dimensions has exactly two linearly independent components, i.e.,  $Q^{12}$  and  $Q^{11} - Q^{22}$ . We will work with the first components, i.e.,  $Q^{12}$  and will omit the top index 12.

## 8.1 True tensor integrals of motion

Up to equivalence transformation (22) we can specify the following inequivalent linear combinations of true scalars (133) and the universal block (110):

$$Q_1 = Q_1^{(0)} + P_1 P_2 + \eta_1, \quad (138)$$

$$Q_2 = Q_2^{(0)} \{K_1, K_2\} \pm 2P_1 P_2 + \eta_2, \quad (139)$$

$$Q_3 = Q^{(0)} + \{P_1, P_2 \pm K_2\} + \eta_3, \quad (140)$$

$$Q_4 = Q^{(0)} + \{P_1 \pm K_1, P_2 \pm K_2\} + \eta_4. \quad (141)$$

The determining equations (14) for symmetry (138) are simple and have the following form:

$$\partial_2 M + \partial_1 N = 0, \quad \partial_1 M + \partial_2 N = 0, \quad \partial_3 N = 0. \quad (142)$$

Taking into account that the generic form of functions  $M$  and  $N$  is given by the following equation

$$M = M(\tilde{r}, x_3), \quad N = x_1 x_2 \tilde{N}(\tilde{r}, x_3)$$

the system (142) is easily integrated and solved by the following functions:

$$M = c_1 r^2 + F(x_3), \quad N = c_1 x_1 x_2.$$

The corresponding PDM system appears to be superintegrable, see Item 2 of Table 4.

Let us consider one more integral of motion which is specified in (141). Denoting  $N = x_1 x_2 \tilde{N}(\tilde{r}^2, x_3)$  we come to the following corresponding equations (14):

$$\partial_4((x_4 - x_3^2 \pm 1)^2 M) + x_3(x_4 - x_3^2 \pm 1)\partial_3 M + \frac{1}{2}\partial_4(x_4 \tilde{N}) = 0,$$

$$x_3(x_4 - x_3^2 - 1)\partial_4 M + \partial_3(x_3^2 M) + \frac{1}{8}\partial_3 \tilde{N}$$

which are perfectly solved by the following functions:

$$M = \frac{(r^2 \pm 1)F\left(\frac{r^2 \mp 1}{x_3}\right) - c_1 r^2}{(r^4 - 1)^2}, \quad \tilde{N} = \frac{c_1 x_1 x_2}{(r^2 \pm 1)^2}.$$

The obtained results are represented in Item 6 of Table 2.

We will not discuss the determining equations for symmetries (139) and (140) but mention that both of them can be solved exactly also. However, for symmetry (139) the related solutions are functions of  $x_3$  only, but for symmetry (140) we obtain the mass function which depends only on  $r$ . Such systems are out of the scope of the present paper since they were discussed in papers [35] and [39].

Table 2. Inverse masses, potentials, vector and tensor integrals of motion for integrable PDM systems

No	$f$	$V$	Integrals of motion
1	$\frac{x_3^2}{c_1+c_2(r^2\pm 1)F_{\pm}^{\frac{1}{2}}+c_3x_3^2(r^2\mp 1)^{-2}}$	$\frac{c_4+c_5(r^2\pm 1)F_{\pm}^{\frac{1}{2}}+c_6x_3^2(r^2\mp 1)^{-2}}{c_1+c_2(r^2\pm 1)F_{\pm}^{\frac{1}{2}}+c_3x_3^2(r^2\mp 1)^{-2}}$	$\{(K_3 \pm P_3), L_\alpha\} + 3\varepsilon_{\alpha\nu}x_\nu$ $+ \frac{2\varepsilon_{\alpha\nu}x_\nu(c_4(r^2\pm 1)+c_5(r^2\mp 1)^2)}{x_3^2}$ $- \left( \frac{2\varepsilon_{\alpha\nu}x_\nu(c_1(r^2\pm 1)+c_2(r^2\mp 1)^2)}{x_3^2} \cdot H \right)$
2	$\frac{x_3^2\tilde{F}_\pm}{c_2(r^2\mp 1)+\tilde{F}_\pm(c_3x_3^2(r^2\pm 1)^{-2}+c_1)}$	$\frac{c_5(r^2\mp 1)+\tilde{F}_\pm(c_6x_3^2(r^2\pm 1)^{-2}+c_4)}{c_2(r^2\mp 1)+\tilde{F}_\pm(c_3x_3^2(r^2\pm 1)^{-2}+c_1)}$	$\{(K_\alpha \pm P_\alpha), D\} - 15x_\alpha$ $\mp \varepsilon_{\alpha\nu}\{L_3, (K_\nu \mp P_\nu)\} \pm 3x_\alpha$ $- 2x_\alpha \left( c_5 \frac{r^2\mp 1}{(r^2\pm 1)^2} \mp \frac{2c_6}{\tilde{F}_\pm} \right)$ $+ 2 \left( x_\alpha \left( c_2 \frac{r^2\mp 1}{(r^2\pm 1)^2} \mp \frac{2c_3}{\tilde{F}_\pm} \right) \cdot H \right)$
3	$\frac{x_3^6}{c_1(x_3^2+4\tilde{r}^2)+c_2x_3^4+c_3x_3^6}$	$\frac{(c_4x_3^2+4\tilde{r}^2)+c_5x_3^4+c_6x_3^6}{c_1(x_3^2+4\tilde{r}^2)+c_2x_3^4+c_3x_3^6}$	$\{P_\alpha, D\} + \varepsilon_{\alpha\nu}\{L_3, P_\nu\}$ $+ 2 \left( \frac{x_\alpha(c_1-c_3x_3^4)}{x_3^4} \cdot H \right)$ $- 2 \frac{x_\alpha(c_4-c_6x_3^4)}{x_3^4}$
4	$\frac{r(r^2\pm 1)^2x_3^2}{c_1r(r^2\pm 1)^2+c_2rx_3^2+c_3(r^2\mp 1)x_3^2}$	$\frac{c_4r(r^2\pm 1)^2+c_5rx_3^2+c_6(r^2\mp 1)x_3^2}{c_1r(r^2\pm 1)^2+c_2rx_3^2+c_3(r^2\mp 1)x_3^2}$	$\{D, (K_\alpha \pm P_\alpha)\} - \frac{2x_\alpha c_5(r^2\mp 1)}{(r^2\pm 1)^2}$ $- \frac{x_\alpha c_6((r^2\mp 1)\mp 4r^2)}{r(r^2\pm 1)^2} - 15x_\alpha$ $+ \left( \frac{2x_\alpha c_2((r^2\mp 1))}{(r^2\pm 1)^2} \cdot H \right)$ $+ \left( \frac{x_\alpha c_3((r^2\mp 1)^2+4r^2)}{r(r^2\pm 1)^2} \cdot H \right)$
5	$\frac{x_3^2}{x_3^2F(r)+c_1}$	$\frac{x_3^2G(r)+c_2}{x_3^2F(r)+c_1}$	$\{L_1, L_2\} + \left( \frac{c_1x_1x_2}{x_3^2} \cdot H \right) - \frac{c_2x_1x_2}{x_3^2},$ $L_1^2 - L_2^2 + \left( \frac{c_1(x_1^2-x_2^2)}{2x_3^2} \cdot h \right) - \frac{c_2(x_1^2-x_2^2)}{2x_3^2}$
6	$\frac{(r^4-1)^2}{(r^2\pm 1)^2F\left(\frac{r^2\mp 1}{x_3}\right)-c_1r^2}$	$\frac{(r^2\pm 1)^2G\left(\frac{r^2\mp 1}{x_3}\right)+c_2r^2}{(r^2\pm 1)^2F\left(\frac{r^2\mp 1}{x_3}\right)-c_1r^2}$	$(K_1 \pm P_1)^2 - (K_2 \pm P_2)^2$ $+ 15(x_1^2 - x_2^2)$ $+ c_1 \left( \frac{x_1^2 - x_2^2}{(r^2\pm 1)^2} \cdot H \right) + c_2 \frac{x_1^2 - x_2^2}{(r^2\pm 1)^2},$ $\{(K_1 \pm P_1), (K_2 \pm P_2)\}$ $+ c_2 \frac{x_1x_2}{(r^2\pm 1)^2} + 15x_1x_2$
7	$\frac{(r^2\pm 1)^2F_\pm^2}{c_1G^\pm+2c_2x_3^2(r^2\pm 1)+c_3F_\pm^2}$	$\frac{c_4G^\pm+2c_5x_3^2(r^2\pm 1)+2c_6F_\pm^2}{c_1G^\pm+2c_2x_3^2(r^2\pm 1)+c_3F_\pm^2}$	$\{(K_1 \pm P_1), L_1\}$ $- \{(K_2 \pm P_2), L_2\}$ $+ 2 \left( x_1x_2 \left( \frac{c_1x_3(r^2\pm 1)}{F_\pm^2} + c_2 \right) \cdot H \right)$ $- 2x_1x_2 \left( \frac{c_4x_3(r^2\pm 1)}{F_\pm^2} + c_5 \right),$ $\{(K_1 \pm P_1), L_2\}$ $+ \{(K_2 \pm P_2), L_1\}$ $+ \left( (x_1^2 - x_2^2) \left( \frac{c_1x_3(r^2\pm 1)}{F_\pm^2} + c_2 \right) \cdot H \right)$ $- (x_1^2 - x_2^2) \left( \frac{c_4x_3(r^2\pm 1)}{F_\pm^2} + c_5 \right)$

## 8.2 Pseudo tensor integrals of motion

Such integrals should include linear combinations of the terms presented in (137). Up to the equivalence we can restrict ourselves to two types of such linear combinations:

$$Q_6 = Q^{(0)} + \{P_1, L_2\} + \{P_2, L_1\} + \eta_6 \quad (143)$$

and

$$Q_7 = Q^{(0)} + \{(P_1 \pm K_1), L_2\} + \{(P_2 \pm K_2), L_1\} + \eta_7. \quad (144)$$

Let us denote  $N = (x_1^2 - x_2^2)\tilde{N}(r, x_3)$  then symmetry (143) generates the following version of the determining equations (14):

$$\begin{aligned} 2(x_1^2 - x_2^2)\partial_4\tilde{N} + 4x_3\partial_4M - \partial_3M + 2\tilde{N} &= 0, \\ 2\partial_4M - \partial_3N &= 0. \end{aligned}$$

which are solved by the following functions:

$$M = c_1 + c_2(\tilde{r}^2 + 4x_3^2 + 2c_3x_3), \tilde{N} = -2c_2x_3 - c_3.$$

The corresponding PDM system appears to be maximally superintegrable, see Item 2 of Table 5.

The next (and the last in this section) symmetry which we consider is specified in (144). The corresponding determining equations (14) are reduced to the following system:

$$\begin{aligned} x_3(x_4 - x_3^2 - 1)\partial_4M + \frac{1}{4}(1 + 3x_3^2 - x_4)\partial_3M - \frac{1}{2}(x_4\tilde{N}) + x_3M &= 0, \\ (x_4 + 3x_3^2 + 1)\partial_3M - 4x_3(x_3^2 + 1)\partial_4M + 4x_3M - 2\tilde{N}, \\ (x_4 - 3x_3^2 - 1)\partial_4M + 2\partial_3(x_3M) + \frac{1}{2}\tilde{N} &= 0 \end{aligned}$$

which is solved by the following functions:

$$\begin{aligned} M &= c_1 \frac{(\tilde{r}^2 - 2x_3^2)(r^2 \pm 1)^2 + 4x_3^2\tilde{r}^2}{(r^2 \pm 1)^2((r^2 \pm 1)^2 \pm 4x_3^2)} + 2c_2 \frac{x_3^2}{(r^2 \pm 1)((r^2 \pm 1)^2 \pm 4x_3^2)} + \frac{c_3}{(r^2 \pm 1)^2}, \\ \tilde{N} &= c_1 \frac{x_3}{((r^2 \pm 1)^2 \pm 4x_3^2)} + c_2. \end{aligned}$$

The related PDM system is integrable, see Item 7 of Table 2.

## 9 Superintegrable systems

Thus we have specified all nonequivalent cylindrically invariant PDM systems which admit a second order integral of motion. By definition such systems admit two integrals of motion one of which is the generator of rotations around the third coordinate axis, and so they are integrable. All the systems admitting two (but no more) integrals of motion are presented in Tables 1 and 2.

However, some of the found systems automatically admit more than two integrals of motion. In addition, for some particular versions of the arbitrary elements, i.e., arbitrary functions and integration constants we also can find additional integrals of motion. The related systems are superintegrable or even maximally superintegrable.

The next (and final) step of our analysis is the specifications of just these systems. It can be done in two ways.

The first way presupposes solving of the functional equations. Let you have two systems any of which admit one second order integral of motion, and the related mass functions are  $M^{(1)}$  and  $M^{(2)}$ . Then we have to equate to zero their linear combination:

$$k_1 M^{(1)} + k_2 M^{(2)} = 0. \quad (145)$$

and this condition generates a functional equation for the arbitrary elements present in  $M^{(1)}$  and  $M^{(2)}$ .

The second way is to solve the extended systems of the determining equations (14) corresponding to the inequivalent pairs of integrals of motion. In all cases when such extended systems have nontrivial solutions we can obtain the mass functions of the related superintegrable systems.

The same speculations with the inequivalent triplets of integrals of motion and the related mass functions make it possible to specify the inequivalent maximally superintegrable systems. We will not present the complete set of the corresponding calculations which do not include new elements in comparison with the given above, but restrict ourselves to considering few examples.

## 9.1 Superintegrable systems admitting scalar integrals of motion

The systems admitting scalar integrals of motion are defined up to arbitrary functions, and to specify such of them which admit as minimum two such integrals it is preferable to use the second approach, i.e. to search for such cases when arbitrary elements  $M$  and  $N$  satisfy more than one system of the determining equations specified in (74), (76), (77), (80) and (81)-(88).

Let the PDM system includes arbitrary elements satisfying determining equations (81), see Item 5 of Table 1. The corresponding position dependent mass include two arbitrary functions. Let us search for such the arbitrary functions that the related PDM system admits one more second order integral of motion. It is the case when functions  $M$  and  $N$  satisfies one more system of the determining equations mentioned in the above.

Solving step by step all pairs of equations including (81) and one of of equations (74), (76), (77), (80) and (82)-(88) we obtain the following results. Equation (81) is incompatible with equations (76), i.e, the corresponding systems of equations have no solutions. The remaining pairs of equations are consistent, but only three of them, namely, (81) and (82), (81) and (83), (81) and (79) have good solutions, presented in Items 1, 3 and 3 of Table 3. Solutions for the other pairs of equations are invariant with respect to two parametric Lie groups and so can be ignored as being classified in our previous papers.

Considering the doublets of equations which include (82) we recognize two more inequivalent systems with good solutions. Those are the doublets including systems (84) or (88). One of the mentioned solutions is represented in Item 4 of Table 3. The other solutions are too cumbersome to be placed in the table, and we fix them in equations (147).

Table 3. Inverse masses, potentials and scalar integrals of motion for superintegrable systems

No	$f$	$V$	Integrals of motion
1	$\frac{\tilde{r}^2}{\tilde{r}^2(c_1+2c_2x_3+c_3(\tilde{r}^2+4x_3^2))+c_4}$	$\frac{\tilde{r}^2(c_5+2c_6x_3+c_7(\tilde{r}^2+4x_3^2))+c_8}{\tilde{r}^2(c_1+2c_2x_3+c_3(\tilde{r}^2+4x_3^2))+c_4}$	$P_3^2 + 2c_6x_3 + 4c_7x_3^2 + c_4\frac{x_3}{\tilde{r}^2}$ $- ((2c_2x_3 + 4c_3x_3^2) \cdot H),$ $\{P_3, D\} - P_n x_3 P_n$ $+ ((-c_1x_3 - c_2(2x_3^2 + \tilde{r}^2)$ $- c_3x_3(4x_3^2 + 3\tilde{r}^2)) \cdot H)$ $+ c_5x_3 + c_6(2x_3^2 + \tilde{r}^2)$ $- c_8\frac{x_3}{\tilde{r}^2} + c_7x_3(4x_3^2 + 3\tilde{r}^2)$
2	$\frac{r\tilde{r}^2}{c_1r\tilde{r}^2+c_2r+c_3x_3+c_7\tilde{r}^2}$	$\frac{c_4r\tilde{r}^2+c_5r+c_6x_3+c_8\tilde{r}^2}{c_1r\tilde{r}^2+c_2r+c_3x_3+c_7\tilde{r}^2}$	$\{P_3, D\} - P_n x_3 P_n$ $+ c_2 \left( \frac{c_2x_3+c_3r}{\tilde{r}^2} \cdot H \right)$ $- \frac{c_5x_3+c_6r}{\tilde{r}^2},$ $L_1^2 + L_2^2$ $- \left( \frac{r^2}{\tilde{r}^2} (c_2 + c_3 \frac{x_3}{r}) \cdot H \right)$ $+ (c_5 + c_6 \frac{x_3}{r}) \frac{r^2}{\tilde{r}^2}$
3	$\frac{\tilde{r}^2 x_3^2}{c_1x_3^2+c_2\tilde{r}^2+c_3x_3^2\tilde{r}^2+c_4r^2\tilde{r}^2x_3^2}$	$\frac{c_5x_3^2+c_6\tilde{r}^2+c_7x_3^2\tilde{r}^2+c_8r^2\tilde{r}^2x_3^2}{c_1x_3^2+c_2\tilde{r}^2+c_3x_3^2\tilde{r}^2+c_4r^2\tilde{r}^2x_3^2}$	$\{P_3, K_3\} + \frac{2c_6\tilde{r}^2}{x_3^2} - 2\omega x_3^2$ $+ 2 \left( (c_3x_3^2 - c_2\frac{\tilde{r}^2}{x_3^2}) \cdot H \right),$ $P_3^2 - \left( \left( \frac{c_2}{x_3^2} + c_4x_3^2 \right) \cdot H \right)$ $+ \frac{c_6}{x_3^2} + c_8x_3^2$
4	$\frac{\tilde{r}^2 x_3^2}{c_1\tilde{r}^2+x_3^2(c_3-2c_2\ln(\tilde{r}))}$	$\frac{x_3^2(2c_5\ln(\tilde{r})+c_6)+c_4\tilde{r}^2}{c_1\tilde{r}^2+x_3^2(c_3-2c_2\ln(\tilde{r}))}$	$P_3^2 - \left( \left( \frac{c_1}{x_3^2} \cdot H \right) + \frac{c_4}{x_3^2}, \right.$ $\left. DL_3 + c_2(\varphi \cdot H) - c_5\varphi \right)$
5	$\frac{\tilde{r}^2 x_3^2 \sqrt{F_{\pm}}}{c_1x_3^2\sqrt{F_{\pm}}+c_2r^2\sqrt{F_{\pm}}+c_3\tilde{r}^2(r^2\pm 1)}$	$\frac{c_4x_3^2\sqrt{F_{\pm}}+c_5r^2\sqrt{F_{\pm}}+c_6\tilde{r}^2(r^2\pm 1)}{c_1x_3^2\sqrt{F_{\pm}}+c_2r^2\sqrt{F_{\pm}}+c_3\tilde{r}^2(r^2\pm 1)}$	$(K_3 \pm P_3)^2 + \frac{F_{\pm}(c_5+c_6(r^2\pm 1))}{x_3^2}$ $- \left( \frac{F_{\pm}(c_2+c_3(r^2\pm 1))}{x_3^2} \cdot H \right) - \hat{\eta},$ $\{P_3, (K_3 \pm P_3)\} + c_5 \frac{(\tilde{r}^2\pm 1)}{x_3^2}$ $+ c_6 \frac{x_3^2(\tilde{r}^2\mp 1)+(\tilde{r}^2\pm 1)^2}{x_3^2\sqrt{F_{\pm}}} - \left( c_2 \frac{(\tilde{r}^2\pm 1)}{x_3^2} \cdot H \right)$ $- \left( c_3 \frac{x_3^2(\tilde{r}^2\mp 1)+(\tilde{r}^2\pm 1)^2}{x_3^2\sqrt{F_{\pm}}} \cdot H \right)$
6	$\frac{\tilde{r}^2 x_3^2 \sqrt{F_{\pm}}}{c_1x_3^2\sqrt{F_{\pm}}+c_3r^2\sqrt{F_{\pm}}+c_3(r^2\pm 1)\tilde{r}^2}$	$\frac{c_4x_3^2\sqrt{F_{\pm}}+c_5r^2\sqrt{F_{\pm}}+c_6(r^2\pm 1)\tilde{r}^2}{c_1x_3^2\sqrt{F_{\pm}}+c_2r^2\sqrt{F_{\pm}}+c_3(r^2\pm 1)\tilde{r}^2}$	$K_3^2 - P_3^2 - \hat{\eta} + \frac{c_5(r^4-1)}{x_3^2}$ $+ \frac{c_6(r^2\mp 1)(r^4\pm 1+2\tilde{r}^2)}{x_3^2\sqrt{F_{\pm}}}$ $- \left( \frac{c_2(r^4-1)}{x_3^2} \cdot H \right)$ $- \left( \frac{c_3(r^2\mp 1)(r^4\pm 1+2\tilde{r}^2)}{x_3^2\sqrt{F_{\pm}}} \cdot H \right),$ $(K_3 \pm P_3)^2 - \hat{\eta} + \frac{c_5(r^2\pm 1)}{x_3^2}$ $+ \frac{c_6(r^2\pm 1)^2\sqrt{F_{\pm}}}{x_3^2} - \left( \frac{c_2(r^2\pm 1)}{x_3^2} \cdot H \right)$ $- \left( \frac{c_3(r^2\pm 1)^2\sqrt{F_{\pm}}}{x_3^2} \cdot H \right)$

We will not represent the following steps and the calculation details since any of the related systems includes four or more equations for two unknowns depending on two variables which make them rather simple solvable. The obtained results are represented in Table 3 and equations (148)-(151). Let us however represent the examples of solutions which are missing in the table and mentioned formulae.

An example of solutions which are not "good" since the related PDM system is invariant with respect to two parametric group is:

$$M = \frac{c_1}{\tilde{r}^2} + \frac{c_2 x_3}{r \tilde{r}^2}. \quad (146)$$

Function (146) solves the system of equations (82), (88). However, the related PDM Schrödinger equation is invariant with respect to two parametric Lie group including rotations around the third coordinate axis and dilatations. Such systems were discussed in [39].

One more nice solution looks as:

$$M = \frac{c_1}{x_3^2} + \frac{1}{\tilde{r}^2} \left( c_2 + c_3 \ln \left( \frac{\tilde{r}}{r^2} \right) \right).$$

The corresponding integrals of motion are  $\hat{Q}_4$  (54) and  $K_3^2 + P_a q(\tilde{r}, x_3) P_a + \eta$ . We will not represent the explicit form of the related functions  $q(\tilde{r}, x_3), \eta$  and  $q_4(\tilde{r}, x_3), \tilde{\eta}_4$  in (54) since the latter integral of motion can be reduced to  $Q_1$  via transformation (22). Notice that this transformation keeps operator  $\hat{Q}_4$  invariant.

$$\begin{aligned} f &= \frac{\tilde{F}_\pm^2 (r^2 \mp 1)^2 \tilde{r}^2}{c_1 (r^2 \mp 1)^2 \tilde{F}_\pm^2 + c_2 x_3 \tilde{r}^2 (r^4 \pm 1) (r^2 \mp 1) + c_3 \Phi_\pm + c_4 \Lambda_\pm}, \\ V &= \frac{c_5 (r^2 \mp 1) \tilde{F}_\pm^2 + c_6 x_3 \tilde{r}^2 (r^4 \pm 1) (r^2 \mp 1) + c_7 \Phi_\pm + c_8 \Lambda_\pm}{c_1 (r^2 \mp 1) \tilde{F}_\pm^2 + c_2 x_3 \tilde{r}^2 (r^4 \pm 1) (r^2 \mp 1) + c_3 \Phi_\pm + c_4 \Lambda_\pm}, \\ \Phi_\pm &= (r^2 \pm 1)^2 ((r^2 \mp 1)^2 \pm r) + 4x_3 (4r^2 - \tilde{r}^2), \\ \Lambda_\pm &= (\tilde{r}^2 + 4x_3^2) (r^2 \pm 1)^2 - 4x_3^2 \tilde{r}^2, \\ (K_3 \pm P_3)^2 - \hat{\eta} - \frac{1}{\tilde{F}_\pm^2} (c_6 x_3 (r^2 \pm 1) F_- + 4(c_8 - 3c_7) x_3^2 (r^2 \pm 1)^2) \\ &+ \left( \frac{1}{\tilde{F}_\pm^2} (c_2 x_3 (r^2 \pm 1) F_\pm + 4(c_4 - 3c_3) x_3^2 (r^2 \pm 1)^2 \cdot H \right), \\ \{D, (K_3 \mp P_3)\} - 15x_3 - \frac{c_6 F_\pm}{8\tilde{F}_\pm^2} + \left( -\frac{c_2 F_\pm}{8\tilde{F}_\pm^2} \cdot H \right) \\ &- \frac{c_7 x_3 (r^2 \mp 1)^4 + (c_8 - 2c_7 x_3) ((r^2 \pm 1)^2 + 2\tilde{r}^2) (2r^2 - \tilde{r}^2) (r^2 \pm 1)}{\tilde{F}_\pm^2 (r^2 \mp 1)^2} \\ &+ \left( \frac{c_3 x_3 (r^2 \mp 1)^4 + (c_4 - 2c_3 x_3) ((r^2 \pm 1)^2 + 2\tilde{r}^2) (2r^2 - \tilde{r}^2) (r^2 \pm 1)}{\tilde{F}_\pm^2 (r^2 \mp 1)^2} \cdot H \right); \end{aligned} \quad (147)$$

$$\begin{aligned}
f &= \frac{(r^4 \pm 1)^2 x_3^2 \tilde{r}^2}{x_3^2 (c_1 (r^4 \mp 1)^2 \tilde{r}^2 + c_3 \tilde{r}^2 (r^4 \pm 1) + c_4 r^2 \tilde{r}^2) + c_2 (r^4 \pm 1 \mp 2r^2 \tilde{r}^2) \tilde{r}^2}, \\
V &= \frac{x_3^2 (c_5 (r^4 \mp 1)^2 \tilde{r}^2 + c_7 \tilde{r}^2 (r^4 \pm 1) + c_8 r^2 \tilde{r}^2) + c_6 (r^4 \pm 1 \mp 2r^2 \tilde{r}^2) \tilde{r}^2}{x_3^2 (c_1 (r^4 \mp 1)^2 \tilde{r}^2 + c_3 \tilde{r}^2 (r^4 \pm 1) + c_4 r^2 \tilde{r}^2) + c_2 (r^4 \pm 1 \mp 2r^2 \tilde{r}^2) \tilde{r}^2}, \\
Q_1 &:= K_3^2 \pm P_3^2 - \hat{\eta} - \frac{(1 \pm r^4)(c_8 x_3^4 + c_6((r^4 \mp 1)^2 \pm 4x_3^4) + 2c_7 r^2 x_3^4)}{x_3^2 (r^4 \mp 1)^2} \\
&\quad + \left( \frac{(1 \pm r^4)(c_4 x_3^4 + c_2((r^4 \mp 1)^2 \pm 4x_3^4) + 2c_3 r^2 x_3^4)}{x_3^2 (r^4 \mp 1)^2} \cdot H \right),
\end{aligned} \tag{148}$$

$$\begin{aligned}
Q_2 &= \{K_3, P_3\} + \frac{c_6(\tilde{r}^2(r^4 \mp 1)^2 \pm 2x_3^4 r^2) - c_7(r^4 \pm 1)x_3^4 - 2c_8 r^2 x_3^4}{2x_3^2 (r^4 \mp 1)^2} \\
&\quad - \left( \frac{c_2(\tilde{r}^2(r^4 \mp 1)^2 - 2x_3^4 r^2) - c_3(r^4 \pm 1)x_3^4 - 2c_4 r^2 x_3^4}{2x_3^2 (r^4 \mp 1)^2} \cdot H \right);
\end{aligned}$$

$$\begin{aligned}
f &= \frac{\tilde{r}^2 (r^2 \pm 1)^2 \sqrt{F_{\pm}}}{\sqrt{F_{\pm}} (c_1 (r^2 \pm 1)^2 + c_2 (r^2 \pm 1)^2 - 2\tilde{r}^2) + c_3 (r^4 - 1)(r^2 \pm 1) + c_4 x_3 \tilde{r}^2}, \\
V &= \frac{c_3 \tilde{F}_{\pm}^2 + c_4 x_3 \tilde{r}^2 (r^2 \mp 1)}{c_1 \tilde{F}_{\pm}^2 + c_2 x_3 \tilde{r}^2 (r^2 \mp 1)}, \\
Q_1 &= (K_3 \pm P_3)^2 - \hat{\eta} + \frac{8c_6 x_3^2 + c_8 x_3 \sqrt{F_{\pm}}}{(r^2 \pm 1)^2} - \left( \frac{8c_2 x_3^2 + c_4 x_3 \sqrt{F_{\pm}}}{(r^2 \pm 1)^2} \cdot H \right),
\end{aligned} \tag{149}$$

$$\begin{aligned}
Q_2 &= \{D, K_3 \pm P_3\} - 15x_3 - 4 \left( \left( \frac{c_3 x_3}{\sqrt{F_{\pm}}} + \frac{c_1 x_3 (r^2 \mp 1)}{\sqrt{F_{\pm}} (r^2 \pm 1)^2} \right) \cdot H \right) \\
&\quad - \frac{1}{4} \left( \frac{c_4 (r^2 \mp 1) ((r^2 \pm 1)^2 \mp 8x_3^2)}{(r^2 \pm 1)^2 \sqrt{F_{\pm}}} \cdot H \right) \\
&\quad + 4 \left( \frac{c_7 x_3}{\sqrt{F_{\pm}}} + \frac{c_5 x_3 (r^2 \mp 1)}{(r^2 \pm 1)^2} \right) + \frac{c_8 (r^2 \mp 1) ((r^2 \pm 1)^2 \mp 8x_3^2)}{4(r^2 \pm 1)^2 \sqrt{F_{\pm}}};
\end{aligned}$$

$$\begin{aligned}
f &= \frac{\tilde{r}^2 \sqrt{F_{\pm}}}{c_1 \sqrt{F_{\pm}} + c_2 (x_3 - 1) \sqrt{r^2 \pm 1 + 2x_3} + c_3 (x_3 + 1) \sqrt{r^2 \pm 1 - 2x_3} + c_4 (r^2 \mp 1)}, \\
V &= \frac{c_5 \sqrt{F_{\pm}} + c_6 (x_3 - 1) \sqrt{r^2 \pm 1 + 2x_3} + c_7 (x_3 + 1) \sqrt{r^2 \pm 1 - 2x_3} + c_8 (r^2 \mp 1)}{c_1 \sqrt{F_{\pm}} + c_2 (x_3 - 1) \sqrt{r^2 \pm 1 + 2x_3} + c_3 (x_3 + 1) \sqrt{r^2 \pm 1 - 2x_3} + c_4 (r^2 \mp 1)}, \\
Q_1 &= \{D, K_3 \pm P_3\} - 15x_3 + \frac{c_6 (r^2 \pm 1)}{\sqrt{r^2 \pm 1 - 2x_3}} + \frac{c_7 (r^2 \pm 1)}{\sqrt{r^2 \pm 1 + 2x_3}} + \frac{4c_8}{\sqrt{F_{\pm}}} \\
&\quad - \left( \left( \frac{c_2 (r^2 \pm 1)}{\sqrt{r^2 \pm 1 - 2x_3}} + \frac{4c_4}{\sqrt{F_{\pm}}} \right) \cdot H \right) - \left( \frac{c_3 (r^2 \pm 1)}{\sqrt{r^2 \pm 1 + 2x_3}} \cdot H \right), \\
Q_2 &= \{P_3, K_3 \pm P_3\} + \frac{c_6 x_3}{\sqrt{r^2 \pm 1 - 2x_3}} + \frac{c_7 x_3}{\sqrt{r^2 \pm 1 + 2x_3}} + \frac{c_8 (r^2 \pm 1)}{\sqrt{F_{\pm}}} \\
&\quad - \left( \frac{c_3 x_3}{\sqrt{r^2 \pm 1 + 2x_3}} \cdot H \right) - \left( \left( \frac{c_2 x_3}{\sqrt{r^2 \pm 1 - 2x_3}} + \frac{c_4 (r^2 \pm 1)}{\sqrt{F_{\pm}}} \right) \cdot H \right);
\end{aligned} \tag{150}$$

$$\begin{aligned}
f &= \frac{(r^2 \pm 1)^2 r \tilde{r}^2}{c_1(r^2 \pm 1)^2 + c_2 r \tilde{r}^2 + c_3 x_3 (r^2 \pm 1)^2 + c_4 \tilde{r}^2 (r^2 \mp 1)}, \\
V &= \frac{c_5 (r^2 \pm 1)^2 + c_6 r \tilde{r}^2 + c_7 x_3 (r^2 \pm 1)^2 + c_8 \tilde{r}^2 (r^2 \mp 1)}{c_1 (r^2 \pm 1)^2 + c_2 r \tilde{r}^2 + c_3 x_3 (r^2 \pm 1)^2 + c_4 \tilde{r}^2 (r^2 \mp 1)}, \\
Q_1 &= \{K_3, P_3\} + \frac{c_7 x_3 (r^2 \pm 1)^2 - x_3^2 (c_6 r + c_8 (r^2 \mp 1))}{r (r^2 \pm 1)^2} \\
&\quad - \left( \frac{c_3 x_3 (r^2 \pm 1)^2 - x_3^2 (c_2 r + c_4 (r^2 \mp 1))}{r (r^2 \pm 1)^2} \cdot H \right), \\
Q_2 &= \{D, (K_3 \pm P_3)\} - \hat{\eta} + \frac{2c_6 x_3 (r^2 \mp 1)}{(r^2 \pm 1)^2} + \frac{c_8 (r^4 \mp r^2 + 1) - c_7 (r^4 - 1) (r^2 \pm 1)}{r (r^2 \pm 1)^2} \\
&\quad - \left( \frac{2c_2 x_3 r (r^2 \mp 1) - c_3 (r^4 - 1) (r^2 \pm 1) + c_4 (r^4 \mp r^2 + 1)}{r (r^2 \pm 1)^2} \cdot H \right).
\end{aligned} \tag{151}$$

$$\begin{aligned}
f &= \frac{\tilde{F}_\pm^2 (r^2 \mp 1)^2 \tilde{r}^2}{c_1 (r^2 \mp 1)^2 \tilde{F}_\pm^2 + c_2 x_3 \tilde{r}^2 (r^4 \pm 1) (r^2 \mp 1) + c_3 \Phi_\pm + c_4 \Lambda_\pm}, \\
V &= \frac{c_5 (r^2 \mp 1) \tilde{F}_\pm^2 + c_6 x_3 \tilde{r}^2 (r^4 \pm 1) (r^2 \mp 1) + c_7 \Phi_\pm + c_8 \Lambda_\pm}{c_1 (r^2 \mp 1) \tilde{F}_\pm^2 + c_2 x_3 \tilde{r}^2 (r^4 \pm 1) (r^2 \mp 1) + c_3 \Phi_\pm + c_4 \Lambda_\pm}, \\
\Phi_\pm &= (r^2 \pm 1)^2 ((r^2 \mp 1)^2 \pm r) + 4x_3 (4r^2 - \tilde{r}^2), \\
\Lambda_\pm &= (\tilde{r}^2 + 4x_3^2) (r^2 \pm 1)^2 - 4x_3^2 \tilde{r}^2, \\
Q_1 &= (K_3 \pm P_3)^2 - \hat{\eta} - \frac{1}{\tilde{F}_\pm^2} (c_6 x_3 (r^2 \pm 1) F_- + 4(c_8 - 3c_7) x_3^2 (r^2 \pm 1)^2) \\
&\quad + \left( \frac{1}{\tilde{F}_\pm^2} (c_2 x_3 (r^2 \pm 1) F_\pm + 4(c_4 - 3c_3) x_3^2 (r^2 \pm 1)^2 \cdot H) \right), \\
Q_2 &= \{D, (K_3 \mp P_3)\} - 15x_3 - \frac{c_6 F_\pm}{8\tilde{F}_\pm^2} + \left( -\frac{c_2 F_\pm}{8\tilde{F}_\pm^2} \cdot H \right) \\
&\quad - \frac{c_7 x_3 (r^2 \mp 1)^4 + (c_8 - 2c_7 x_3) ((r^2 \pm 1)^2 + 2\tilde{r}^2) (2r^2 - \tilde{r}^2) (r^2 \pm 1)}{\tilde{F}_\pm^2 (r^2 \mp 1)^2} \\
&\quad + \left( \frac{c_3 x_3 (r^2 \mp 1)^4 + (c_4 - 2c_3 x_3) ((r^2 \pm 1)^2 + 2\tilde{r}^2) (2r^2 - \tilde{r}^2) (r^2 \pm 1)}{\tilde{F}_\pm^2 (r^2 \mp 1)^2} \cdot H \right).
\end{aligned} \tag{152}$$

Summarizing, we find all inequivalent superintegrable PDM systems admitting scalar integrals of motion. The number of such systems appears to be rather extended, see Table 3 and the following formulae (147)-(151). The latter formulae collect such expressions which are too cumbersome to be placed in a table.

## 9.2 Superintegrable systems with combined symmetries

The next class of superintegrable PDM systems which we consider are those ones which admit both the scalar and vector or tensor integrals of motion. Since the systems admitting the non-scalar integral of motion up to the only exception has a rather simple shape defined up to arbitrary parameters, it is relatively easy to solve the functional equations (145) (where  $M^{(1)}$  and  $M^{(2)}$  are position dependent masses of the systems with different types of symme-

tries) in order to select the cases when there exist additional integrals of motion and so the related systems are superintegrable. To verify the completeness of the obtained in this way list of superintegrable systems it is reasonable to verify the consistence of the systems of the determining equations corresponding to the combinations of symmetries missing in this list.

Table 4. Inverse masses, potentials and combined integrals of motion for superintegrable systems

No	$f$	$V$	Integrals of motion
1	$\frac{\tilde{r}x_3^2}{\tilde{r}F\left(\frac{r^2\pm 1}{x_3}\right)+c_1x_3^2(r^2\mp 1)F_{\pm}}$	$\frac{\tilde{r}G\left(\frac{r^2\pm 1}{x_3}\right)+c_2x_3^2(r^2\mp 1)F_{\pm}}{\tilde{r}F\left(\frac{r^2\pm 1}{x_3}\right)+c_1x_3^2(r^2\mp 1)F_{\pm}}$	$\{L_3, (K_{\alpha} \pm P_{\alpha})\} + 3\varepsilon_{\alpha\nu}x_{\nu}$ $+ 2c_1\left(\frac{\varepsilon_{\alpha\nu}x_{\nu}}{\tilde{r}} \cdot H\right) - 2c_2\frac{\varepsilon_{\alpha\nu}x_{\nu}}{\tilde{r}}$ , $(K_3 \pm p_3)^2 + \frac{G\left(\frac{r^2\pm 1}{x_3}\right)F_{\pm}}{x_3^2}$ $- \left(\frac{F\left(\frac{r^2\pm 1}{x_3}\right)F_{\pm}}{x_3^2} \cdot H\right)$
2	$\frac{(r^4-1)^2x_3^2}{(r^2\pm 1)^2(c_1(r^2\mp 1)^2+c_2x_3^2)-c_3x_3^2r^2}$	$\frac{(r^2\pm 1)^2(c_4(r^2\mp 1)^2+c_5x_3^2)-c_6x_3^2r^2}{(r^2\pm 1)^2(c_1(r^2\mp 1)^2+c_2x_3^2)-c_1x_3^2r^2}$	$(K_1 \pm P_1)^2 - (K_2 \pm P_2)^2$ $+ 15(x_1^2 - x_2^2)$ $+ c_1\left(\frac{x_1^2-x_2^2}{(r^2\pm 1)^2} \cdot H\right) + c_2\frac{x_1^2-x_2^2}{(r^2\pm 1)^2}$ , $\{(K_1 \pm P_1), (K_2 \pm P_2)\} + 15x_1x_2$ $+ c_1\left(\frac{x_1x_2}{(r^2\pm 1)^2} \cdot H\right) + c_2\frac{2x_1x_2}{(r^2\pm 1)^2}$ , $\{L_1, L_2\} - \frac{c_4x_1x_2}{x_3^2} + \left(\frac{c_1x_1x_2}{x_3^2} \cdot H\right)$ , $L_1^2 - L_2^2 - \frac{c_4(x_1^2-x_2^2)}{2x_3^2}$ $+ \left(\frac{c_1(x_1^2-x_2^2)}{2x_3^2} \cdot H\right)$
3	$\frac{1}{c_1\tilde{r}^2+F(x_3)}$	$\frac{c_2\tilde{r}^2+G(x_3)}{c_1\tilde{r}^2+F(x_3)}$	$P_1P_2 - (c_1x_1x_2 \cdot H) + c_2x_1x_2$ , $P_3^2 - (F(x_3) \cdot H) + G(x_3)$ , $P_1^2 - P_2^2 + c_2(x_1^2 - x_2^2) -$ $(c_1(x_1^2 - x_2^2) \cdot H)$
4	$\frac{x_3^2}{x_3^2(c_1r^2+c_2)+c_1}$	$\frac{x_3^2(c_3r^2+c_2)+c_4}{x_3^2(c_1r^2+c_2)+c_1}$	$L_1^2 - L_2^2 - \frac{c_4(x_1^2-x_2^2)}{2x_3^2} + \left(\frac{c_1(x_1^2-x_2^2)}{2x_3^2} \cdot H\right)$ , $\{L_1, L_2\} - \frac{c_4x_1x_2}{x_3^2} + \left(\frac{c_1x_1x_2}{x_3^2} \cdot H\right)$ $P_3^2 + c_3x_3^2 + \frac{c_4}{x_3^2} - \left((c_1x_3^2 + \frac{c_2}{x_3^2}) \cdot H\right)$
5	$\frac{\tilde{r}}{F(x_3)\tilde{r}+c_1}$	$\frac{G(x_3)\tilde{r}+c_2}{F(x_3)\tilde{r}+c_1}$	$\{L_3, P_{\alpha}\} - c_2\frac{\varepsilon_{\alpha\nu}x_{\nu}}{\tilde{r}} + c_1\varepsilon_{\alpha\nu}\left(\frac{x_{\nu}}{\tilde{r}} \cdot H\right)$ $P_3^2 - (F(x_3) \cdot H) + G(x_3)$
6	$\frac{x_3^2r}{(c_1x_3^2+c_3)r+2c_2x_3^2}$	$\frac{(c_5x_3^2+c_4)r+c_6x_3^2}{(c_1x_3^2+c_3)r+2c_2x_3^2}$	$\{P_{\alpha}, D\} - 2\left(x_{\alpha}(c_1 + \frac{c_2}{r}) \cdot H\right)$ $+ 2x_{\alpha}(2c_5 + \frac{c_6}{r})$ , $P_3^2 - \left(\frac{c_2}{x_3^2} \cdot H\right) + \frac{c_4}{x_3^2}$
7	$((r^2 \pm 1)^2 \mp 4x_3^2)^{\frac{1}{2}}$	$c_1$	$\{(K_3 \pm P_3, P_{\alpha}) + P_n x_3 x_{\alpha} P_n,$ $\{P_3, (K_3 \pm P_3)\} + P_{\alpha} x_3^2 P_{\alpha}$

In this way we find the superintegrable systems presented in Table 4 which collects the systems admitting integrals of motion which transform in different way under rotation trans-

formations. More exactly, there are such systems which admit both scalar and vector or tensor integrals of motion. However, the system represented in Item 2 admits only tensorial integrals of motion.

The final step is to classify the maximally superintegrable system. It can be done in the same way as in the superintegrable case. The classification results are presented in Table 5 which includes two parts.

Table 5. Inverse masses, potentials and integrals of motion for maximally superintegrable systems

No	$f$	$V$	Integrals of motion
1	$\frac{r^2\tilde{r}}{\tilde{r}F(r)+c_1x_3}$	$\frac{\tilde{r}(G(r)+c_2x_3)}{\tilde{r}F(r)+c_1x_3}$	$\{L_3, L_\alpha\} - 2\left(\frac{c_1x_\alpha}{\tilde{r}} \cdot H\right) + \frac{2c_2x_\alpha}{\tilde{r}},$ $L_1^2 + L_2^2 - c_1\left(\frac{x_3}{\tilde{r}} \cdot H\right) + \frac{c_2x_3}{\tilde{r}},$ $\{K_3, P_\alpha\} + 2\varepsilon_{\alpha\nu}DL_\nu$ $+ \left(\frac{x_\alpha}{r^2}(x_3F(r) - c_1\tilde{r}) \cdot H\right)$ $- \frac{x_\alpha}{r^2}(G(r) - c_2\tilde{r}) + P_\alpha x_3 x_\alpha P_\alpha$
2	$\frac{1}{c_1+2c_2x_3+c_3(\tilde{r}^2+4x_3^2)}$	$\frac{c_4+2c_5x_3+c_6(\tilde{r}^2+4x_3^2)}{c_1+2c_2x_3+c_3(\tilde{r}^2+4x_3^2)}$	$P_3^2 + 2c_5x_3 + 4c_6x_3^2$ $- ((2c_2x_3 + 4c_3x_3^2) \cdot H),$ $\{P_3, D\} - P_n x_3 P_n + c_4x_3 + c_5(2x_3^2 + \tilde{r}^2)$ $+ c_6x_3(4x_3^2 + 3\tilde{r}^2) - ((c_1x_3 + c_2(2x_3^2 + \tilde{r}^2)$ $+ c_3x_3(4x_3^2 + 3\tilde{r}^2)) \cdot H),$ $P_1L_1 - P_2L_2$ $-(x_1x_2(2c_3x_3 + c_2) \cdot H) + x_1x_2(2c_6x_3 + c_5),$ $\{P_1, L_2\} + \{P_2, L_1\}$ $- ((x_1^2 - x_2^2)(2c_3x_3 + c_2) \cdot H)$ $+ (x_1^2 - x_2^2)(2c_6x_3 + c_5)$
3	$\frac{\tilde{r}^2}{c_1+c_2x_3\tilde{r}}$	$\frac{c_3+c_4x_3\tilde{r}}{c_1+c_2x_3\tilde{r}}$	$\{L_3, P_\alpha\} + c_1\varepsilon_{\alpha\nu}\left(\frac{x_\nu}{\tilde{r}} \cdot H\right) - c_3\frac{x_2}{\tilde{r}},$ $P_3^2 - (c_2x_3 \cdot H) + c_4x_3,$ $\{P_3, D\} + \left(\left(\frac{c_1}{x_3} - c_2(x_3^2 + \frac{\tilde{r}^2}{2}) \cdot H\right)\right.$ $\left. - \frac{c_3}{x_3} - c_4(x_3^2 + \frac{\tilde{r}^2}{2})\right)$
4	$\frac{1}{c_1r^2+2c_2x_3+c_3}$	$\frac{c_4r^2+2c_5x_3+c_6}{c_1r^2+2c_2x_3+c_3}$	$P_3P_\alpha - (x_\alpha(c_1x_3 + c_2) \cdot H)$ $+ x_\alpha(c_4x_3 + c_5),$ $P_3^2 - ((c_1x_3^2 + 2c_2x_3) \cdot H)$ $+ c_4x_3^2 + 2c_5x_3,$ $P_1P_2 - c_1(x_1x_2 \cdot H) + c_4x_1x_2,$ $(P_1^2 - P_2^2) + c_4(x_1^2 - x_2^2)$ $- c_1((x_1^2 - x_2^2) \cdot H)$
5	$\frac{x_3^2}{c_1x_3^2r^2+c_2}$	$\frac{c_3x_3^2r^2+c_4}{c_1x_3^2r^2+c_2}$	$P_1P_2 - c_1(x_1x_2 \cdot H) + c_3x_1x_2,$ $P_1^2 - P_2^2 + c_3(x_1^2 - x_2^2) - c_1((x_1^2 - x_2^2) \cdot H),$ $\{L_1, L_2\} + \left(\frac{2c_2x_1x_2}{x_3} \cdot H\right) - \frac{2c_4x_1x_2}{x_3^2},$ $L_1^2 - L_2^2 + \left(\frac{c_2(x_1^2 - x_2^2)}{x_3} \cdot H\right) - \frac{c_4(x_1^2 - x_2^2)}{x_3^2},$ $P_3^2 - \left((c_1x_3^2 + \frac{c_2}{x_3}) \cdot H\right) + c_3x_3^2 + \frac{c_4}{x_3}$

Table 5 (continued). Inverse masses, potentials and integrals of motion for maximally superintegrable systems

No	$f$	$V$	Integrals of motion
6	$\frac{\tilde{r}^2 \sqrt{F_{\pm}}}{c_1 \sqrt{F_{\pm} + c_2(r^2 \mp 1)}}$	$\frac{c_3 \sqrt{F_{\pm} + c_4(r^2 \mp 1)}}{c_1 \sqrt{F_{\pm} + c_2(r^2 \mp 1)}}$	$K_3^2 - P_3^2 - \hat{\eta} - \frac{2c_4(r^2 \mp 1)}{\sqrt{F_{\pm}}} + \left( \frac{2c_2(r^2 \mp 1)}{\sqrt{F_{\pm}}} \cdot H \right),$ $\{D, (K_3 \pm P_3)\} - 15x_3 \mp \frac{4c_4x_3}{\sqrt{F_{\pm}}} \pm \left( \frac{4c_2x_3}{\sqrt{F_{\pm}}} \cdot H \right),$ $(K_3 \pm P_3)^2 - \hat{\eta}$
7	$\frac{r^2 \tilde{r}^2}{c_1 r^2 + c_2 r^4 \tilde{r}^2 + c_3 r^2 \tilde{r}^2}$	$\frac{c_4 r^2 + c_5 r^4 \tilde{r}^2 + c_6 r^2 \tilde{r}^2}{c_1 r^2 + c_2 r^4 \tilde{r}^2 + c_3 r^2 \tilde{r}^2}$	$\{P_3, K_3\} - 2P_a x_3 P_a$ $+ \left( (c_2 x_3^2 - c_3 x_3^2) \cdot H \right)$ $-\frac{c_5 x_3^2}{r^2} + c_6 x_3^2,$ $L_1^2 + L_2^2 - (c_1 \frac{r^2}{\tilde{r}^2} \cdot H) + c_4 \frac{r^2}{\tilde{r}^2},$ $P_3^2 - (c_2 x_3^2 \cdot H) + c_5 x_3^2$
8	$\frac{1}{c_2(\tilde{r}^2 + 4x_3^2) + 2c_3x_3 + c_1}$	$\frac{c_6x_3 + c_5(\tilde{r}^2 + 4x_3^2) + c_4}{c_2(\tilde{r}^2 + 4x_3^2) + 2c_3x_3 + c_1}$	$P_1 L_1 - P_2 L_2 + x_1 x_2 (2c_5 x_3 + c_6)$ $-(x_1 x_2 (2c_2 x_3 + c_3) \cdot H),$ $\{P_1, L_2\} + \{P_2, L_1\} + (x_1^2 - x_2^2)(2c_5 x_3 + c_6)$ $-((x_1^2 - x_2^2)(2c_2 x_3 + c_3) \cdot H)$ $P_3^2 - 2((2c_2 x_3^2 + c_3 x_3) \cdot H)$ $+ 4c_5 x_3^2 + c_6 x_3,$ $P_1 P_2 - c_2(x_1 x_2 \cdot H) + c_5 x_1 x_2,$ $P_1^2 - P_2^2 - c_1((x_1^2 - x_2^2) \cdot H) + c_4(x_1^2 - x_2^2)$
9	$\frac{x_3^2}{c_1 x_3^2(x_3^2 + 4\tilde{r}^2) + c_2 + c_3 x_3^2}$	$\frac{c_4 + c_5 x_3^2(x_3^2 + 4\tilde{r}^2) + c_6 x_3^2}{c_1 x_3^2(x_3^2 + 4\tilde{r}^2) + c_2 + c_3 x_3^2}$	$\{P_3, L_\alpha\} + 2 \frac{\varepsilon_{\alpha\nu} x_\nu (c_4 - c_5 x_3^4)}{x_3^2} + 2 \left( \frac{\varepsilon_{\alpha\nu} x_\nu (c_1 x_3^4 - c_2)}{x_3^2} \cdot H \right),$ $P_3^2 - \left( (c_1 \frac{x_3}{x_3^2} + \frac{c_2}{x_3^2}) \cdot H \right) + \frac{c_4}{x_3^2} + \frac{c_5}{x_3^2},$ $P_1 P_2 - c_1(x_1 x_2 \cdot H) + c_4 x_1 x_2,$ $P_1^2 - P_2^2 - c_1((x_1^2 - x_2^2) \cdot H) + c_4(x_1^2 - x_2^2)$
10	$\frac{x_3^2(r^2 \pm 1)}{c_1(r^2 \pm 1) + c_2 x_3^2}$	$\frac{c_3(r^2 \pm 1) + c_4 x_3^2}{c_1(r^2 \pm 1) + c_2 x_3^2}$	$\{D, (K_\alpha \pm P_\alpha)\} - 15x_\alpha - \frac{2x_\alpha c_4(r^2 \mp 1)}{(r^2 \pm 1)^2}$ $+ \left( \frac{2x_\alpha c_1(r^2 \mp 1)}{(r^2 \pm 1)^2} \cdot H \right),$ $\{(K_3 \pm P_3), L_\alpha\} + 3\varepsilon_{\alpha\nu} x_\nu - \frac{2c_4 \varepsilon_{\alpha\nu} x_\nu (r^2 \pm 1)}{x_3^2}$ $+ 2c_1 \left( \frac{\varepsilon_{\alpha\nu} x_\nu (r^2 \pm 1)}{x_3^2} \cdot H \right),$ $\{L_1, L_2\} + \left( \frac{2c_1 x_1 x_2}{x_3^2} \cdot H \right) - \frac{2c_3 x_1 x_2}{x_3^2},$ $L_1^2 - L_2^2 + \left( \frac{2c_1(x_1^2 - x_2^2)}{x_3^2} \cdot H \right) - \frac{2c_3(x_1^2 - x_2^2)}{x_3^2}$

At this point the classification of cylindrically invariant PDM system admitting second order integrals of motion has been completed.

## 10 Discussion

In contrast with the cases of the standard Schrödinger equations and 2D Schrödinger equations with position dependent mass we still do not have the completed description of second order integrals of motion for 3D PDM Schrödinger equations. However, some steps to such description have been already made: the maximally superintegrable systems and separable systems have been already classified [25, 26], there are successes in the classification of nondegenerate and semidegenerate systems [28, 30]. Moreover, the systems invariant w.r.t. three- and two-parametric Lie groups are classified completely [35, 39].

In the present paper we make the next step to the complete classification of the mentioned integrals of motion. Namely, we present all inequivalent quantum mechanical PDM systems which, in addition to the second order integrals of motion, admit the fixed one parameter Lie symmetry group.

As it was shown in [17] there are six inequivalent one parametric Lie groups which can be possessed by the PDM systems. We start with one of them, namely, with the group of rotations around the fixed axis. In other words, we deal with the cylindrically symmetric PDM systems and classify such of them which admit at least one second order integral of motion.

Let us mention that the PDM systems with cylindric symmetry possess a rather extended collection of the mentioned integrals. Namely, we have fixed as much as 66 inequivalent systems and presented their integrals of motion. In particular we specify 18 superintegrable and 10 maximally superintegrable systems. They are collected in five tables. In addition, the most cumbersome of them are presented separately in formulae (147)-(151). Notice that any item including the terms "±" in fact represents two systems one of which corresponds to the sign "+" and the other to the sign "-".

To optimize calculations we separate the integrals of motion to three qualitatively different subclasses in accordance with their transformation properties with respect to the rotations which by definition leaves the PDM Hamiltonians invariant. The mentioned subclasses include the scalar, vector and tensor versions of the integrals. Moreover, the scalar and tensor integrals of motion can be effectively separated in accordance with their parity properties.

The systems admitting one scalar integral of motion are defined up to arbitrary functions which depend on specific variables. Such (integrable) systems are presented in Table 1.

The majority of systems admitting vector or tensor integrals of motion are defined more strictly and includes only arbitrary parameters. The reason of it is that for these subclasses the related PDM system is supposed to admit as minimum two linearly independent integrals of motion. The same is true for the case of superintegrable systems admitting integrals of motion of arbitrary type.

Thus we have made an essential step to the complete classification of the 3D PDM systems admitting second order integrals of motion. In spite of that we consider only one out of six inequivalent one parametric Lie groups which can be accepted by such systems, this step is very important since the number of found systems is very large, maybe more large than the total number of all systems admitting the other inequivalent Lie symmetries. The latter statement is supported by our computing experiments.

We believe that the presented classification is complete. However, the determining equations which we solve to find the inequivalent systems and their integrals of motion are rather complicated systems of partial differential equations with variable coefficients, and there is a danger to overlook some special solutions additional to the found generic ones. That is why we

present the mentioned determining equations explicitly. And the absolutely rigorous statement (which, however, is not constructive) is that the discussed systems should include the arbitrary elements which are functions solving these equations.

The next planned steps to the complete classification of the integrals of motion admitted by the 3D PDM systems presuppose the classifications of the systems which possess symmetries with respect to the remaining inequivalent one parametric Lie groups for such systems specified in [17]. Finally, we plan to classify such integrals of motion for the systems which have no Lie symmetry. The latter problem appears not to be catastrophically complicated thanks to the existence of rather strong equivalence group.

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