Analytic weak-stable manifolds in unfoldings of saddle-nodes

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Dedicated to the memory of Claudia Wulff. A dear friend and a respected colleague.

Abstract

Any attracting, hyperbolic and proper node of a two-dimensional analytic vector-field has a unique strong-stable manifold. This manifold is analytic. The corresponding weak-stable manifolds are, on the other hand, not unique, but in the nonresonant case there is a unique weak-stable manifold that is analytic. As the system approaches a saddle-node (under parameter variation), a sequence of resonances (of increasing order) occur. In this paper, we give a detailed description of the analytic weak-stable manifolds during this process. In particular, we relate a "flapping-mechanism", corresponding to a dramatic change of the position of the analytic weak-stable manifold as the parameter passes through the infinitely many resonances, to the lack of analyticity of the center manifold at the saddle-node. Our work is motivated and inspired by the work of Merle, Raphaël, Rodnianski, and Szeftel, where this flapping mechanism is the crucial ingredient in the construction of C^∞-smooth self-similar solutions of the compressible Euler equations.

Keywords: analytic weak-stable manifolds, center manifolds, Gevrey properties, saddle-nodes;
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Contents

1 Introduction

Consider an analytic and generic ϵ-family of two-dimensional vector-fields unfolding a saddle-node bifurcation at $\epsilon = 0$. Then we can assume that the saddle-node singularity splits for $\epsilon > 0$ into an attracting node and a saddle, both hyperbolic. Moreover, the following system

$$\begin{aligned} \dot{x} &= (x-\epsilon)x, \\ \dot{y} &= -y(1+a^\epsilon x) + g^\epsilon(x,y), \end{aligned} \tag{1}$$

where $g^\epsilon(x,y) = \mathcal{O}(x^2, x^2y, xy^2)$, is by [13, Theorem 2.2] (see also Theorem 2.1 below) an analytic normal form for this situation with $\epsilon \geq 0$. The functions a^ϵ and g^ϵ depend continuously on $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 > 0$. The saddle is at $(\epsilon, 0)$ whereas the node is at the origin for any $\epsilon \in (0, \epsilon_0)$. It is well-known that the saddle's stable and unstable manifolds, W^s and W^u, are analytic. The linearization of the node has eigenvalues $-\epsilon$ and -1. It is therefore resonant for $\epsilon^{-1} \in \mathbb{N}$. When the node is nonresonant ($\epsilon^{-1} \notin \mathbb{N}$) it is known [6, Theorem 2.15] that the node can be linearized locally by an analytic change of coordinates to the form

$$\begin{aligned} \dot{x} &= -\epsilon x, \\ \dot{y} &= -y. \end{aligned}$$

Here $x = 0$ is the strong-stable manifold W^{ss}, which is analytic. The invariant curves $y = c|x|^{\epsilon^{-1}}$, $c \neq 0$, tangent to the weak eigendirection at $x = 0$, are all weak-stable invariant manifolds with finite smoothness at $x = 0$. The set $y = 0$ is therefore the unique analytic weak-stable manifold W^{ws}. At a resonance $\epsilon^{-1} = N \in \mathbb{N}$, the node can be brought into the analytic normal form

$$\begin{aligned} \dot{x} &= -N^{-1}x, \\ \dot{y} &= -y + bx^N, \end{aligned}$$

see [6, Theorem 2.15]. In the generic case $b \neq 0$, all weak-stable manifolds have finite smoothness at the origin (due to logarithms). Specifically, there is no analytic weak-stable manifold in this case. Note that this classification in the context of the normal form (1) is (clearly) nonuniform with respect to $\epsilon > 0$.

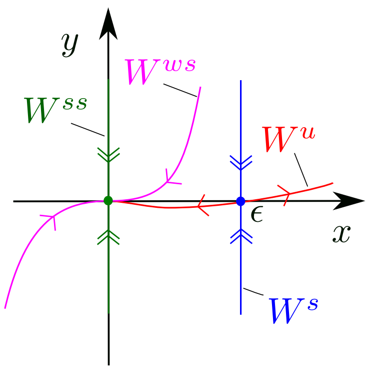

Figure 1: Phaseportrait of (1) for $\epsilon > 0$, $\epsilon^{-1} \notin \mathbb{N}$, with a hyperbolic saddle at $(\epsilon, 0)$ with stable and unstable manifolds (W^s and W^u in blue and red, respectively) and a hyperbolic proper node at the origin. The node always has a unique strong-stable manifold (W^{ss} in green) and in the nonresonant case ($\epsilon^{-1} \notin \mathbb{N}$) a unique analytic weak-stable manifold (W^{ws} in magenta).

In the present paper, we provide a detailed description of the analytic weak-stable manifold W^{ws} of (1) for all $0 < \epsilon \ll 1$ (see our Assumptions 1 and 2 below). Our overall strategy follows [10]. Here the authors constructed C^∞-smooth invariant manifolds (for a specific polynomial system) by matching a global unstable manifold with an analytic weak-stable manifold close to a saddle-node $\epsilon\to 0$. These invariant manifolds correspond to C^∞-smooth self-similar solutions of the isentropic ideal compressible Euler equations that were used in [11] to determine finite time energy blowup solutions of Navier-Stokes equations (isentropic ideal compressible), see also [9] for applications to the defocusing nonlinear Schrödinger equation.

In order to control the analytic weak-stable manifolds, the authors of [10] first apply a new approach for the center manifold W^c at $\epsilon = 0$. In particular, they define a number S^0_∞, which depends on the nonlinearity (in our case, it will depend on the full jet of g^0), and show that if this quantity is nonzero $S^0_\infty \neq 0$, then a "leading order term" of the analytic weak-stable manifold can be determined. The proof of the main result of [10] is not based upon dynamical systems theory but rather on careful estimation and boot-strapping arguments in order to bound the growth of the coefficients of certain series expansions. In this paper, we consider a general case (1), as opposed to a specific polynomial g^ϵ as in [10], and – being inspired by the use of Nagumo-norms in e.g. [3–5] – set up fix-point equations for the series expansions of the center manifold W^c and the analytic weak-stable manifold W^{ws}. For the center manifold, we solve the relevant fix-point equation using Banach's fix-point theorem on a Banach space of formal series $\sum_{k=2}^\infty m_k^0 x^k$ with the norm defined by $\sup_{k\ge 2}\frac{|m_k^0|}{\Gamma(k+a^0)}$. Here Γ is the gamma function. This approach has the benefit that it provides the bound on the fix-point directly. We will consider a simplified setting, where

$$g^\epsilon(x,y) - g^\epsilon(x,0) = \mathcal{O}(\mu), \tag{2}$$

with $0 \le \mu < \mu_0$ small enough, see further details below. Here μ_0 is independent of $\epsilon \ge 0$. We conjecture that our results are true without this assumption (i.e. for any analytic and generic unfolding of a saddle-node with $a^0 > -2$, see Assumption 1 below), but leave this to future work (see Section 5). We feel that (2) gives a suitable forum to present the phenomenon in an accessible way.

The condition $S^0_\infty \neq 0$ will imply that the center manifold is nonanalytic, and a consequence of our results is also that

$$S^0_\infty \neq 0 \Longrightarrow W^{ws}\cap W^u = \emptyset \quad \forall\, 0<\epsilon\ll 1,\, \epsilon^{-1}\notin\mathbb{N}, \tag{3}$$

see Fig. 1. This result is in line with the statement in [14, Example 3.1, p. 13], which says that

$$W^c \notin C^\omega \Longrightarrow W^{ws}\cap W^u = \emptyset \quad \forall\, 0<\epsilon\ll 1,\, \epsilon^{-1}\notin\mathbb{N}.$$

This is of course the generic situation. However, our results for $S^0_\infty \neq 0$ allow us to determine on what side of W^u the analytic weak-stable manifold W^{ws} lands.

1.1 Further background

A first glimpse of the phenomenon that we want to study, can be obtained from the following simple example:

$$\begin{aligned}\dot x &= -\epsilon x,\\ \dot y &= -y + u(x),\end{aligned} \tag{4}$$

with

$$u(x) = \sum_{k=2}^\infty u_k x^k, \quad |u_k| \le B\rho^{-k},$$

being analytic on the open disc $|x|<\rho$. Here $x=0$ is the strong stable manifold W^{ss} and for $\epsilon^{-1}\notin\mathbb{N}$ the analytic weak-stable manifold W^{ws} exists and takes the graph form

$$y = m^\epsilon(x), \quad m(x) = \sum_{k=2}^\infty \frac{u_k}{1-\epsilon k}x^k \quad \forall\, 0 \le |\overline{x}| < \rho. \tag{5}$$

This follows from a simple calculation. Notice that there are small divisors in the expression for m^ϵ for $\epsilon \approx \frac{1}{N}$, $N\in\mathbb{N}$ (the resonances). Let

$$\epsilon^{-1} = N^\epsilon + \alpha^\epsilon \notin \mathbb{N}, \quad N^\epsilon := \lfloor \epsilon^{-1}\rfloor, \quad \alpha^\epsilon \in (0,1),$$

and define

$$V^\epsilon(x) = N^\epsilon \frac{u_{N^\epsilon}}{\alpha^\epsilon} x^{N^\epsilon} - (N^\epsilon+1)\frac{u_{N^\epsilon+1}}{1-\alpha^\epsilon} x^{N^\epsilon+1}. \tag{6}$$

Then the sum of the terms in (5) with $k = N^\epsilon$ and $k = N^\epsilon + 1$ takes the following form

$$\begin{aligned}
\sum_{k=N^\epsilon}^{N^\epsilon+1} \frac{u_k}{1-\epsilon k} x^k &= \epsilon^{-1}\frac{u_{N^\epsilon}}{\alpha^\epsilon} x^{N^\epsilon} - \epsilon^{-1}\frac{u_{N^\epsilon+1}}{1-\alpha^\epsilon} x^{N^\epsilon+1} \\
&= \left(N^\epsilon \frac{u_{N^\epsilon}}{\alpha^\epsilon} + u_{N^\epsilon}\right) x^{N^\epsilon} - \left((N^\epsilon+1)\frac{u_{N^\epsilon+1}}{1-\alpha^\epsilon} - u_{N^\epsilon+1}\right) x^{N^\epsilon+1} \\
&= V^\epsilon(x) + u_{N^\epsilon} x^{N^\epsilon} + u_{N^\epsilon+1} x^{N^\epsilon+1}.
\end{aligned}$$

It follows that $B^\epsilon := m^\epsilon - V^\epsilon$ is uniformly bounded with respect to $\alpha^\epsilon \in [0,1)$, and for any $\upsilon > 0$ there is a $\delta > 0$ such that

$$|B^\epsilon(x)| \le \upsilon \quad \forall\, 0 \le |x| \le \delta,\ \alpha^\epsilon \in (0,1). \tag{7}$$

The function V^ϵ, on the other hand, is not uniformly bounded if $u_{N^\epsilon} \neq 0$ or $u_{N^\epsilon+1} \neq 0$. Specifically, if $u_{N^\epsilon} u_{N^\epsilon+1} \neq 0$ then it follows that *we can track W^{ws} : $y = m^\epsilon(x)$ through $y = V^\epsilon(x)$ for $x \neq 0$, $\alpha^\epsilon \to 0^+$ and $\alpha^\epsilon \to 1^-$* (since $V^\epsilon(x)$, $x \neq 0$, goes unbounded in these limits). Here by "track" we will mean that the position of W^{ws} can qualitatively be determined as follows:

Lemma 1.1. *Fix $N^\epsilon \in \mathbb{N}$, $K > 0$, suppose that $u_{N^\epsilon} \neq 0$ and define $s = \operatorname{sign}(u_{N^\epsilon})$. Let W^{ws} : $y = m^\epsilon(x)$, $0 \le |x| < \rho$, denote the analytic weak-stable manifold of (4), $\epsilon^{-1} \notin \mathbb{N}$. Then the following holds true for all $0 < \delta \ll 1$:*

The position of W^{ws} for all $0 < \alpha^\epsilon < N^\epsilon \frac{|u_{N^\epsilon}|}{K}\delta^{N^\epsilon}$ can be determined as follows:

1. *Suppose that N^ϵ is even. Then W^{ws} intersects $\{y = \frac{sK}{2}\}$ for both $-\delta < x < 0$ and $0 < x < \delta$.*
2. *Suppose that N^ϵ is odd. Then W^{ws} intersects $\{y = -\frac{sK}{2}\}$ for $-\delta < x < 0$ and $\{y = \frac{sK}{2}\}$ for $0 < x < \delta$.*

Proof. For $u_{N^\epsilon} \neq 0$, we have

$$\left| N^\epsilon \frac{u_{N^\epsilon}}{\alpha^\epsilon}\delta^{N^\epsilon}\right| > K \quad \forall\, 0 < \alpha^\epsilon < N^\epsilon \frac{|u_{N^\epsilon}|}{K}\delta^{N^\epsilon}.$$

Then from (6), we obtain that the following holds true for $x = -\delta$ and $x = \delta$:

$$|V^\epsilon(x)| \ge \frac{3K}{4} \quad \forall\, 0 < \alpha^\epsilon < N^\epsilon \frac{|u_{N^\epsilon}|}{K}\delta^{N},\ 0 < \delta \ll 1.$$

Then by using $m^\epsilon = B^\epsilon + V^\epsilon$ and (7) with $0 < \upsilon \ll K$ for $0 < \delta \ll 1$ the result follows. ☐

A similar result holds for $0 < 1 - \alpha^\epsilon \ll 1$ if $u_{N^\epsilon+1} \neq 0$, which we state without proof.

Lemma 1.2. *Fix $N^\epsilon \in \mathbb{N}$, $K > 0$, suppose that $u_{N^\epsilon+1} \neq 0$ and define $s = \operatorname{sign}(u_{N^\epsilon+1})$. Let W^{ws} : $y = m^\epsilon(x)$, $0 \le |x| < \rho$, denote the analytic weak-stable manifold of (4), $\epsilon^{-1} \notin \mathbb{N}$. Then the following holds for all $0 < \delta \ll 1$:*

The position of W^{ws} for all $0 < 1 - \alpha^\epsilon < (N^\epsilon + 1)\frac{|u_{N^\epsilon+1}|}{K}\delta^{N^\epsilon+1}$ can be determined as follows:

1. *Suppose that N^ϵ is even. Then W^{ws} intersects $\{y = \frac{sK}{2}\}$ for $-\delta < x < 0$ and $\{y = -\frac{sK}{2}\}$ for $0 < x < \delta$.*
2. *Suppose that N^ϵ is odd. Then W^{ws} intersects $\{y = -\frac{sK}{2}\}$ for both $-\delta < x < 0$ and $0 < x < \delta$.*

By Lemma 1.1 and Lemma 1.2, we obtain a "flapping phenomenon" when $u_{N^\epsilon} u_{N^\epsilon+1} \neq 0$, *whereby the position of W^{ws} (at least on one side of the node) changes dramatically as α^ϵ transverses the interval* $(0,1)$. We illustrate this flapping phenomena in Fig. 2 for $u_{N^\epsilon} > 0, u_{N^\epsilon+1} < 0$ (which is relevant for (27) with $0 < \epsilon \ll 1$, please compare with Fig. 5). It is essentially this flapping mechanism that (together with

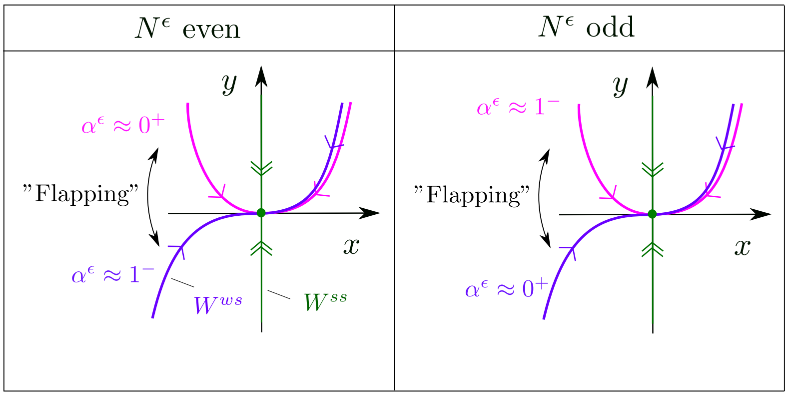

Figure 2: The “flapping mechanism” of the analytic weak-stable manifolds (W^{ws} in magenta and purple) of (4) for $u_{N^\epsilon} > 0, u_{N^\epsilon+1} < 0$.

a basic continuity argument) allows the authors of [10] to connect their analytic weak-stable manifolds with a global analytic manifold (that does not “flap”) and construct C^∞-smooth self-similar solutions close to resonances (and close to a saddle-node where the resonances accumulate).

For a general (fully nonlinear) analytic system, quantities corresponding to u_{N^ϵ} and $u_{N^\epsilon+1}$ for a hyperbolic node can in principle be computed for any fixed N^ϵ in terms of the jet of the nonlinearity (through normal form computations [6, Chapter 2]). But in the context of (1), our results show that the condition $u_{N^\epsilon}u_{N^\epsilon+1} \neq 0$ can be related to the lack of analyticity of the center manifold $y = m^0(x)$ of the origin for $\epsilon = 0$:

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y(1+a^0x) + g^0(x,y). \end{aligned} \tag{8}$$

We therefore now review some basic facts about center manifolds. It is known that m^0 is C^∞-smooth in this planar context, see [6, Theorem 2.19]. It is also known that it is in general nonunique (see Fig. 3 for $x < 0$) and nonanalytic. As an example of a nonanalytic center manifold, consider $a^0 = 0$, $g^0(x,y) = x^2$ for $\epsilon = 0$. As a first order system

$$x^2 \frac{dy}{dx} = -y + x^2,$$

this y-linear case corresponds to Euler's famous example written in the form of (8). Here one can easily show that $y = \sum_{k=2}^\infty m_k^0 x^k$ leads to

$$m_k^0 = (-1)^k (k-1)! \quad \forall\, k \geq 2.$$

Consequently, we have $m_k^0 x^k \not\to 0$ as $k \to \infty$ for any $x \neq 0$ and the center manifold is therefore nonanalytic.

In general, it is known, see e.g. [3], that the expansion of the center manifold $y = m^0(x)$ for (8) as a formal series

$$y = \sum_{k=2}^\infty m_k^0 x^k, \tag{9}$$

is Gevrey-1:

$$|m_k^0| \leq C D^{-k} k! \quad \forall\, k \geq 2, \tag{10}$$

for some $C, D > 0$. This formal series is 1-summable along any sector that is not centered along the negative real axis, see [1, Chapter 3] and [3] for further details.

We will in this paper show (under the conditions of Assumption 1 (on $a^0 > -2$) and Assumption 2 (on the nonlinearity, see also (2))) that the bound on m_k^0 can be improved such that

$$|m_k^0| \leq F\Gamma(k + a^0) \quad \forall\, k \geq 2, \tag{11}$$

for some $F>0$ independent of k. Here Γ is the gamma function, see Section 1.2 below, and by Stirling's formula (see (22) below):

$$\Gamma(k+a^0) = (1+o(1))k!k^{a^0-1}, \tag{12}$$

for $k \gg 1$. Therefore (11) is Gevrey-1 (see (10)) for any $D<1$, also $D=1$ if $a^0 \le 1$. The bound (11) agrees with [10, Lemma 5.4] on their specific nonlinearity.

To illustrate how the bound (11) occur, consider the case $g^0(x,y)=f^0(x)$ (where g^0 is independent of y) in (8) written in the equivalent form:

$$x^2\frac{dy}{dx} + y(1+a^0x) = f^0(x), \quad f^0(x) = \sum_{k=2}^{\infty} f_k^0 x^k. \tag{13}$$

This differential equation for $y=y(x)$ is linear in y and one can solve explicitly for the m_k^0's of the formal series. Indeed, inserting the formal series (9) into (13) leads to

$$\sum_{k=2}^{\infty}\left((k+a^0)m_k x^{k+1} + m_k^0 x^k\right) = \sum_{k=2}^{\infty} f_k x^k,$$

and therefore to the recursion relation:

$$m_k^0 + (k-1+a^0)m_{k-1}^0 = f_k^0 \quad \forall\, k\ge 2, \tag{14}$$

with $m_1^0=0$.

Lemma 1.3. *Suppose that $a^0>-2$ and define*

$$S_k^0 := \sum_{j=2}^{k}\frac{(-1)^j f_j^0}{\Gamma(j+a^0)}. \tag{15}$$

Then the solution of the recursion relation (14) *with $m_1^0=0$ is*

$$m_k^0 = (-1)^k\Gamma(k+a^0)S_k^0. \tag{16}$$

Proof. The result can easily be proven by induction using the base case $m_1^0=0$ and the basic property of the gamma function: $\Gamma(z+1)=z\Gamma(z)$, see (19), in the induction step. □

Seeing that f^0 is analytic, we have

$$|f_k^0| \le B\rho^{-k},$$

for some $B>0$, $\rho>0$, and the sum

$$S_\infty^0 := \lim_{k\to\infty} S_k^0 = \sum_{j=2}^{\infty}\frac{(-1)^j f_j^0}{\Gamma(j+a^0)},$$

see (15), is therefore absolutely convergent for any $a^0>-2$:

$$|S_\infty^0| \le \sum_{j=2}^{\infty}\frac{|f_j^0|}{\Gamma(j+a^0)} \le F := B\sum_{j=2}^{\infty}\frac{\rho^{-j}}{\Gamma(j+a^0)} < \infty.$$

The property (11) therefore follows from (16) in the context of (13).

Notice that if $S_\infty^0 \ne 0$, then by (12) we have

$$m_k^0 = (-1)^k\Gamma(k+a^0)S_k^0 = (-1)^k(1+o(1))S_\infty^0\Gamma(k+a^0) = (-1)^k(1+o(1))S_\infty^0 k^{a^0-1}k! \tag{17}$$

for all $k\gg 1$. This implies that the center manifold is nonanalytic. In the linear case (13), it is also possible to go the other way. We collect this in the following lemma:

Lemma 1.4. *Suppose that $a^0 > -2$. Then the center manifold of the linear system* (13) *is analytic if and only if $S_\infty^0 = 0$.*

Proof. $\Rightarrow$: From (17), we have that for any $x \neq 0$, $m_k^0 x^k \not\to 0$ for $k \to 0$. Consequently, if $S_\infty^0 \neq 0$ then the center manifold is nonanalytic.

$\Leftarrow$: If $S_\infty^0 = 0$ then

$$|S_k^0| = |S_\infty^0 - S_k^0| \leq B \sum_{j=k+1}^{\infty} \frac{\rho^{-j}}{\Gamma(j+a^0)},$$

and for any $k \geq k_0(a^0)$, $j \in \mathbb{N}_0$:

$$\frac{\Gamma(k+a^0)}{\Gamma(k+1+j+a^0)} = \frac{(k-1)!}{(k+j)!}(1+o_{k_0\to\infty}(1))\left(\frac{k}{k+1+j}\right)^{a^0} \leq \frac{2}{k^j}\left(\frac{k}{k+1+j}\right)^{-2} \leq 8(1+j)^2 k^{-j},$$

with $k_0 \gg 1$, using Stirling's formula (see (22) below) and

$$\left(\frac{k+1+j}{k}\right)^2 \leq \left(\frac{2k(1+j)}{k}\right)^2 = 4(1+j)^2.$$

Then upon using $\sum_{j=0}^{\infty} 2^{-j}(1+j)^2 = 12$ it follows that

$$|m_k^0| \leq 8B\rho^{-k-1}\sum_{j=0}^{\infty}(\rho k)^{-j}(1+j)^2 \leq 96B\rho^{-k-1} \quad \forall\, k \geq k_0 \geq 2\rho^{-1}.$$

We conclude that $\sum_{k=2}^{\infty} m_k^0 x^k$ converges absolutely for all $0 \leq |x| < \rho$ if $S_\infty^0 = 0$. ☐

A first important step of our approach is to carry the classification of the analyticity of the center manifold for $\epsilon = 0$ over to the nonlinear case. For this, we will use a fix-point argument in an appropriate Banach space of formal series. This leads to the definition of S_∞^0 for a nonlinearity g^0, satisfying the Assumptions 1 and 2 below. If $S_\infty^0 \neq 0$ then the center manifold will be nonanalytic.

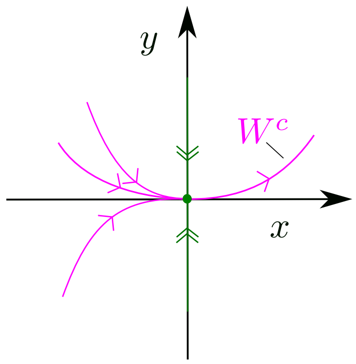

Figure 3: The saddle node for $\epsilon = 0$. The center manifold (W^c in magenta) is only unique on the positive side of $x = 0$.

Subsequently, for $\epsilon > 0$ and $S_\infty^0 \neq 0$, we (essentially) expand the analytic weak-stable invariant manifold $y = m^\epsilon(x)$ into the form

$$m^\epsilon = B^\epsilon + (-1)^{N^\epsilon} S_\infty^0 V^\epsilon, \quad N^\epsilon = \lfloor \epsilon^{-1} \rfloor,$$

on a subset $x \in I^\epsilon$, where (in essence, see Theorem 2.5 for details) *only B^ϵ is uniformly bounded with respect to $\alpha^\epsilon = \epsilon^{-1} - \lfloor \epsilon^{-1} \rfloor \in (0,1)$*. We will therefore *track $y = m^\epsilon(x)$ for $S_\infty^0 \neq 0$ using $y = (-1)^{N^\epsilon} S_\infty^0 V^\epsilon(x)$ for $\alpha^\epsilon \to 0^+$ and $\alpha^\epsilon \to 1^-$* as in the example (4) above. (It would be more accurate to say that the tracking will first be done in scaled coordinates, see (36), and that $V^\epsilon(x) = \epsilon \overline{V}^\epsilon(\epsilon^{-1}x)$, see (39). Moreover, $x > 0$ and $x < 0$ will be treated slightly different, but we refer the reader to further details and the precise statements below.) In this context, it is worth pointing out that $S^0_\infty \neq 0$ essentially ensures that a condition like $u_{N^\epsilon} u_{N^\epsilon+1} \neq 0$ holds true near all resonances $\epsilon^{-1} \in \mathbb{N}$ for $0 < \epsilon \ll 1$, see Theorem 2.5 and Corollary 2.6.

While the discovery of the phenomenon in a specific problem is due to [10], our treatment of the underlying general mechanism is novel and more in the spirit of dynamical systems theory. We also feel that our proof streamlines the approach of [10], used for their specific nonlinearity (rational with numerator cubic, denominator quadratic, see [10, Eqs. (1.9)–(1.10)]). Moreover, we will perform the important estimates not by brute force calculations but by using appropriate fix-point arguments in suitable normed spaces.

1.2 Basic properties of the gamma function

The gamma function $z \mapsto \Gamma(z)$, defined for $\mathrm{Re}(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{18}$$

will play a crucial role in the following. We therefore collect a few well-known facts (see e.g. [12, Chapter 5]) that will be used throughout the manuscript.

First, we recall that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$, which follows from $\Gamma(1) = 1$ and the basic property

$$\Gamma(z+1) = z\Gamma(z) \quad \forall\, \mathrm{Re}(z) > 0. \tag{19}$$

The gamma function can be analytically extended to the whole complex plane except zero and the negative integers (which are all simple poles); specifically,

$$\lim_{x\to 0} x\Gamma(x) = \Gamma(1) = 1, \tag{20}$$

In this paper, we will use Stirling's well-known formula:

$$\Gamma(x+1) = (1+o(1))\sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \tag{21}$$

for $x \to \infty$. In fact, we will often use it in the following form:

$$\frac{\Gamma(x+b)}{\Gamma(x)} = (1+o(1))x^b, \tag{22}$$

for $b \in \mathbb{R}$ and $x \to \infty$, which can be obtained directly from (21). We will also use the reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \forall\, z \notin \mathbb{Z}. \tag{23}$$

and the Euler integral of the first kind:

$$\int_0^1 (1-v)^{x-1} v^{y-1} dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \forall\, x, y > 0. \tag{24}$$

Finally, the digamma function ϕ is defined as the logarithmic derivative of the gamma function:

$$\phi(z) := \frac{\Gamma'(z)}{\Gamma(z)}. \tag{25}$$

It has a unique positive zero at $z \approx 1.4616312...$ and $\phi(z)$ is positive for all z-values larger than this number. It will be particularly important to us that ϕ is an increasing function of $z > 0$:

$$\phi'(z) > 0. \tag{26}$$

2 Main results

We first state a general result (based upon [13, Theorem 2.2]) on saddle-nodes.

Theorem 2.1. *For any analytic and generic family of two-dimensional vector-fields unfolding a saddle-node, there exists a locally defined analytic change of coordinates, depending continously on a parameter* $\epsilon \in [0, \epsilon_0)$, $0 < \epsilon_0 \ll 1$, *such that on the singularity-side* ($\epsilon \geq 0$) *of the bifurcation, the system takes the following normal form:*

$$\begin{aligned} \dot{x} &= (x - \epsilon)x, \\ \dot{y} &= -y(1 + a^\epsilon x) + g^\epsilon(x, y), \end{aligned} \tag{27}$$

where

$$\begin{gathered} g^\epsilon(x, y) = f^\epsilon(x) + u^\epsilon(x, y) \\ f^\epsilon(x) = \sum_{k=2}^{\infty} f_k^\epsilon x^k, \quad u^\epsilon(x, y) = \sum_{k=2}^{\infty} u_{k,1}^\epsilon x^k y + \sum_{k=1}^{\infty} \sum_{l=2}^{\infty} u_{k,l}^\epsilon x^k y^l. \end{gathered} \tag{28}$$

In particular, the following holds regarding the absolutely convergent power series expansions of f^ϵ *and* u^ϵ *for all* $\rho > 0$ *small enough: Let*

$$D_1 := [0, \epsilon_0) \times \{0 \leq |x| < \rho\}, \quad D_2 := [0, \epsilon_0) \times \{0 \leq |x| < \rho\} \times \{0 \leq |y| < \rho\},$$

and define

$$B := \sup_{(\epsilon,x) \in D_1} |f^\epsilon(x)|, \quad \mu := \sup_{(\epsilon,x,y) \in D_2} |u^\epsilon(x, y)|. \tag{29}$$

Then

$$|f_k^\epsilon| \leq B\rho^{-k}, \quad |u_{k,l}^\epsilon| \leq \mu\rho^{-k-l} \quad \textit{and} \quad u_{k,1}^0 = 0 \quad \forall\, k, l \in \mathbb{N},\, \epsilon \in [0, \epsilon_0) \tag{30}$$

The proof of Theorem 2.1 (available in Appendix A) is obtained by applying three elementary transformations to the normal form in [13, Theorem 2.2].

In the remainder of the paper, we will assume the following regarding (27) and (28):

Assumption 1. *The following inequality holds true:*

$$a^0 := \lim_{\epsilon \to 0} a^\epsilon > -2.$$

Assumption 2. B *and* $\rho > 0$ *are fixed and* $\mu > 0$ *in* (29) *is a parameter that is small enough (see details below).*

Following Assumption 2, we will henceforth write

$$u^\epsilon = \mu h^\epsilon \quad \text{and} \quad u_{k,l}^\epsilon = \mu h_{k,l}^\epsilon,$$

so that g^ϵ in (27) becomes

$$g^\epsilon(x, y) =: f^\epsilon(x) + \mu h^\epsilon(x, y), \tag{31}$$

where

$$f^\epsilon(x) = \sum_{k=2}^{\infty} f_k^\epsilon x^k, \quad h^\epsilon(x, y) = \sum_{k=2}^{\infty} h_{k,1}^\epsilon x^k y + \sum_{k=1}^{\infty} \sum_{l=2}^{\infty} h_{k,l}^\epsilon x^k y^l, \tag{32}$$

with

$$|f_k^\epsilon| \leq B\rho^{-k}, \quad |h_{k,l}^\epsilon| \leq \rho^{-k-l} \quad \text{and} \quad h_{k,1}^0 = 0 \quad \forall\, k, l \in \mathbb{N},\, \epsilon \in [0, \epsilon_0). \tag{33}$$

The results below will be stated for (27) **with** g^ϵ **given by** (31) **for** $0 < \mu \ll 1$ **(in accordance with Assumption 2).**

The reference [10] also assumes a condition like Assumption 1 (see [10, Eq. 5.3]) in the context of their specific rational example of an analytic unfolding, see [10, Eqs. (1.9)–(1.10)]. On the other hand, a condition like Assumption 2, which can also be viewed as (2), does not appear in [10]. We conjecture that our results are true without Assumption 2 (and therefore holds for any analytic and generic unfolding of a saddle-node with $a^0 > -2$), but leave this extension to future work. We will discuss the matter further in Section 5.

Remark 1. *Assumption 2 is only an assumption on the nonlinearity in y. Indeed, by the last equality in* (33) *and continuity with respect to ϵ, we have $h^{\epsilon}_{k,1} = o(1)$ for $\epsilon \to 0$ uniformly in $k \in \mathbb{N}$.*

Remark 2. *Obviously, from* (29) *we have $\mu = \mathcal{O}(\rho^3)$ as $\rho \to 0$ in general (since g^{ϵ} starts with cubic terms) and in this sense one can achieve μ small by taking $\rho > 0$ small. But this will not be helpful to us (and we do not expect it to be useful in general). This is in contrast to arguments based upon Nagumo norms (see e.g. [5]), where the size of the domain can be used as a small parameter to obtain the appropriate contraction of a fix-point formulation of Gevrey-properties of formal series. At this stage, our approach in the present paper requires B and $\rho > 0$ fixed and $\mu > 0$ small enough, as stated in Assumption 2.*

Our first main result relates to the center manifold.

Theorem 2.2. *Consider* (27) *with g^{ϵ} given by* (31) *for $\epsilon = 0$:*

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y(1 + a^0 x) + g^0(x, y), \end{aligned} \tag{34}$$

and suppose that Assumptions 1 and 2 hold true. Let $W^c : y = m^0(x)$, $m^0(0) = 0$, with m^0 defined in a neighborhood of $x = 0$, denote the center manifold of $(x, y) = (0, 0)$. Then there is a $\mu_0 > 0$ such that for all $0 \le \mu < \mu_0$ the following statements hold true:

1. *There exists a number S^0_{∞}, which depends upon the full jet of g^0, such that if $S^0_{\infty} \neq 0$ then the following holds:*

   (a)

   $$\left| \frac{1}{k!} \frac{d^k}{dx^k} m^0(0) \right| = (1 + o_{k\to\infty}(1)) S_{\infty} \Gamma(k + a^0) \quad \textit{for} \quad k \to \infty.$$

   (b) The center manifold W^c is nonanalytic.

2. *$S^0_{\infty} = S^0_{\infty}(f^0_2)$ is a C^1-function with respect to f^0_2 (as well as all other parameters of the system), recall* (32), *satisfying*

   $$\frac{\partial S^0_{\infty}}{\partial f^0_2}(f^0_2) = \frac{1}{\Gamma(2 + a^0)} + \mathcal{O}(\mu) \neq 0.$$

The second statement shows that the center manifold being nonanalytic for (27), under the Assumptions (1) and (2), is a generic condition. We exemplify this as follows:

Corollary 2.3. *Suppose that the conditions of Theorem 2.2 hold true, in particular $0 < \mu \ll 1$ so that $\frac{\partial S^0_{\infty}}{\partial f^0_2}(f^0_2) \neq 0$, and suppose that the center manifold of* (34) *is analytic. Then the center manifold of the perturbed system*

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y(1 + a^0 x) + g^0(x, y) + q x^2, \end{aligned}$$

is nonanalytic for all $q \neq 0$ small enough.

Proof. This should be clear enough. $\square$

Remark 3. *The generic property of the nonanalyticity of the center manifold could also be stated (more abstractly) in terms of $S^0_{\infty} = 0$ being the zero set of an analytic function of the parameters.*

Our next result relates to the analytic weak-stable invariant manifold W^{ws} of (27). To present this, we will first write (27) in the equivalent form

$$x(x - \epsilon) \frac{dy}{dx} + y(1 + a^{\epsilon} x) = g^{\epsilon}(x, y). \tag{35}$$

For all $\epsilon^{-1} \notin \mathbb{N}$, W^{ws} takes the graph form

$$y = m^\epsilon(x) = \sum_{k=2}^{\infty} m_k^\epsilon x^k;$$

with the last equality valid locally $x \in (-\delta^\epsilon, \delta^\epsilon)$, $\lim_{\epsilon\to 0} \delta^\epsilon = 0$. In particular, $y = m^\epsilon(x)$ is a (locally defined) solution of (35).

Now, the blowup transformation defined by

$$x = \epsilon\overline{x}, \quad y = \epsilon\overline{y}, \tag{36}$$

for all $\epsilon > 0$, separates the node and the saddle, so that the latter is at $(\overline{x}, \overline{y}) = (1, 0)$. Inserting (36) into (35) gives

$$\epsilon\overline{x}(\overline{x} - 1)\frac{d\overline{y}}{d\overline{x}} + \overline{y}(1 + \epsilon a^\epsilon \overline{x}) = \epsilon^{-1} g^\epsilon(\epsilon\overline{x}, \epsilon\overline{y}), \tag{37}$$

where

$$\epsilon^{-1} g^\epsilon(\epsilon\overline{x}, \epsilon\overline{y}) =: \epsilon\overline{f}^\epsilon(\overline{x}) + \epsilon\mu\overline{h}^\epsilon(\overline{x}, \overline{y}),$$
$$\overline{f}^\epsilon(\overline{x}) = \sum_{k=2}^{\infty} f_k^\epsilon \epsilon^{k-2}\overline{x}^k, \quad \overline{h}^\epsilon(\overline{x}, \overline{y}) = \sum_{k=2}^{\infty} h_{k,1}^\epsilon \epsilon^{k-1}\overline{x}^k\overline{y} + \sum_{k=1}^{\infty}\sum_{l=2}^{\infty} h_{k,l}^\epsilon \epsilon^{k+l-2}\overline{x}^k\overline{y}^l.$$

In these coordinates, (37) is a singularly perturbed system with respect to $0 < \epsilon \ll 1$ and W^{ws} takes the following form

$$\overline{y} = \overline{m}^\epsilon(\overline{x}) := \epsilon^{-1} m^\epsilon(\epsilon\overline{x}) = \sum_{k=2}^{\infty} \epsilon^{k-1} m_k^\epsilon \overline{x}^k,$$

where the last equality again holds true locally ($\overline{x} \in (-\epsilon^{-1}\delta^\epsilon, \epsilon^{-1}\delta^\epsilon)$). In the language of Geometric Singular Perturbation Theory (GSPT) [7,8], the set $\{\overline{y} = 0\}$ is a normally hyperbolic and attracting critical manifold in the $(\overline{x}, \overline{y})$-plane of $(37)_{\epsilon=0}$. Therefore there is a (nonunique) slow manifold as a graph $\overline{y} = \mathcal{O}(\epsilon)$ over a compact subset $\overline{x} \in I$. This slow manifold only has finite smoothness in general, see [7]. However, the unstable manifold W^u of the saddle $(\overline{x}, \overline{y}) = (1, 0)$ in the $(\overline{x}, \overline{y})$-coordinates is an example of an analytic slow manifold of the following graph form:

$$W^u: \quad \overline{y} = \epsilon\overline{H}^\epsilon(\overline{x}) \quad \overline{x} \in (0, 2] \quad \overline{H}^\epsilon(0^+) = 0; \tag{38}$$

here $\overline{H}^\epsilon(\overline{x})$ extends C^k-smoothly ($1 \le k < \infty$, specifically not analytically, see Corollary 2.6 item 1) to $\overline{x} = 0$. We will also need the following lemma (which we prove in Section 4.5).

Lemma 2.4. *Suppose that $\epsilon^{-1} \notin \mathbb{N}$, $0 < \epsilon \ll 1$, and write*

$$\epsilon^{-1} =: N^\epsilon + \alpha^\epsilon, \quad N^\epsilon := \lfloor \epsilon^{-1} \rfloor \quad \textit{and} \quad \alpha^\epsilon \in (0, 1).$$

Then the following holds true.

1. *The series*

$$\overline{V}^\epsilon(\overline{x}) := \frac{\Gamma(\alpha^\epsilon)\Gamma(1 - \alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})} \sum_{k=N^\epsilon}^{\infty} \frac{\Gamma(k + a^\epsilon)}{\Gamma(k + 1 - \epsilon^{-1})}\overline{x}^k, \tag{39}$$

is absolutely convergent for all $0 \le |\overline{x}| < 1$; in particular $\overline{V}^\epsilon(0) = 0$,

$$\overline{V}^\epsilon(\overline{x}) > 0, \quad \frac{d}{d\overline{x}}\overline{V}^\epsilon(\overline{x}) > 0 \quad \forall\, \overline{x} \in (0, 1), \tag{40}$$

2. *Lower bound:*

$$\overline{V}^\epsilon(\overline{x}) \ge \epsilon \left(\frac{\overline{x}}{1 - \overline{x}}\right)^{N^\epsilon + 1} \quad \forall\, 0 \le \overline{x} \le \frac{3}{4}. \tag{41}$$

3. At the same time, for any $0 < |\overline{x}| < 1$,

$$|\overline{V}^\epsilon(\overline{x})| \to \infty \quad \textit{for} \quad \alpha^\epsilon \to 0^+ \textit{ and } 1^-.$$

4. Asymptotics for $\overline{x} = \mathcal{O}(\epsilon)$*: Let* $\overline{x} = \epsilon\overline{x}_2 \in [-\epsilon\delta_2, \epsilon\delta_2]$*,* $\delta_2 > 0$ *fixed. Then for all* $0 < \epsilon \ll 1$*,* $\epsilon^{-1} \notin \mathbb{N}$*:*

$$\begin{aligned}\overline{V}^\epsilon(\epsilon\overline{x}_2) &= (1+o(1))\Gamma(\alpha^\epsilon)(N^\epsilon)^{\alpha^\epsilon+1-\alpha^\epsilon}(\epsilon\overline{x}_2)^{N^\epsilon} \\ &\times \left(1 + \frac{\overline{x}_2}{1-\alpha^\epsilon}\left[1 + \overline{x}_2\int_0^1 e^{(1-v)\overline{x}_2} v^{1-\alpha^\epsilon} dv + o(1)\right]\right),\end{aligned} \tag{42}$$

with each $o(1)$ *being uniform with respect to* $\alpha^\epsilon \in (0,1)$*.*

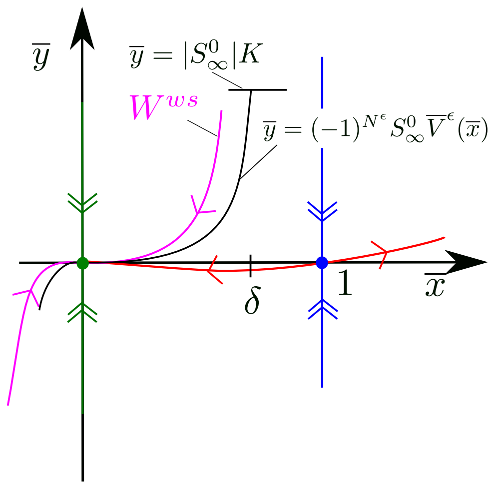

Figure 4: Phaseportrait of (37) for $\epsilon > 0$, $\epsilon^{-1} \notin \mathbb{N}$ (and $(-1)^{N^\epsilon} S^0_\infty > 0$); please compare with Fig. 1. Theorem 2.5 says that if $S^0_\infty \neq 0$ then we can track W^{ws} (in magenta) by the graph $\overline{y} = (-1)^{N^\epsilon} S^0_\infty \overline{V}^\epsilon(\overline{x})$, see also Corollary 2.6 and further details in Theorem 2.5.

Our main result on the analytic weak-stable manifold then takes the following form (see Fig. 4).

Theorem 2.5. *Fix* $K > 0$*,* $\delta_2 > 0$*,* $0 < \upsilon \ll K$ *and consider* (37) *with* g^ϵ *given by* (31)*, satisfying Assumptions 1 and 2. Then the quantity* S^0_∞ *from Theorem 2.2 is well-defined. We suppose that*

$$S^0_\infty \neq 0, \tag{43}$$

so that the center manifold is nonanalytic.

Now, consider the convergent series $\overline{V}^\epsilon$ *defined in* (39)*. Then the following holds for all* $0 < \epsilon \ll 1$*,* $\epsilon^{-1} \notin \mathbb{N}$*: Let* $W^{ws} : \overline{y} = \overline{m}^\epsilon(\overline{x})$*, with* $\overline{m}^\epsilon$ *defined in a neighborhood of the origin, denote the analytic weak-stable manifold in the* $(\overline{x}, \overline{y})$*-coordinates, see* (36)*, and let* $I \subset \left[-\delta_2\epsilon, \frac{3}{4}\right]$ *be an interval so that*

$$|\overline{V}^\epsilon(\overline{x})| \leq K \quad \forall \overline{x} \in I. \tag{44}$$

Then $I \subset \mathrm{domain}(\overline{m}^\epsilon)$ *and*

$$|\overline{m}^\epsilon(\overline{x}) - (-1)^{N^\epsilon} S^0_\infty \overline{V}^\epsilon(\overline{x})| \leq \upsilon \quad \forall \overline{x} \in I. \tag{45}$$

In other words, when (43) holds true, then by taking $0 < \epsilon \ll 1$, *we can track* $W^{ws} : \overline{y} = \overline{m}^\epsilon(\overline{x})$ *through* $\overline{y} = (-1)^{N^\epsilon} S^0_\infty \overline{V}^\epsilon(\overline{x})$. More precisely, we have the following result, which we illustrate for $S^0_\infty > 0$ in Fig. 5; the weak manifold has to be reflected about the x-axis for $S^0_\infty < 0$.

Corollary 2.6. *Fix* $c > 0$ *small enough, suppose that Assumptions 1 and 2 hold true and that* $S^0_\infty \neq 0$*. Put* $s = \mathrm{sign}(S^0_\infty)$ *and let* W^{ws} *denote the analytic weak-stable manifold. Then the following holds true regarding the position of* W^{ws} *for all* $N^\epsilon = \lfloor \epsilon^{-1} \rfloor \gg 1$*:*

Intersections of W^{ws} ***with*** $\{y = \pm c\}$ ***for*** $x > 0$***:***

1. *W^{ws} does not intersect W^u. More precisely, we have the following:*

   (a) *Suppose that N^ϵ is even. Then W^{ws} intersects $\{y = sc\}$ for $x > 0$.*

   (b) *Suppose that N^ϵ is odd. Then W^{ws} intersects $\{y = -sc\}$ for $x > 0$.*

***Intersections of** W^{ws} **with** $\{y = \pm c\}$ **for** $x < 0$:*

Define

$$\underline{\alpha}(N^\epsilon) := (N^\epsilon)^{a^0 - N^\epsilon}, \quad 1 - \overline{\alpha}(N^\epsilon) := (N^\epsilon)^{a^0 - 1 - N^\epsilon}. \tag{46}$$

2. *Suppose that N^ϵ is even. Then the following holds:*

   (a) *W^{ws} intersects $\{y = sc\}$ for $x < 0$ for all $0 < \alpha^\epsilon \leq \underline{\alpha}(N^\epsilon)$.*

   (b) *W^{ws} intersects $\{y = -sc\}$ for $x < 0$ for all $0 < 1 - \alpha^\epsilon \leq 1 - \overline{\alpha}(N^\epsilon)$.*

3. *Suppose that N^ϵ is odd. Then the following holds:*

   (a) *W^{ws} intersects $\{y = sc\}$ for $x < 0$ for all $0 < \alpha^\epsilon \leq \underline{\alpha}(N^\epsilon)$.*

   (b) *W^{ws} intersects $\{y = -sc\}$ for $x < 0$ for all $0 < 1 - \alpha^\epsilon \leq 1 - \overline{\alpha}(N^\epsilon)$.*

Proof. We let $K > 0$ be large enough and take $0 < \upsilon \ll K$ small enough and first consider the intersections of W^{ws} with $\{y = \pm c\}$ for $x > 0$ (proving items 1a and 1b). We let $0 < \epsilon \ll 1$ be so that $\overline{V}^\epsilon(\frac{3}{4}) > K$, see (41). Then since $\overline{V}^\epsilon(\overline{x})$ is an increasing function of $\overline{x}$, see (40), $\delta \in (0, \frac{3}{4})$ defined by the equation

$$\overline{V}^\epsilon(\delta) = K,$$

is uniquely determined. We then apply Theorem 2.5 with $I = [0, \delta]$. In particular, from (45) we conclude that W^{ws} intersects $\{\overline{y} = \pm\frac{1}{2}S^0_\infty K\}$ for $\overline{x} \in (0, \delta)$ when N^ϵ is even/odd, respectively. From $\{\overline{y} = \pm\frac{1}{2}S^0_\infty K\}$, we undo the scaling (36) and return to (27) and apply the backward flow, see Fig. 4. This completes the proof of items 1a and 1b. Finally, for item 1 we notice that W^u does not intersect $\{y = \pm c\}$ for all $0 < \epsilon \ll 1$, recall (38).

We then turn to the intersection of W^{ws} with $\{y = \pm c\}$ for $x < 0$. For this purpose, we again let $K > 0$ be large enough, put $\delta_2 = 1$ (for concreteness), take $0 < \upsilon \ll K$ small enough, $I = [-\delta, 0]$ with $0 < \delta \leq \epsilon$ and use the expansion (42) for $-\delta_2\epsilon \leq \overline{x} \leq 0$ to obtain the following for all $N^\epsilon \gg 1$: Consider $\underline{\alpha}(N^\epsilon)$ and $\overline{\alpha}(N^\epsilon)$ defined in (46). Then for any $0 < \alpha^\epsilon \leq \underline{\alpha}(N^\epsilon) \ll 1$, $\overline{V}^\epsilon(\epsilon\overline{x}_2)$ is given by

$$\frac{1}{\alpha^\epsilon}(N^\epsilon)^{a^\epsilon + 1 - N^\epsilon}\overline{x}_2^{N^\epsilon} e^{\overline{x}_2}, \tag{47}$$

to leading order, whereas for any $0 < 1 - \alpha^\epsilon \leq 1 - \overline{\alpha}(N^\epsilon) \ll 1$, $\overline{V}^\epsilon(\epsilon\overline{x}_2)$ is given by

$$\frac{1}{1 - \alpha^\epsilon}(N^\epsilon)^{a^\epsilon - N^\epsilon}\overline{x}_2^{N^\epsilon + 1} e^{\overline{x}_2}, \tag{48}$$

to leading order. We have here used (20),

$$\left[1 + \overline{x}_2 \int_0^1 e^{(1-v)\overline{x}_2} v dv\right] = \frac{e^{\overline{x}_2} - 1}{\overline{x}_2} \quad \text{and} \quad \left[1 + \overline{x}_2 \int_0^1 e^{(1-v)\overline{x}_2} dv\right] = e^{\overline{x}_2}.$$

In both cases ((47) and (48)), there are remainder terms that we can assume are bounded by $\upsilon > 0$, uniformly with respect to α^ϵ (whenever (47) and (48) do not exceed K in absolute value); this characterization will be adequate for our purposes.

We now further claim that for any $0 < \alpha^\epsilon \leq \underline{\alpha}(N^\epsilon)$, then (47) with $\overline{x}_2 = -1$ exceeds $K > 0$ in absolute value. To show this, we just estimate

$$\left|\frac{1}{\alpha^\epsilon}(N^\epsilon)^{a^\epsilon + 1 - N^\epsilon}(-1)^{N^\epsilon} e^{-1}\right| \geq \frac{1}{\underline{\alpha}(N^\epsilon)}(N^\epsilon)^{a^\epsilon + 1 - N^\epsilon} e^{-1} = (N^\epsilon)^{1 + a^\epsilon - a^0} e^{-1} \geq (N^\epsilon)^{\frac{1}{2}} e^{-1} > K,$$

using that $|a^\epsilon - a^0| \le \frac{1}{2}$ for all $N^\epsilon \gg 1$. A similar result holds for (48) for all $0 < 1-\alpha^\epsilon \le 1-\overline{\alpha}(N^\epsilon)$. We leave out the details in this case.

Consider now items 2a and 3a regarding $\alpha^\epsilon \to 0$. We then have by (47) (which is continuous and monotone with respect to $\overline{x}_2 \in [-1,0)$) and (45) that for any $0 < \alpha^\epsilon \le \underline{\alpha}(N^\epsilon)$, the equation $|\overline{m}^\epsilon(\overline{x})| = \frac{1}{2}|S_\infty^0|K$ has a solution $\overline{x}_- \in (-\epsilon, 0)$. The sign of $\overline{m}^\epsilon(\overline{x}_-)$ is determined by $(-1)^{N^\epsilon} s\overline{x}_-^{N^\epsilon}$, cf. (45) and (47). From $\{\overline{y} = \pm\frac{1}{2}|S_\infty^0|K\}$, we undo the scaling (36) and return to (27). Then the proof of items 2a and 3a is completed by using the backward flow. Indeed, W^{ws} aligns itself with one side of W^{ss} in this case and we can therefore just use W^{ss} as a guide for the backward flow up until W^{ss}'s transverse intersection with $\{y = \pm c\}$, see Fig. 4. The case $\alpha^\epsilon \to 1$ (items 2b and 3b) is similar and we therefore leave out further details.

□

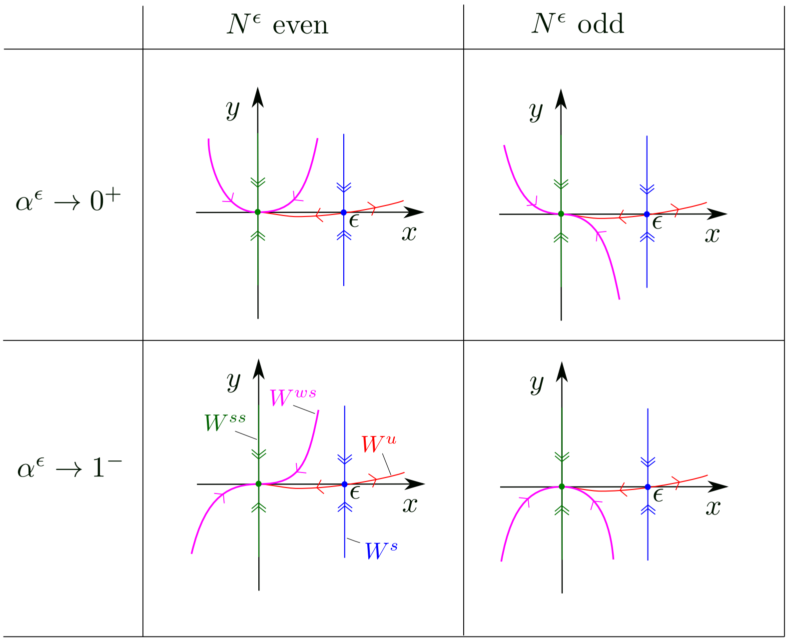

Figure 5: Illustration of the results of Theorem 2.5, see also Corollary 2.6. The strong stable manifold W^{ss} in green, the analytic weak-stable manifold W^{cs} in magenta, the stable manifold of the saddle W^s in blue, and finally the unstable manifold of the saddle W^u in red. The diagram assumes $S_\infty^0 > 0$; if $S_\infty^0 < 0$ then the diagram should be reflected about the x-axis. The analytic weak-stable manifold W^{ws} "flaps" on the $x < 0$-side of the node as α^ϵ transverses $(0,1)$, aligning on $y > 0$ or $y < 0$ with W^{ss} as either $\alpha^\epsilon \to 0^+$ or $\alpha^\epsilon \to 1^-$. On the $x > 0$-side, W^{ws} remains on one side of W^u for all $\alpha^\epsilon \in (0,1)$ and only "flaps" as N^ϵ varies. In particular, W^{ws} and W^u do not intersect.

2.1 Overview

We prove Theorem 2.2 in Section 3. Theorem 2.5 is proven in Section 4, see also Section 4.5 where Lemma 2.4 is proven. The strategy of the proof of Theorem 2.5 follows [10] insofar that we write $\overline{y} = \overline{m}^\epsilon(\overline{x})$ as a finite sum $\overline{y} = \sum_{k=2}^{N^\epsilon} \overline{m}_k^\epsilon \overline{x}^k$, up until "before the resonance", plus a remainder $\overline{M}^\epsilon(\overline{x}) = \mathcal{O}(\overline{x}^{N^\epsilon+1})$ that we solve by setting up a fix-point equation using an integral operator $\mathcal{T}^\epsilon$, see Lemma 4.17. A main difficult lies in estimating the growth of coefficients in the series expansion of $\overline{g}^\epsilon$ when composed with the finite sum $\overline{y} = \sum_{k=2}^{N^\epsilon} \overline{m}_k^\epsilon \overline{x}^k$ (with the number of terms going unbounded as $\epsilon \to 0$). This is covered by the novel Lemma 4.7 (which does not depend upon Assumption 2). Our treatment of $\overline{M}^\epsilon$ is also novel (and also does not rely on Assumption 2) insofar that we view the integral operator $\mathcal{T}^\epsilon$ as a bounded operator on a certain Banach space $\mathcal{D}_\delta^\epsilon$ of analytic functions $H = H(\overline{x})$ with $H(\overline{x}) = \mathcal{O}(\overline{x}^{N^\epsilon+1})$, see (132). We conclude the paper in Section 5.

3 The center manifold W^c: The proof of Theorem 2.2

In this section, we consider $\epsilon = 0$ and (34) in the equivalent form

$$x^2 \frac{dy}{dx} + y(1 + a^0 x) = g^0(x, y), \tag{49}$$

where

$$g^0(x, y) = f^0(x) + \mu h^0(x, y) = \sum_{k=2}^{\infty} f_k^0 x^k + \mu \sum_{k=1}^{\infty} \sum_{l=2}^{\infty} h_{k,l}^0 x^k y^l,$$

cf. (32) and (33). Let

$$\widehat{m}^0(x) = \sum_{k=2}^{\infty} m_k^0 x^k, \tag{50}$$

denote the formal series expansion of the center manifold $y = m^0(x)$. We define

$$w_k^0 := \Gamma(k + a^0) \quad \forall\, k \geq 2, \tag{51}$$

and a norm

$$\|y\| = \sup_{k \geq 2} \frac{|y_k|}{w_k^0}, \tag{52}$$

on the space $\mathcal{D}^0$ of formal series $y = \sum_{k=2}^{\infty} y_k x^k$. Notice that (51) is well-defined by virtue of Assumption 1 and that $\mathcal{D}^0$ is a Banach space (due to the sequence space l^∞ being Banach). For any $C > 0$, we also define

$$\mathcal{B}^C := \{y \in \mathcal{D}^0 \,:\, \|y\| \leq C\}, \tag{53}$$

as the closed ball of radius C. Moreover, for $y(x) = \sum_{k=2}^{\infty} y_k x^k \in \mathcal{D}^0$ we define $(g^0(\cdot, y(\cdot)))_k$ by

$$g^0(x, y(x)) = \sum_{k=2}^{\infty} (g^0(\cdot, y(\cdot)))_k x^k;$$

$(h^0(\cdot, y(\cdot)))_k$ is defined in a similar way. By (31), we have $(h^0(\cdot, y(\cdot)))_k = 0$ for $k = 2, 3$ and 4 and therefore

$$\begin{cases} (g^0(\cdot, y(\cdot)))_2 &= f_2^0, \\ (g^0(\cdot, y(\cdot)))_3 &= f_3^0, \\ (g^0(\cdot, y(\cdot)))_4 &= f_4^0, \\ (g^0(\cdot, y(\cdot)))_k &= f_k^0 + \mu (h^0(\cdot, y(\cdot)))_k, \quad k \geq 5. \end{cases} \tag{54}$$

Lemma 3.1. *Let $y \in \mathcal{D}^0$ and define $(y^l)_k$, $k \geq 2l$, by*

$$y(x)^l =: \sum_{k=2l}^{\infty} (y^l)_k x^k.$$

Then

$$(h^0(\cdot, y(\cdot)))_k = \sum_{l=2}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{j=2l}^{k-1} h_{k-j,l}^0 (y^l)_j \quad \forall\, k \geq 5. \tag{55}$$

Proof. We use the expansion of h^0 in (32) and Cauchy's product rule:

$$\sum_{k=0}^{\infty} q_k \sum_{l=0}^{\infty} p_l = \sum_{k=0}^{\infty}\sum_{j=0}^{k} q_{k-j}p_j. \tag{56}$$

We have

$$\begin{aligned} h^0(x,y(x)) = \sum_{l=2}^{\infty}\sum_{k=1}^{\infty} h^0_{k,l}x^k y(x)^l &= \sum_{l=2}^{\infty}\left(\sum_{k=1}^{\infty} h^0_{k,l}x^k\right)\left(\sum_{j=2l}^{\infty}(y^l)_j x^j\right) \\ &= \sum_{l=2}^{\infty}\sum_{k=2l+1}^{\infty}\left(\sum_{j=2l}^{k-1} h^0_{k-j,l}(y^l)_j\right)x^k \\ &= \sum_{k=5}^{\infty}\sum_{l=2}^{\lfloor \frac{k-1}{2}\rfloor}\left(\sum_{j=2l}^{k-1} h^0_{k-j,l}(y^l)_j\right)x^k. \end{aligned}$$

□

Proposition 3.2. *Let $y \in \mathcal{B}^C$. Then $g^0(x,y) \in \mathcal{D}^0$. In particular, there is a constant $K = K(a^0,\rho,C)$ such that*

$$|(g^0(\cdot,y(\cdot)))_k| \le B\rho^{-k} + \mu K w^0_{k-2} \quad \forall\, k \ge 5. \tag{57}$$

Moreover, $y \mapsto h^0(x,y)$ is C^1 and

$$|(D(h^0(\cdot,y(\cdot))(z))_k| \le K w^0_{k-2}\|z\| \quad \forall\, z \in \mathcal{D}^0, \tag{58}$$

recall the definition of $\|\cdot\|$ in (52)*.*

We prove this proposition in Section 3.1 below. First we need some intermediate results.

Lemma 3.3. *Consider w^0_k defined in* (51) *for all $k \ge 2$ and suppose $a^0 > -2$ (Assumption 1). Then the following holds.*

1. *Convolution estimate: There exists a $C = C(a^0) > 0$ such that*

$$\sum_{j=2}^{k-2} w^0_j w^0_{k-j} \le C w^0_{k-2} \quad \forall k \ge 4.$$

2. *Let $\rho > 0$. Then there exists a $C = C(a^0,\rho) > 0$ such that*

$$\sum_{j=2}^{k-2} \rho^{j-k+2} w^0_j \le C w^0_{k-2} \quad \forall\, k \ge 4.$$

3. *Let $\xi > 0$. Then there exists a $C = C(a^0,\xi) > 0$ such that*

$$\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor} \xi^{l-2} w^0_{k-2(l-1)} \le C w^0_{k-2} \quad \forall\, k \ge 4.$$

Proof. We prove the items 1–3 successively in the following.

Proof of item 1. We first notice that

$$\sum_{j=2}^{k-2} w^0_j w^0_{k-j} \le 2\sum_{j=2}^{\lfloor \frac{k}{2}\rfloor} w^0_j w^0_{k-j},$$

with $k \geq 4$. The result follows once we have shown that

$$2\sum_{j=2}^{\lfloor \frac{k}{2} \rfloor} w_j^0 w_{k-j}^0 \leq C w_{k-2}^0 \quad \forall\, k \geq 4, \tag{59}$$

for some $C = C(a^0)$. We believe that this result, which is a result on gamma functions, is known, but for completeness we will present a simple proof that will form the basis for proofs of similar statements later on.

The starting point for this approach is to define $\Phi_1^0(j)$ for $j \in [2, k-2]$ by

$$w_j^0 w_{k-j}^0 = \exp(\Phi_1^0(j)). \tag{60}$$

We have

$$\frac{d}{dj}\Phi_1^0(j) = \phi(j + a^0) - \phi(k - j + a^0), \quad \frac{d^2}{dj^2}\Phi_1^0(j) = \phi'(j + a^0) + \phi'(k - j + a^0),$$

using (25). Since the digamma function $\phi(z)$ is strictly increasing for $z > 0$, see (26), and since $a^0 > -2$ (recall Assumption 1), we conclude that $\Phi_1^0(j)$, $j \in [2, k-2]$, is convex, having a single minimum at $j = \frac{k}{2}$. We therefore have that

$$\Phi_1^0(j) \leq Q_1^0(j-2) + P_1^0 \quad \forall\, j \in [2, \tfrac{k}{2}], \tag{61}$$

where

$$\exp(P_1^0) = w_2^0 w_{k-2}^0 \quad \text{and} \quad Q_1^0 = \frac{1}{\frac{k}{2} - 2}\log\frac{(w_{k/2}^0)^2}{w_2^0 w_{k-2}^0} < 0; \tag{62}$$

in particular equality holds in (61) for $j = 2$ and $j = \frac{k}{2}$ so that also

$$\exp\left(Q_1^0\left(\frac{k}{2} - 2\right) + P_1^0\right) = w_{k/2}^0.$$

We illustrate the situation in Fig. 6. Then by (21), a simple calculation shows that

$$Q_1^0 = -\log 4 + o(1) \quad \text{and} \quad (w_{k/2}^0)^2 / w_{k-2}^0 \to 0 \quad \text{for} \quad k \to \infty. \tag{63}$$

Therefore

$$\begin{aligned}\sum_{j=2}^{\lfloor \frac{k}{2} \rfloor} w_j^0 w_{k-j}^0 &\leq w_2^0 w_{k-2}^0 + \int_2^\infty e^{Q_1^0(j-2)+P_1^0}\, dj \\ &\leq (1 + \log^{-1} 4) w_2^0 w_{k-2}^0 \left(1 + o_{k_0\to\infty}(1)\right),\end{aligned}$$

using (62) and (63), for all $k \geq k_0$ large enough. This finishes the proof of item 1.

Proof of item 2. We proceed as in the proof of item 1: Let $w_j^0 = \exp(\Phi_2^0(j))$ for $j \in [2, k-2]$. Then Φ_2^0 is convex; in fact $\frac{d}{dj}\Phi_2^0(j) = \phi(j + a^0)$ (positive for $j \geq 4$ since $a^0 > -2$), $\frac{d^2}{dj^2}\Phi_2^0(j) = \phi'(j + a^0) > 0$, see (25) and (26). We conclude that

$$\Phi_2^0(j) \leq Q_2^0(j-2) + P_2^0 \quad \forall\, j \in [2, k-2], \tag{64}$$

where

$$\exp(P_2^0) = w_2^0 \quad \text{and} \quad Q_2^0 = \frac{1}{k-4}\log\frac{w_{k-2}^0}{w_2^0} > 0; \tag{65}$$

in particular equality holds in (64) for $j = 2$ and $j = k-2$. By (21), we find that

$$Q_2^0 = \log k - 1 + o(1) \quad \text{for} \quad k \to \infty. \tag{66}$$

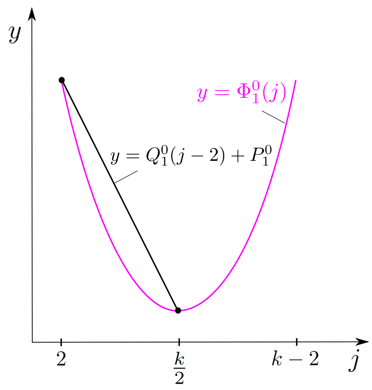

Figure 6: Graph of the function Φ_1^0 (magenta) and the secant $Q_1^0(j-2) + P_1^0$ (in black), see (60) and (62). Since Φ_1^0 is convex, (61) holds.

We can therefore estimate

$$\sum_{j=2}^{k-2} \rho^{j-k+2} w_j^0 \leq \rho^{-k+2} e^{-2Q_2^0+P_2^0} \sum_{j=2}^{k-2} (\rho e^{Q_2^0})^j.$$

By (66), there is a $k_0 \gg 1$ such that

$$\rho e^{Q_2^0} \geq 2 \quad \forall\, k \geq k_0,$$

and therefore by estimating the geometric sum and using (65), we find that

$$\sum_{j=2}^{k-2} \rho^{j-k+2} w_j^0 \leq 2e^{Q_2^0(k-4)+P_2^0} = 2w_{k-2}^0 \quad \forall\, k \geq k_0.$$

It follows that

$$C := \sup_{k\geq 4} \left(\frac{1}{w_{k-2}^0} \sum_{j=2}^{k-2} \rho^{j-k+2} w_j^0 \right) < \infty,$$

is well-defined.

Proof of item 3. We use

$$w_j^0 \leq e^{Q_2^0(j-2)+P_2^0} \quad \forall j \in [2, k],$$

with Q_2^0 and P_2^0 defined in (65), to estimate

$$\sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \xi^{l-2} w_{k-2(l-1)}^0 \leq e^{Q_2^0 k+P_2^0} \xi^{-2} \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} (\xi e^{-2Q_2^0})^l.$$

By (66), there is a $k_0 \gg 1$ such that

$$\xi e^{-2Q_2^0} \leq \frac{1}{2} \quad \forall\, k \geq k_0,$$

and therefore by estimating the geometric sum and using (65), we find that

$$\sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \xi^{l-2} w_{k-2(l-1)}^0 \leq 2e^{Q_2^0(k-4)+P_2^0} = 2w_{k-2}^0 \quad \forall\, k \geq k_0.$$

It follows that

$$C := \sup_{k\geq 4}\left(\frac{1}{w^0_{k-2}}\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\xi^{l-2}w^0_{k-2(l-1)}\right) < \infty,$$

is well-defined. $\square$

Remark 4. *The strategy used in the proof of Lemma 3.3, based on the convexity of the functions* $\Phi_i^0(j)$*, see e.g.* (60) *and Fig. 6, will also be used for* $\epsilon > 0$ *below, see Lemma 4.5 .*

Lemma 3.4. *If* $G \in \mathcal{D}^0$ *and* $H \in \mathcal{D}^0$ *then* $GH \in \mathcal{D}^0$*. In particular, there is a constant* $C = C(a^0)$ *such that*

$$|(GH)_k| \leq C\|G\|\|H\|w^0_{k-2} \quad \forall\, k \geq 4. \tag{67}$$

Proof. Notice that (67) implies the first statement since

$$\frac{w^0_{k-2}}{w^0_k} = \frac{1}{(k-1+a^0)(k-2+a^0)} \quad \forall k \geq 4.$$

using (19). Next regarding (67), we use (56): $(GH)_k = \sum_{j=2}^{k-2} G_j H_{k-j} \quad \Longrightarrow$

$$|(GH)_k| \leq \|G\|\|H\|\sum_{j=2}^{k-2} w^0_j w^0_{k-j} \leq C\|G\|\|H\|w^0_{k-2},$$

by Lemma 3.3 item 1. $\square$

A consequence of this result is that

$$|(y^l)_k| \leq \|y\|^l C^{l-1} w^0_{k-2(l-1)} \quad \forall\, k \geq 2l, \tag{68}$$

for all $l \geq 2$. This follows by induction. Indeed, having already established the base case, $l = 2$, in Lemma 3.4, we can proceed analogously for any l by writing

$$(y^l)_k = \sum_{j=2(l-1)}^{k-2} (y^{l-1})_j y_{k-j},$$

and using

$$\sum_{j=2}^{k-2(l-1)} w^0_j w^0_{k-2(l-2)-j} \leq C w^0_{k-2(l-1)},$$

cf. Lemma 3.3 item 1. We also emphasize the following:

$$(y^l)_k,\ k \geq 2l, \quad \text{only depends upon} \quad y_2, \cdots, y_{k-2(l-1)} \quad \forall\, l \in \mathbb{N}. \tag{69}$$

3.1 Proof of Proposition 3.2

We now turn to the proof of Proposition 3.2 (with $k \geq 5$). By (33), (54), (55) and Lemma 3.4, we have

$$\begin{aligned}
|(g^0(\cdot, y(\cdot)))_k| &\leq B\rho^{-k} + \mu \sum_{l=2}^{\lfloor \frac{k-1}{2}\rfloor} \|y\|^l \rho^{-l} C^{l-1} \sum_{j=2l}^{k-1} \rho^{j-k} w^0_{j-2(l-1)} \\
&\leq B\rho^{-k} + \mu \sum_{l=2}^{\lfloor \frac{k}{2}\rfloor} \|y\|^l \rho^{-l} C^{l-1} \sum_{j=2l}^{k} \rho^{j-k} w^0_{j-2(l-1)};
\end{aligned}$$

the last estimate, due to

$$\sum_{l=2}^{\lfloor \frac{k-1}{2}\rfloor}(\cdots)\sum_{j=2l}^{k-1}(\cdots)\leq \sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}(\cdots)\sum_{j=2l}^{k}(\cdots),$$

is not important, but it streamlines some estimates for $\epsilon=0$ with similar ones for $\epsilon>0$ later on (see e.g. (115)). We focus on the final term:

$$\mu\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\|y\|^l\rho^{-l}C^{l-1}\sum_{j=2l}^{k}\rho^{j-k}w^0_{j-2(l-1)}$$

By Lemma 3.3 item 2 with $k\to k-2(l-2)$, we can conclude that

$$\sum_{j=2l}^{k}\rho^{j-k}w^0_{j-2(l-1)}\leq Cw^0_{k-2(l-1)},$$

where $C>0$ is large enough but independent of l and k. We are therefore left with

$$\mu\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\|y\|^l\rho^{-l}C^{l}w^0_{k-2(l-1)},$$

upon increasing $C>0$ if necessary. This sum is bounded by $\mu K w_{k-2}$, with $K=K(\|y\|)>0$, for all $k\geq 5$ by Lemma 3.3 item 3. This completes the proof of (57).

The proof of (58) proceeds completely analogously. In particular, we find that

$$(D(h^0(\cdot,y(\cdot))(z))_k=\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\sum_{j=2l}^{k-1}h^0_{k-j,l}l(y^{l-1}z)_j\quad \forall\, z(x)=\sum_{k=2}^{\infty}z_kx^k\in\mathcal{D}^0,\ k\geq 5,$$

which is well-defined by Lemma 3.4. We therefore leave out further details. □

3.2 Solving for the center manifold

We are now ready to show that $\widehat{m}^0\in\mathcal{D}^0$. We define the nonlinear operator $T^0:\mathcal{D}^0\to\mathcal{D}^0$ by

$$T^0\left(y\right)(x)=\sum_{k=2}^{\infty}(-1)^kw^0_k\left(\sum_{j=2}^{k}\frac{(-1)^j(g^0(\cdot,y(\cdot))_j}{w^0_j}\right)x^k. \tag{70}$$

Lemma 3.5. *Let $\widehat{m}^0(x)=\sum_{k=2}^{\infty}m^0_kx^k$ be the formal series expansion of the center manifold $y=m^0(x)$ of (34). Then $\widehat{m}^0\in\mathcal{D}^0$ if and only if $\widehat{m}^0$ is a fix-point of $T^0:\mathcal{D}^0\to\mathcal{D}^0$:*

$$T^0(\widehat{m}^0)=\widehat{m}^0.$$

Proof. It follows from Lemma 1.3 that the m^0_k's of the formal series of the center manifold $\widehat{m}^0(x)=\sum_{k=2}^{\infty}m^0_kx^k$ are given by

$$m^0_k=(-1)^kw^0_kS^0_k,\quad S^0_k:=\sum_{j=2}^{k}\frac{(-1)^j(g^0(\cdot,\widehat{m}^0(\cdot)))_j}{w^0_j}; \tag{71}$$

here the right hand side only depends upon $m^0_2,\ldots,m^0_{k-3}$ (which is a simple consequence of (55) and (69)). Consequently,

$$\widehat{m}^0=T^0(\widehat{m}^0),$$

and the statement follows. □

Proposition 3.6. *Let*

$$F := B\sum_{j=2}^{\infty}\frac{\rho^{-j}}{w_j^0} < \infty.$$

Then there is a $\mu_0 > 0$ sufficiently small such that $T^0 : \mathcal{B}^{2F} \to \mathcal{B}^{2F}$ (recall (53)) is C^1 and a contraction for all $0 \le \mu < \mu_0$ small enough.

Proof. We have

$$\begin{aligned}\|T^0(y)\| = \sup_{k\in[2,\infty)}\left|S_k^0\right| &\le \sum_{j=2}^{\infty}\frac{|(g^0(\cdot,y(\cdot)))_j|}{w_j^0}\\ &\le \sum_{j=2}^{\infty}\frac{|f_j^0|}{w_j^0} + \mu\sum_{j=5}^{\infty}\frac{|(h^0(\cdot,y(\cdot)))_j|}{w_j^0} \le F + \mathcal{O}(\mu) \le 2F,\end{aligned} \tag{72}$$

for all $y \in \mathcal{B}^{2F}$, provided that $\mu > 0$ is small enough. Here we have used that

$$\sum_{j=2}^{\infty}\frac{|f_j^0|}{w_j^0} \le F,\quad \sum_{j=5}^{\infty}\frac{|(h^0(\cdot,y(\cdot)))_j|}{w_j^0} \le K\sum_{j=5}^{\infty}\frac{1}{(j-1+a^0)(j-2+a^0)} < \infty,\quad K = K(F), \tag{73}$$

by Proposition 3.2, see (57), and $a^0 > -2$, recall Assumption 1. T^0 is also C^1 and $D(T^0(y)) = \mathcal{O}(\mu)$ cf. (58). □

For any $0 \le \mu < \mu_0$, we have $\widehat{m}^0 \in \mathcal{D}^0$, being the unique fix-point of the mapping $T^0 : \mathcal{B}^{2F} \to \mathcal{B}^{2F}$. We then define

$$S_\infty^0 := \lim_{k\to\infty} S_k^0 = \sum_{j=2}^{\infty}\frac{(-1)^j(g^0(\cdot,\widehat{m}^0(\cdot)))_j}{w_j^0}, \tag{74}$$

see (71).

Lemma 3.7. *Consider the assumptions of Proposition 3.6. Then the series S_∞^0 is absolutely convergent and $|S_\infty^0| \le 2F$.*

Proof. We have

$$|S_\infty^0| \le \sum_{j=2}^{\infty}\frac{|(g^0(\cdot,\widehat{m}^0(\cdot)))_j|}{w_j^0} \le 2F,$$

by (72). □

In turn, if $S_\infty^0 \neq 0$ then

$$|m_k^0| = (1+o(1))S_\infty^0 w_k^0 \quad \text{for} \quad k \to \infty, \tag{75}$$

cf. (71) and (74), and there are constants $0 < C_1 < C_2$ such that

$$C_1(k-1)!k^{a^0} \le |m_k^0| \le C_2(k-1)!k^{a^0}, \tag{76}$$

for all k large enough. Here we have used (22):

$$\Gamma(k+a^0) = \Gamma(k)(1+o(1))k^{a^0} = (k-1)!(1+o(1))k^{a^0}.$$

In this way, we obtain our first result.

Lemma 3.8. *If $S_\infty^0 \neq 0$ then $\widehat{m}^0 \in \mathcal{D}^0$ is not convergent for any $x \neq 0$ and the center manifold of $(x,y) = (0,0)$ for (49) is therefore not analytic.*

Proof. From (76) we have that $m_k^0 x^k \not\to 0$ as $k \to 0$ for any $x \neq 0$. □

Remark 5. *We do not know whether the converse:*

$$\text{“if } S^0_\infty = 0 \text{ holds then the center manifold is analytic”,}$$

which holds in the linear case (recall Lemma 1.4), is true in general. We leave this as an open problem for future work.

Lemma 3.9. *For all $\mu > 0$ sufficiently small, the following holds*

$$\frac{\partial S^0_\infty}{\partial f^0_2} = \frac{1}{w^0_2} + \mathcal{O}(\mu) \neq 0.$$

Proof. We have

$$S^0_\infty = \sum_{j=2}^\infty \frac{f^0_j}{w^0_j} + \mu \sum_{j=5}^\infty \frac{(-1)^j (h^0(\cdot, \widehat{m}^0(\cdot)))_j}{w^0_j}. \tag{77}$$

The nonlinear operator T^0 depends on f^0_2 in a C^1-way and so does its fix-point $\widehat{m}^0 \in \mathcal{D}^0$. In particular, from $T^0(\widehat{m}^0) = \widehat{m}^0$, with T^0 given by (70), we conclude that $\frac{\partial \widehat{m}^0}{\partial f^0_2} \in \mathcal{D}^0$ satisfies the fix-point equation:

$$\frac{\partial \widehat{m}^0}{\partial f^0_2} = \frac{1}{w^0_2} \sum_{k=2}^\infty (-1)^k w^0_k x^k + \mu \sum_{k=5}^\infty (-1)^k w^0_k \left(\sum_{j=5}^k \frac{(-1)^j (D(h^0(\cdot, \widehat{m}^0(\cdot)))(\frac{\partial \widehat{m}^0}{\partial f^0_2}))_j}{w^0_j} \right) x^k.$$

which, with $\widehat{m}^0 \in \mathcal{D}^0$ given, we can solve by Banach's fixed theorem. The result then follows by differentiation of (77).

□

3.3 Completing the proof of Theorem 2.2

Theorem 2.2 item 1 follows from Lemma 3.8, see also (75) with $m^0_k = \frac{1}{k!}\frac{d^k}{dx^k} m^0(0)$, $w^0_k = \Gamma(k + a^0)$. Finally, Lemma 3.9 is precisely the statement in Theorem 2.2 item 2.

4 The analytic weak-stable manifold W^{ws}: The proof of Theorem 2.5

To study (35) and the analytic weak-stable manifold for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$, we use the scalings (36), repeated here for convenience:

$$x = \epsilon \overline{x}, \quad y = \epsilon \overline{y}.$$

Inserting this into (27) leads to the singularly perturbed system

$$\begin{aligned} \dot{\overline{x}} &= \epsilon \overline{x}(\overline{x} - 1), \\ \dot{\overline{y}} &= -\overline{y}(1 + \epsilon a^\epsilon \overline{x}) + \epsilon \overline{g}^\epsilon(\overline{x}, \overline{y}), \end{aligned} \tag{78}$$

or equivalently

$$\epsilon \overline{x}(\overline{x} - 1) \frac{d\overline{y}}{d\overline{x}} + \overline{y}(1 + \epsilon a^\epsilon \overline{x}) = \epsilon \overline{g}^\epsilon(\overline{x}, \overline{y}), \tag{79}$$

where

$$\overline{g}^\epsilon(\overline{x}, \overline{y}) := \epsilon^{-2} g^\epsilon(\epsilon \overline{x}, \epsilon \overline{y}) = \underbrace{\epsilon^{-2} f^\epsilon(\epsilon \overline{x})}_{:= \overline{f}^\epsilon(\overline{x})} + \mu \underbrace{\epsilon^{-2} h^\epsilon(\epsilon \overline{x}, \epsilon \overline{y})}_{:= \overline{h}^\epsilon(\overline{x}, \overline{y})}, \tag{80}$$

using (31). Here we have also defined $\overline{f}^\epsilon$ and $\overline{h}^\epsilon$ in the last equality. By (32), we obtain the absolutely convergent power series expansion of $\overline{f}^\epsilon$ and $\overline{h}^\epsilon$:

$$\overline{f}^\epsilon(\overline{x}) = \sum_{k=2}^\infty f^\epsilon_k \epsilon^{k-2} \overline{x}^k, \quad \overline{h}^\epsilon(\overline{x}, \overline{y}) = \sum_{k=2}^\infty h^\epsilon_{k,1} \epsilon^{k-1} \overline{x}^k \overline{y} + \sum_{k=1}^\infty \sum_{l=2}^\infty h^\epsilon_{k,l} \epsilon^{k+l-2} \overline{x}^k \overline{y}^l. \tag{81}$$

For all $\epsilon^{-1} \notin \mathbb{N}$, $0 < \epsilon \ll 1$, $(x, y) = (0, 0)$ is a nonresonant hyperbolic node of (78) (the eigenvalues being $-\epsilon$ and -1). Consequently, there is an analytic weak-stable manifold:

$$W^{ws}: \quad \overline{y} = \overline{m}^\epsilon(\overline{x}), \quad \overline{m}^\epsilon(\overline{x}) = \sum_{k=2}^{\infty} \overline{m}_k^\epsilon \overline{x}^k, \quad \overline{x} \in (-\delta, \delta), \tag{82}$$

$\delta = \delta(\epsilon) > 0$, see e.g. [6, Theorem 2.14]. Now, for any formal series $\overline{y}(\overline{x}) = \sum_{k=2}^{\infty} \overline{y}_k \overline{x}^k$, we define $(\overline{g}^\epsilon(\cdot, \overline{y}(\cdot)))_k$ as above:

$$\overline{g}^\epsilon(\overline{x}, \overline{y}(\overline{x})) = \sum_{k=2}^{\infty} (\overline{g}^\epsilon(\cdot, \overline{y}(\cdot)))_k \overline{x}^k.$$

Again, $(\overline{h}^\epsilon(\cdot, \overline{y}(\cdot)))_k$ is defined in the same way, recall (80). We have $(\overline{h}^\epsilon(\cdot, \overline{y}(\cdot)))_k = 0$ for $k = 2$ and 3 and therefore

$$\begin{cases} (\overline{g}^\epsilon(\cdot, \overline{y}(\cdot)))_2 &= f_2^\epsilon, \\ (\overline{g}^\epsilon(\cdot, \overline{y}(\cdot)))_3 &= f_3^\epsilon \epsilon, \\ (\overline{g}^\epsilon(\cdot, \overline{y}(\cdot)))_k &= f_k^\epsilon \epsilon^{k-2} + \mu(\overline{h}^\epsilon(\cdot, \overline{y}(\cdot)))_k, \quad k \geq 4. \end{cases} \tag{83}$$

Lemma 4.1. *Let $\overline{y} = \sum_{k=2}^{\infty} \overline{y}_k \overline{x}^k$ and define $(\overline{y}^l)_k$, $k \geq 2l$, by*

$$\overline{y}(\overline{x})^l =: \sum_{k=2l}^{\infty} (\overline{y}^l)_k \overline{x}^k.$$

Then

$$(\overline{h}^\epsilon(\cdot, \overline{y}(\cdot)))_k = \sum_{j=2}^{k-2} h_{k-j,1}^\epsilon \epsilon^{k-j-1} \overline{y}_j + \sum_{l=2}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{j=2l}^{k-1} h_{k-j,l}^\epsilon \epsilon^{k-j+l-2} (\overline{y}^l)_j, \quad k \geq 4. \tag{84}$$

(The last sum is zero for $k = 4$.)

Proof. The proof is identical to the proof of Lemma 3.1 and further details are therefore left out. $\square$

Lemma 4.2. *Suppose that $\epsilon^{-1} \notin \mathbb{N}$, $0 < \epsilon \ll 1$ and let (82) denote the analytic weak-stable manifold. Then the $\overline{m}_k^\epsilon$'s satisfy the recursion relation:*

$$(1 - \epsilon k)\overline{m}_k^\epsilon + \epsilon(k - 1 + a^\epsilon)\overline{m}_{k-1}^\epsilon = (\epsilon \overline{g}^\epsilon(\cdot, \overline{m}^\epsilon(\cdot)))_k \quad \forall\, k \geq 2; \tag{85}$$

here we define $\overline{m}_1^\epsilon = 0$. In particular, the right side of (85) only depends upon $\overline{m}_2^\epsilon, \ldots, \overline{m}_{k-2}^\epsilon$.

Proof. Simple calculation. $\square$

Lemma 4.3. *For $\epsilon^{-1} \notin \mathbb{N}$, $0 < \epsilon \ll 1$, define*

$$\overline{w}_k^\epsilon := \frac{\Gamma\left(\epsilon^{-1} - k\right) \Gamma\left(k + a^\epsilon\right)}{\epsilon \Gamma\left(\epsilon^{-1}\right)} \quad \forall\, k \geq 2, \tag{86}$$

and

$$\overline{S}_k^\epsilon := \sum_{j=2}^{k} \frac{(-1)^j (\epsilon \overline{g}^\epsilon(\cdot, \overline{m}^\epsilon(\cdot)))_j}{\overline{w}_j^\epsilon (1 - \epsilon j)} \quad \forall\, k \geq 2.$$

Then $\overline{S}_k^\epsilon$ depends upon $\overline{m}_2^\epsilon, \ldots, \overline{m}_{k-2}^\epsilon$ for each $k \geq 4$ and $\overline{m}_k^\epsilon$ satisfies

$$\overline{m}_k^\epsilon = (-1)^k \overline{w}_k^\epsilon \overline{S}_k^\epsilon \quad \forall\, k \geq 2. \tag{87}$$

Proof. The result follows from induction on k, with the base case being $k=2$, upon using (85) and the recursion relation

$$(1-\epsilon k)\overline{w}_k^\epsilon = \epsilon(k-1+a^\epsilon)\overline{w}_{k-1}^\epsilon,$$

for the $\overline{w}_k^\epsilon$'s in the induction step. $\square$

Lemma 4.4. *Write*

$$\overline{m}_k^\epsilon =: \epsilon^{k-1} m_k^\epsilon, \tag{88}$$

and let $\widehat{m}^0(x) = \sum_{k=2}^\infty m_k^0 x^k$ *denote the formal series expansion of the center manifold for* $\epsilon = 0$*, recall* (71)*. Then for any fixed* k*,*

$$m_k^\epsilon \to m_k^0,$$

as $\epsilon \to 0$*.*

Proof. Inserting (88) into (85), it is straightforward to obtain

$$m_k^\epsilon(1-\epsilon k) + (k-1+a^\epsilon)m_{k-1}^\epsilon = (g^\epsilon(\cdot, m^\epsilon(\cdot)))_k \to m_k^0 + (k-1+a^0)m_{k-1}^0 = (g^0(\cdot, m^0(\cdot)))_k,$$

as $\epsilon \to 0$. The result then follows from induction on k. $\square$

4.1 Growth properties of $\overline{m}_k^\epsilon$

We now study the formal series (82) and the growth properties of $\overline{m}_k^\epsilon$, $k = 2, \dots N^\epsilon$. For this purpose, the following lemma, on the properties of the $\overline{w}_k^\epsilon$'s, defined in (86), will be crucial.

Lemma 4.5. *Suppose that* $a^0 > -2$*, that* $\epsilon^{-1} \notin \mathbb{N}$ *and write*

$$\epsilon^{-1} =: N^\epsilon + \alpha^\epsilon, \quad N^\epsilon := \lfloor \epsilon^{-1} \rfloor, \quad \alpha^\epsilon \in (0,1), \tag{89}$$

Then the following can be said about $\overline{w}_k^\epsilon$*, defined in* (86)*, for all* $2 \le k \le N^\epsilon$*:*

1. *For fixed* $k \ge 2$*:*

$$\overline{w}_k^\epsilon = \epsilon^{k-1}(1+o(1))\Gamma(k+a^\epsilon),$$

as $\epsilon \to 0$*.*

2. *Lower bound of* $\overline{w}_k^\epsilon(1-\epsilon k)$*:*

$$\overline{w}_k^\epsilon(1-\epsilon k) \ge \Gamma(k+a^\epsilon)\epsilon^{k-1} \quad \forall\, 2 \le k \le N^\epsilon + 1.$$

3. *Convolution estimate: There is a* $C = C(a^0)$ *such that*

$$\sum_{j=2}^{k-2} \overline{w}_j^\epsilon \overline{w}_{k-j}^\epsilon \le C \overline{w}_2^\epsilon \overline{w}_{k-2}^\epsilon \quad \forall\, 4 \le k \le N^\epsilon + 1, \tag{90}$$

and

$$\sum_{j=k-(N^\epsilon-1)}^{N^\epsilon-1} \overline{w}_j^\epsilon \overline{w}_{k-j}^\epsilon \le C \overline{w}_{N^\epsilon-1}^\epsilon \overline{w}_{k-(N^\epsilon-1)}^\epsilon \quad \forall\, N^\epsilon+1 \le k \le 2(N^\epsilon-1). \tag{91}$$

4. *Define*

$$\begin{aligned} Q_4^\epsilon &:= \frac{1}{N^\epsilon - 3} \log \frac{\overline{w}_{N^\epsilon-1}^\epsilon}{\overline{w}_2^\epsilon}, \\ P_4^\epsilon &:= \log(\overline{w}_2^\epsilon) \end{aligned} \tag{92}$$

Then

$$\begin{aligned} Q_4^\epsilon &= \frac{1}{N^\epsilon - 3}\left((a^\epsilon + 1 - \alpha^\epsilon)\log N^\epsilon + \log\frac{\Gamma(1+\alpha^\epsilon)}{\Gamma(2+a^\epsilon)} + o(1)\right), \\ P_4^\epsilon &= \log\epsilon + \log\Gamma(2+a^\epsilon) + o(1) \end{aligned} \tag{93}$$

and

$$\overline{w}_k^\epsilon \le e^{Q_4^\epsilon(k-2)+P_4^\epsilon} \quad \forall\, 2 \le k \le N^\epsilon - 1, \tag{94}$$

for all $0 < \epsilon \ll 1$. *In particular,*

$$\sum_{k=2}^{N^\epsilon-1} \overline{w}_k^\epsilon \delta^k \le \sum_{k=2}^{N^\epsilon-1} e^{Q_4^\epsilon(k-2)+P_4^\epsilon}\delta^k \le \delta^2 C\epsilon \quad \forall\, 0<\delta\le\frac{3}{4}.$$

for some $C > 0$ *and all* $0 < \epsilon \ll 1$.

5. *For fixed* $\epsilon^{-1} \notin \mathbb{N}$,

$$\begin{aligned} \overline{w}_k^\epsilon &= \frac{(-1)^{N^\epsilon-k}\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})}\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})} \\ &= \mathcal{O}(1)k^{\epsilon^{-1}+a^\epsilon-1}, \end{aligned} \tag{95}$$

with respect to $k \to \infty$.

6. *Let* $\xi > 0$. *Then there is a constant* $C = C(a^\epsilon, \xi)$ *such that*

$$\sum_{j=2}^{k-2} (\xi^{-1}\epsilon)^{k-2-j}\overline{w}_j^\epsilon \le C\overline{w}_{k-2}^\epsilon \quad \forall\, 4 \le k \le N^\epsilon + 1, \tag{96}$$

for all $0 < \epsilon \ll 1$.

7. *Let* $\xi > 0$. *Then there is a constant* $C = C(a^\epsilon, \xi)$ *such that*

$$\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor} (\xi^{-1}\epsilon)^{l-2}(\overline{w}_2^\epsilon)^{l-1}\overline{w}_{k-2(l-1)}^\epsilon \le C\overline{w}_2^\epsilon\overline{w}_{k-2}^\epsilon \quad \forall\, 4 \le k \le N^\epsilon + 1, \tag{97}$$

for all $0 < \epsilon \ll 1$.

Proof. We prove the items 1-7 successively in the following.

Proof of item 1. For fixed $k \in \mathbb{N}$, we have

$$\overline{w}_k^\epsilon = \frac{\Gamma(\epsilon^{-1}-k)}{\epsilon\Gamma(\epsilon^{-1})}\Gamma(k+a^\epsilon) = \frac{\Gamma(\epsilon^{-1})\epsilon^{k-1}}{\Gamma(\epsilon^{-1})}(1+o(1))\Gamma(k+a^\epsilon) = \mathcal{O}(\epsilon^{k-1}),$$

using (22), (86) and the definiton of the gamma function.

Proof of item 2. We calculate

$$\frac{\overline{w}_k^\epsilon(1-\epsilon k)}{\Gamma(k+a^\epsilon)} = \frac{\Gamma(\epsilon^{-1}-k+1)}{\Gamma(\epsilon^{-1})} = \frac{1}{\Pi_{j=1}^{k-1}(\epsilon^{-1}-j)} = \epsilon^{k-1}\Pi_{j=1}^{k-1}\frac{1}{1-j\epsilon} \ge \epsilon^{k-1},$$

using (19) and $1-(k-1)\epsilon \ge \alpha^\epsilon > 0$ for $2 \le k \le N^\epsilon + 1$.

Proof of item 3. We first focus on (90) and notice from item 1 that the claim holds true for all $4 \le k \le k_0$ with $k_0 > 0$ fixed and all $0 < \epsilon \ll 1$. We therefore consider $k_0 < k \le N^\epsilon + 1$ with $k_0 > 0$ fixed large. We write

$$\sum_{j=2}^{k-2}\overline{w}_j^\epsilon\overline{w}_{k-j}^\epsilon \le 2\sum_{j=2}^{\lfloor\frac{k}{2}\rfloor}\overline{w}_j^\epsilon\overline{w}_{k-j}^\epsilon =: 2\sum_{j=2}^{\lfloor\frac{k}{2}\rfloor} e^{\Phi_{31}^\epsilon(j)}.$$

By proceeding as in the proof of Lemma 3.4, a simple computation, using (25) and (26), shows that $\Phi^\epsilon_{31}(j)$, $j \in [2, k-2]$, is convex, having a unique minimum at $j = \frac{k}{2}$. Therefore

$$\Phi^\epsilon_{31}(j) \le Q^\epsilon_{31}(j-2) + P^\epsilon_{31}, \tag{98}$$

where Q^ϵ_{31} and P^ϵ_{31} are chosen such that

$$Q^\epsilon_{31} = \frac{\Phi^\epsilon_{31}\left(\frac{k}{2}\right) - \Phi^\epsilon_{31}(2)}{\frac{k}{2}-2} = \frac{1}{\frac{k}{2}-2}\log\frac{\left(\overline{w}^\epsilon_{\frac{k}{2}}\right)^2}{\overline{w}^\epsilon_2\overline{w}^\epsilon_{k-2}}, \quad P^\epsilon_{31} = \Phi^\epsilon_{31}(2).$$

In particular, equality holds for $j = 2$ and $j = \frac{k}{2}$ in (98) and consequently

$$\left(\overline{w}^\epsilon_{\frac{k}{2}}\right)^2 = e^{Q^\epsilon_{31}(\frac{k}{2}-2)+P^\epsilon_{31}}, \quad \overline{w}^\epsilon_2\overline{w}^\epsilon_{k-2} = e^{P^\epsilon_{31}}.$$

Using (86) and (21), a simple calculation shows that

$$Q^\epsilon_{31} < \frac{1}{\frac{k}{2}-2}\log\frac{\Gamma\left(\frac{k}{2}+a^\epsilon\right)^2}{\Gamma(2+a^\epsilon)\Gamma(k-2+a^\epsilon)} = -\log 4(1+o_{k_0\to\infty}(1)),$$

for all $k_0 < k \le N^\epsilon + 1$, uniformly in $0 < \epsilon \ll 1$. Then proceeding as in the proof of Lemma 3.4, we have

$$\begin{aligned}\sum_{j=2}^{k-2}\overline{w}^\epsilon_j\overline{w}^\epsilon_{k-j} &\le 2\overline{w}^\epsilon_2\overline{w}^\epsilon_{k-2} + \int_2^\infty e^{Q^\epsilon_{31}(j-2)+P^\epsilon_{31}}dj \\ &\le 2(1+\log 4)\overline{w}^\epsilon_2\overline{w}^\epsilon_{k-2}(1+o(1)),\end{aligned}$$

which completes the proof of (90).

The inequality (91) is proven in a similar way. First, we put $k = 2(N^\epsilon - 1) - p$ and use (86) and (22) to obtain

$$\overline{w}^\epsilon_{N^\epsilon-1-p+j}\overline{w}^\epsilon_{N^\epsilon-1-j} = \Gamma(\alpha^\epsilon+1+p-j)\Gamma(\alpha^\epsilon+1+j)(N^\epsilon)^{2(a^\epsilon-\alpha^\epsilon)-p}(1+o(1)).$$

Therefore

$$\begin{aligned}\sum_{j=k-(N^\epsilon-1)}^{N^\epsilon-1}\overline{w}^\epsilon_j\overline{w}^\epsilon_{k-j} &= \sum_{j=0}^{p}\overline{w}^\epsilon_{N^\epsilon-1-p+j}\overline{w}^\epsilon_{N^\epsilon-1-j} \\ &= (N^\epsilon)^{2(a^\epsilon-\alpha^\epsilon)-p}\sum_{j=0}^{p}\Gamma(\alpha^\epsilon+1+p-j)\Gamma(\alpha^\epsilon+1+j)(1+o(1)).\end{aligned}$$

Here

$$\sum_{j=0}^{p}\Gamma(\alpha^\epsilon+1+p-j)\Gamma(\alpha^\epsilon+1+j) \le C\Gamma(\alpha^\epsilon+1)\Gamma(\alpha^\epsilon+1+p),$$

cf. (59) and therefore (91) holds true for all $2(N^\epsilon-1)-p \le k \le 2(N^\epsilon-1)$ and any $p > 0$ provided that $0 < \epsilon \ll 1$.

We therefore proceed to consider $N^\epsilon + 1 \le k \le 2(N^\epsilon-1)-p$ with $p > 0$ fixed large. We write

$$\sum_{j=k-(N^\epsilon-1)}^{N^\epsilon-1}\overline{w}^\epsilon_j\overline{w}^\epsilon_{k-j} \le 2\sum_{j=k-(N^\epsilon-1)}^{\lfloor\frac{k}{2}\rfloor}\overline{w}^\epsilon_j\overline{w}^\epsilon_{k-j} := 2\sum_{j=k-(N^\epsilon-1)}^{\lfloor\frac{k}{2}\rfloor}e^{\Phi^\epsilon_{32}(j)}.$$

As above, $\Phi^\epsilon_{32}(j)$, $j \in [2, k-2]$, is convex, having a unique minimum at $j = \frac{k}{2}$, so that

$$\Phi^\epsilon_{32}(j) \le Q^\epsilon_{32}(j-2) + P^\epsilon_{32}, \tag{99}$$

where Q^ϵ_{32} and P^ϵ_{32} are now chosen such that

$$
\begin{aligned}
Q^\epsilon_{32} &= \frac{\Phi_{32}\left(\frac{k}{2}\right) - \Phi_{32}(k-(N^\epsilon-1))}{N^\epsilon - 1 - \frac{k}{2}} = \frac{1}{N^\epsilon - 1 - \frac{k}{2}} \log \frac{\left(\overline{w}^\epsilon_{\frac{k}{2}}\right)^2}{\overline{w}^\epsilon_{k-(N^\epsilon-1)}\overline{w}^\epsilon_{N^\epsilon-1}},\\
P^\epsilon_{32} &= \Phi^\epsilon(k-(N^\epsilon-1)).
\end{aligned}
$$

Equality holds in (99) for $j = k-(N^\epsilon-1)$ and $j = \frac{k}{2}$. Now, using (86) and (21) a simple calculation shows that

$$
Q^\epsilon_{32} < \frac{1}{N^\epsilon - 1 - \frac{k}{2}} \log \frac{\Gamma\left(\epsilon^{-1} - \frac{k}{2}\right)^2}{\Gamma(1+\alpha^\epsilon)\Gamma(\epsilon^{-1} - (k-(N^\epsilon-1)))} \leq -\log 4(1 + o_{p\to\infty}(1)),
$$

for all $N^\epsilon + 1 \leq k \leq 2(N^\epsilon - 1) - p$, uniformly in $0 < \epsilon \ll 1$. We can now complete the proof by proceeding in the exact same way that we did in the proof of (90).

Proof of item 4. First, we write

$$
\overline{w}^\epsilon_k = e^{\Phi^\epsilon_4(k)},
$$

where

$$
\Phi^\epsilon_4(k) = \log \frac{\Gamma(\epsilon^{-1} - k)\Gamma(k + a^\epsilon)}{\epsilon \Gamma(\epsilon^{-1})}.
$$

Again, $\Psi^\epsilon_4(k)$ is convex on $k \in [2, N^\epsilon - 1]$ (having a minimum at $k = k_m(\epsilon) := \frac{1}{2\epsilon} - \frac{a^\epsilon}{2}$). Next, Q^ϵ_4 and P^ϵ_4, defined by (92), are chosen such that

$$
Q^\epsilon_4 = \frac{\Psi^\epsilon_4(N^\epsilon - 1) - \Psi^\epsilon_4(2)}{N^\epsilon - 3}, \quad P^\epsilon_4 = \Psi^\epsilon_4(2),
$$

specifically

$$
\Psi^\epsilon_4(k) \leq Q^\epsilon_4(k-2) + P^\epsilon_4,
$$

for all $k \in [2, N^\epsilon - 1]$ with equality for $k = 2$ and $k = N^\epsilon - 1$:

$$
\overline{w}^\epsilon_2 = e^{P^\epsilon_4}, \quad \overline{w}^\epsilon_{N^\epsilon - 1} = e^{Q^\epsilon_4(N^\epsilon - 3) + P^\epsilon_4}. \tag{100}
$$

Moreover, $Q^\epsilon_4 = o(1)$, see (93) which we prove below. Consequently, for all $0 < \delta \leq \frac{3}{4}$, we have

$$
\begin{aligned}
\sum_{k=2}^{N^\epsilon - 1} \overline{w}^\epsilon_k \delta^k &\leq \sum_{k=2}^{N^\epsilon - 1} e^{Q^\epsilon_4(k-2) + P^\epsilon_4} \delta^k\\
&\leq e^{P^\epsilon_4}\delta^2 + \int_2^\infty e^{Q^\epsilon_4(k-2) + P^\epsilon_4}\delta^k dk\\
&\leq C\delta^2 e^{P^\epsilon_4}.
\end{aligned}
$$

Here we have used that

$$
\int_2^\infty e^{a(k-2)}\delta^k dk = \frac{1}{\log \delta^{-1} - a}\delta^2 \quad \forall\, 0 < \delta < e^{-a}.
$$

To complete the proof of item 4, we just have to prove the asymptotics in (93). The asymptotics of P^ϵ_4 follows from item 1, so we focus on Q^ϵ_4. For this, we use Stirling's approximation in the form (22) for $N^\epsilon \gg 1$:

$$
\begin{aligned}
Q^\epsilon_4 &= \frac{1}{N^\epsilon - 3} \log \frac{\overline{w}^\epsilon_{N^\epsilon - 1}}{\overline{w}^\epsilon_2}\\
&= \frac{1}{N^\epsilon - 3} \log \left(\frac{\Gamma(1+\alpha^\epsilon)}{\Gamma(2+a^\epsilon)} \frac{\Gamma(N^\epsilon - 1 + a^\epsilon)}{\Gamma(N^\epsilon + \alpha^\epsilon - 2)} \right)\\
&= \frac{1}{N^\epsilon - 3} \log \left(\frac{\Gamma(1+\alpha^\epsilon)}{\Gamma(2+a^\epsilon)} (1 + o(1))(N^\epsilon)^{a^\epsilon + 1 - \alpha^\epsilon} \right),
\end{aligned} \tag{101}
$$

using (86), $\epsilon^{-1} = N^\epsilon + \alpha$ and

$$\frac{\Gamma(N^\epsilon - 1 + a^\epsilon)}{\Gamma(N^\epsilon + \alpha^\epsilon - 2)} = (1 + o(1))\frac{(N^\epsilon)^{a^\epsilon - 1}}{(N^\epsilon)^{\alpha^\epsilon - 2}},$$

in the last equality of (101).

Proof of item 5. For (5), we use the reflection formula (23) and Sterling's approximation in the form (22) for $k \to \infty$.

Proof of item 6. It is easy to verify the claim for all $4 \le k \le k_0$ for any $k_0 > 0$ fixed and $0 < \epsilon \ll 1$ by using item 1. We therefore consider $k_0 < k \le N^\epsilon + 1$ with $k_0 > 0$ fixed large and write

$$\overline{w}_j^\epsilon =: e^{\Phi_6^\epsilon(j)}.$$

Again, Φ_6^ϵ is convex for any $j \in [2, N^\epsilon - 1]$ and therefore

$$\Phi_6^\epsilon(j) \le Q_6^\epsilon (j-2) + P_6^\epsilon,$$

where Q_6^ϵ and P_6^ϵ are chosen such that equality holds for $j = 2$ and $j = k - 2$:

$$Q_6^\epsilon = \frac{1}{k-4}\log\frac{\overline{w}_{k-2}^\epsilon}{\overline{w}_2^\epsilon}, \quad e^{P_6^\epsilon} = \overline{w}_2^\epsilon. \tag{102}$$

By the convexity of Φ_6^ϵ it follows that Q_6^ϵ is increasing. Therefore by item 1 and (22)

$$\begin{aligned} Q_6^\epsilon &\ge \log\epsilon + \mathcal{O}(\epsilon) + \log\left(\frac{\Gamma(k_0 - 2 + a^\epsilon)}{\Gamma(2 + a^\epsilon)}\right)^{\frac{1}{k_0 - 4}} \\ &= \log\epsilon + \mathcal{O}(\epsilon) + \log k_0 (1 + o_{k_0 \to \infty}(1)), \end{aligned}$$

for all $k_0 \le k \le N^\epsilon - 1$. In turn, we can assume that

$$\xi\epsilon^{-1} e^{Q_6^\epsilon} \ge 2 \quad \forall\, k \in [k_0, N^\epsilon - 1].$$

This allow us to estimate the sum as a geometric sum:

$$\begin{aligned} \sum_{j=2}^{k-2} (\xi^{-1}\epsilon)^{k-2-j}\overline{w}_j^\epsilon &\le (\xi^{-1}\epsilon)^{k-2} e^{P_6^\epsilon} \sum_{j=2}^{k-2} (\xi\epsilon^{-1} e^{Q_6^\epsilon})^j \\ &\le 2(\xi^{-1}\epsilon)^{k-2} e^{P_6^\epsilon} (\xi\epsilon^{-1} e^{Q_6^\epsilon})^{k-2} \\ &\le 2e^{Q_6^\epsilon (k-2) + P_6^\epsilon} \\ &= 2\overline{w}_{k-2}^\epsilon. \end{aligned}$$

Proof of item 7. It is easy to verify the claim for all $4 \le k < k_0$ for any $k_0 > 0$ and $0 < \epsilon \ll 1$ by using item 1. We therefore consider $k_0 \le k \le N^\epsilon + 1$ with $k_0 > 0$ fixed large and write

$$\overline{w}_j^\epsilon =: e^{\Phi_6^\epsilon(j)},$$

as in the proof of item 6, with

$$\Phi_6^\epsilon(j) \le Q_6^\epsilon (j-2) + P_6^\epsilon,$$

for all $j \in [2, k-2]$ with equality for $j = 2$ and $j = k - 2$. We may assume that $k_0 > 0$ is such that

$$(\xi^{-1}\epsilon e^{-2Q_6^\epsilon + P_6^\epsilon}) \le \frac{1}{2} \quad \forall\, k \in [k_0, N^\epsilon + 1],$$

for all $0 < \epsilon \ll 1$. In this way, we estimate can estimate the sum as a geometric sum

$$\begin{aligned} \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} (\xi^{-1}\epsilon)^{l-2} (\overline{w}_2^\epsilon)^{l-1} \overline{w}_{k-2(l-1)}^\epsilon &\le (\xi\epsilon^{-1})^2 \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} (\xi^{-1}\epsilon)^l e^{P_6^\epsilon (l-1)} e^{Q_6^\epsilon (k-2l) + P_6^\epsilon} \\ &\le (\xi\epsilon^{-1})^2 e^{Q_6^\epsilon k} \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} \left(\xi^{-1}\epsilon e^{-2Q_6^\epsilon + P_6^\epsilon}\right)^l \\ &\le 2(\xi\epsilon^{-1})^2 e^{Q_6^\epsilon k} \left(\xi^{-1}\epsilon e^{-2Q_6^\epsilon + P_6^\epsilon}\right)^2 \\ &= 2\overline{w}_2^\epsilon \overline{w}_{k-2}^\epsilon, \end{aligned}$$

for all $k_0 < k \le N^\epsilon + 1$. Here we have used (102) in the last equality.

$\square$

In contrast to the analysis of the center manifold for $\epsilon = 0$, we are in the present case of $\epsilon > 0$ only interested in estimating the partial sum of (82):

$$\overline{y} = \sum_{k=2}^{N^\epsilon-1} \overline{m}_k^\epsilon \overline{x}^k, \quad m_k^\epsilon = (-1)^k \overline{w}_k^\epsilon \overline{S}_k^\epsilon, \quad \overline{S}_k^\epsilon = \sum_{j=2}^{k} \frac{(-1)^j (\epsilon \overline{g}^\epsilon(\cdot, \overline{m}^\epsilon(\cdot)))_j}{\overline{w}_j^\epsilon (1-\epsilon j)},$$

where

$$N^\epsilon = \lfloor \epsilon^{-1} \rfloor,$$

recall (89). (We will deal with the remainder later, see Section 4.3). We therefore define the semi-norm

$$\| \sum_{k=2}^{\infty} \overline{y}_k \overline{x}^k \| := \sup_{k \in [2, N^\epsilon - 1]} \frac{|\overline{y}_k|}{\overline{w}_k^\epsilon}, \tag{103}$$

on the set of formal series $\overline{y} = \sum_{k=2}^{\infty} \overline{y}_k \overline{x}^k$.

Lemma 4.6. *Consider $\overline{y}(\overline{x}) = \sum_{k=2}^{N^\epsilon-1} \overline{y}_k \overline{x}^k$ and define $(\overline{y}^l)_k, k = 2l, \ldots, l(N^\epsilon - 1)$ by*

$$\overline{y}(\overline{x})^l =: \sum_{k=2l}^{l(N^\epsilon-1)} (\overline{y}^l)_k \overline{x}^k. \tag{104}$$

Then there exists a $C = C(a^0) > 0$ such that for any $l \in \mathbb{N} \setminus \{1\}$ and all $1 \le p \le l$ the following holds true:

$$\begin{aligned} |(\overline{y}^l)_k| &\le \|y\|^l \binom{l-1}{p-1} C^{l-1} (\overline{w}_2^\epsilon)^{l-p} (\overline{w}_{N^\epsilon-1}^\epsilon)^{p-1} (\overline{w}_{k-(p-1)(N^\epsilon-1)-2(l-p)}^\epsilon), \\ &\forall\, k \in [(p-1)(N^\epsilon-1) + 2(l-p+1), p(N^\epsilon-1) + 2(l-p)], \end{aligned} \tag{105}$$

for all $0 < \epsilon \ll 1$. Here $\binom{l-1}{p-1}$ is the binomial coefficient, "$l-1$ choose $p-1$", for any $1 \le p \le l$.

In particular, for $p = 1$:

$$\begin{aligned} |(\overline{y}^l)_k| &\le \|y\|^l C^{l-1} (\overline{w}_2^\epsilon)^{l-1} \overline{w}_{k-2(l-1)}^\epsilon, \\ &\forall\, k \in [2l, N^\epsilon - 1 + 2(l-1)], \end{aligned} \tag{106}$$

for all $0 < \epsilon \ll 1$.

Proof. The claim is proven by induction, with the base case being $l = 2$, $p = 1$ and $p = 2$.

The base case: $(l, p) = (2, 1), (2, 2)$. For $l = 2$, we have by Cauchy's product formula:

$$|(\overline{y}^2)_k| \le \|y\|^2 \sum_{j=\max(2, k-(N^\epsilon-1))}^{\min(k-2, N^\epsilon-1)} \overline{w}_j^\epsilon \overline{w}_{k-j}^\epsilon.$$

We first consider $p = 1$: $4 \le k \le (N^\epsilon - 1) + 2 = N^\epsilon + 1$. Then by item 3 of Lemma 4.5, see (90), we conclude that

$$|(\overline{y}^2)_k| \le \|y\|^2 C \overline{w}_2^\epsilon \overline{w}_{k-2}^\epsilon.$$

Next, for $p = 2$:

$$|(\overline{y}^2)_k| \le \|y\|^2 \sum_{j=k-(N^\epsilon-1)}^{N^\epsilon-1} \overline{w}_j^\epsilon \overline{w}_{k-j}^\epsilon \le \|y\|^2 C \overline{w}_{N^\epsilon-1}^\epsilon \overline{w}_{k-(N^\epsilon-1)}^\epsilon,$$

using (91).

Induction step. The induction proceeds in two steps: We assume that the claim is true for all $l \in \mathbb{N} \setminus \{1\}$ and all $1 \le p \le l$. We then first proof that it is true for $l+1$, $1 \le p \le l$. Subsequently, we consider $p = l+1$.

We assume that (105) holds true. Then by using Cauchy's product formula we find that

$$(\overline{y}^{l+1})_k = \sum_{j=\max(2l,k-(N^\epsilon-1))}^{\min(k-2,l(N^\epsilon-1))} (\overline{y}^l)_j \overline{y}_{k-j}.$$

For $p = 1$ and $k \in [2(l+1), N^\epsilon - 1 + 2l]$, we find

$$\begin{aligned}
|(\overline{y}^{l+1})_k| &\le \|y\|^{l+1} C^{l-1} (\overline{w}_2^\epsilon)^{l-1} \sum_{j=2l}^{k-2} \overline{w}^\epsilon_{j-2(l-1)} \overline{w}^\epsilon_{k-j} \\
&\le \|y\|^{l+1} C^{l-1} (\overline{w}_2^\epsilon)^{l-1} \sum_{j=2}^{k-2l} \overline{w}^\epsilon_j \overline{w}^\epsilon_{(k-2(l-1))-j} \\
&\le \|y\|^{l+1} C^{l} (\overline{w}_2^\epsilon)^{l} \overline{w}^\epsilon_{k-2l},
\end{aligned}$$

using (90), which proves (105) with $l \to l+1$ and $p = 1$. Next, for $2 \le p \le l$, we find completely analogously that

$$\begin{aligned}
|(\overline{y}^{l+1})_k| \le & \sum_{j=k-(N^\epsilon-1)}^{k-2} |(\overline{y}^l)_j| |\overline{y}_{k-j}| \\
\le & \sum_{j=k-(N^\epsilon-1)}^{(p-1)(N^\epsilon-1)+2(l-p+1)} |(\overline{y}^l)_j| |\overline{y}_{k-j}| + \sum_{j=(p-1)(N^\epsilon-1)+2(l-p+1)}^{k-2} |(\overline{y}^l)_j| |\overline{y}_{k-j}|,
\end{aligned}$$

for

$$k \in [(p-1)(N^\epsilon - 1) + 2(l-p+1), p(N^\epsilon - 1) + 2(l-p+1)].$$

Therefore by (105) (for (l,p) and $(l,p) \to (l,p-1)$):

$$\begin{aligned}
|(\overline{y}^{l+1})_k| \le & \|\overline{y}\|^{l+1} \binom{l-1}{p-2} C^{l-1} (\overline{w}_2^\epsilon)^{l-p+1} (\overline{w}^\epsilon_{N^\epsilon-1})^{p-2} \\
& \times \sum_{j=k-(N^\epsilon-1)}^{(p-1)(N^\epsilon-1)+2(l-p+1)} \overline{w}^\epsilon_{j-(p-2)(N^\epsilon-1)-2(l-p+1)} \overline{w}^\epsilon_{k-j} \\
& + \|y\|^{l+1} \binom{l-1}{p-1} C^{l-1} (\overline{w}_2^\epsilon)^{l-p} (\overline{w}^\epsilon_{N^\epsilon-1})^{p-1} \\
& \times \sum_{j=(p-1)(N^\epsilon-1)+2(l-p+1)}^{k-2} \overline{w}^\epsilon_{j-(p-1)(N^\epsilon-1)-2(l-p)} \overline{w}^\epsilon_{k-j} \\
& \le \|y\|^{l+1} C^{l} (\overline{w}_2^\epsilon)^{l-p+1} (\overline{w}^\epsilon_{N^\epsilon-1})^{p-1} \left(\binom{l-1}{p-2} + \binom{l-1}{p-1} \right) \\
& \times \overline{w}^\epsilon_{k-(p-1)(N^\epsilon-1)-2(l-p+1)},
\end{aligned}$$

using (90) and (91) to estimate the two sums. Then as

$$\binom{l-1}{p-2} + \binom{l-1}{p-1} = \binom{l}{p-1} \tag{107}$$

the claim follows.

We are left with proving that the claim holds true for $p = l+1$ and

$$k \in [l(N^\epsilon - 1) + 2, (l+1)(N^\epsilon - 1)],$$

where

$$|(\overline{y}^{l+1})_k| \leq \sum_{j=k-(N^\epsilon-1)}^{l(N^\epsilon-1)} |(\overline{y}^l)_j||\overline{y}_{k-j}|.$$

By the induction assumption, we have

$$|(\overline{y}^l)_k| \leq \|\overline{y}\|^l C^{l-1}(\overline{w}^\epsilon_{N^\epsilon-1})^{l-1}\overline{w}^\epsilon_{k-(l-1)(N^\epsilon-1)},$$

for all

$$k \in [(l-1)(N^\epsilon-1)+2, l(N^\epsilon-1)],$$

see (105) with $p=l$. Therefore

$$\begin{aligned}|(\overline{y}^{l+1})_k| &\leq \|\overline{y}\|^{l+1}C^{l-1}(\overline{w}^\epsilon_{N^\epsilon-1})^{l-1}\sum_{j=k-(N^\epsilon-1)}^{l(N^\epsilon-1)} \overline{w}^\epsilon_{j-(l-1)(N^\epsilon-1)}\overline{w}^\epsilon_{k-j}\\ &\leq \|\overline{y}\|^{l+1}C^{l}(\overline{w}^\epsilon_{N^\epsilon-1})^{l}\overline{w}^\epsilon_{k-l(N^\epsilon-1)},\end{aligned}$$

using (91). This proves (105) with $l \to l+1$ and $p = l+1$ and completes the proof.

□

By using Lemma 4.5 item 4, we obtain the following uniform bound on $(\overline{y}^l)_k$

Lemma 4.7. *Consider $\overline{y}(\overline{x}) = \sum_{k=2}^{N^\epsilon-1}\overline{y}_k\overline{x}^k$ and recall the definition of $(\overline{y}^l)_k$ in (104). Then there is a new $C>0$ such that*

$$|(\overline{y}^l)_k| \leq \|\overline{y}\|^l C^{l-1}e^{(-2Q_4^\epsilon+P_4^\epsilon)l}e^{Q_4^\epsilon k} \quad \forall\, 2l \leq k \leq l(N^\epsilon-1), \tag{108}$$

for all $0<\epsilon\ll 1$. Here Q_4^ϵ and P_4^ϵ are defined in (92).

Proof. We will use (105), repeated here for convenience:

$$\begin{aligned}|(\overline{y}^l)_k| &\leq \|y\|^l\begin{pmatrix}l-1\\p-1\end{pmatrix}C^{l-1}\underline{(\overline{w}^\epsilon_2)^{l-p}(\overline{w}^\epsilon_{N^\epsilon-1})^{p-1}\overline{w}^\epsilon_{k-(p-1)(N^\epsilon-1)-2(l-p)}},\\ &\forall\, k\in[(p-1)(N^\epsilon-1)+2(l-p+1), p(N^\epsilon-1)+2(l-p)],\end{aligned} \tag{109}$$

where $1\leq p\leq l$. Using (92) we have

$$\overline{w}^\epsilon_2 = e^{P_4^\epsilon}, \quad \overline{w}^\epsilon_{N^\epsilon-1} = e^{Q_4^\epsilon(N^\epsilon-3)+P_4^\epsilon} \quad \text{and} \quad \overline{w}^\epsilon_k \leq e^{Q_4^\epsilon(k-2)+P_4^\epsilon} \quad \forall\, 2\leq k\leq N^\epsilon+1,$$

and we can therefore estimate the underlined factor in (109) as follows:

$$\underline{(\overline{w}^\epsilon_2)^{l-p}(\overline{w}^\epsilon_{N^\epsilon-1})^{p-1}\overline{w}^\epsilon_{k-(p-1)(N^\epsilon-1)-2(l-p)}} \leq e^{P_4^\epsilon(l-p)}e^{(Q_4^\epsilon(N^\epsilon-3)+P_4^\epsilon)(p-1)}e^{(Q_4^\epsilon(k-\{\cdots\}-2)+P_4^\epsilon)},$$

where $\{\cdots\} = (p-1)(N^\epsilon-1)+2(l-p)$. By simplifying, we obtain

$$\underline{(\overline{w}^\epsilon_2)^{l-p}(\overline{w}^\epsilon_{N^\epsilon-1})^{p-1}\overline{w}^\epsilon_{k-(p-1)(N^\epsilon-1)-2(l-p)}} \leq e^{(-2Q_4^\epsilon+P_4^\epsilon)l}e^{Q_4^\epsilon k}. \tag{110}$$

Subsequently, we use

$$\begin{pmatrix}l-1\\p-1\end{pmatrix} \leq \sum_{q=0}^{l-1}\begin{pmatrix}l-1\\q\end{pmatrix} = 2^{l-1}, \tag{111}$$

for all $1\leq p\leq l$. Therefore (108) follows from (109), (110) and (111). □

Lemma 4.8. *Recall the definition of the semi-norm $\|\cdot\|$ in (103) and suppose that $\|\overline{y}\|\leq C$ with $C>0$. Then there is an $\overline{K}=\overline{K}(C)>0$, independent of μ and ϵ, such that*

$$\begin{cases}|(\epsilon\overline{g}^\epsilon(\cdot,\overline{y}(\cdot)))_2| & \leq B\rho^{-2}\epsilon,\\ |(\epsilon\overline{g}^\epsilon(\cdot,\overline{y}(\cdot)))_3| & \leq B\rho^{-3}\epsilon^2,\\ |(\epsilon\overline{g}^\epsilon(\cdot,\overline{y}(\cdot)))_k| & \leq B\rho^{-k}\epsilon^{k-1}+\mu\epsilon^2\overline{K}\overline{w}^\epsilon_{k-2} \quad \forall\, 4\leq k\leq N^\epsilon+1,\end{cases}$$

for all $0<\epsilon\ll 1$.

Proof. We use (83):

$$|(\overline{g}^\epsilon(\cdot,\overline{y}(\cdot)))_k| \le |(\overline{f}^\epsilon(\cdot))_k| + \mu|(\overline{h}^\epsilon(\cdot,\overline{y}(\cdot)))_k| \quad \forall\, k\ge 2.$$

The first term on the right hand side is directly estimated by (33):

$$|(\overline{f}^\epsilon(\cdot))_k| \le B\rho^{-k}\epsilon^{k-2},$$

for all $k\ge 2$. We therefore focus on the second term, which vanishes for $k=2$ and $k=3$. By using (33), (84),

$$|\overline{y}_j| \le \|\overline{y}\|\overline{w}_j^\epsilon \quad \forall\, j\in[2,N^\epsilon-1],$$

and (106), we obtain

$$\begin{aligned}
|(\overline{h}^\epsilon(\cdot,\overline{y}(\cdot)))_k| &\le \sum_{j=2}^{k-2}\rho^{-k+j-1}\epsilon^{k-j-1}\|\overline{y}\|\overline{w}_j^\epsilon\\
&+\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\sum_{j=2l}^{k}\rho^{-k+j-l}\epsilon^{k-j+l-2}\|\overline{y}\|^l C^{l-1}(\overline{w}_2^\epsilon)^{l-1}\overline{w}_{j-2(l-1)}^\epsilon\\
&=\|\overline{y}\|\rho^{-1}\epsilon\sum_{j=2}^{k-2}(\rho^{-1}\epsilon)^{k-2-j}\overline{w}_j^\epsilon\\
&+\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\rho^{-l}\epsilon^{l-2}\|\overline{y}\|^l C^{l-1}(\overline{w}_2^\epsilon)^{l-1}\sum_{j=2}^{k-2(l-1)}(\rho^{-1}\epsilon)^{k-2(l-1)-j}\overline{w}_j^\epsilon,
\end{aligned}$$

for all $4\le k\le N^\epsilon+1$. We now use (96) and (97), respectively:

$$\begin{aligned}
|(\overline{h}^\epsilon(\cdot,\overline{y}(\cdot)))_k| &\le \|\overline{y}\|\rho^{-1}\epsilon C\overline{w}_{k-2}^\epsilon\\
&+\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\rho^{-l}\epsilon^{l-2}\|\overline{y}\|^l C^l(\overline{w}_2^\epsilon)^{l-1}\overline{w}_{k-2(l-1)}^\epsilon\\
&\le \|\overline{y}\|\rho^{-1}\epsilon C\overline{w}_{k-2}^\epsilon\\
&+\rho^{-2}\|\overline{y}\|^2C^2\sum_{l=2}^{\lfloor\frac{k}{2}\rfloor}\left(\rho^{-1}\|\overline{y}\|C\epsilon\right)^{l-2}(\overline{w}_2^\epsilon)^{l-1}\overline{w}_{k-2(l-1)}^\epsilon\\
&\le \overline{K}\epsilon\overline{w}_{k-2}^\epsilon,
\end{aligned}$$

with $\overline{K}=\overline{K}(\|y\|,a^0,\rho)>0$ large enough. Here we have used that $\overline{w}_2^\epsilon=\mathcal{O}(\epsilon)$ cf. Lemma 4.5 item 1. □

This leads to the following important estimate:

Lemma 4.9. *Fix $C>0$ and define*

$$F = B\sum_{j=2}^{\infty}\frac{\rho^{-j}}{\Gamma(j+a^0)}.$$

Then the following holds for all $0\le\mu<\mu_0$ with $\mu_0>0$ small enough:

$$\left|\sum_{j=2}^{k}\frac{(-1)^j(\epsilon\overline{g}^\epsilon(\cdot,\overline{y}(\cdot)))_j}{\overline{w}_j^\epsilon(1-\epsilon j)}\right| \le 2F \quad \forall\, 2\le k\le N^\epsilon,\ \|y\|\le C,$$

for all $0<\epsilon\ll 1$.

Proof. Let $\overline{K}=\overline{K}(C)>0$ be the constant in Lemma 4.8. We then estimate

$$\left|\sum_{j=2}^{k}\frac{(-1)^j(\epsilon\overline{g}^\epsilon(\cdot,\overline{y}(\cdot)))_j}{\overline{w}_j^\epsilon(1-\epsilon j)}\right| \le B\sum_{j=2}^{N^\epsilon}\frac{\rho^{-j}\epsilon^{j-1}}{\overline{w}_j^\epsilon(1-\epsilon j))} + \mu\overline{K}\sum_{j=4}^{N^\epsilon}\frac{\epsilon^2\overline{w}_{j-2}^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)},$$

for all $2 \leq k \leq N^\epsilon$ using Lemma 4.8. Then by the definition of $\overline{w}_k^\epsilon$ (86) and (19), we have

$$\begin{aligned}\frac{\overline{w}_{j-2}^\epsilon}{\overline{w}_j^\epsilon} &= \frac{\Gamma(\epsilon^{-1}-j+2)\Gamma(j-2+a^\epsilon)}{\Gamma(\epsilon^{-1}-j)\Gamma(j+a^\epsilon)}\\ &= \frac{(\epsilon^{-1}-j+1)(\epsilon^{-1}-j)}{(j-1+a^\epsilon)(j-2+a^\epsilon)}.\end{aligned}$$

Therefore by Lemma 4.5 item 2 we find that

$$\begin{aligned}\left|\sum_{j=2}^{k}\frac{(-1)^j(\epsilon\overline{g}^\epsilon(\cdot,\overline{y}(\cdot)))_j}{\overline{w}_j^\epsilon(1-\epsilon j)}\right| &\leq B\sum_{j=2}^{N^\epsilon}\frac{\rho^{-j}}{\Gamma(j+a^\epsilon)}+\mu\overline{K}\sum_{j=4}^{N^\epsilon}\frac{(1-\epsilon(j-1))(1-\epsilon j)}{(j-1+a^\epsilon)(j-2+a^\epsilon)(1-\epsilon j)}\\ &\leq B\sum_{j=2}^{\infty}\frac{\rho^{-j}}{\Gamma(j+a^\epsilon)}+\mu\overline{K}\sum_{j=4}^{\infty}\frac{1}{(j-1+a^\epsilon)(j-2+a^\epsilon)}.\end{aligned} \tag{112}$$

The result now follows. ◻

Following Lemma 4.3, we have that $\overline{y} = \overline{m}^\epsilon(\overline{x})$ (as a power series) is a fix-point of the nonlinear operator T^ϵ defined by

$$T^\epsilon(\overline{y}) = \sum_{k=2}^{\infty}(-1)^k\overline{w}_k^\epsilon\sum_{j=2}^{k}\frac{(-1)^j(\epsilon\overline{g}^\epsilon(\cdot,\overline{y}(\cdot))_j}{\overline{w}_j^\epsilon(1-\epsilon j))}x^k. \tag{113}$$

By Lemma 4.8 and Lemma 4.9, we have that there is a $\mu_0 > 0$ such that for all $0 \leq \mu < \mu_0$ the following estimate holds:

$$\|T^\epsilon(\overline{y})\| \leq 2F \quad \forall\, \|\overline{y}\| \leq 2F,\ 0 < \epsilon \ll 1,$$

with respect to the semi-norm $\|\cdot\|$ defined in (103). This directly leads to the following:

Proposition 4.10. *There is a $\mu_0 > 0$, such that for all $0 \leq \mu < \mu_0$ the following holds true:*

1. *The analytic weak-stable manifold satisfies the following estimate*

$$\|\overline{m}^\epsilon\| \leq 2F,$$

for all $0 < \epsilon \ll 1$.

2. *The numbers*

$$\overline{S}_k^\epsilon := \sum_{j=2}^{k}\frac{(-1)^j(\epsilon\overline{g}^\epsilon(\cdot,\overline{m}^\epsilon(\cdot)))_j}{\overline{w}_j^\epsilon(1-\epsilon j)} \quad 2 \leq k \leq N^\epsilon,$$

are uniformly bounded with respect to $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$.

Lemma 4.11. *Let $0 \leq \mu < \mu_0$ with $\mu_0 > 0$ small enough so that Proposition 4.10 applies and so that the series S_∞^0 from Lemma 3.7 is well-defined and absolutely convergent. Then*

$$\overline{S}_{N^\epsilon}^\epsilon \to S_\infty^0 \quad \textit{for} \quad N^\epsilon \to \infty.$$

Proof. The proof is elementary, but since this result is crucial to the whole construction, we provide the full details:

Write

$$(\epsilon\overline{g}^\epsilon(\cdot,\overline{m}^\epsilon(\cdot)))_j =: \epsilon\overline{g}_j^\epsilon, \quad (g^0(\cdot,m^0(\cdot)))_j =: \overline{g}_j^0.$$

By Lemma 4.9, $|\overline{S}_{N^\epsilon}^\epsilon| \leq 2F$. Moreover, $S_\infty^0 = \sum_{j=2}^\infty \frac{(-1)^j g_j^0}{w_j^0}$ is absolutely convergent, recall Lemma 3.7.

For fixed j we have (recall item 1 of Lemma 4.5)

$$\overline{w}_j^\epsilon = \Gamma\left(j+a^0\right)\epsilon^{j-1}(1+o(1)) = w_j^0\epsilon^{j-1}(1+o(1)).$$

Moreover, by Lemma 4.4 we have

$$\epsilon^{1-j}\epsilon\overline{g}_j^\epsilon \to g_j^0,$$

and therefore

$$\frac{(-1)^j(\epsilon\overline{g}_j^\epsilon)}{\overline{w}_j^\epsilon(1-\epsilon j)} \to \frac{(-1)^j g_j^0}{\Gamma(j+a^0)}, \tag{114}$$

as $\epsilon \to 0$ (fixed j).

Next, we estimate

$$\begin{aligned}
\left|\sum_{j=2}^{N^\epsilon}\frac{(-1)^j\epsilon\overline{g}_j^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)} - \sum_{j=2}^{\infty}\frac{(-1)^j g_j^0}{w_j^0}\right| &\le \left|\sum_{j=2}^{J}\left(\frac{(-1)^j\epsilon\overline{g}_j^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)} - \frac{(-1)^j g_j^0}{w_j^0}\right)\right| \\
&\quad + \left|\sum_{j=J+1}^{N^\epsilon}\frac{(-1)^j\epsilon\overline{g}_j^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)}\right| + \left|\sum_{j=J+1}^{\infty}\frac{(-1)^j g_j^0}{w_j^0}\right| \\
&\le \sum_{j=2}^{J}\left|\frac{(-1)^j\epsilon\overline{g}_j^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)} - \frac{(-1)^j g_j^0}{w_j^0}\right| \\
&\quad + \sum_{j=J+1}^{\infty}\left(\frac{B\rho^{-j}}{\Gamma(j+a^\epsilon)} + \frac{\mu\overline{K}}{(j-1+a^\epsilon)(j-2+a^\epsilon)}\right) \\
&\quad + \sum_{j=J+1}^{\infty}\left(\frac{B\rho^{-j}}{\Gamma(j+a^0)} + \frac{\mu K}{(j-1+a^0)(j-2+a^0)}\right),
\end{aligned} \tag{115}$$

for any $2 \le J \le N^\epsilon$, using $w_j^0 = \Gamma(j+a^0)$, Lemma 4.8 (see also (112)) and Proposition 3.2 (see also (73)). Consequently, we have

$$\begin{aligned}
\left|\sum_{j=2}^{N^\epsilon}\frac{(-1)^j\epsilon\overline{g}_j^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)} - \sum_{j=2}^{\infty}\frac{(-1)^j g_j^0}{w_j^0}\right| &\le \sum_{j=2}^{J}\left|\frac{(-1)^j\epsilon\overline{g}_j^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)} - \frac{(-1)^j g_j^0}{w_j^0}\right| \\
&\quad + 2\mu(\overline{K}+K)\sum_{j=J+1}^{\infty}\frac{1}{(j-2+a^0)(j-1+a^0)} + 3B\sum_{j=J+1}^{\infty}\frac{\rho^{-j}}{\Gamma(j+a^0)},
\end{aligned} \tag{116}$$

for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$. Now, for any $\upsilon > 0$, we take $J \gg 1$ (independent of $\epsilon > 0$) so that each of the last two convergent series on the right hand side of (116) are less than $\upsilon/3$. Subsequently, we then take $\epsilon > 0$ small enough so that the first term on the right hand side of (116) (using (114)) is less than $\upsilon/3$. In total, we have

$$\left|\sum_{j=2}^{N^\epsilon}\frac{(-1)^j\epsilon\overline{g}_j^\epsilon}{\overline{w}_j^\epsilon(1-\epsilon j)} - \sum_{j=2}^{\infty}\frac{(-1)^j g_j^0}{w_j^0}\right| \le \upsilon,$$

and the result follows. □

4.2 Estimating the finite sum

Let $j^n[H]$ denote the nth-order Taylor jet/partial sum of $H(\overline{x}) = \sum_{k=2}^\infty H_k\overline{x}^k$:

$$j^n[H] := \sum_{k=2}^{n} H_k(\cdot)^k \quad \forall\, n \in \mathbb{N}.$$

Moreover, we define the nth-order remainder by

$$r^n[H] = (I - j^n)[H] := \sum_{k=n+1}^{\infty} H_k(\cdot)^k \quad \forall\, n \in \mathbb{N}.$$

Lemma 4.12. *Consider the partial sum*

$$j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) = \sum_{k=2}^{N^\epsilon-1} \overline{m}_k^\epsilon \overline{x}^k,$$

of the series $\overline{m}^\epsilon(\overline{x}) = \sum_{k=2}^\infty \overline{m}_k^\epsilon \overline{x}^k$. *Then there is a constant* $C > 0$ *such that*

$$|j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x})| \le C\epsilon \quad \forall \overline{x} \in \left[-\frac{3}{4}, \frac{3}{4}\right], \tag{117}$$

for all $0 < \epsilon \ll 1$.

Proof. The estimate (117) follows from item 4 of Lemma 4.5 with $\delta = \frac{3}{4}$. □

Lemma 4.13. *For any* $\overline{D} > 0$, *we define*

$$\widetilde{g}^\epsilon(\overline{x}, q) := \overline{g}^\epsilon(\overline{x}, j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) + q) \quad \forall \overline{x} \in \left[-\frac{3}{4}, \frac{3}{4}\right], \, q \in (-\overline{D}, \overline{D}). \tag{118}$$

It is well-defined for all $0 < \epsilon \ll 1$ *and has the absolutely convergent power series expansion*

$$\widetilde{g}^\epsilon(\overline{x}, q) = \widetilde{g}_0^\epsilon(\overline{x}) + \overline{x}^2 \widetilde{g}_1^\epsilon(\overline{x}) q + \overline{x} \sum_{l=2}^\infty \widetilde{g}_l^\epsilon(\overline{x}) q^l, \tag{119}$$

with

$$\widetilde{g}_0^\epsilon(\overline{x}) = \sum_{k=2}^\infty \widetilde{g}_{k,0}^\epsilon \overline{x}^k, \quad \widetilde{g}_{k,0}^\epsilon = (\overline{g}^\epsilon(\cdot, j^{N^\epsilon-1}[\overline{m}^\epsilon](\cdot)))_k.$$

Moreover, we have the following estimates (Q_4^ϵ *is defined in* (92)):

$$|\widetilde{g}_{k,0}^\epsilon| \le C(\overline{w}_k^\epsilon)^2 e^{Q_4^\epsilon(k-4)} \quad \forall \, k \ge N^\epsilon + 1; \tag{120}$$

specifically, for $k = N^\epsilon + 1$:

$$|\widetilde{g}_{N^\epsilon+1,0}^\epsilon| \le C \overline{w}_2^\epsilon \overline{w}_{N^\epsilon-1}^\epsilon, \tag{121}$$

and

$$|\widetilde{g}_l^\epsilon(\overline{x})| \le \mu C \overline{D}^{-l+1} \quad \forall \, l \ge 1, \tag{122}$$

for all $0 < \epsilon \ll 1$, $\overline{x} \in [-\frac{3}{4}, \frac{3}{4}]$. *Here* $C > 0$ *is some constant that is independent of* $\overline{D}$ *and* ϵ.

Proof. The expansion of $\widetilde{g}^\epsilon$ follows from composition of analytic functions. For the property of the convergence radius in (122), we use the binomial theorem to obtain

$$\widetilde{g}_l^\epsilon(\overline{x}) = \mu \sum_{n=l}^\infty \left(\sum_{m=1}^\epsilon h_{m,n}^\epsilon \epsilon^{m-1} \overline{x}^m \right) \epsilon^{n-1} \binom{n}{l} \left(j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) \right)^{n-l}, \quad l \ge 2, \tag{123}$$

cf. (80) and (81), and use (117), (111) and (33). This gives

$$|\widetilde{g}_l^\epsilon(\overline{x})| \le \frac{3}{2} \mu \rho^{-1} \sum_{n=l}^\infty \epsilon^{n-1} \rho^{-n} 2^n (C\epsilon)^{n-l} \le 3\mu\rho^{-1}(2\rho^{-1})^l \epsilon^{l-1} \le 6\mu\rho^{-2} \overline{D}^{-l+1},$$

for all $\overline{x} \in [-\frac{3}{4}, \frac{3}{4}]$, $\overline{D} < (2\rho^{-1}\epsilon)^{-1}$ and $0 < \epsilon \ll 1$, upon estimating the geometric sums.

Next, we notice that (121) follows from (120) upon using (100). We therefore turn to proving (120). For this purpose, we use (83) and focus on estimating $(\overline{h}^\epsilon(\cdot, j^{N^\epsilon-1}[\overline{m}^\epsilon](\cdot)))_k$. By (84), (33), $\|\overline{m}^\epsilon\| \le 2F$ in the seminorm (103) (cf. Proposition 4.10) and Lemma 4.7, we obtain that

$$
\begin{aligned}
|(\overline{h}^\epsilon(\cdot, j^{N^\epsilon-1}[\overline{m}^\epsilon](\cdot)))_k| \le 2F &\sum_{j=2}^{\min(k-2,N^\epsilon-1)} \rho^{-k+j-1}\epsilon^{k-j-1}\overline{w}_j^\epsilon + \\
&+\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\sum_{j=2l}^{\min(k,l(N^\epsilon-1))} \rho^{-k+j-l}\epsilon^{k-j+l-2}(2F)^l C^{l-1} e^{(-2Q_4^\epsilon+P_4^\epsilon)l}e^{Q_4^\epsilon j}\\
&\le 2F\rho^{-3}\epsilon(\rho^{-1}\epsilon)^{k-(N^\epsilon+1)}\sum_{j=2}^{N^\epsilon-1}(\rho^{-1}\epsilon)^{N^\epsilon-1-j}\overline{w}_j^\epsilon\\
&+e^{Q_4^\epsilon k}\epsilon^{-2}C^{-1}\sum_{l=2}^{\lfloor \frac{k}{2}\rfloor}\left(2\rho^{-1}\epsilon F C e^{-2Q_4^\epsilon+P_4^\epsilon}\right)^l \sum_{j=2l}^{\min(k,l(N^\epsilon-1))}\left(\rho^{-1}\epsilon e^{-Q_4^\epsilon}\right)^{k-j}.
\end{aligned}
$$

Here

$$
0<\rho^{-1}\epsilon e^{-Q_4^\epsilon}\ll 1,\quad 0<2\rho^{-1}\epsilon F C e^{-2Q_4^\epsilon+P_4^\epsilon}\ll 1,
$$

for all $0<\epsilon\ll 1$, recall (93). But then, by estimating the geometric series and using $\exp(P_4^\epsilon)=\overline{w}_2^\epsilon$ (see (100)), we conclude that

$$
|(\overline{h}^\epsilon(\cdot, j^{N^\epsilon-1}[\overline{m}^\epsilon](\cdot)))_k| \le \overline{C}e^{Q_4^\epsilon k}e^{-4Q_4^\epsilon+2P_4^\epsilon} = \overline{C}(\overline{w}_2^\epsilon)^2 e^{Q_4^\epsilon(k-4)},
$$

for some $\overline{C}>0$ large enough. This gives the desired estimates (upon $\overline{C}\to C$). □

We now turn to estimating $j^{N^\epsilon}[\overline{m}^\epsilon]$; in contrast to $j^{N^\epsilon-1}[\overline{m}^\epsilon]$, it is not uniformly bounded with respect to $\alpha^\epsilon \in (0,1)$.

Lemma 4.14. *Suppose that $S_\infty^0\ne 0$. Then*

$$
|\overline{m}_{N^\epsilon}^\epsilon \overline{x}^{N^\epsilon}| \le (1+o(1))S_\infty^0 \overline{w}_{N^\epsilon}^\epsilon \delta^{N^\epsilon}\quad \forall\, \overline{x}\in[-\delta,\delta], \tag{124}
$$

for all $0<\epsilon\ll 1$. Moreover, fix any $K>0$ and suppose for $N^\epsilon\gg 1$ and $\alpha^\epsilon\in(0,1)$ that

$$
\delta \le \min\left(\frac{3}{4},\left(\frac{K}{2|S_\infty^0||\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}}\right)^{\frac{1}{N^\epsilon}}\right). \tag{125}
$$

(For fixed α^ϵ, the expression on the right hand side of (125) converges to $\frac{3}{4}$ for $N^\epsilon\to\infty$). Then

$$
|j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x})|\le K\quad \forall\,\overline{x}\in[-\delta,\delta]. \tag{126}
$$

Proof. We estimate

$$
\begin{aligned}
|\overline{m}_{N^\epsilon}^\epsilon \overline{x}^{N^\epsilon}| &\le |\overline{S}_{N^\epsilon}^\epsilon|\overline{w}_{N^\epsilon}^\epsilon\delta^{N^\epsilon}\\
&=(1+o(1))|S_\infty^0|\frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon)}{\Gamma(\epsilon^{-1}-1)}\delta^{N^\epsilon}\\
&=(1+o(1))|S_\infty^0|\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}\delta^{N^\epsilon},
\end{aligned}\tag{127}
$$

using (87), (19), $S_\infty^0\ne 0$ and Stirling's approximation (in the form (22)) on the factor

$$
\frac{\Gamma(N^\epsilon+a^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})} = \frac{\Gamma(N^\epsilon+a^\epsilon)}{(1-\epsilon)\Gamma(\epsilon^{-1}-1)} = (1+o(1))\frac{\Gamma(N^\epsilon)(N^\epsilon)^{a^\epsilon}}{\Gamma(N^\epsilon)(N^\epsilon)^{\alpha^\epsilon-1}} = (1+o(1))(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}, \tag{128}
$$

for $N^\epsilon\to\infty$; in particular the $o(1)$-terms in (127) are uniform with respect to α^ϵ. Using (125), we have

$$
(1+o(1))|S_\infty^0|\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+1-\alpha^\epsilon}\delta^{N^\epsilon}\le \frac{1}{2}K(1+o(1))
$$

The result then follows from $j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x}) = j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x}) + \overline{m}_{N^\epsilon}^\epsilon\overline{x}^{N^\epsilon}$. □

If we take $\overline{D}>C>0$ and $0<\epsilon\ll 1$, then it follows from Lemma 4.12 that $\widetilde{g}^\epsilon(x,\overline{m}_{N^\epsilon}^\epsilon\overline{x}^{N^\epsilon})$ is well-defined for all $\overline{x}\in[\delta,\delta]$ with $\delta>0$ satisfying (125).

4.3 The operator $\mathcal{T}^\epsilon$

Define $H \mapsto \mathcal{T}^\epsilon[H]$ by

$$\begin{aligned}\mathcal{T}^\epsilon[H](\overline{x}) &:= \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^{\overline{x}} \frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{v^{\epsilon^{-1}+1}}H(v)dv \\ &:= \frac{|\overline{x}|^{\alpha^\epsilon}\overline{x}^{N^\epsilon}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\int_0^{\overline{x}} \frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{|v|^{\alpha^\epsilon}v^{N^\epsilon+1}}H(v)dv \quad \forall -1<\overline{x}<1.\end{aligned} \tag{129}$$

It is well-defined on analytic functions H with $j^{N^\epsilon}[H]=0$, see also [10, Section 7].

Lemma 4.15. *Suppose that $\epsilon^{-1}\notin\mathbb{N}$. Then the following statements hold true:*

1. *For any analytic H with $j^{N^\epsilon}[H]=0$, $G=\mathcal{T}^\epsilon[H]$ is the unique solution of*

$$\epsilon\overline{x}(1-\overline{x})\frac{dG}{d\overline{x}} - (1+\epsilon a^\epsilon\overline{x})G = \epsilon H \quad \textit{and} \quad j^{N^\epsilon}[G]=0. \tag{130}$$

2. *$\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]$ has an absolutely convergent power series representation for $-1<\overline{x}<1$:*

$$\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) = \frac{\Gamma(1-\alpha^\epsilon)}{\Gamma(N^\epsilon+1+a^\epsilon)}\sum_{k=N^\epsilon+1}^{\infty}\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k. \tag{131}$$

Proof. To prove item 1, we define $\mathcal{J}(\overline{x}) := \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}$ and subsequently $\mathcal{I}(\overline{x}) := \int_0^{\overline{x}}\frac{1}{\mathcal{J}(v)v(1-v)}H(v)dv$. Then

$$\mathcal{T}^\epsilon[H] = \mathcal{J}\mathcal{I} \quad \text{and} \quad \mathcal{J}(\overline{x})I'(\overline{x}) = \frac{1}{\overline{x}(1-\overline{x})}H(\overline{x}).$$

Moreover,

$$\begin{aligned}\mathcal{J}'(\overline{x}) &= \mathcal{J}(\overline{x})\left(\epsilon^{-1}\overline{x}^{-1} + (\epsilon^{-1}+a^\epsilon)(1-\overline{x})^{-1}\right) \\ &= \mathcal{J}(\overline{x})\frac{1+\epsilon a^\epsilon\overline{x}}{\epsilon\overline{x}(1-\overline{x})},\end{aligned}$$

and therefore

$$\epsilon\overline{x}(1-\overline{x})\mathcal{T}^\epsilon[H]'(\overline{x}) = (1+\epsilon a^\epsilon\overline{x})\,\mathcal{J}(\overline{x})\mathcal{I}(\overline{x}) + \epsilon H(\overline{x}).$$

Consequently,

$$\epsilon\overline{x}(1-\overline{x})\mathcal{T}^\epsilon[H]'(\overline{x}) - (1+\epsilon a^\epsilon\overline{x})\mathcal{T}^\epsilon[H](\overline{x}) = \epsilon H(\overline{x}),$$

as desired.

Next, to prove item 2, we use item 1 and the fact that the solution is unique. Then Lemma 4.3 with

$$(\epsilon\overline{g}^\epsilon(\cdot,\overline{m}^\epsilon(\cdot)))_k = \begin{cases}-\epsilon & \text{for } k=N^\epsilon+1,\\ 0 & \text{else,}\end{cases}$$

allow us to write $\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) = \sum_{k=N^\epsilon+1}^\infty \overline{m}_k^\epsilon\overline{x}^k$ as an absolutely convergent power series; notice the change of sign when comparing (130) and (79). In particular, we find that

$$\overline{S}_k^\epsilon = \frac{(-1)^{N^\epsilon+1}}{\overline{w}_{N^\epsilon+1}^\epsilon(1-\alpha^\epsilon)} \quad \forall\, k\geq N^\epsilon+1\,(\text{zero otherwise}),$$

and therefore

$$\overline{m}_k^\epsilon = \frac{1}{1-\alpha^\epsilon}\frac{(-1)^k\overline{w}_k^\epsilon}{(-1)^{N^\epsilon+1}\overline{w}_{N^\epsilon+1}^\epsilon} \quad \forall\, k\geq N^\epsilon+1\,(\text{zero otherwise}),$$

by (87). Subsequently, we then use item 5 of Lemma 4.5 to write

$$\begin{aligned}\frac{(-1)^k\overline{w}_k^\epsilon}{(-1)^{N^\epsilon+1}\overline{w}_{N^\epsilon+1}^\epsilon} &= \frac{\Gamma(2-\alpha^\epsilon)}{\Gamma(N^\epsilon+1+a^\epsilon)}\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\\ &= \frac{(1-\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)}{\Gamma(N^\epsilon+1+a^\epsilon)}\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}.\end{aligned}$$

This gives the desired expression in (2).

□

We will view $\mathcal{T}^\epsilon$ on the Banach space $\mathcal{D}_\delta^\epsilon$ of analytic functions $H:[0,\delta]\to\mathbb{R}$ with $|H(\overline{x})\overline{x}^{-N^\epsilon-1}|$ bounded at $\overline{x}=0$. More specifically, we define

$$\mathcal{D}_\delta^\epsilon := \{H:[0,\delta]\to\mathbb{R}\text{ analytic }:\ \|\!|H|\!\|_\delta<\infty\},$$

with the Banach norm

$$\|\!|H|\!\|_\delta := \sup_{\overline{x}\in(0,\delta]}|H(\overline{x})|\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})};\tag{132}$$

here we have used that $\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) = \mathcal{O}(\overline{x}^{N^\epsilon+1})$ as $\overline{x}\to 0$ and that $\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})>0$ for all $\overline{x}\in(0,1)$, cf. Lemma 4.15 item 2.

The case $\overline{x}<0$ has to be treated slightly different (we will have to take $0\le-\overline{x}\le\delta_2\epsilon$); we will consider this case at the end of Section 4.6 below.

A nice property of the Banach norm (132) is highlighted in the following Lemma.

Lemma 4.16. *Define*

$$\|H\|_\delta := \sup_{\overline{x}\in[0,\delta]}|H(\overline{x})|.\tag{133}$$

Then the following estimate holds:

$$\|H\|_\delta \le \|\!|H|\!\|_\delta \quad \forall\, H\in\mathcal{D}_\delta^\epsilon.$$

Proof. The proof is elementary. Indeed, for any $\overline{x}\in(0,\delta]$ we have

$$|H(\overline{x})|\le\left(|H(\overline{x})|\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\right)\times\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}\le\|\!|H|\!\|_\delta\times\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)},\tag{134}$$

and therefore

$$|H(\overline{x})|\le\|\!|H|\!\|_\delta\times 1,\tag{135}$$

since $\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})$ is an increasing function of $\overline{x}\in[0,1)$. As $H(0)=0$ the inequality (135) holds for all $\overline{x}\in[0,\delta]$, completing the proof. □

Notice also the (obvious) fact that

$$\|\!|H|\!\|_{\delta'}\le\|\!|H|\!\|_\delta,$$

for any $0<\delta'<\delta$. This also holds with $\|\!|$ replaced by $\|$, recall (133). We will use these properties without further mention in the following. The following set of equalities

$$\|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]\|_\delta = \|\!|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]|\!\|_\delta = \mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta)\quad\forall\,0<\delta<1,\tag{136}$$

are consequences of $\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})$ being an increasing function of $\overline{x}\in[0,1)$, and they will also be important.

It turns out that

$$0<\delta\le\frac{3}{4},\tag{137}$$

will be adequate for our purposes.

Lemma 4.17. *Fix any $\delta_2 > 0$, suppose that $\epsilon^{-1} \notin \mathbb{N}$ and that (137) holds. Then there exists a $K_2 = K_2(\delta_2, a^0)$ such that the following holds true.*

1. *Define $\sigma_\epsilon : [-\delta_2\epsilon, \frac{3}{4}] \to \mathbb{R}_+$ by*

$$\sigma_\epsilon(\overline{x}) := \begin{cases} 1 & \forall\, 0 \le |\overline{x}| \le \delta_2\epsilon \\ (\overline{x}^{-1}\delta_2\epsilon)^{1-\alpha^\epsilon} & \forall\, \delta_2\epsilon < \overline{x} \le \frac{3}{4}, \end{cases} \tag{138}$$

so that

$$\left(\frac{4}{3}\delta_2\epsilon\right)^{1-\alpha^\epsilon} \le \sigma_\epsilon(\overline{x}) \le 1 \quad \forall \overline{x} \in \left[-\delta_2\epsilon, \frac{3}{4}\right]. \tag{139}$$

Then there are constants $0 < C_1 < C_2$, $C_i = C_i(\delta_2, a^0)$, such that the following holds for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$:

$$C_1 \left(\frac{|\overline{x}|}{1-\overline{x}}\right)^{N^\epsilon+1} \sigma_\epsilon(\overline{x}) \le (1-\alpha^\epsilon) \left| \mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) \right| \le C_2 \left(\frac{|\overline{x}|}{1-\overline{x}}\right)^{N^\epsilon+1} \sigma_\epsilon(\overline{x}), \tag{140}$$

for all $-\delta_2\epsilon \le \overline{x} \le \frac{3}{4}$.

2. *Asymptotics for $\overline{x} = \mathcal{O}(\epsilon)$: For any $\overline{x} = \epsilon\overline{x}_2$, $\overline{x}_2 \in [-\delta_2, \delta_2]$, the following asymptotics hold true*

$$\mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right](\epsilon\overline{x}_2) = \frac{1}{1-\alpha^\epsilon}(\epsilon\overline{x}_2)^{N^\epsilon+1}\left[1 + \overline{x}_2 \int_0^1 e^{(1-v)\overline{x}_2} v^{1-\alpha^\epsilon} dv + \mathcal{O}(\epsilon)\right] \quad \forall \overline{x}_2 \in [-\delta_2, \delta_2],$$

with $\mathcal{O}(\epsilon)$ being uniform with respect to $\alpha^\epsilon \in (0,1)$.

3. *$\mathcal{T}^\epsilon : \mathcal{D}^\epsilon_\delta \to \mathcal{D}^\epsilon_\delta$ is a bounded operator. In particular, let*

$$|||\mathcal{T}^\epsilon|||_\delta := \sup_{|||H|||_\delta = 1} |||\mathcal{T}^\epsilon[H]|||_\delta,$$

denote the operator norm. Then

$$|||\mathcal{T}^\epsilon|||_\delta \le \frac{K_2}{1-\alpha^\epsilon}(1 + \log \sigma_\epsilon(\delta)^{-1}).$$

In particular, the operator norm is uniformly bounded with respect to $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$ if $0 < \delta \le \delta_2\epsilon$ (cf. (138)).

4. *The following holds for any $i \in \mathbb{N}$:*

$$|||\mathcal{T}^\epsilon\left[(\cdot)^i H\right]|||_\delta \le \frac{K_2\delta^i}{i} |||H|||_\delta \quad \forall\, H \in \mathcal{D}^\epsilon_\delta. \tag{141}$$

5. *The following holds for any $l \in \mathbb{N}$:*

$$|||\mathcal{T}^\epsilon\left[(\cdot)^{lN^\epsilon+1}\right]|||_\delta \le \delta^{(l-1)N^\epsilon} \mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta). \tag{142}$$

6. *Suppose that $E, R > 0$, $0 < \delta < R$ and consider*

$$H(\overline{x}) = \sum_{k=N^\epsilon+1}^{\infty} H_k \overline{x}^k \quad \forall \overline{x} \in [0, \delta],$$

with $|H_k| \le ER^k$. Then

$$|||\mathcal{T}^\epsilon[H]|||_\delta \le \frac{ER^{N^\epsilon+1}}{1-R\delta} \mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta),$$

for all $0 < \epsilon \ll 1$.

7. *Let $C := C_2 C_1^{-1} > 1$, with $C_i > 0$ defined in* (140)*, $E, R > 0$, and suppose that $\overline{Y} \in \mathcal{D}_\delta^\epsilon$ and*

$$H(\overline{x}, \overline{Y}) = \sum_{l=2}^{\infty} H_l(\overline{x}) \overline{Y}^l \quad \forall \overline{x} \in [0, \delta],\, 0 \le \|\overline{Y}\|_\delta < R^{-1}, \tag{143}$$

with

$$\|H_l\|_\delta \le E R^{l-1} \quad \forall l \ge 2,$$

recall (133)*. Then*

$$\|\mathcal{T}^\epsilon \left[H(\cdot, \overline{Y}(\cdot))\right]\|_\delta \le 4\epsilon E K_2 C R \|Y\|_\delta^2 \quad \forall 0 \le \|\overline{Y}\|_\delta < \frac{1}{2}(CR)^{-1},$$

uniformly in $\alpha^\epsilon \in (0,1)$. In particular, $\overline{Y} \mapsto \mathcal{T}^\epsilon \left[h(\cdot, \overline{Y}(\cdot))\right]$ is C^1 and for all $0 < \epsilon \ll 1$, it is a contraction:

$$\|D_{\overline{Y}}(\mathcal{T}^\epsilon \left[H(\cdot, \overline{Y}(\cdot))\right]) (\overline{Z})\|_\delta \le \mathcal{O}(\epsilon) \|\overline{Z}\|_\delta \quad \forall \overline{Z} \in \mathcal{D}_\delta^\epsilon,\, \|\overline{Y}\|_\delta < \frac{1}{2}(CR)^{-1}. \tag{144}$$

Proof. We prove the items 1–7 successively in the following.

Proof of item 1. The result follows from [10, Lemma 7.2], see [10, Eq. (7.10)], and it is based on the integral representation for $\mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right]$:

$$\begin{aligned}\mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) &= \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}} \int_0^{\overline{x}} (1-v)^{\epsilon^{-1}+a^\epsilon-1} v^{-\alpha^\epsilon} dv \\ &= \frac{|\overline{x}|^{\alpha^\epsilon} \overline{x}^{N^\epsilon}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}} \int_0^{\overline{x}} (1-v)^{\epsilon^{-1}+a^\epsilon-1} |v|^{-\alpha^\epsilon} dv\end{aligned} \tag{145}$$

For completeness, we include the details (which will also be important later): Firstly, for $\overline{x} = \epsilon \overline{x}_2 \in [-\epsilon\delta_2, \epsilon\delta_2]$, $\delta_2 > 0$ fixed, we use:

$$(1-\overline{x})^{\epsilon^{-1}} = e^{\epsilon^{-1}\log(1-\overline{x})} = e^{-\overline{x}_2}\left(1 + \mathcal{O}(\epsilon \overline{x}_2^2)\right) = \mathcal{O}(1) \gtrless \begin{cases} e^{-2\delta_2} \\ e^{2\delta_2} \end{cases} \tag{146}$$

for all $0 < \epsilon \ll 1$. Consequently, for $\overline{x} \in [-\epsilon\delta_2, \epsilon\delta_2]$

$$\begin{aligned}\mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) &= \left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1} (1+o(1)) |\overline{x}|^{\alpha^\epsilon} \overline{x}^{-1} \int_0^{\overline{x}} \mathcal{O}(1) |v|^{-\alpha^\epsilon} dv \\ &= \left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1} (1+o(1)) \frac{\mathcal{O}(1)}{1-\alpha^\epsilon},\end{aligned}$$

where $0 < C_1 < \mathcal{O}(1) < C_2$. Next, for $\delta_2 \epsilon < \overline{x} \le \frac{3}{4}$, we use a separate set of estimates:

$$\begin{aligned}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})| &\le \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}} \int_0^1 (1-v)^{\epsilon^{-1}+a^\epsilon-1} v^{-\alpha^\epsilon} dv \\ &= \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}} \frac{\Gamma(\epsilon^{-1}+a^\epsilon)\Gamma(1-\alpha^\epsilon)}{\Gamma(\epsilon^{-1}+a^\epsilon+1-\alpha^\epsilon)} \\ &= \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}} (1+o(1)) \Gamma(1-\alpha^\epsilon) \epsilon^{1-\alpha^\epsilon} \\ &\le C_2 \left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1} (\overline{x}^{-1}\delta_2\epsilon)^{1-\alpha^\epsilon} (1-\alpha^\epsilon)^{-1},\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})| &\ge \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}} \int_0^{\delta_2\epsilon} (1-v)^{\epsilon^{-1}+a^\epsilon-1} v^{-\alpha^\epsilon} dv \\ &\ge C_1 \left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1} \frac{(\overline{x}^{-1}\delta_2\epsilon)^{1-\alpha^\epsilon}}{1-\alpha^\epsilon}.\end{aligned}$$

for some $C_1 = C_1(\delta_2, a^0)$ small enough, cf. (146). Here we have also used (24), (22) and (20).

Proof of item 2. For item 2, we use (145) with the substitution $v = \overline{x}\tilde{v}$ and (146) with $\overline{x} = \epsilon\overline{x}_2 \in [-\epsilon\delta_2, \epsilon\delta_2]$. This gives

$$\begin{aligned}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) &= \overline{x}^{N^\epsilon+1}e^{\overline{x}_2}(1+\mathcal{O}(\epsilon))\int_0^1 e^{-(1+\epsilon(a^\epsilon-1))\tilde{v}\overline{x}_2}(1+\mathcal{O}(\epsilon\tilde{v}^2))\tilde{v}^{-\alpha^\epsilon}d\tilde{v}\\ &= \overline{x}^{N^\epsilon+1}e^{\overline{x}_2}(1+\mathcal{O}(\epsilon))\left(\int_0^1 e^{-(1+\epsilon(a^\epsilon-1))v\overline{x}_2}v^{-\alpha^\epsilon}dv + \mathcal{O}(\epsilon)\right),\end{aligned}$$

with each $\mathcal{O}(\epsilon)$ uniform with respect to α^ϵ. We now use integration by parts on the remaining integral:

$$\begin{aligned}\int_0^1 e^{-(1+\epsilon(a^\epsilon-1))t\overline{x}_2}v^{-\alpha^\epsilon}dv &= \frac{e^{-(1+\epsilon(a^\epsilon-1))\overline{x}_2}}{1-\alpha^\epsilon}\left[1+(1+\epsilon(a^\epsilon-1))\overline{x}_2\int_0^1 e^{(1+\epsilon(a^\epsilon-1))(1-v)\overline{x}_2}v^{1-\alpha^\epsilon}dv\right]\\ &= \frac{e^{-\overline{x}_2}}{1-\alpha^\epsilon}\left[1+\overline{x}_2\int_0^1 e^{(1-v)\overline{x}_2}v^{1-\alpha^\epsilon}dv + \mathcal{O}(\epsilon)\right].\end{aligned}$$

This completes the proof.

Proof of item 3. We estimate using (129), (134) and (140)

$$\begin{aligned}\left|\mathcal{T}^\epsilon[H]\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\right| &\le \frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\frac{1}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\int_0^{\overline{x}}\frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{v^{\epsilon^{-1}+1}}\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](v)dv|||H|||_\delta\\ &\le C_2C_1^{-1}\frac{\overline{x}^{\epsilon^{-1}}}{(1-\overline{x})^{\epsilon^{-1}+a^\epsilon}}\left(\frac{1-\overline{x}}{\overline{x}}\right)^{N^\epsilon+1}\sigma_\epsilon(\overline{x})^{-1}\\ &\quad\times\int_0^{\overline{x}}\frac{(1-v)^{\epsilon^{-1}+a^\epsilon-1}}{v^{\epsilon^{-1}+1}}\left(\frac{v}{1-v}\right)^{N^\epsilon+1}\sigma_\epsilon(v)dv|||H|||_\delta\\ &\le C_2C_1^{-1}(1-\overline{x})^{1-\alpha^\epsilon-a^\epsilon}\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\\ &\quad\times\int_0^{\overline{x}}(1-v)^{\alpha^\epsilon+a^\epsilon-2}v^{-\alpha^\epsilon}\sigma_\epsilon(v)dv|||H|||_\delta\\ &\le K_2\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}v^{-\alpha^\epsilon}\sigma_\epsilon(v)dv|||H|||_\delta\quad\forall\, 0<\overline{x}\le\delta\le\frac{3}{4},\end{aligned}$$

for some $K_2 = K_2(\delta_2, a^0) > 0$. Here we have used uniform bounds on

$$(1-\overline{x})^{1-\alpha^\epsilon-a^\epsilon}\quad\text{and}\quad(1-\overline{x})^{\alpha^\epsilon+a^\epsilon-2}\quad\text{for}\quad\overline{x}\in\left[0,\frac{3}{4}\right].$$

Due to (138), we estimate $0 < \overline{x} \le \delta_2\epsilon$ and $\delta_2\epsilon < \overline{x} \le \delta$ separately. The former gives an estimate

$$\left|\mathcal{T}^\epsilon[H]\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\right| \le \frac{K_2}{1-\alpha^\epsilon}|||H|||_\delta\quad\forall\, 0<\overline{x}\le\delta_2\epsilon,$$

directly. We therefore consider $\delta_2\epsilon < \overline{x} \le \delta \le \frac{3}{4}$ and find

$$\begin{aligned}\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}v^{-\alpha^\epsilon}\sigma_\epsilon(v)dv &= \frac{1}{(\delta_2\epsilon)^{1-\alpha^\epsilon}}\int_0^{\delta_2\epsilon}v^{-\alpha^\epsilon}dv + \int_{\delta_2\epsilon}^{\overline{x}}v^{-1}dv\\ &= \frac{1}{1-\alpha^\epsilon} - \log(\overline{x}^{-1}\delta_2\epsilon)\\ &= \frac{1}{1-\alpha^\epsilon}(1+\log\sigma_\epsilon(\overline{x})^{-1}).\end{aligned}$$

This completes the proof.

Proof of item 4. Proceeding as in the proof of item 3, we find

$$\left|\mathcal{T}^\epsilon\left[(\cdot)^i H\right]\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\right|\le K_2\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}v^{i-\alpha^\epsilon}\sigma_\epsilon(v)dv\|\!|H|\!\|_\delta\quad\forall\, 0<\overline{x}\le\delta. \tag{147}$$

As above, we estimate $0<\overline{x}\le\delta_2\epsilon$ and $\delta_2\epsilon<\overline{x}\le\delta$ separately. In the former case, we directly obtain that

$$\begin{aligned}\left|\mathcal{T}^\epsilon[(\cdot)^i H]\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\right|&\le\frac{K_2\overline{x}^i}{i+1-\alpha^\epsilon}\|\!|H|\!\|_\delta\\&\le\frac{K_2\overline{x}^i}{i}\|\!|H|\!\|_\delta\quad\forall\, 0<\overline{x}\le\delta_2\epsilon.\end{aligned}$$

We are therefore left with $\delta_2\epsilon<\overline{x}\le\delta$ where

$$\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}v^{i-\alpha^\epsilon}\sigma_\epsilon(v)dv\le\int_0^{\overline{x}}v^{i-1}dv=\frac{\overline{x}^i}{i}\quad\forall\,\epsilon\delta_2<\overline{x}\le\delta,$$

Here we have used that

$$\frac{1}{(\delta_2\epsilon)^{1-\alpha^\epsilon}}\int_0^{\delta_2\epsilon}v^{i-\alpha^\epsilon}dv\le\int_0^{\delta_2\epsilon}v^{i-1}dv.$$

This completes the proof.

Proof of item 5. This case is easy:

$$\left|\mathcal{T}^\epsilon\left[(\cdot)^{lN^\epsilon+1}\right](\overline{x})\right|\le\delta^{(l-1)N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})\quad\forall\, 0\le\overline{x}\le\delta,$$

and therefore

$$\|\!|\mathcal{T}^\epsilon\left[(\cdot)^{lN^\epsilon+1}\right]|\!\|_\delta\le\delta^{(l-1)N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta).$$

Proof of item 6. We have

$$|H(\overline{x})|\le ER^{N^\epsilon+1}\overline{x}^{N^\epsilon+1}\sum_{k=0}^\infty R^k\delta^k=\frac{ER^{N^\epsilon+1}\overline{x}^{N^\epsilon+1}}{1-R\delta}\quad\forall 0\le\overline{x}\le\delta,$$

and consequently

$$|\mathcal{T}^\epsilon[H](\overline{x})|\le\frac{ER^{N^\epsilon+1}}{1-R\delta}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})\quad\forall\, 0<\overline{x}\le\delta.$$

The result follows.

Proof of item 7. Now, for item 7 we use the linearity of $\mathcal{T}^\epsilon$ and first estimate each of the terms of the sum $\sum_{l\ge2}\mathcal{T}^\epsilon\left[H_l(\cdot)\overline{Y}^l\right]$. By (129), (134) and (140), we find that

$$\begin{aligned}\left|\mathcal{T}^\epsilon\left[\overline{Y}^l\right](\overline{x})\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})}\right|&\le K_2\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}v^{-\alpha^\epsilon}\sigma_\epsilon(v)\left(\frac{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](v)}{\mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta)}\right)^{l-1}dv\|\!|\overline{Y}|\!\|_\delta^l\\&\le K_2C^{l-1}\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}dv\|\!|\overline{Y}|\!\|_\delta^l\quad\forall\, 0<\overline{x}\le\delta,\end{aligned} \tag{148}$$

with $C=C_2/C_1$. We claim that

$$\|\!|\mathcal{T}^\epsilon\left[\overline{Y}^l\right]|\!\|_\delta\le\frac{2\epsilon}{l-1}K_2C^{l-1}\|\!|\overline{Y}|\!\|_\delta^l. \tag{149}$$

In order to prove this, we only have to show that

$$\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}}\frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}dv\le\frac{2\epsilon}{l-1}\quad\forall\, 0<\overline{x}\le\delta, \tag{150}$$

cf. (148). Consider first the simplest case $0 < \overline{x} \le \delta \le \delta_2\epsilon$. Then $\sigma_\epsilon(\overline{x}) = \sigma_\epsilon(v) = \sigma_\epsilon(\delta) = 1$, for all $0 \le \overline{x} \le v$, and we have

$$\begin{aligned}
\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}} \frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}dv &= \frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}\left(\frac{\overline{x}}{\delta}\right)^{(l-1)(N^\epsilon+1)}\\
&\le \frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}\\
&\le \frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon-l(1-\alpha^\epsilon)}\\
&= \frac{1}{(l-1)(N^\epsilon+\alpha^\epsilon)}\\
&= \frac{\epsilon}{l-1},
\end{aligned}$$

and (150) follows. We are left with $0 < \overline{x} \le \delta_2\epsilon < \delta$ and $\delta_2\epsilon < \overline{x} \le \delta$. For the former, we have $\sigma_\epsilon(\overline{x}) = \sigma_\epsilon(v) = 1$, $\sigma_\epsilon(\delta) = (\delta^{-1}\delta_2\epsilon)^{1-\alpha^\epsilon}$ and

$$\begin{aligned}
\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}} \frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}dv &= \overline{x}^{\alpha^\epsilon-1}\int_0^{\overline{x}} \frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}}{\delta^{(l-1)(N^\epsilon+1)}(\delta^{-1}\delta_2\epsilon)^{(l-1)(1-\alpha^\epsilon)}}dv\\
&= \frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}\left(\frac{\overline{x}}{\delta}\right)^{(l-1)(N^\epsilon+\alpha^\epsilon)}\left(\frac{\overline{x}}{\delta_2\epsilon}\right)^{(l-1)(1-\alpha^\epsilon)}\\
&\le \frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon},\\
&\le \frac{\epsilon}{l-1},
\end{aligned}$$

and (150) follows. We finally consider $\delta_2\epsilon < \overline{x} \le \delta$:

$$\begin{aligned}
\frac{\overline{x}^{\alpha^\epsilon-1}}{\sigma_\epsilon(\overline{x})}\int_0^{\overline{x}} \frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}\sigma_\epsilon(v)^l}{\delta^{(l-1)(N^\epsilon+1)}\sigma_\epsilon(\delta)^{l-1}}dv &= \frac{1}{(\delta_2\epsilon)^{1-\alpha^\epsilon}}\int_0^{\delta_2\epsilon} \frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon}}{\delta^{(l-1)(N^\epsilon+\alpha^\epsilon)}(\delta_2\epsilon)^{(l-1)(1-\alpha^\epsilon)}}dv\\
&\quad+\int_{\delta_2\epsilon}^{\overline{x}} \frac{v^{(l-1)(N^\epsilon+1)-\alpha^\epsilon-l(1-\alpha^\epsilon)}}{\delta^{(l-1)(N^\epsilon+\alpha^\epsilon)}}dv\\
&\le \frac{1}{(l-1)(N^\epsilon+1)+1-\alpha^\epsilon}+\frac{1}{(l-1)(N^\epsilon+\alpha^\epsilon)}\\
&\le \frac{2}{(l-1)(N^\epsilon+\alpha^\epsilon)}\\
&= \frac{2\epsilon}{l-1},
\end{aligned}$$

and (150) follows. Here we have used $\delta \ge \delta_2\epsilon$ in the denominator of the first integral on the right hand side. In turn, we obtain (149) and therefore

$$\begin{aligned}
\left\|\left|\mathcal{T}^\epsilon\left[\sum_{l=2}^\infty H_l(\cdot)\overline{Y}^l\right]\right\|\right|_\delta &\le 2\epsilon E K_2 C^{-1}R^{-1}\sum_{l=2}^\infty (CR|||\overline{Y}|||_\delta)^l\\
&= 2\epsilon E K_2\frac{CR|||\overline{Y}|||_\delta^2}{1-CR|||\overline{Y}|||_\delta}\\
&\le 4\epsilon E K_2 CR|||\overline{Y}|||_\delta^2 \quad \forall\, |||\overline{Y}|||_\delta < \frac{1}{2}(CR)^{-1}.
\end{aligned}$$

The fact that the mapping $\overline{Y} \mapsto \mathcal{T}^\epsilon[\sum_{l=2}^\infty H_l(\cdot)\overline{Y}^l]$ is C^1 and a contraction for all $\epsilon > 0$ small enough follows from identical computations. Further details are therefore left out.

□

The following will also be important: Define

$$\overline{U}^\epsilon(\overline{x}) := (N^\epsilon + a^\epsilon)\overline{w}^\epsilon_{N^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}); \tag{151}$$

this quantity corresponds to M_0 in [10, Lemma 7.2].

Lemma 4.18. *Fix $\delta_2 > 0$ and suppose that $\epsilon^{-1} \in \mathbb{N}$. Then we have the following statements regarding $\overline{U}$:*

1. *$\overline{U}$ has an absolutely convergent power series representation*

$$\overline{U}^\epsilon(\overline{x}) = \frac{\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})}\sum_{k=N^\epsilon+1}^\infty \frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k \quad \forall -1 < \overline{x} < 1, \tag{152}$$

2. *For all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$, $0 < \delta \le \frac{3}{4}$ the following estimate holds:*

$$\|\overline{U}^\epsilon\|_\delta = |||\overline{U}^\epsilon|||_\delta \gtrless \frac{1}{1-\alpha^\epsilon}\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\left(\frac{\delta}{1-\delta}\right)^{N^\epsilon+1}\sigma_\epsilon(\delta)\begin{cases}C_1,\\ C_2.\end{cases} \tag{153}$$

Here $C_i = C_i(\delta_2, a^0) > 0$, $i = 1, 2$.

3. *Asymptotics for $\overline{x} = \mathcal{O}(\epsilon)$: Let $\overline{x} = \epsilon\overline{x}_2 \in [-\epsilon\delta_2, \epsilon\delta_2]$, $\delta_2 > 0$ fixed. Then:*

$$\overline{U}^\epsilon(\epsilon\overline{x}_2) = \frac{\Gamma(\alpha^\epsilon)}{1-\alpha^\epsilon}(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}(\epsilon\overline{x}_2)^{N^\epsilon+1}\left[1 + \overline{x}_2\int_0^1 e^{(1-v)\overline{x}_2}v^{1-\alpha^\epsilon}dv + o(1)\right], \tag{154}$$

with $o(1)$ uniform with respect to $\alpha^\epsilon \in (0,1)$.

Proof. We prove the items 1–3 successively in the following.
Proof of item 1. For (152) we use (2) and (95):

$$\begin{aligned}
&\overline{w}^\epsilon_{N^\epsilon} \times (N^\epsilon + a^\epsilon)\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x})\\
&= \frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})} \times \frac{(N^\epsilon+a^\epsilon)\Gamma(1-\alpha^\epsilon)}{\Gamma(N^\epsilon+1+a^\epsilon)}\sum_{k=N^\epsilon+1}^\infty \frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k\\
&= \frac{\Gamma(\alpha^\epsilon)\Gamma(1+\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})}\sum_{k=N^\epsilon+1}^\infty \frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k.
\end{aligned} \tag{155}$$

Proof of item 2. Next, regarding (153) we use (136), (86), (19), (22),

$$\begin{aligned}
\overline{U}^\epsilon &= \frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon+1)}{\epsilon\Gamma(\epsilon^{-1})}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right] = \frac{\Gamma(\alpha^\epsilon)\Gamma(N^\epsilon+a^\epsilon+1)}{(1-\epsilon)\Gamma(N^\epsilon+\alpha^\epsilon-1)}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right]\\
&= \Gamma(\alpha^\epsilon)(1+o(1))(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right],
\end{aligned} \tag{156}$$

and subsequently (140).
Proof of item 3. Finally, for (154) we use (156) and Lemma 4.17 item 2:

$$\begin{aligned}
\overline{U}^\epsilon(\epsilon\overline{x}_2) &= \Gamma(\alpha^\epsilon)(1+o(1))(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\epsilon\overline{x}_2)\\
&= \Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\frac{1}{1-\alpha^\epsilon}(\epsilon\overline{x}_2)^{N^\epsilon+1}\left[1 + \overline{x}_2\int_0^1 e^{(1-v)\overline{x}_2}v^{1-\alpha^\epsilon}dv + o(1)\right].
\end{aligned}$$

□

4.4 Solving for the analytic weak-stable manifold

In the following, we write $\overline{m}^\epsilon$ as

$$\overline{m}^\epsilon = j^{N^\epsilon}[\overline{m}^\epsilon] + \overline{M}^\epsilon, \quad r^{N^\epsilon}[\overline{m}^\epsilon] =: \overline{M}^\epsilon; \tag{157}$$

we will use $\mathcal{T}^\epsilon$ to set up a fix-point equation for $\overline{M}^\epsilon$. For this purpose, let

$$\overline{G}^\epsilon(\overline{x},\overline{Y}) := \overline{g}^\epsilon(\overline{x}, j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x}) + \overline{Y}) - \overline{g}^\epsilon(\overline{x}, j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x})).$$

We clearly have

$$\overline{G}^\epsilon(\overline{x},\overline{Y}) = \overline{x}^2\overline{G}_1^\epsilon(\overline{x})\overline{Y} + \sum_{l\geq 2}\overline{G}_l^\epsilon(\overline{x})\overline{Y}^l, \tag{158}$$

and for any $\overline{D}>0$:

$$\|\overline{G}_l^\epsilon\|_\delta \leq \mu C\overline{D}^{-l+1} \quad \forall\, l\in\mathbb{N},$$

with $C>0$ independent of μ and ϵ, provided that (125) hold and that $0<\epsilon\ll 1$. This is essentially identical to the computation leading to (123) (with $j^{N^\epsilon-1}$ replaced by j^{N^ϵ}), using $\|j^{N^\epsilon}[\overline{m}^\epsilon]\|_\delta \leq K$, see (126).

The following result corresponds to [10, Lemma 7.3].

Lemma 4.19. *$\overline{Y} = M^\epsilon$ satisfies the fix-point equation:*

$$\begin{aligned}\overline{Y}(\overline{x}) &= (N^\epsilon + a^\epsilon)(-1)^{N^\epsilon}\overline{w}_{N^\epsilon}^\epsilon \overline{S}_{N^\epsilon}^\epsilon \mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\overline{x}) \\ &\quad - \mathcal{T}^\epsilon\left[r^{N^\epsilon}\left[\overline{g}^\epsilon((\cdot), j^{N^\epsilon}[\overline{m}^\epsilon](\cdot))\right]\right](\overline{x}) - \mathcal{T}^\epsilon\left[\overline{G}^\epsilon((\cdot),\overline{Y}(\cdot))\right](\overline{x}).\end{aligned} \tag{159}$$

Proof. With $\overline{m}_k^\epsilon$ given by (87), we obtain

$$\begin{aligned}\epsilon\overline{x}(\overline{x}-1)\frac{d\overline{M}^\epsilon}{d\overline{x}} + (1+\epsilon a^\epsilon\overline{x})\overline{M}^\epsilon &= -\epsilon(N^\epsilon + a^\epsilon)\overline{m}_{N^\epsilon}^\epsilon \overline{x}^{N^\epsilon+1} \\ &\quad + \epsilon\left(r^{N^\epsilon}\left[\overline{g}^\epsilon((\cdot), j^{N^\epsilon}[\overline{m}^\epsilon](\cdot))\right](\overline{x}) + \overline{G}^\epsilon(\overline{x},\overline{M}^\epsilon)\right).\end{aligned}$$

Using $\overline{m}_{N^\epsilon}^\epsilon = (-1)^{N^\epsilon}\overline{w}_{N^\epsilon}^\epsilon\overline{S}_{N^\epsilon}^\epsilon$, (151) and Lemma 4.15 item 1, the result follows; notice again the change of sign when comparing (130) and (79). □

We denote the right hand side of (159) by $\mathcal{F}(\overline{Y})(\overline{x})$:

$$\begin{aligned}\mathcal{F}(\overline{Y}) : &= (N^\epsilon + a^\epsilon)(-1)^{N^\epsilon}\overline{w}_{N^\epsilon}^\epsilon \overline{S}_{N^\epsilon}^\epsilon \mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right] \\ &\quad - \mathcal{T}^\epsilon\left[r^{N^\epsilon}\left[\overline{g}^\epsilon((\cdot), j^{N^\epsilon}[\overline{m}^\epsilon](\cdot))\right]\right] - \mathcal{T}^\epsilon\left[\overline{G}^\epsilon((\cdot),\overline{Y}(\cdot))\right],\end{aligned} \tag{160}$$

and define the closed ball

$$\mathcal{B}^C := \mathcal{D}_\delta^\epsilon \cap \{|\!|\!|Y|\!|\!|_\delta \leq C\},$$

of radius $C>0$.

Proposition 4.20. *Suppose that $S_\infty^0 \neq 0$ and $\mu>0$ small enough. Then for any $K>0$, $\alpha^\epsilon\in(0,1)$, $N^\epsilon\gg 1$, there is an $0<\overline{\delta}\leq\frac{3}{4}$ such that for any $0<\delta\leq\overline{\delta}$ the following holds:*

1. *Boundedness of the "leading order term":*

$$\|\overline{w}_{N^\epsilon}^\epsilon(\cdot)^{N^\epsilon}\|_\delta + \|\overline{U}^\epsilon\|_\delta \leq \frac{K}{|S_\infty^0|}, \tag{161}$$

2. *$\mathcal{F}:\mathcal{B}_\delta^{2K}\to\mathcal{B}_\delta^{2K}$, defined by (160), is a contraction.*

Proof. The first claim in item 1 is obvious as $\overline{U}^\epsilon(0) = 0$, see also Lemma 4.18 and (124). We therefore proceed to proof item 2 regarding $\mathcal{F} : \mathcal{B}_\delta^{2K} \to \mathcal{B}_\delta^{2K}$ being a contraction. For this purpose, we estimate each of the three terms on the right hand side (159) in the norm $|||\cdot|||_\delta$, recall (132).

The first term:

$$(N^\epsilon + a^\epsilon)(-1)^{N^\epsilon} \overline{w}_{N^\epsilon}^\epsilon \overline{S}_{N^\epsilon}^\epsilon \mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right](\overline{x}).$$

We write the first term as

$$(1 + o(1))(-1)^{N^\epsilon} S_\infty^0 \overline{U}^\epsilon(\overline{x}),$$

using Lemma 4.11 and (151). Then by using (152) and the assumption on δ, see (161), we have

$$|||(N^\epsilon + a^\epsilon)(-1)^{N^\epsilon} \overline{w}_{N^\epsilon}^\epsilon \overline{S}_{N^\epsilon}^\epsilon \mathcal{T}^\epsilon \left[(\cdot)^{N^\epsilon+1}\right] |||_\delta \le \frac{4}{3}K.$$

The second term:

$$-\mathcal{T}^\epsilon \left[r^{N^\epsilon} \left[\overline{g}^\epsilon((\cdot), j^{N^\epsilon}[\overline{m}^\epsilon](\cdot))\right]\right](\overline{x}).$$

For the second term, we first use (118):

$$\overline{g}^\epsilon(\overline{x}, j^{N^\epsilon}[\overline{m}^\epsilon](\overline{x}))) = \widetilde{g}^\epsilon(\overline{x}, \overline{m}_{N^\epsilon}^\epsilon \overline{x}^{N^\epsilon}).$$

Therefore

$$\begin{aligned} r^{N^\epsilon} \left[\overline{g}^\epsilon((\cdot), j^{N^\epsilon}[\overline{m}^\epsilon](\cdot)))\right](\overline{x}) = &\sum_{k=N^\epsilon+1}^\infty \tilde{g}_{k,0}^\epsilon \overline{x}^k + \widetilde{g}_1^\epsilon(\overline{x}) \overline{m}_{N^\epsilon}^\epsilon \overline{x}^{N^\epsilon+2} \\ &+ \sum_{l=2}^\infty \widetilde{g}_l^\epsilon(\overline{x})(\overline{m}_{N^\epsilon}^\epsilon)^l \overline{x}^{lN^\epsilon+1}, \end{aligned}$$

using (119). We now use item 6 of Lemma 4.17:

$$|||\mathcal{T}^\epsilon \left[\sum_{k=N^\epsilon+1}^\infty \tilde{g}_{k,0}^\epsilon (\cdot)^k\right] |||_\delta \le \frac{ER^{N^\epsilon+1}}{1 - R\delta} \mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta).$$

Here

$$E = C(\overline{w}_2^\epsilon)^2 e^{-4Q_4^\epsilon}, \quad R = e^{Q_4^\epsilon},$$

cf. (120) and (92) Consequently,

$$ER^{N^\epsilon+1} = C(\overline{w}_2^\epsilon)^2 e^{Q_4^\epsilon(N^\epsilon-3)} = C\overline{w}_2^\epsilon \overline{w}_{N^\epsilon-1}^\epsilon,$$

using (92) and therefore $R\delta \le \frac{4}{5}$ for $0 < \delta \le \frac{3}{4}$ for all $0 < \epsilon \ll 1$. Therefore we arrive at

$$\begin{aligned} |||\mathcal{T}^\epsilon \left[\sum_{k=N^\epsilon+1}^\infty \tilde{g}_{k,0}^\epsilon (\cdot)^k\right] |||_\delta &\le 5C\overline{w}_2^\epsilon \overline{w}_{N^\epsilon-1}^\epsilon \mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta) \\ &\le \epsilon^2 \overline{C} K, \end{aligned}$$

with $\overline{C} > 0$ large enough, for all $0 < \epsilon \ll 1$. Here we have also used (153), (161), (155), $\Gamma(\alpha^\epsilon) > 1$ and $\overline{w}_2^\epsilon = \mathcal{O}(\epsilon)$ to estimate

$$5C\overline{w}_2^\epsilon \overline{w}_{N^\epsilon-1}^\epsilon \mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta) \le \epsilon^2 \overline{C} |S_\infty^0| |||\overline{U}^\epsilon|||_\delta \le \epsilon^2 \overline{C} K,$$

in the last inequality. Proceeding completely analogously, we obtain

$$|||\mathcal{T}^\epsilon \left[\widetilde{g}_1^\epsilon(\cdot) \overline{m}_{N^\epsilon}^\epsilon (\cdot)^{N^\epsilon+2}\right] |||_\delta \le \mu C |S_\infty^0| \overline{w}_{N^\epsilon}^\epsilon \delta \mathcal{T}\left[(\cdot)^{N^\epsilon+1}\right](\delta) \le \mu\epsilon\overline{C},$$

and by Lemma 4.17 item 5

$$
\begin{aligned}
|||\mathcal{T}^\epsilon\left[\sum_{l=2}^\infty \widetilde{g}_l^\epsilon(\cdot)(\overline{m}_{N^\epsilon}^\epsilon)^l(\cdot)^{lN^\epsilon+1}\right]|||_\delta &\leq \mu C K_2 \sum_{l=2}^\infty \overline{D}^{-l+1}|\overline{m}_{N^\epsilon}|^l\delta^{N^\epsilon(l-1)}\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta)\\
&= \mu C K_2 \sum_{l=2}^\infty \left(\overline{D}^{-1}|\overline{m}_{N^\epsilon}|\delta^{N^\epsilon}\right)^{l-1}|\overline{m}_{N^\epsilon}|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta)\\
&= \mu C K_2 \frac{\overline{D}^{-1}|\overline{m}_{N^\epsilon}|\delta^{N^\epsilon}}{1-\overline{D}^{-1}|\overline{m}_{N^\epsilon}|\delta^{N^\epsilon}}|\overline{m}_{N^\epsilon}|\mathcal{T}^\epsilon\left[(\cdot)^{N^\epsilon+1}\right](\delta)\\
&\leq \mu\epsilon\overline{C},
\end{aligned}
$$

for some $\overline{C}>0$ independent of μ and ϵ. Here we have again used (161) and taken $\overline{D}>0$ large enough. In total, we conclude that

$$
|||-\mathcal{T}^\epsilon\left[r^{N^\epsilon}\left[\overline{g}^\epsilon((\cdot),j^{N^\epsilon}[\overline{m}^\epsilon](\cdot))\right]\right]|||_\delta \leq \mathcal{O}(\epsilon).
$$

It follows that the second term is bounded by $\frac{1}{3}K$ for all $0<\epsilon\ll 1$.

The third term:

$$
-\mathcal{T}^\epsilon\left[\overline{G}^\epsilon((\cdot),\overline{Y}(\cdot))\right](\overline{x}).
$$

For the third term, we also use items 4 (with $i=2$) and 7 of Lemma 4.17. Specifically, by (158), we directly obtain

$$
|||-\mathcal{T}^\epsilon\left[\overline{G}^\epsilon((\cdot),\overline{Y}(\cdot))\right]|||_\delta = O(\mu)\leq \frac{1}{3}K,
$$

for $|||Y|||_\delta\leq 2K$ by taking $\overline{D}>0$ large enough.

In total, $\mathcal{F}:\mathcal{B}_\delta^{2K}\to\mathcal{B}_\delta^{2K}$ is well-defined when (161) holds and $0<\epsilon\ll 1$, $\epsilon^{-1}\notin\mathbb{N}$ for $\mu>0$ small enough. The fact that $\mathcal{F}$ is a contraction follows directly from using item 7 of Lemma 4.17 on the third term, see (144). In fact, we find that the Lipschitz constant is $O(\mu)$ for all $0<\epsilon\ll 1$ (independently of $\alpha^\epsilon\in(0,1)$). ☐

Proposition 4.21. *Assume that the conditions of Proposition 4.20 hold true and that* (161) *holds. Let* $\overline{M}^\epsilon$ *denote the solution of the fix-point equation* (159). *Then*

$$
\overline{m}^\epsilon(\overline{x}) = \underbrace{\left\{j^{N^\epsilon-1}[\overline{m}^\epsilon](\overline{x})-\mathcal{T}^\epsilon\left[\cdots\right](\overline{x})\right\}}_{:=\overline{B}^\epsilon(\overline{x})}+(1+o(1))(-1)^{N^\epsilon}S_\infty^0\underbrace{\left\{\overline{w}_{N^\epsilon}^\epsilon\overline{x}^{N^\epsilon}+\overline{U}^\epsilon(\overline{x})\right\}}_{:=\overline{V}^\epsilon(\overline{x})},\tag{162}
$$

for all $0\leq\overline{x}\leq\delta$, *with this equation defining* $\overline{B}^\epsilon$ *and* $\overline{V}^\epsilon$, *and where*

$$
[\cdots] = r^{N^\epsilon}\left[\overline{g}^\epsilon((\cdot),j^{N^\epsilon}[\overline{m}^\epsilon](\cdot))\right](\overline{x})+\overline{G}^\epsilon(\overline{x},\overline{M}^\epsilon(\overline{x})).
$$

Moreover, the following estimates hold true:

$$
\|\overline{B}^\epsilon\|_\delta = \mathcal{O}(\epsilon),\quad \|\overline{V}^\epsilon(\overline{x})\|_\delta\leq\frac{K}{|S_\infty|}.\tag{163}
$$

Proof. (162) follows directly from (157) with $\overline{Y}=\overline{M}^\epsilon$ given by (159). Subsequently, the estimate for $\overline{B}^\epsilon$ follows from (133) and the proof of Proposition 4.20 (see the second and third terms). Finally for the estimate for $\overline{V}^\epsilon$, we use that $\overline{U}^\epsilon$ is an increasing function of $\overline{x}\in[0,\delta]$ and

$$
\|\overline{V}^\epsilon\|_\delta = \|\overline{w}_{N^\epsilon}^\epsilon(\cdot)^{N^\epsilon}+\overline{U}^\epsilon\|_\delta = \overline{w}_{N^\epsilon}^\epsilon\delta^{N^\epsilon}+\overline{U}^\epsilon(\delta) = \|\overline{w}_{N^\epsilon}^\epsilon(\cdot)^{N^\epsilon}\|_\delta+\|\overline{U}^\epsilon\|_\delta\leq\frac{K}{|S_\infty^0|},
$$

using (161) in the final inequality. ☐

$\overline{V}^{\epsilon}$ has the following absolutely convergent power series expansion

$$\overline{V}^{\epsilon}(\overline{x}) = \frac{\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)}{\epsilon\Gamma(\epsilon^{-1})} \sum_{k=N^\epsilon}^{\infty} \frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})}\overline{x}^k \quad -1<\overline{x}<1. \tag{164}$$

This follows from (152).

Remark 6. *The quantity $\overline{V}^{\epsilon}$ corresponds to Θ_{main} in [10, Eq. (7.25)].*

4.5 Completing the proof of Lemma 2.4

We prove the items 1–4 successively in the following.

Proof of item 1. The power series expansion (164) of $\overline{V}^{\epsilon}$ in (162) was proven in Proposition 4.21. The absolute convergence of this power series expansion follows from (22):

$$\frac{\Gamma(k+a^\epsilon)}{\Gamma(k+1-\epsilon^{-1})} = (1+o(1))k^{\epsilon^{-1}+a^\epsilon-1}. \tag{165}$$

Each term of the series $\overline{V}^{\epsilon}$ is positive for $\overline{x}>0$, which implies (40).

Proof of item 2. For the lower bound, we use $\overline{w}_{N^\epsilon}^{\epsilon}>0$, the definition of $\overline{V}^{\epsilon}$:

$$\overline{V}^{\epsilon}(\overline{x}) = \overline{w}_{N^\epsilon}^{\epsilon}\overline{x}^{N^\epsilon} + \overline{U}^{\epsilon}(\overline{x}) \geq \overline{U}^{\epsilon}(\overline{x}) \quad \forall\, 0\leq \overline{x}<1,$$

and the lower bound of $\overline{U}^{\epsilon}$ (see (153)):

$$\overline{V}^{\epsilon}(\overline{x}) \geq C_1\frac{1}{1-\alpha^\epsilon}\Gamma(\alpha^\epsilon)(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon}\left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1}\sigma_\epsilon(\overline{x}) \quad \forall\, 0\leq\overline{x}\leq\frac{3}{4}.$$

We take $\delta_2=\frac{3}{4}$ in (139) and obtain

$$\sigma_\epsilon(\overline{x}) \geq \epsilon^{1-\alpha^\epsilon} \quad \forall\, 0\leq\overline{x}\leq\frac{3}{4},$$

which together with

$$(N^\epsilon)^{a^\epsilon+2-\alpha^\epsilon} \geq \frac{1}{2}\epsilon^{\alpha^\epsilon-a^\epsilon-2} \quad \text{and} \quad \frac{1}{1-\alpha^\epsilon}\Gamma(\alpha^\epsilon)>1 \quad \forall\,\alpha^\epsilon\in(0,1),\ 0<\epsilon\ll 1,$$

leads to

$$\overline{V}^{\epsilon}(\overline{x}) \geq \frac{1}{2}C_1\epsilon^{-a^\epsilon-1}\left(\frac{\overline{x}}{1-\overline{x}}\right)^{N^\epsilon+1} \quad \forall\, 0\leq\overline{x}\leq\frac{3}{4}.$$

We now use that $a^0>-2$, recall Assumption 1, to conclude that

$$\frac{1}{2}C_1\epsilon^{-a^\epsilon-1}\geq\epsilon \Longleftrightarrow \frac{1}{2}C_1\geq\epsilon^{2+a^\epsilon} \quad \forall\, 0<\epsilon\ll 1.$$

Indeed, let $\nu=2+a^0>0$. Then by taking $0<\epsilon\ll 1$, we have that $|a^\epsilon-a^0|\leq\frac{1}{2}\nu$ and hence

$$\epsilon^{2+a^\epsilon}\leq\epsilon^{\nu}\epsilon^{-\frac{1}{2}\nu}=\epsilon^{\frac{1}{2}\nu}\to 0 \quad \text{for} \quad \epsilon\to 0.$$

This completes the proof of item 2.

Proof of item 3. The divergence with respect to $\alpha^\epsilon\to 0^+$ and 1^- is a direct consequence of the factors $\Gamma(\alpha^\epsilon)\Gamma(1-\alpha^\epsilon)$ in the definition of $\overline{V}^{\epsilon}$, recall (20).

Proof of item 4. Finally, in order to obtain (42) we use the form in (162):

$$\overline{V}^{\epsilon}(\overline{x}) = \overline{w}_{N^\epsilon}^{\epsilon}\overline{x}^{N^\epsilon} + \overline{U}^{\epsilon}(\overline{x}),$$

(154) and

$$\overline{w}^{\epsilon}_{N^{\epsilon}}\overline{x}^{N^{\epsilon}} = (1+o(1))\Gamma(\alpha^{\epsilon})(N^{\epsilon})^{a^{\epsilon}+1-\alpha^{\epsilon}}\overline{x}^{N^{\epsilon}}.$$

This gives

$$\begin{aligned}\overline{V}^{\epsilon}(\epsilon\overline{x}_2) &= (1+o(1))\Gamma(\alpha^{\epsilon})(N^{\epsilon})^{a^{\epsilon}+1-\alpha^{\epsilon}}(\epsilon\overline{x}_2)^{N^{\epsilon}}\\ &+\frac{\Gamma(\alpha^{\epsilon})}{1-\alpha^{\epsilon}}(N^{\epsilon})^{a^{\epsilon}+2-\alpha^{\epsilon}}(\epsilon\overline{x}_2)^{N^{\epsilon}+1}\left[1+\overline{x}_2\int_0^1 e^{(1-v)\overline{x}_2}v^{1-\alpha^{\epsilon}}dv+o(1)\right]\\ &= (1+o(1))\Gamma(\alpha^{\epsilon})(N^{\epsilon})^{a^{\epsilon}+1-\alpha^{\epsilon}}(\epsilon\overline{x}_2)^{N^{\epsilon}}\\ &\times\left(1+\frac{\overline{x}_2}{1-\alpha^{\epsilon}}\left[1+\overline{x}_2\int_0^1 e^{(1-v)\overline{x}_2}v^{1-\alpha^{\epsilon}}dv+o(1)\right]\right).\end{aligned}$$

4.6 Completing the proof of Theorem 2.5

We consider I of the forms $[0,\delta]$, $0<\delta\le\frac{3}{2}$, and $[-\delta,0]$, $0<\delta\le\delta_2\epsilon$, separately in the following and prove the statement of Theorem 2.5 in these cases. It will then follow that the statement is true for any $I\subset[-\delta_2\epsilon,\frac{3}{4}]$ satisfying (44).

The case $I=[0,\delta]$. If $I=[0,\delta]$, $\delta\le\frac{3}{4}$, satisfies (44), then

$$\|\overline{V}^{\epsilon}\|_{\delta} = \|\overline{w}^{\epsilon}_{N^{\epsilon}}(\cdot)^{N^{\epsilon}}\|_{\delta}+\|\overline{U}^{\epsilon}\|_{\delta}\le K.$$

We therefore apply Proposition 4.20 with K replaced by $K|S^0_{\infty}|$. In this case, Theorem 2.5 then follows from Proposition 4.21, see (162) and (163).

The case $I=[-\delta,0]$. Now, we consider case $I=[-\delta,0]$ with $0<\delta\le\delta_2\epsilon$. In this case we adapt the space $\mathcal{D}^{\epsilon}_{\delta}$ and the norm $|||\cdot|||_{\delta}$ in the following way:

$$\mathcal{D}^{\epsilon}_{\delta} := \{H:[-\delta,0]\to\mathbb{R}\text{ analytic}\,:\,|||H|||_{\delta}<\infty\},$$

where

$$|||H|||_{\delta} := \sup_{\overline{x}\in[-\delta,0)}\left|H(u)\frac{\mathcal{T}\left[(\cdot)^{N^{\epsilon}+1}\right](-\delta)}{\mathcal{T}\left[(\cdot)^{N^{\epsilon}+1}\right](\overline{x})}\right|, \tag{166}$$

and importantly:

$$0<\delta\le\delta_2\epsilon. \tag{167}$$

By (140), we have

$$C_1\left(\frac{|\overline{x}|}{1-\overline{x}}\right)^{N^{\epsilon}+1}\le(1-\alpha^{\epsilon})\left|\mathcal{T}^{\epsilon}\left[(\cdot)^{N^{\epsilon}+1}\right](\overline{x})\right|\le C_2\left(\frac{|\overline{x}|}{1-\overline{x}}\right)^{N^{\epsilon}+1}\quad\forall-\delta_2\epsilon\le\overline{x}\le0, \tag{168}$$

see (138).

Lemma 4.22. *Fix $\delta_2>0$ and suppose that* (167) *holds. Then the items 3–7 in Lemma 4.17 also hold true with $|||\cdot|||_{\delta}$ given by* (166).

Proof. The proof of Lemma 4.17 carries over since (168) holds. $\square$

In this way, we obtain a similar versions of Proposition 4.20 (which only relies on the estimates in items 3–7 in Lemma 4.17) and Proposition 4.21 with

$$\|H\|_{\delta} := \sup_{u\in[-\delta,0]}|H(u)|,$$

relying on (168) to estimate $\overline{V}^{\epsilon}$ in the sup-norm. Then, by proceeding as above for $I=[0,\delta]$, we complete the proof of Theorem 2.5 in the case $I=[-\delta,0]$, $0<\delta\le\delta_2\epsilon$,

5 Discussion

In this paper, we have provided a detailed description of analytic weak-stable manifolds near generic and analytic saddle-nodes (under a certain smallness assumption of the quantity $\mu = \sup u^\epsilon$, see Assumption 2). In further details, we have identified the quantity S_∞^0, with the property that $S_\infty^0 \neq 0$ implies the following (cf. Theorems 2.2 and 2.5):

(R1) The center manifold is nonanalytic.

(R2) A certain flapping mechanism of the analytic weak-stable manifold W^{ws}.

(R3) W^{ws} does not intersect the unstable manifold of the saddle W^u for all $0 < \epsilon \ll 1$, $\epsilon^{-1} \notin \mathbb{N}$.

Overall our approach is inspired by [10], proving statement (R2) for a specific (rational) system. In summary, [10] performs a blowup (scaling) transformation, writes the weak analytic manifold as a power series in the scaled coordinates, truncated at an order just below the resonant term, and then treats the remainder through a certain integral operator ($\mathcal{T}^\epsilon$). We also follow this strategy, but have brought the method into a form that is more in tune with dynamical systems theory (through normal forms, center manifolds and fix-point arguments). In this way, we established (R1)–(R3) as a generic phenomena for analytic saddle-nodes (albeit still within the context of Assumptions 1 and 2). We also feel that we have obtained a deeper understanding of the underlying phenomena and also streamlined the method of [10] along the way. For an example of the latter, in our treatment of $\mathcal{T}^\epsilon$ we have used a Banach space of analytic functions $H(\overline{x}) = \mathcal{O}(\overline{x}^{N^\epsilon+1})$ and set up a fix-point argument for the remainder; this approach does not depend upon Assumption 2 (see discussion below).

It is our belief that our results will find use in different areas of dynamical systems, in particular in the area of singularly perturbed systems where weak-stable manifolds play an important role (e.g. in the folded node [2,15]). At the same time, we are confident that our results can be generalized. In particular, we conjecture that Assumption 2 can be removed so that our statements hold true for any analytic and generic unfolding of a saddle-node with $a^0 > -2$ (by virtue of Assumption 1 and the normal form, see Theorem 2.1).

Let us first emphasize where Assumption 2 is needed: It is used in the proof of $\widehat{m}^0 \in \mathcal{D}^0$, see Proposition 3.6, and in the proof of the uniform boundedness of $\overline{m}^\epsilon$ in the semi-norm (103), see Proposition 4.10.

At the same time, it is important to emphasize that *it is NOT needed in the treatment of the remainder, see Proposition 4.20.* To see this, notice that in the proof of Proposition 4.20, we only need that $\overline{G}$ has small Lipschitz-norm, see **The third term**. To show this, we can first use the final condition of (33), see Remark 1, to estimate the $\overline{Y}$-linear part of $\overline{G}$, and for the nonlinear part of $\overline{G}$ we can use Lemma 4.17 item 7 (it is, in particular, $\mathcal{O}(\epsilon)$ in the norm $\|\!|\cdot|\!\|_\delta$ for all $\mu > 0$).

In other words, in order to remove Assumption 2 we just need to find alternative proofs of Proposition 3.6 and Proposition 4.10.

Let us focus on the former and

$$\widehat{m}^0(x) = \sum_{k=2}^{\infty} m_k^0 x^k \in \mathcal{D}^0 \Longleftrightarrow \sup_{k\geq 2} \frac{|m_k^0|}{w_k^0} < \infty.$$

In [10], the authors show that their corresponding sequence $\frac{m_k^0}{w_k^0}$, $k \geq 2$, is bounded by essentially setting up a majorant equation for S_k^0 defined by

$$m_k^0 =: (-1)^k w_k^0 S_k^0, \tag{169}$$

see [10, Lemma 5.4]. It follows from (14) (with f_k^0 replaced by $(g^0(\cdot, \widehat{m}^0(\cdot)))_k$) that S_k^0 satisfies the following recursion relation

$$S_k^0 = S_{k-1}^0 + \frac{(-1)^k (g^0(\cdot, \widehat{m}^0(\cdot)))_k}{w_k^0}, \quad w_k^0 = \Gamma(k + a^0),$$

with $(g^0(\cdot, \widehat{m}^0(\cdot)))_k$ depending on $S_2^0, \ldots, S_{k-3}^0$ (upon using (169)). The proof of [10, Lemma 5.4] first consists of showing (using the majorant equation for S_k^0) that $|S_k^0| \leq K_0 C_0^k$ for some $K_0 > 0$, $C_0 > 1$ and all k. This is Step 3 of their proof. One can take $K_0 = 1, C_0 = e^n$ for some $n \in \mathbb{N}$.

Then in Step 4 of the proof of [10, Lemma 5.4], the authors show that the exponential bound can be improved: There is some $\delta = \frac{1}{m} > 0$, $m \in \mathbb{N}$ large enough, and a $K_1 > 0$ large enough such that $|S_k^0| \le K_1 (C_0 e^{-\delta})^k = K_1 e^{k\left(n-\frac{1}{m}\right)}$ for all k. Importantly, the authors of [10] show that this process can be iterated (with n and m fixed) in the following sense: For each $l \in \mathbb{N}$ with $n - \frac{l}{m} \ge 0$, there is a $K_l > 0$ such that

$$|S_k^0| \le K_l e^{k(n-l\delta)} = K_l e^{k\left(n-\frac{l}{m}\right)},$$

for all k. Here $l \in \mathbb{N}$ is the number of applications of the improvement. Setting $l = nm$ then gives $|S_k^0| \le K_{nm}$ (uniform bound) for all k as desired.

For our general normal form in Theorem 2.1, it is straightforward to reproduce Step 3 and the existence of C_0 from (55) (without using $\mu > 0$ small). However, we have not been able to reproduce the argument in Step 4 in the general framework. We will pursue this further (along with alternative approaches to majorize S_k^0) in future work. Here we also plan to consider $a^0 \le -2$ (which is excluded in the present work by Assumption 1). Furthermore, we would also like to explore connections to the interesting results of C. Rousseau [13], see also [14] and references therein, on the analytic classification of unfoldings of saddle-nodes.

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A Proof Theorem 2.1

We first use [13, Theorem 2.2], where the following normal form is provided

$$\begin{aligned}\dot{x} &= (x-\epsilon)x,\\ \dot{y} &= -y(1+a^\epsilon x) + g^\epsilon(x,y),\end{aligned} \tag{170}$$

with

$$g^\epsilon(x,y) = x(x-\epsilon)f^\epsilon(x) + y^2\tilde{u}^\epsilon(x,y). \tag{171}$$

In comparison with [13, Theorem 2.2] we have here replaced their $(x, \epsilon, t, g_0, o(y))$ by

$$(-x+\sqrt{\epsilon}, \frac{1}{4}\epsilon^2, -t, -f^\epsilon, u^\epsilon(x,y) = y^2\tilde{u}^\epsilon(x,y)).$$

In order to achieve the desired normal form in Theorem 2.1 we apply three elementary transformations (T1)–(T3) to obtain (170) with g^ϵ given by (28) and satisfying (30), repeated here for convenience:

$$\begin{aligned} g^\epsilon(x,y) &= f^\epsilon(x) + u^\epsilon(x,y)\\ f^\epsilon(x) = \sum_{k=2}^\infty f_k^\epsilon x^k, \quad u^\epsilon(x,y) &= \sum_{k=2}^\infty u_{k,1}^\epsilon x^k y + \sum_{k=1}^\infty \sum_{l=2}^\infty u_{k,l}^\epsilon x^k y^l,\end{aligned} \tag{172}$$

and

$$|f_k^\epsilon| \le B\rho^{-k}, \quad |u_{k,l}^\epsilon| \le \mu\rho^{-k-l} \quad \text{and} \quad u_{k,1}^0 = 0 \quad \forall\, k,l \in \mathbb{N},\ \epsilon \in [0,\epsilon_0), \tag{173}$$

respectively. The purposes of each of these successive transformations are:

(T1) Remove the x-linear term from (171) ($-\epsilon x f^\epsilon(0)$) on the right hand side of the y-equation in (170).

(T2) Remove the y-linear term of the resulting nonlinearity $g^\epsilon = f^\epsilon + u^\epsilon$ obtained after application of (T1) for $\epsilon = 0$: $u_{k,1}^0 = 0$ for all $k \ge 2$, see the last condition in (173).

(T3) Remove the $x = 0$ part of the resulting nonlinearity $g^\epsilon = f^\epsilon + u^\epsilon$ obtained after application of (T2): $u_{0,l}^\epsilon = 0$ for all $l \ge 2$ and all $\epsilon \in [0, \epsilon_0)$, see (172).

For (T1), we define $\tilde{y} = y + \epsilon\frac{f^\epsilon(0)}{1-\epsilon}x$. This gives the following system

$$\begin{aligned}\dot{x} &= (x-\epsilon)x,\\ \dot{\tilde{y}} &= -\tilde{y}(1+\tilde{a}^\epsilon x) + \tilde{g}^\epsilon(x,\tilde{y}),\end{aligned} \tag{174}$$

with

$$\tilde{g}^\epsilon(x,\tilde{y}) = \sum_{k=2}^\infty \tilde{f}_k^\epsilon x^k + \sum_{k=2}^\infty \tilde{u}_{k,1}^\epsilon x^k y + \sum_{k=0}^\infty\sum_{l=2}^\infty \tilde{u}_{k,l}^\epsilon x^k y^l, \tag{175}$$

and

$$\tilde{a}^\epsilon = a^\epsilon + 2\epsilon\frac{f^\epsilon(0)}{1-\epsilon}h^\epsilon(0,0).$$

This completes (T1). We drop the tildes.

For (T2), we introduce a new x-fibered diffeomorphism defined by

$$(x,y)\mapsto \tilde{y} = e^{-\psi(x)}y, \quad \psi(x) := \sum_{k=2}^\infty \frac{u_{k,1}^0}{k-1}x^{k-1} \Longrightarrow x^2\psi'(x) = \sum_{k=2}^\infty u_{k,1}^0 x^k.$$

A simple calculation then shows that in the new $(x,\tilde{y})$-coordinates, we obtain a system of the form (174) with $\tilde{g}^\epsilon$ given by (175), for a new $\tilde{f}^\epsilon$ and a new $\tilde{u}^\epsilon$ now satisfying $u_{k,1}^0 = 0$ for all $k\in\mathbb{N}$, upon dropping the tildes. This completes (T2).

Now, finally for (T3) we analytically linearize the $x = 0$-subsystem: There is a locally defined analytic near-identity diffeomorphism $y\mapsto \tilde{y} = \psi^\epsilon(y)$, $\psi^\epsilon(0) = 0$, $\frac{d}{dy}\psi^\epsilon(0) = 1$, depending continuously on $\epsilon\in[0,\epsilon_0)$, such that

$$\dot{y} = -y + \sum_{l=2}^\infty u_{0,l}^\epsilon y^l \Longrightarrow \dot{\tilde{y}} = -\tilde{y}.$$

In the coordinates $(x,\tilde{y})$, we therefore obtain the desired form (170) with g^ϵ given by (172) and satisfying (173) upon dropping the tildes a final time. In particular, the estimates in (173) follow from Cauchy's estimate for all $\rho > 0$ small enough.

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