# ON THE MAXIMUM PRINCIPLE FOR GENERAL LINEAR ELLIPTIC EQUATIONS. 

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#### Abstract

We consider maximum principles and related estimates for linear second order elliptic partial differential operators in n-dimensional Euclidean space, which improve previous results, with H-J Kuo, through sharp $L^{p}$ dependence on the drift coefficient $b$. As in our previous work, the ellipticity is determined through the principal coefficient matrix $\mathcal{A}$ lying in sub-cones of the positive cone, which are dual cones of the Gårding $k$-cones, for $k=1, \cdots, n$. Our main results are maximum principles for bounded domains, which extend those of Aleksandrov in the case $k=n$, together with extensions to unbounded domains, depending on appropriate integral norms of $\mathcal{A}$, and corresponding local maximum principles. We also consider applications to local estimates in the uniformly elliptic case, including extensions of the Krylov-Safonov Hölder and Harnack estimates.


## 1. Introduction

In this paper we are concerned with maximum principles for general linear second order partial differential elliptic operators of the form,

$$
\begin{equation*}
L u=\mathcal{A} \cdot D^{2} u+b \cdot D u=a^{i j} D_{i j} u+b_{i} D_{i} u, \tag{1.1}
\end{equation*}
$$

where $\mathcal{A}=\left[a^{i j]}\right.$ is a positive measurable mapping from a domain $\Omega$ in Euclidean $n$-space $\mathbb{R}^{n}$ into $\mathbb{S}^{n}$, the linear space of $n \times n$ real symmetric matrices, and $b=\left(b_{1}, \cdots b_{n}\right)$ is a measurable mapping from $\Omega$ into $\mathbb{R}^{n}$. In particular, we improve the corresponding results of Kuo and Trudinger in [7], with respect to the conditions assumed on the drift coefficient $b$, together with their applications to corresponding improvements of the Pucci conjecture bounds in [15].

In accordance with [7, the crucial algebraic quantities in our maximum principles are the elementary symmetric polynomials $S_{k}$ in $\mathbb{R}^{n}$ given by

$$
\begin{equation*}
=\sum_{1 \leq i_{1}<\cdots<i_{k} \leq n} \lambda_{i_{1}} \cdots \lambda_{i_{k}}, \quad k=1, \cdots, n, \tag{1.2}
\end{equation*}
$$

and their associated cones,

$$
\begin{equation*}
\Gamma_{k}=\left\{\lambda \in \mathbb{R}^{n} \mid S_{j}[\lambda]>0, \forall j=1, \cdots, k\right\}, \tag{1.3}
\end{equation*}
$$

and closed dual cones,

$$
\begin{equation*}
\bar{\Gamma}_{k}^{*}=\left\{\lambda \in \mathbb{R}^{n} \mid \lambda . \mu \geq 0, \forall \mu \in \Gamma_{k}\right\} . \tag{1.4}
\end{equation*}
$$

Normalised dual functions $\rho_{k}^{*}$ are defined on $\bar{\Gamma}_{k}^{*}$ as follows:

$$
\begin{equation*}
\rho_{k}^{*}(\lambda)=\inf \left\{\left.\frac{\lambda \cdot \mu}{n} \right\rvert\, \mu \in \Gamma_{k}, S_{k}(\mu) \geq\binom{ n}{k}\right\} . \tag{1.5}
\end{equation*}
$$

[^0]As with the corresponding normalised elementary symmetric functions $\rho_{k}$ on $\bar{\Gamma}_{k}$, given by $\rho_{k}=$ $\left\{S_{k}(\lambda) /\binom{n}{k}\right\}^{1 / k}$, the dual functions $\rho_{k}^{*}$ are nondecreasing along rays and concave.

Our dual cones, corresponding to the open cones $\Gamma_{k}$, are then given by

$$
\begin{equation*}
\Gamma_{k}^{*}:=\left\{\lambda \in \bar{\Gamma}_{k}^{*} \mid \rho_{k}^{*}(\lambda)>0\right\} \tag{1.6}
\end{equation*}
$$

Note that it is readily shown by approximation that $\bar{\Gamma}_{k}^{*}$ is indeed the closure of $\Gamma_{k}^{*}$ and that from the concavity of $\rho_{k}$, it also follows that $\Gamma_{k}^{*}$ is open in $\mathbb{R}^{n}$ for $k>1$; (see Remark 1.1).

Clearly, $\Gamma_{k} \subset \Gamma_{l}$ for $k \geq l$ and $\Gamma_{1}$ is the half-space $\left\{\sum \lambda_{i}>0\right\}$, while $\Gamma_{n}$ is the positive cone. Consequently the dual cones $\Gamma_{k}^{*} \subset \Gamma_{l}^{*}$ for $k \leq l$, with $\Gamma_{1}^{*}$ the open ray through $(1, \ldots, 1)$ and $\Gamma_{n}^{*}=\Gamma_{n}$. Note also that our notation here differs from that in [7], where we have used $\Gamma_{k}^{*}$ to denote the closed cone $\bar{\Gamma}_{k}^{*}$.

Next we say that a symmetric $n \times n$ matrix $A \in \Gamma_{k},\left(\Gamma_{k}^{*}, \bar{\Gamma}_{k}, \bar{\Gamma}_{k}\right)$ if its eigenvalues $\lambda \in \Gamma_{k},\left(\Gamma_{k}^{*}, \bar{\Gamma}_{k}, \bar{\Gamma}_{k}\right)$ and define $\rho_{k}^{*}(A)=\rho_{k}^{*}(\lambda)$. It follows that $\rho_{k}^{*}(A) \geq \rho_{l}^{*}(A)$ for $k \geq l, A \in \Gamma_{k}^{*}$ while $\left.\rho_{n}^{*} A\right)=(\operatorname{det} A)^{1 / n}$ and $\rho_{1}^{*}(A)=\lambda_{\text {min }}(A)$, where $\lambda_{\text {min }}(A)$ denotes the minimum eigenvalue of $A$. Expressing the eigenvalues $\lambda(\mathcal{A})=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ in non-decreasing order, so that $\lambda_{\text {min }}=\lambda_{1}$, we also have the following estimate, for $\mathcal{A} \in \Gamma_{k}^{*}$ :

$$
\begin{equation*}
\rho_{k}^{*}(\mathcal{A}) \leq \frac{k}{n}\binom{n}{k}^{1 / k}\left(\lambda_{1} \ldots \lambda_{k}\right)^{1 / k} \tag{1.7}
\end{equation*}
$$

with equality in the cases $k=1$ and $k=n$. To prove (1.7), we fix a vector $\mu=\left(\mu_{1}, \ldots, \mu_{n}\right)$ in the definition (1.5) by setting

$$
\mu_{i}=\binom{n}{k}^{1 / k}\left(\lambda_{1} \ldots \lambda_{k}\right)^{1 / k} \lambda_{i}^{-1},
$$

for $i=1, \ldots k$, and $\mu_{i}=0$ for $i=k+1, \ldots n$.
Remark 1.1. We remark here that the inequality (1.7) will also be sharp for $1<k<n$, as equality will hold there along the ray passing through the point where $\lambda_{i}=1$, for $i \leq k$ and $\lambda_{i}=k$, for $i>k$. This can be proved by showing that the above choice of $\mu$ will then be the unique critical point for the associated minimisation problem, which is a consequence of the concavity of $\rho_{k}$. Similarly we can use the uniqueness of an optimal $\mu=\mu(\lambda)$ in (1.5) for $\lambda \in \Gamma_{k}^{*}$ to verify the openness of $\Gamma_{k}^{*}$.

The following maximum principle, for bounded $\Omega$, now extends that in Theorem 1.1 of [7] to general linear operators $L$, in the cases, $k \geq n / 2$.

Theorem 1.1. Let $L$ be an operator of the form (1.1) with coefficients $\mathcal{A}(\Omega) \subset \Gamma_{k}^{*}, b / \rho_{k}^{*}(\mathcal{A}) \in L^{p}(\Omega)$ for $k \geq n / 2$ and some $p>n$, $\left(\geq n\right.$ if $k=n$ ). Then if $u \in W_{\text {loc }}^{2, q}(\Omega) \cap C^{0}(\bar{\Omega}),(L u)^{-} / \rho_{k}^{*}(\mathcal{A}) \in L^{q}(\Omega)$, for $q \geq k$ if $k>n / 2$ and $q>n / 2$ if $k=n / 2$, we have the estimate,

$$
\begin{equation*}
\sup _{\Omega} u \leq \sup _{\partial \Omega} u+C(\operatorname{diam} \Omega)^{2-n / q}\left\|\frac{(L u)^{-}}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}(\Omega)}, \tag{1.8}
\end{equation*}
$$

where $C$ is a constant depending on $n, k, p, q$ and $(\operatorname{diam} \Omega)^{1-n / p}\left\|b / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{p}(\Omega)}$.

The case $k=n$ is the well known maximum principle, due originally to Aleksandrov for operators in the general form (1.1); (see for example [1, 2, 3]).

By adapting the proof of Corollary 4.2 in [7], we can obtain, from the proof of Theorem 1.1, the following variant, where the dependence on the diameter of $\Omega$ is replaced by that on the $L_{q}$ norm of $\mathcal{A} / \rho_{k}^{*}(\mathcal{A})$, thereby providing an extension to unbounded domains. To fit better with later applications, we will express our assumed bounds on $\mathcal{A}$ in terms of $\mathcal{T}=\mathcal{T}(\mathcal{A}):=\operatorname{trace} \mathcal{A}$.

Theorem 1.2. Assume additionally in Theorem 1.1, that $k>n / 2$ and $\mathcal{A} / \rho_{k}^{*}(\mathcal{A}) \in L^{k}(\Omega)$. Then we have the estimate

$$
\begin{equation*}
\sup _{\Omega} u \leq \sup _{\partial \Omega} u+C R^{2-n / q}\left\|\frac{(L u)^{-}}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}(\Omega)} \tag{1.9}
\end{equation*}
$$

where $R=\left\|\mathcal{T} / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{k}(\Omega)}^{k / n}$ and $C$ is a constant depending on $n, k, p$ and $R^{1-n / p}\left\|b / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{p}(\Omega)}$.
We will consider the cases, $k<n / 2$ in Theorem 1.1 and $k \leq n / 2$ in Theorem 2.2 , in conjunction with our treatment in Sections 2 and 3. We just remark here that Theorems 1.1 and 1.2 extend to these cases with $q>n / 2$ and the conditions on the coefficient $b$ replaced by $b / \sqrt{\rho_{k}^{*}(\mathcal{A}) \lambda_{\min }(\mathcal{A})} \in L^{p}(\Omega)$ for $p>n$.

As further consequences of Theorem 1.1, we also obtain, in Section 4, corresponding improvements of the local estimates for uniformly elliptic operators in 7 which then provide further extensions of the fundamental Hölder and Harnack estimates of Krylov and Safonov to the ellipticity cones $\Gamma_{k}^{*}$. The reader is also referred to [4], [5] and [10] for extensions of the case $k=n$ to exponents $q \leq n$, with $q$ at least close to $n$ and $p=n$.

Finally, we remark here that Theorems 1.1 and 1.2 , extend immediately to more general operators of the form,

$$
\begin{equation*}
L u=a^{i j} D_{i j} u+b_{i} D_{i} u+c u \tag{1.10}
\end{equation*}
$$

with scalar coefficient $c \leq 0$ satisfying $c / \rho^{*}(\mathcal{A}) \in L^{q}(\Omega)$.

## 2. Proof of Theorem 1.1

Theorem 1.1 is a consequence of a sharper version of the special case of operators $L_{0}$ of the form,

$$
\begin{equation*}
L_{0} u:=\mathcal{A} \cdot D^{2} u \tag{2.1}
\end{equation*}
$$

proved in Section 2 of [7], and the gradient estimate from Theorem 4.1 in [17]. First we recall the upper $k$-contact set of a function $u \in L^{\infty}(\Omega)$ is defined by

$$
\begin{equation*}
\Omega_{k}^{+}=\left\{x_{0} \in \Omega \mid \exists k \text {-convex } v \in C^{2}(\Omega), \text { satisfying } u \leq-v \text { in } \Omega, u\left(x_{0}\right)=-v\left(x_{0}\right)\right\} \tag{2.2}
\end{equation*}
$$

where a function $v \in C^{2}(\Omega)$ is called $k$-convex if $D^{2} v \in \bar{\Gamma}_{k}(\Omega)$. Then from Section 2 of [7], it follows that if $u \in C^{2}(\Omega) \cap C^{0}(\bar{\Omega})$ satisfies $u \leq 0$ on $\partial \Omega$, and $\Omega_{0}$ is a domain containing $\Omega$, with $u$ extended to vanish on $\Omega_{0}-\Omega$, so that $\Omega_{0, k}^{+} \subset \Omega$, we have the estimate,

$$
\begin{equation*}
\sup _{\Omega} u \leq C\left(\operatorname{diam} \Omega_{0}\right)^{2-n / q}\left\|\frac{\left(L_{0} u\right)^{-}}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}\left(\Omega_{0, k}^{+}\right)} \tag{2.3}
\end{equation*}
$$

where $C$ is a constant depending on $n, k$ and $q$.
To proceed further we need the extended notion of $k$-convex function introduced in [17], namely an upper semi-continous function $v$ in a domain $\Omega$ is called $k$-convex if any quadratic polynomial $\varphi$ for which the difference $v-\varphi$ has a local maximum in $\Omega$ is $k$-convex. As in [17] we denote the linear space of $k$-convex functions in $\Omega$ by $\Phi^{k}(\Omega)$. Equivalent characterisations are also provided in Lemmas 2.1 and 2.2 in [17]. From Theorem 4.1 and Lemma 2.3 in [17], it then follows that $\Phi^{k}(\Omega)$ is contained in the Sobolev space $W_{\text {loc }}^{1, r}(\Omega)$ for $r<\frac{n k}{n-k}$ and moreover, we have the estimate:

$$
\begin{equation*}
\|D v\|_{L^{r}\left(\Omega^{\prime}\right)} \leq C(\operatorname{diam} \Omega)^{(n-r) / r} \sup _{\Omega}|v| \tag{2.4}
\end{equation*}
$$

for any $v \in \Phi^{k}(\Omega) \cap L^{\infty}(\Omega)$ and subdomain $\Omega^{\prime}$ satisfying $\operatorname{dist}\left(\Omega^{\prime}, \partial \Omega\right) \geq \kappa(\operatorname{diam} \Omega)$, for some $\kappa \in(0,1 / 2)$, with constant $C$ depending on $n, k, r$ and $\kappa$.

Now, defining the upper $k$-envelope of $u$ on $\Omega_{0}$ by

$$
\begin{equation*}
u_{0}=\inf \left\{-v \mid v \in \Phi^{k}\left(\Omega_{0}\right), u \leq-v \text { in } \Omega_{0}\right\} \tag{2.5}
\end{equation*}
$$

we then have $u_{0} \in \Phi^{k}\left(\Omega_{0}\right), 0 \leq u \leq u_{0}$ in $\Omega_{0}$ and $u_{0}=u$ in $\Omega_{0, k}^{+}$. To apply the estimate (2.3), we suppose $\Omega$ lies in a ball of radius $R$ and take $\Omega_{0}$ to be the concentric ball of radius $2 R$. Using the coordinate transformation $x \rightarrow\left(x-x_{0}\right) / R$, where $x_{0}$ denotes the centre of $\Omega_{0}$, we can then assume $R=1$. For convenience, by dividing $L$ by $\rho_{k}^{*}(\mathcal{A})$, we can also assume $\rho_{k}^{*}(\mathcal{A})=1$. From the estimate (2.4), applied to the function $u_{0}$ in $\Omega_{0}$, with $\kappa=1 / 4$ and $r$ chosen so that

$$
\frac{1}{r}=\frac{1}{q}-\frac{1}{p},
$$

we then have the estimate

$$
\begin{equation*}
\|b . D u\|_{L^{q}\left(\Omega_{0, k}^{+}\right)} \leq C\|b\|_{L^{p}\left(\Omega_{0, k}^{+}\right)} \sup u, \tag{2.6}
\end{equation*}
$$

where $C$ is a constant depending on $n, k, p$ and $q$. By combining (2.3) and (2.6), we can obtain the desired estimate (1.6), in the case when $\|b\|_{p}$ is sufficiently small. To proceed to the general case, we adapt the argument used by Pucci [9] in his treatment of the case $k=n$. Writing $M=\sup _{\Omega} u$, we let $\mathcal{U}_{m}=\mathcal{U}_{m, k}$, for $m \in[0, M)$ denote the $k$-upper contact set of the function $u-m$ in $\Omega_{0}$, so that $\mathcal{U}_{0}=\Omega_{0, k}^{+}$in our previous notation. For $m=M$, we set $\mathcal{U}_{M}=\{x \in \Omega \mid u=M\}$. For a sufficiently large positive integer $N$, we then let $m_{0}, \ldots, m_{N}$ be an increasing sequence in $[0, M]$, such that $m_{0}=0$, $m_{N}=M$ and

$$
\begin{equation*}
\int_{\mathcal{U}_{i-1}-\mathcal{U}_{i}}|b|^{p}=\frac{1}{N^{p / q}} \int_{\mathcal{U}_{0}-\mathcal{U}_{M}}|b|^{p} \tag{2.7}
\end{equation*}
$$

for $i=1, \ldots, N$, and $\mathcal{U}_{i}=\mathcal{U}_{m_{i}}$. Now we define

$$
y_{i}:=\int_{\mathcal{U}_{i-1}-\mathcal{U}_{i}}|b . D u|^{q},
$$

so that

$$
\Sigma_{j=i}^{N} y_{j}=\int_{\mathcal{U}_{i-1}}|b \cdot D u|^{q}
$$

for $i=1, \ldots, N$, since $D u=0$ a.e on $\mathcal{U}_{N}$. Applying the estimates, (2.3) and (2.6), to the functions $u-m_{i}$ and using (2.7), we then obtain, for $i=1, \ldots, N$, the estimates

$$
\begin{equation*}
y_{i} \leq \frac{C_{0}}{N}\|b\|_{L^{p}\left(\mathcal{U}_{0}\right)}^{q}\left\{\Sigma_{j=i}^{N} y_{j}+\left\|(L u)^{-}\right\|_{L^{q}\left(\mathcal{U}_{0}\right)}^{q}\right\} \tag{2.8}
\end{equation*}
$$

for a further constant $C_{0}$ depending on $n, k, p$ and $q$. Now setting

$$
\theta=C_{0}\|b\|_{L^{p}\left(\mathcal{U}_{0}\right)}^{q}, \alpha=\theta /(N-\theta), \quad \mu=\left\|(L u)^{-}\right\|_{L^{q}\left(\mathcal{U}_{0}\right)}^{q}
$$

and taking $N>\theta$, we can rewrite (2.8) as

$$
y_{i} \leq \alpha\left(\Sigma_{j=i+1}^{N} y_{j}+\mu\right)
$$

In accordance with the discrete Gronwall inequality, we then have, by iteration,

$$
y_{i} \leq \alpha(1+\alpha)^{(N-i)} \mu
$$

for $i=1, \ldots, N$. Summing over $i$, we now obtain

$$
\int_{\mathcal{U}_{0}}|b \cdot D u|^{q} \leq\left\{(1+\alpha)^{N}-1\right\} \mu=\left\{\left(\frac{N}{N-\theta}\right)^{N}-1\right\} \mu
$$

so that letting $N \rightarrow \infty$, we obtain the estimate

$$
\begin{equation*}
\|b . D u\|_{L^{q}\left(\mathcal{U}_{0}\right)} \leq\left(e^{\theta}-1\right)^{1 / q}\left\|(L u)^{-}\right\|_{L^{q}\left(\mathcal{U}_{0}\right)} \tag{2.9}
\end{equation*}
$$

Note that we can also infer (2.9) directly from (2.8) and the Lemma in Section 3 of [9], which corresponds to a cruder version of our preceding estimate with finite $N>\theta$.

Combining (2.3) and (2.9) and returning to our original coordinates, with $u$ replaced by $u-\sup _{\partial \Omega} u$, we obtain the following sharper version of our desired estimate (1.7) for functions $u \in C^{2}(\Omega) \cap C^{0}(\bar{\Omega})$,

$$
\begin{equation*}
\left.\sup _{\Omega} u \leq \sup _{\partial \Omega} u+C d^{2-n / q} \exp \left(C_{0} d^{(1-n / p) q} \| b / \rho_{k}^{*}(\mathcal{A})\right) \|_{L^{p}\left(\Omega_{k}^{+}\right)}^{q}\right)\left\|\frac{(L u)^{-}}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}\left(\Omega_{k}^{+}\right)}, \tag{2.10}
\end{equation*}
$$

where $d=\operatorname{diam} \Omega, C$ and $C_{0}$ are constants depending respectively on $n, k, q$ and $n, k, p, q$. By following the approximation argument given, for example, in the proof of the case $k=n$ in [3], we may extend the estimate to functions $u \in W_{l o c}^{2, q}(\Omega) \cap C^{0}(\bar{\Omega})$, thereby completing the proof of Theorem 1.1.

We can extend the estimate (2.10) to the cases $k<n / 2$ through a completely different approach by adapting that used for divergence structure operators in [11]. Under our previous normalisations, $R=\rho_{k}^{*}(\mathcal{A})=1$, we consider the function $w$ given by

$$
\begin{equation*}
w=\log \frac{M+\mu}{M-u+\mu} \tag{2.11}
\end{equation*}
$$

where $\mu=\left\|(L u)^{-}\right\|_{L^{q}(\Omega)}$ for some $q>n / 2$, so that

$$
\begin{aligned}
L_{0} w & =\mathcal{A} D w \cdot D w-b \cdot D w+\frac{1}{M-u+\mu} L u \\
& \geq-\frac{1}{M-u+\mu}(L u)^{-}-\frac{|b|^{2}}{4 \lambda_{\min }(\mathcal{A})}
\end{aligned}
$$

Applying the estimate $(2.3)$, with $\Omega_{0}=\Omega$, we then obtain

$$
\sup w \leq C\left(1+\left\|b^{2} / 4 \lambda_{\min }(\mathcal{A})\right\|_{L^{q}(\Omega)}\right)
$$

and hence, from (2.11) and taking account of our normalisations, we obtain, in place of (2.10),

$$
\begin{equation*}
\sup _{\Omega} u \leq \sup _{\partial \Omega} u+d^{2-n / q} \exp C\left\{1+d^{2 q-n)}\left\|b^{2} / \rho_{k}^{*}(\mathcal{A}) \lambda_{\min }(\mathcal{A})\right\|_{L^{q}(\Omega}^{q}\right\}\left\|\frac{(L u)^{-}}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}(\Omega)} \tag{2.12}
\end{equation*}
$$

for all $k$ and any $q>n / 2$, where $C$ is a constant depending on $n, k$, and $q$.

## 3. Proof of Theorem 1.2

Unlike the case $L=L_{0}$ in [7], Theorem 1.2 does not immediately follow from the corresponding local maximum principle. However the approach is still similar and we can combine the proofs, with the resultant local maximum principle formulated in the next section as Theorem 3.1. Accordingly we fix a ball $B:=B_{R}:=B_{R}(y)$ in $\mathbb{R}^{n}$ of centre $y$ and radius $R$, such that the intersection $\Omega \cap B$ is non-empty. Under the hypotheses of Theorem 1.2 and our previous normalisations, $B=B_{1}(0)$ and $\rho_{k}^{*}(\mathcal{A})=1$, we define a cut-off function $\eta$ by

$$
\begin{equation*}
\eta=\left[\left(1-|x|^{2}\right)^{+}\right]^{\beta} \tag{3.1}
\end{equation*}
$$

for some $\beta \geq 1$, to be chosen. Then setting $v=\eta\left(u^{+}\right)^{2}$, for $u \in C^{2}(\Omega) \cap C^{0}(\bar{\Omega})$ satisfying $u \leq 0$ on $\partial \Omega \cap B$, we now compute, in $\Omega \cap B \cap\{u>0\}$,

$$
\begin{align*}
L_{0} v:=a^{i j} D_{i j} v & =u^{2} a^{i j} D_{i j} \eta+4 u a^{i j} D_{i} \eta D_{j} u+2 \eta a^{i j} D_{i} u D_{j} u+2 \eta u L_{0} u \\
& \geq-C \mathcal{T} \eta^{1-2 / \beta} u^{2}-\eta b . D u^{2}+2 \eta u L u \tag{3.2}
\end{align*}
$$

where $C$ is a constant depending on $n$ and $\beta$. Now we fix the point $x_{0}$ so that $u\left(x_{0}\right)=\sup _{\Omega} u=\sup _{\Omega} v$ and take $\beta=2$. Applying the estimate (2.3) with $\Omega_{0}=B_{2}$, we then obtain

$$
\begin{equation*}
\sup _{\Omega} v \leq C \|\left(\eta^{1-2 / \beta} \mathcal{T} u^{2}+\eta\left(b . D u^{2}\right)^{-}+\eta u(L u)^{-} \|_{L^{q}\left(\mathcal{U}_{0}\right)}\right. \tag{3.3}
\end{equation*}
$$

for a constant $C$ depending on $n, k, q$ and $\beta$, where now $\mathcal{U}_{0} \subset B \cap \Omega$ denotes the upper $k$-contact set of the function $v$ in $\Omega_{0}$.
To prove the global estimate, (1.8), we now fix the point $x_{0}$ so that $u\left(x_{0}\right)=\sup _{\Omega} u=\sup _{\Omega} v$ and take $\beta=2$ in (3.3). Consequently, if

$$
\begin{equation*}
\|\mathcal{T}\|_{L^{q}(\Omega)} \leq 1 / 2 C, \tag{3.4}
\end{equation*}
$$

where $C$ is the constant in (3.3), with $\beta=2$, we have

$$
\sup _{\Omega} v \leq 2 C\left\|\eta\left(b . D u^{2}\right)^{-}+\eta u(L u)^{-}\right\|_{L^{q}\left(\mathcal{U}_{0}\right)}
$$

so that we can adapt the proof of Theorem 1.1 to handle the coefficient $b$. Here we let $\mathcal{U}_{m}=\mathcal{U}_{m, k}$, for $m \in[0, M)$, denote the $k$-upper contact set of the function $v_{m}:=\eta\left(u^{2}-m^{2}\right)$ in $\Omega_{0}$ and define

$$
y_{i}:=\int_{\mathcal{U}_{i-1}-\mathcal{U}_{i}}\left(\eta\left|b \cdot D u^{2}\right|\right)^{q},
$$

so that now

$$
\Sigma_{j=i}^{N} y_{j}=\int_{\mathcal{U}_{i-1}}\left(\eta\left|b \cdot D u^{2}\right|\right)^{q}
$$

for $i=1, \ldots, N$.
To proceed further we also need to estimate $D \eta / \eta$ in $\mathcal{U}_{m}$. Letting $v_{0}$ denote the upper $k$-envelope of $v_{m}$ in $\Omega_{0}$, it follows that

$$
\left(1-(|x| / 2)^{2-n / k} \sup v_{m} \leq v_{0} \leq \eta \leq\left(1-|x|^{2}\right) \sup v_{m}\right.
$$

in $\mathcal{U}_{m}$, which implies

$$
|D \eta| / \eta \leq \frac{\beta}{1-(1 / 2)^{(2 k / n)-1}}:=\gamma
$$

in $\mathcal{U}_{m}$.
Now denoting $v^{i}=v_{m_{i-1}}$, we then have, in the upper contact set $\mathcal{U}_{i-1}$,

$$
\begin{aligned}
\eta\left|b . D u^{2}\right| & \leq|b|\left\{\left|D v^{i}\right|+|D \eta|\left(u^{2}-m_{i-1}^{2}\right)\right\} \\
& \leq|b|\left\{\left|D v^{i}\right|+\gamma \sup v^{i}\right\},
\end{aligned}
$$

so that using the estimates (2.3) and (2.6) we obtain again the fundamental inequality (2.8), for a constant $C$ depending on $n, k, p, q$ and $\beta$. We remark that the case $p=k=n=q$ is much simpler here in that we can just use concavity of the $k$-upper envelope of $v^{i}$ to estimate $\left|D v^{i}\right| \leq v^{i}$ in $\mathcal{U}_{i-1}$. The proof can then be further simplified by taking $v=\eta u^{+}$.

To conclude the estimate (1.8), we then need to fix $\beta=2$ and an appropriate radius $R$ satisfying the condition (3.4) in our original coordinates which is done by taking

$$
\begin{equation*}
R=C\left\|\mathcal{T} / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{q}(\Omega)}^{q / n} \tag{3.5}
\end{equation*}
$$

for a further constant $C$ depending on $n, k$ and $q$.
Accordingly we obtain an estimate of the form (2.10), with the diameter $d$ replaced by $\left\|\mathcal{T} / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{q}(\Omega)}^{q / n}$, thereby completing the proof of Theorem 1.2.

In order to embrace the cases $k<n / 2$, we revisit the corresponding extension of Theorem 1.1 and set $v=\eta w$, where $w$ is given by (2.11) and as above, the cut-off function $\eta$ is given by (3.1), with $\beta=2$. Then we obtain, in place of (3.2),

$$
\begin{align*}
L_{0} v & =w a^{i j} D_{i j} \eta+2 a^{i j} D_{i} \eta D_{j} w+\eta a^{i j} D_{i} w D_{j} w-\eta b . D w+\frac{\eta}{M-u+\mu} L u \\
& \geq-C \mathcal{T} w-\frac{|b|^{2} \eta}{4 \lambda_{\min }(\mathcal{A})}-\frac{\eta}{M-u+\mu}(L u)^{-} . \tag{3.6}
\end{align*}
$$

Now choosing $R$ as in (3.5) and applying the estimate (2.3) in $B_{2 R}$, we obtain an estimate of the form (2.12) with $d$ replaced by $\left\|\mathcal{T} / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{q}(\Omega)}^{q / n}$.

## 4. Local estimates

We begin with an extension of the local maximum principle in Theorem 4.1 in [7], under a slight sharpening of the hypotheses of Theorem 1.2.

Theorem 4.1. Assume additionally in Theorem 1.1, that $\mathcal{A} / \rho_{k}^{*}(\mathcal{A}) \in L^{r}(\Omega)$ for $r>\max \{k, n / 2\}$. Then for any ball $B=B_{R}=B_{R}(y)$ intersecting $\Omega$, concentric ball $B_{\sigma R}, 0<\sigma<1$, and $\kappa>0$ we have the estimate

$$
\begin{equation*}
\sup _{\Omega \cap B_{\sigma}} u \leq \sup _{\partial \Omega \cap B} u+C\left\{\left[R^{-n} \int_{\Omega \cap B}\left(u^{+}\right)^{\kappa}\right]^{1 / \kappa}+R^{2-n / q}\left\|\frac{(L u)^{-}}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}(\Omega \cap B)}\right\}, \tag{4.1}
\end{equation*}
$$

where $C$ is a constant depending on $n, k, p, q, r, \sigma, \kappa, R^{1-n / p}\left\|b / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{p}(\Omega \cap B)}$ and $R^{-n / r}\left\|\mathcal{T} / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{r}(\Omega \cap B)}$ If $k<n / 2$, the estimate (4.1) holds with $b / \rho_{k}^{*}$ replaced by $b / \sqrt{\rho_{k}^{*} \lambda_{\min }}$.

To obtain the estimate (4.1), in the cases $k \geq n / 2$, we can simply return to the proof of Theorem 1.2 , in Section 3, and not use condition (3.4) in the estimate (3.3). We then obtain the estimate

$$
\begin{equation*}
\sup _{\Omega \cap B} v \leq C \exp \left(C_{0}\|b\|_{L^{p}(\Omega \cap B)}^{q}\right)\left\|\eta^{1-2 / \beta} \mathcal{T}\left(u^{+}\right)^{2}+\eta u^{+}(L u)^{-}\right\|_{L^{q}(\Omega \cap B)} \tag{4.2}
\end{equation*}
$$

where, corresponding to (2.10), $C$ and $C_{0}$ are constants depending respectively on $n, k, \beta$ and $n, k, \beta, p$. Since

$$
\left\|\eta^{1-2 / \beta} \mathcal{T}\left(u^{+}\right)^{2}+\eta u^{+}(L u)^{-}\right\|_{q} \leq\|\mathcal{T}\|_{r}\left\|v^{1-2 / \beta}\left(u^{+}\right)^{4 / \beta}\right\|_{r^{*}}+\left\|v^{1 / 2}(L u)^{-}\right\|_{q}
$$

where

$$
\frac{1}{r^{*}}=\frac{1}{q}-\frac{1}{r}>0
$$

we can then infer (4.1), by taking $\beta=4 r^{*} / \kappa$. The cases $k<n / 2$ are essentially already proved in [7, as we can estimate from (3.2),

$$
L_{0} v \geq-C\left(\mathcal{T}+\frac{|b|^{2}}{\lambda_{\min }(\mathcal{A})}\right) \eta^{1-2 / \beta} u^{2}+2 \eta u L u
$$

where $C$ depends on $n$ and $\beta$. Instead of (4.2), we now obtain the estimate

$$
\sup _{\Omega \cap B} v \leq C\left\|\eta^{1-2 / \beta}\left(\mathcal{T}+\frac{|b|^{2}}{\lambda_{\min }(\mathcal{A})}\right)\left(u^{+}\right)^{2}+\eta u^{+}(L u)^{-}\right\|_{L^{q}(\Omega \cap B)},
$$

where $C$ depends on $n, k, q, \beta$. Fixing $p=2 r>2 q$ and $r^{*}$ as before, we then infer the cases $k<n / 2$ of Theorem 4.1.

From Theorem 1.1 also follow corresponding extensions of the Hölder and Harnack estimates in Theorem 4.3 in $[7]$ for uniformly elliptic operators, where the condition on $\mathcal{A}$ in Theorem 3.1 is strengthened to $\mathcal{A} / \rho_{k}^{*}(\mathcal{A}) \in L^{\infty}(\Omega)$. Defining

$$
a_{k}:=\sup _{\Omega} \frac{\lambda_{\max }}{\rho_{k}^{*}}(\mathcal{A}),
$$

we then have from inequality (1.6), the uniformly elliptic condition,

$$
\frac{\lambda_{\max }}{\lambda_{\min }}(\mathcal{A}) \leq\left(\frac{n}{k}\right)^{k}\binom{n}{k}^{-1} a_{k}^{k} \leq a_{k}^{k},
$$

which is a refinement of inequality (4.3) in [7.
Our Hölder and Harnack estimates now follow as corollaries of the following weak Harnack inequality for supersolutions, which was proved in [12] in the case $k=n, q=2 n$, by adapting the key ideas in [6].

Theorem 4.2. Assume additionally to the conditions on $L$ in Theorem 1.1, that $L$ is uniformly elliptic in $\Omega$. Then, for any ball $B=B_{R}=B_{R}(y) \subset \Omega, u \geq 0, \in W^{2, q}(B)$, for $q=k$ if $k>n / 2, q>n / 2$ if $k \leq n / 2$, and $0<\sigma, \tau<1$, there exists $\kappa>0$ such that

$$
\begin{equation*}
\left[R^{-n} \int_{B_{\sigma R}} u^{\kappa}\right]^{1 / \kappa} \leq C\left\{\inf _{B_{\tau R}} u+R^{2-n / q}\left\|\frac{(L u)^{+}}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}\left(B_{R}\right)}\right\} \tag{4.3}
\end{equation*}
$$

where $\kappa$ and $C$ depend on $n, k, p, q, \sigma, \tau, a_{k}$ and $R^{1-n / p}\left\|b / \rho_{k}^{*}(\mathcal{A})\right\|_{L^{p}(\Omega}$.
Corollary 4.1. Assume that $L$ satisfies the hypotheses of Theorem 4.2 and $u \in W^{2, q}(B)$. Then for any concentric ball $B_{\sigma}=B_{\sigma R}(y), 0<\sigma<1$, we have the oscillation estimate

$$
\begin{equation*}
\underset{B_{\sigma}}{\operatorname{osc}^{\prime}} u \leq C \sigma^{\alpha}\left\{\underset{B}{\text { osc }} u+R^{2-n / q}\left\|\frac{L u}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}\left(B_{R}\right)}\right\} \tag{4.4}
\end{equation*}
$$

where $\alpha>0$ and $C$ depend on $n, k, p, q, a_{k}$ and $R^{1-n / p}\left\|b \rho_{k}^{*}(\mathcal{A})\right\|_{L^{p}(B)}$. Furthermore if $u \geq 0$, we have the Harnack inequality

$$
\begin{equation*}
\sup _{B_{\sigma}} u \leq C\left\{\inf _{B_{\sigma}} u+R^{2-n / q}\left\|\frac{L u}{\rho_{k}^{*}(\mathcal{A})}\right\|_{L^{q}\left(B_{R}\right)}\right\} \tag{4.5}
\end{equation*}
$$

where $C$ depends on the same quantities as in (4.3).

Note that the case $p=k=q=n$ is proved in [10] and its derivation from the corresponding case in Theorem 1.1 is more technically delicate than the cases where $p>n$. The cases $k \leq n / 2$, where we can take $p=2 q$, are essentially already covered in [7]. Here we will just indicate how the cases $p>n$ follow by simple modifications of previous approaches, when $k=n$. For convenience we will just adapt the presentation in [13], although, as remarked above the main ideas go back to [6]. The first step in the proof of Theorem 4.2 corresponds to a weak form of the estimate (4.3), where $\sigma<\tau$ and the left hand side is replaced by $\inf _{B_{\sigma R}} u$. Normalising $R=\lambda_{\min }(\mathcal{A})=1$, we define a comparison function $v$ by

$$
v(x):=\frac{|x|^{-\beta}-1}{\sigma^{-\beta}-1} \inf _{B_{\sigma}} u
$$

for a constant $\beta \geq \sup \mathcal{T}-2$. We then obtain

$$
L v \geq-\frac{\beta|x|^{-\beta-2}}{\sigma^{-\beta}-1}(b . x) \inf _{B_{\sigma}} u
$$

in $B-B_{\sigma}$, so that by applying Theorem 1.1 to the function $v-u$, and assuming $\sigma \leq 1 / 2$, we obtain

$$
u(x)-v(x) \geq-C\left\|\left\lvert\, \frac{|b|}{\sigma} \inf _{B_{\sigma}} u+(L u)^{+}\right.\right\|_{L^{q}\left(B-B_{\sigma}\right)},
$$

for $\sigma<|x|<1$, where $C$ depends on $n, k, p, q, \sup \mathcal{T}$ and $\|b\|_{L^{p}(B)}$.
By removing the normalisation, $R=1$, and taking account of the more explicit estimate (2.10), we then obtain an estimate

$$
\begin{equation*}
\inf _{B_{\sigma R}} u \leq C\left\{\inf _{B_{\tau R}} u+R^{2-n / p}\left\|(L u)^{+}\right\|_{L^{p}\left(B_{R}\right)}\right\}, \tag{4.6}
\end{equation*}
$$

provided $R$ is sufficiently small, in the sense that

$$
R^{1-n / p}\|b\|_{L^{p}\left(B_{R}\right)} \leq 1 / C,
$$

where now $C$ depends on $n, k, p, q, \sigma, \tau$ and $a_{k}$.
The extension to general $R$ can then be achieved by standard covering arguments. In particular, after normalising $R=1$ as before, we can apply the estimate (4.6), with $\sigma=1 / 3, \tau=2 / 3$, to a finite chain of overlapping balls $B_{\rho}\left(y_{i}\right)$ for $i=1, \ldots N$ satisfying

$$
\rho^{1-n / p}\|b\|_{L^{p}(B)} \leq 1 / C,
$$

whose centres $y_{i}$ are equally spaced along a line segment joining $y_{1} \in B_{\sigma-\rho / 3}$ to $y_{N} \in B_{\tau}$ satisfying $\left|y_{i+1}-y_{i}\right|=\rho / 3, i=1, \ldots N-1,\left|y_{N}-y_{0}\right| \in(\rho / 3,2 \rho / 3)$, where $y_{0} \in B_{\tau}$ satisfies $u\left(y_{0}\right)=\inf _{B_{\tau}} u$. By iteration, we then obtain an estimate (4.6), under the hypotheses of Theorem 4.2, (with $\left.\lambda_{\min }(\mathcal{A})=1\right)$, where now the constant $C$ depends additionally on $R^{1-n / p}\|\mid b\|_{L^{p}\left(B_{R}\right)}$.

The weak Harnack inequality, Theorem 4.2 then follows by combining the estimate (4.6) with the local maximum principle in Theorem 4.1 applied to the function $1-u$, and using the key measure theoretic argument of Krylov and Safonov, as presented for example in [13.

The results of this section also extend to operators of the form (1.10) with $c / \rho^{*}(\mathcal{A}) \in L^{q}(\Omega)$

## 5. Application to the Pucci conjecture

We conclude this paper by indicating the application to the corresponding improvement of the condition on the drift term in [15] in the case $q=k>n / 2$. First we note that, by defining a uniform ellipticity constant $a_{0}$ for the operator $L$ by

$$
a_{0}=\sup _{\Omega} \frac{\mathcal{T}}{\lambda_{\min }}(\mathcal{A}),
$$

it follows from Section 3 in [15], that $\mathcal{A} \in \Gamma_{k}^{*}$ if

$$
\begin{equation*}
\chi:=k-n\left[1-(n-1) / a_{0}\right]>0 . \tag{5.1}
\end{equation*}
$$

Note here that we are not normalising $\mathcal{T}=1$ as in [15] and $a_{0}=1 / \delta$ where $\delta$ is the ellipticity constant used in [8] and [15].

Moreover from the first inequalities on page 116 there, with $k$ replaced by $k-\chi$, we can obtain an explicit estimate from below for $\rho_{k}^{*}(\mathcal{A})$. As there, we first note that for $\lambda_{i}>0, i=1, \ldots, n$, the inner product $\lambda . \mu$ does not increase when we rearrange the components of $\lambda$ in increasing order and those of $\mu$ in decreasing order. Consequently if $\lambda=\lambda(\mathcal{A})$ and $\mu \in \Gamma_{k}$ are so ordered, with $\mu_{n} \leq 0$, we obtain

$$
\begin{aligned}
\lambda . \mu & \geq \lambda_{1}\left\{\sum_{i<n} \mu_{i}+\left[a_{0}-(n-1)\right] \mu_{n}\right\} \\
& \geq \lambda_{1}\left\{\sum_{i<n} \mu_{i}+\frac{(k-\chi)(n-1)}{(n-k+\chi)} \mu_{n}\right\} \\
& =\lambda_{1}\left\{S_{1}(\mu)+\frac{n(k-1-\chi)}{n-k+\chi} \mu_{n}\right\} .
\end{aligned}
$$

Now, using the fundamental inequality,

$$
\begin{equation*}
(n-k) S_{1}(\mu)+n(k-1) \mu_{n}>0 \tag{5.2}
\end{equation*}
$$

for $\mu \in \Gamma_{k}$, we then obtain, for $k>1$,

$$
\begin{aligned}
\lambda . \mu & \geq \lambda_{1} S_{1}(\mu) \frac{(n-1) \chi}{(n-k+\chi)} \\
& =\lambda_{1} S_{1}(\mu) \frac{\chi a_{0}}{n(k-1)},
\end{aligned}
$$

and hence our desired estimate,

$$
\begin{equation*}
\rho_{k}^{*}(\mathcal{A}) \geq \frac{\chi a_{0}}{n(k-1)} \lambda_{\min } \tag{5.3}
\end{equation*}
$$

for $k>1$. Note that we can cover the case when $\mu_{n}>0$ by assuming $\chi a_{0} \leq(k-1) /(n-1)$ and the argument above extends to the case $k=1$, with $\chi=0$, and is consistent with the equality $\rho_{1}^{*}=\lambda_{\text {min }}$.

Consequently Theorems 1.1,1.2, together with their extensions to $1<k \leq n / 2$, and Theorem 4.1, for $k>1$, extend to uniformly elliptic operators satisfying (5.1), in place of $\mathcal{A} \in \Gamma_{k}^{*}$, with $\rho_{k}^{*}(\mathcal{A})$ replaced by $\lambda_{\min }(\mathcal{A}) \chi a_{0}$. Similarly, Theorem 41 and Corollary 4.2 extend with additionally $a_{k}$ replaced by $1 / \chi$.

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