DIMENSION CONSERVATION OF HARMONIC MEASURES IN PRODUCTS OF HYPERBOLIC SPACES

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ABSTRACT. We show that the harmonic measure on a product of boundaries satisfies dimension conservation for a random walk with non-elementary marginals on a countable group acting on a product of hyperbolic spaces under finite first moment condition.

1. INTRODUCTION

Let Γ and Γ^* be non-elementary hyperbolic groups. We study a random walk on the product group $\Gamma := \Gamma \times \Gamma^*$ and establish a dimension formula for the harmonic measure on the product of (Gromov) boundaries. After stating our results in this special case, we consider a countable group of isometries of a product of two hyperbolic metric spaces.

Let π be a probability measure on Γ such that marginals μ and μ^* are non-elementary, i.e., their supports generate non-elementary subgroups in Γ and in Γ^* as groups respectively. For such a π , a **harmonic measure** ν_{π} is defined on the product of boundaries $\partial \Gamma \times \partial \Gamma^*$ (cf. Section 2.1). It is a unique probability measure satisfying that

$$u_{\pi} = \pi * \nu_{\pi}, \quad \text{where } \pi * \nu_{\pi} = \sum_{\boldsymbol{x} \in \Gamma} \pi(\boldsymbol{x}) \boldsymbol{x} \nu_{\pi} \text{ and } \boldsymbol{x} \nu_{\pi} := \nu_{\pi} \circ \boldsymbol{x}^{-1}.$$

In the above, we consider the natural action of $\Gamma = \Gamma \times \Gamma^*$ on $\partial \Gamma \times \partial \Gamma^*$. The harmonic measure ν_{π} has marginals ν_{μ} and $\nu_{\mu^{\star}}$ on $\partial\Gamma$ and on $\partial\Gamma^{\star}$ respectively, and these are determined by μ and by μ^* . The measure ν_{μ} (or ν_{μ^*}) for a single hyperbolic group and its generalization has attracted intensive studies, including the dimension, e.g., [Kai98, Led01], more recently, [BHM11, HS17, Tan19, DY23]. However, the harmonic measure ν_{π} for a product group has been studied only in a few cases (see Vol21) for a special case of products of hyperbolic free product groups). Dimensional properties of such a measure exhibit new features since it contains different behaviors depending on the factors. This manifests an additional difficulty, which arises in a higher rank setting—in that case, the boundaries are assembled in further intricate ways such as flag varieties. See [KLP11] for a thorough discussion on this matter of subject, and recent works [Les21, Rap21, LL23]. For related results on ergodic invariant measures for affine iterated function systems, see e.g., [KP96, FH09, Fen23]. We study the harmonic measure ν_{π} itself and the conditional measure ν_{π}^{η} of ν_{π} on $\partial \Gamma \times \partial \Gamma^{\star}$ for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \Gamma^{\star}$. It is shown that these are exact dimensional under a finite first moment condition. The dimension formula is a sum of ratios asymptotic entropy over drift corrected with differential entropy. For the conditional measures, we provide a sufficient condition for the strict positivity of dimension.

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Let us consider left invariant hyperbolic metrics d and d^* quasi-isometric to word metrics in Γ and in Γ^* , respectively. They induce quasi-metrics q and q^* in the compactified spaces $\Gamma \cup \partial \Gamma$ and $\Gamma^* \cup \partial \Gamma^*$, respectively, and let $q(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) := \max\{q(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2), q^*(\eta_1, \eta_2)\}$ for $\boldsymbol{\xi}_i = (\boldsymbol{\xi}_i, \eta_i) \in (\Gamma \cup \partial \Gamma) \times (\Gamma^* \cup \partial \Gamma^*)$ and i = 1, 2 (cf. Section 2.1). Let us assume that π has a finite first moment, i.e., $\sum_{\boldsymbol{x} \in \Gamma} \boldsymbol{d}(\operatorname{id}, \boldsymbol{x}) < \infty$ for the identity element id and $\boldsymbol{d}(\boldsymbol{x}_1, \boldsymbol{x}_2) := \max\{d(x_1, x_2), d^*(y_1, y_2)\}$ for $\boldsymbol{x}_i = (x_i, y_i) \in \Gamma$ and i = 1, 2. The asymptotic entropy $h(\pi)$ is defined as the limit for *n*-fold convolutions $\pi_n := \pi^{*n}$,

$$h(\pi) = \lim_{n \to \infty} \frac{1}{n} \sum_{\boldsymbol{x} \in \boldsymbol{\Gamma}} -\pi_n(\boldsymbol{x}) \log \pi_n(\boldsymbol{x}).$$

The **drift** $l(\Gamma, \mu)$ is defined as the limit for $\mu_n := \mu^{*n}$,

$$l(\Gamma, \mu) = \lim_{n \to \infty} \frac{1}{n} \sum_{x \in \Gamma} d(\mathrm{id}, x) \mu_n(x).$$

Similarly for $l(\Gamma^*, \mu^*)$. These are positive since μ and μ^* are non-elementary (cf. Section 2.3). For every real r > 0 and and every $\boldsymbol{\xi} \in \partial \Gamma \times \partial \Gamma^*$, let $\boldsymbol{B}(\boldsymbol{\xi}, r)$ denote the open ball of radius r centered at $\boldsymbol{\xi}$ in $(\partial \Gamma \times \partial \Gamma^*, \boldsymbol{q})$.

Theorem 1.1. Let Γ and Γ^* be non-elementary hyperbolic groups, and π be a probability measure on $\Gamma = \Gamma \times \Gamma^*$ with finite first moment and non-elementary marginals μ and μ^* on Γ and on Γ^* respectively. Suppose that $l(\Gamma, \mu) \ge l(\Gamma^*, \mu^*)$. Then the harmonic measure ν_{π} on $(\partial \Gamma \times \partial \Gamma^*, q)$ is exact dimensional, i.e., for ν_{π} -almost every $\boldsymbol{\xi} \in \partial \Gamma \times \partial \Gamma^*$,

$$\lim_{r \to 0} \frac{\log \nu_{\pi} \left(\boldsymbol{B}(\boldsymbol{\xi}, r) \right)}{\log r} = \frac{h(\pi) - h(\mu^{\star})}{l(\Gamma, \mu)} + \frac{h(\mu^{\star})}{l(\Gamma^{\star}, \mu^{\star})}.$$

In particular, the Hausdorff dimension of ν_{π} is computed as

$$\dim \nu_{\pi} = \frac{h(\pi) - h(\mu^{\star})}{l(\Gamma, \mu)} + \frac{h(\mu^{\star})}{l(\Gamma^{\star}, \mu^{\star})}$$

Theorem 1.1 is shown in Theorem 4.5 in a more general setting. The above result is based on the exact dimensionality of disintegrated measures: Let ν_{π}^{η} for $\eta \in \partial \mathcal{X}^{*}$ denote a system of conditional measures of ν_{π} on $\partial \Gamma \times \partial \Gamma^{*}$ with respect to the σ -algebra generated by the projection from $\partial \Gamma \times \partial \Gamma^{*}$ to $\partial \Gamma^{*}$.

Theorem 1.2. Let Γ and Γ^* be non-elementary hyperbolic groups, and π be a probability measure on $\Gamma = \Gamma \times \Gamma^*$ with finite first moment and non-elementary marginals μ and μ^* on Γ and on Γ^* respectively. For ν_{μ^*} -almost every $\eta \in \partial \Gamma^*$, the conditional measure ν_{π}^{η} is exact dimensional on $(\partial \Gamma \times \partial \Gamma^*, \mathbf{q})$, i.e., for ν_{π}^{η} -almost every $\boldsymbol{\xi} \in \partial \Gamma \times \partial \Gamma^*$,

$$\lim_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} = \frac{h(\pi) - h(\mu^{\star})}{l(\Gamma, \mu)},$$

In particular, the Hausdorff dimension of ν_{π}^{η} is computed as for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \Gamma^{\star}$,

$$\dim \nu_{\pi}^{\eta} = \frac{h(\pi) - h(\mu^{\star})}{l(\Gamma, \mu)}.$$

Theorem 1.2 is shown in Theorem 1.3 in a more general setting. Following Furstenberg [Fur08, Definition 3.1], we say that a Borel probability measure ν on a product of compact metric space $\mathcal{M} \times \mathcal{M}^*$ satisfies **dimension conservation** if the following holds. Let us consider the pushforward ν^* and a system of conditional measures ν^{η} for $\eta \in \mathcal{M}^*$ of ν associated with the projection $\mathcal{M} \times \mathcal{M}^* \to \mathcal{M}^*$: the measures ν and ν^* are exact dimensional with dimension dim ν and dim ν^* respectively, for ν^* -almost every $\eta \in \mathcal{M}^*$, conditional measures ν^{η} are exact dimensional with dimension dim ν^{η} , and

$$\dim \nu = \dim \nu^{\eta} + \dim \nu^{\star}.$$

In the above, we understand that the metric in $\mathcal{M} \times \mathcal{M}^*$ is defined as the maximum of metrics in \mathcal{M} and \mathcal{M}^* (or an arbitrary one bi-Lipschitz to it). It has been shown that ν_{μ^*} on $(\partial \Gamma^*, q^*)$ is exact dimensional with dimension $h(\mu^*)/l(\Gamma^*, \mu^*)$ [Tan19, Theorem 3.8]. Therefore Theorems 1.1 and 1.2 imply that the harmonic measure ν_{π} on $\partial \Gamma \times \partial \Gamma^*$ satisfies dimension conservation. The statement holds in a more general setting; see Section 4. For an extension to a product of more than two hyperbolic groups, see Remark 4.6.

Let (\mathcal{X}, d) and (\mathcal{X}^*, d^*) be proper roughly geodesic hyperbolic metric spaces with bounded growth at some scales (for the definitions, see Section 2). Examples of such spaces include Gromov hyperbolic Riemannian manifolds with sectional curvature bounded from below and from above and Cayley graphs of hyperbolic groups. The space $\mathcal{X} \times \mathcal{X}^*$ is equipped with a base point \boldsymbol{o} and the metric $\boldsymbol{d}(\boldsymbol{x}_1, \boldsymbol{x}_2) := \max\{d(x_1, x_2), d^*(y_1, y_2)\}$ for $\boldsymbol{x}_i = (x_i, y_i) \in \mathcal{X} \times \mathcal{X}^*$ and i = 1, 2. Let us consider a countable subgroup Γ in the product of isometry groups Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$. We say that Γ has a **finite exponential growth** relative to $(\mathcal{X} \times \mathcal{X}^*, \boldsymbol{d})$ if there exists a constant c > 0 such that for all r > 0,

$$\# \{ \boldsymbol{x} \in \boldsymbol{\Gamma} : \boldsymbol{d}(\boldsymbol{o}, \boldsymbol{x} \cdot \boldsymbol{o}) < r \} \le ce^{cr}.$$

In the above, #A denotes the cardinality of a set A. For a probability measure π on Γ , let μ and μ^* denote the marginal on Isom \mathcal{X} and on Isom \mathcal{X}^* respectively. Let supp μ^* denote the support of μ^* . The **differential entropy** of the pair $(\partial \mathcal{X}^*, \mu^*)$ is defined by

$$h(\partial \mathcal{X}^{\star}, \mu^{\star}) := \sum_{x \in \operatorname{supp} \mu^{\star}} \mu^{\star}(x) \int_{\partial \mathcal{X}^{\star}} \log \frac{dx\nu_{\mu^{\star}}}{d\nu_{\mu^{\star}}}(\eta) \, dx\nu_{\mu^{\star}}(\eta)$$

In general, it holds that $h(\partial \mathcal{X}^*, \mu^*) \leq h(\mu^*)$, and the equality holds if and only if $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is a Poisson boundary for the pair (Isom \mathcal{X}^*, μ^*) (cf. Section 2.3). Let $l(\mathcal{X}, \mu)$ be the drift associated with a μ^* -random walk on \mathcal{X}^* . Theorem 1.2 is generalized in this setting.

Theorem 1.3. Let (\mathcal{X}, d) and (\mathcal{X}^*, d^*) be proper roughly geodesic hyperbolic metric spaces with bounded growth at some scale, and Γ be a countable subgroup of Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$ with finite exponential growth relative to $(\mathcal{X} \times \mathcal{X}^*, d)$. If π is a probability measure on Γ with finite first moment and non-elementary marginals μ and μ^* respectively, then the conditional measure ν_{π}^{η} is exact dimensional for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$. In fact, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ and for ν_{π}^{η} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^*$,

$$\lim_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} = \frac{h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star})}{l(\mathcal{X}, \mu)}.$$

In particular, the Hausdorff dimension of ν_{π}^{η} is computed as for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$\dim \nu_{\pi}^{\eta} = \frac{h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star})}{l(\mathcal{X}, \mu)}$$

For the differential entropy, it holds that $h(\partial \mathcal{X}^*, \mu^*) = 0$ if and only if $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is trivial, i.e., ν_{μ^*} is invariant under the action of Γ on $\partial \mathcal{X}^*$. If μ^* is non-elementary, then $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is non-trivial and $h(\partial \mathcal{X}^*, \mu^*) > 0$. In the setting of Theorem 1.3, it can be the case that $h(\pi) = h(\partial \mathcal{X}^*, \mu^*)$ (see Example 1.5 below). The following result provides a sufficient condition under which $h(\pi) > h(\partial \mathcal{X}^*, \mu^*)$, i.e., $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is a proper quotient of the Poisson boundary for the pair (Γ, π) . If this is the case, then the Hausdorff dimension of conditional measures are strictly positive.

Theorem 1.4. Let Γ and Γ^* be countable subgroups in Isom \mathcal{X} and in Isom \mathcal{X}^* respectively, and $\Gamma := \Gamma \times \Gamma^*$. Further let us consider a probability measure π on Γ of the following form: For some $\alpha \in (0, 1]$,

$$\pi = \alpha \lambda \times \lambda^* + (1 - \alpha)\pi_0$$

with non-elementary probability measures λ and λ^* on Γ and on Γ^* respectively, and a probability measure π_0 on Γ . It holds that $h(\pi) - h(\partial \mathcal{X}^*, \mu^*) > 0$, where μ^* is the marginal of π on Γ^* .

Theorem 1.4 is shown in Theorem 5.4; moreover, if in addition $\Gamma = \Gamma \times \Gamma^*$ has a finite exponential growth relative to $(\mathcal{X} \times \mathcal{X}^*, \mathbf{d})$, then for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, the Hausdorff dimension of the conditional measure ν_{π}^{η} is positive (cf. Theorem 1.3).

Example 1.5. Let Γ be a hyperbolic group and μ be a non-elementary probability measure on Γ with finite first moment relative to a word metric. For $\rho \in [0, 1]$, let

$$\pi^{\rho} := \rho \mu \times \mu + (1 - \rho) \mu_{\text{diag}},$$

where $\mu_{\text{diag}}((x, x^*)) := \mu(x)$ if $x = x^*$, and 0 if otherwise. The π^{ρ} -random walk on $\Gamma \times \Gamma$ appears in the study of noise sensitivity problem on groups [BB23, Tan22]. By Theorems 1.2 and 1.4 applied to the case when $\Gamma = \Gamma^*$ and $\mu = \mu^*$, it holds that for all $\rho \in (0, 1]$,

$$\dim \nu_{\pi^{\rho}}^{\eta} = \frac{h(\pi^{\rho}) - h(\mu)}{l(\Gamma, \mu)} > 0 \quad \text{for } \nu_{\mu}\text{-almost every } \eta \in \partial \Gamma.$$

For $\rho = 0$, since $h(\pi^{\rho}) = h(\mu)$, it holds that $\dim \nu_{\pi^{\rho}}^{\eta} = 0$ for ν_{μ} -almost every $\eta \in \partial \Gamma$. Theorem 1.1 shows that for all $\rho \in [0, 1]$,

$$\dim \nu_{\pi^{\rho}} = \frac{h(\pi^{\rho})}{l(\Gamma, \mu)}.$$

This reproduces [Tan22, Theorem 3.1].

Example 1.6. This example is not covered by Theorem 1.4. Suppose that $\Gamma = \Gamma \times \Gamma^*$ for two hyperbolic groups Γ and Γ^* , and that there exists a (proper) surjective homomorphism $\Pi : \Gamma \to \Gamma^*$. Let $\Delta : \Gamma \to \Gamma \times \Gamma^*$ be the diagonal embedding $\Delta(x) = (x, \Pi(x))$ for $x \in \Gamma$. For a non-elementary probability measure μ on Γ with finite first moment, let $\pi := \Delta_* \mu$ be the pushforward of π by Δ . In this case, marginals of π are μ on Γ and $\Pi_* \mu$ on Γ^* respectively, where $\Pi_*\mu$ is the pushforward of μ by Π . Applying to Theorem 1.2 with $\mu^* = \Pi_*\mu$ shows that

$$\dim \nu_{\pi}^{\eta} = \frac{h(\mu) - h(\Pi_{*}\mu)}{l(\Gamma, \mu)} \quad \text{for } \nu_{\Pi_{*}\mu}\text{-almost every } \eta \in \partial \Gamma^{\star}$$

This follows since $h(\pi) = h(\mu)$ and $h(\partial \Gamma^*, \Pi_*\mu) = h(\Pi_*\mu)$. It holds that $h(\mu) = h(\Pi_*\mu)$ if Π is an isomorphism, and it depends on Π whether a strict inequality $h(\mu) > h(\Pi_*\mu)$ holds or not. As a simple explicit example, let $\Gamma = F_{m+1}$ and $\Gamma^* = F_m$ be free groups of rank m + 1 and m respectively for $m \ge 2$, equipped with word metrics associated with free bases. Further let $\Delta : F_{m+1} \to F_m$ be a homomorphism defined by sending the free basis in F_{m+1} to the free basis in F_m . For the uniform distribution μ on the symmetrized free basis in F_{m+1} , the induced distribution on F_m defines a simple random walk on F_m with holding probability 1/(m+1). A computation yields $l(F_{m+1}, \mu) = m/(m+1)$, $l(F_m, \Pi_*\mu) = (m-1)/(m+1)$,

$$h(\mu) = \frac{m}{m+1}\log(2m+1)$$
 and $h(\Pi_*\mu) = \frac{m-1}{m+1}\log(2m-1)$.

Therefore

$$\dim \nu_{\pi}^{\eta} = \log(2m+1) - \frac{m-1}{m}\log(2m-1) \quad \text{for } \nu_{\pi}^{\eta} \text{-almost every } \eta \in \partial F_{m}.$$

Furthermore, Theorem 1.1 shows that

dim
$$\nu_{\pi} = \log(2m+1) + \frac{1}{m}\log(2m-1).$$

Outlines of proofs. Let us briefly mention the proof of Theorem 1.1 for a product of two hyperbolic groups. For a single hyperbolic group Γ with a non-elementary probability measure μ , the corresponding harmonic measure ν_{μ} on $\partial\Gamma$ is exact dimensional [Tan19, Theorem 3.8]. Roughly speaking, it boils down to estimate probabilities that for a μ random walk w_n , an independent μ -random walk w'_n is around w_n within distance o(n)for $n \in \mathbb{Z}_+$. This leads an estimate of the harmonic measure ν_{μ} on the balls $B(w_{\infty}, e^{-ln})$ where $l := l(\Gamma, \mu)$. Here the μ -random walk $\{w_n\}_{n\in\mathbb{Z}_+}$ is for a sampling w_{∞} in $\partial\Gamma$ according to ν_{μ} and the independent μ -random walk $\{w'_n\}_{n\in\mathbb{Z}_+}$ is for the estimate $\nu_{\mu}(B(w_{\infty}, e^{-ln}))$. Since the probability that w'_n is around w_n within distance o(n) is $e^{-h(\mu)n+o(n)}$ by the Shannon theorem for random walks, this explains $\nu_{\mu}(B(w_{\infty}, e^{-ln})) = e^{-h(\mu)n+o(n)}$, which is the exact dimensionality of ν_{μ} with the right dimension $h(\mu)/l$.

The conditional measure ν_{π}^{η} for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \Gamma^{\star}$ is the hitting distribution of a conditional process. This is a Markov chain (although the transition probabilities are not group-invariant) and (one of) the methods developed for a single hyperbolic group in [ibid] applies. The asymptotic entropy of this conditional process equals $h(\pi) - h(\mu^{\star})$ by the Shannon theorem for the conditional process [Kai00]. The conditional measure ν_{π}^{η} is defined on $\partial \Gamma \times \partial \Gamma^{\star}$ but supported on $\partial \Gamma \times \{\eta\}$ for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \Gamma^{\star}$. An analogous discussion to the μ -random walk above works and this leads to estimating the ν_{π}^{η} -measures on the balls in the boundary $B(w_{\infty}, e^{-ln}) \times \{\eta\}$ for ν_{π}^{η} -almost every $(w_{\infty}, \eta) \in \partial \Gamma \times \partial \Gamma^{\star}$. In fact, we obtain $\nu_{\pi}^{\eta}(B(w_{\infty}, e^{-ln}) \times \{\eta\}) = e^{-(h(\pi) - h(\mu^{\star}))n + o(n)}$, deducing Theorem 1.2.

The harmonic measure ν_{π} on $\partial \Gamma \times \partial \Gamma^*$ is, however, analyzed in a completely different way. First of all it requires to take into account the difference between l and l^* where $l^* := l(\Gamma^*, \mu^*)$. If $l \ge l^*$, then

$$\frac{h(\pi) - h(\mu)}{l^{\star}} + \frac{h(\mu)}{l} \le \frac{h(\pi) - h(\mu^{\star})}{l} + \frac{h(\mu^{\star})}{l^{\star}}$$

since $h(\pi) \leq h(\mu) + h(\mu^*)$, and the inequality can be strict. Since the right hand side of the above inequality is the correct value, the dimension upper bound should use the inequality $l \geq l^*$ whereas the dimension lower bound would not need it. Concerning the dimension upper bound, the Shannon theorem for the conditional process shows that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for the conditional process $\{\boldsymbol{w}_n\}_{n \in \mathbb{Z}_+}$,

$$\mathbf{P}^{\eta}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]) \leq e^{h(\mu^{\star})n+o(n)}\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]).$$

In the above, $[\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n]$ denotes the cylinder set. At this point, we keep track the whole trajectory up to time *n* instead of just looking at the position \boldsymbol{w}_n . The argument here is inspired by [LL23, Section 8] (where they refer to [Fen23] for the idea). The ν_{π}^{η} on the balls $\boldsymbol{B}(\boldsymbol{w}_{\infty}, e^{-l^{\star}n}) = B(w_{\infty}, e^{-l^{\star}n}) \times B(w_{\infty}^{\star}, e^{-l^{\star}n})$ estimates by Theorem 1.2,

$$\nu_{\pi}^{\eta}\left(\boldsymbol{B}(\boldsymbol{w}_{\infty}, e^{-l^{\star}n})\right) = \exp\left(-\left(\frac{h(\pi) - h(\mu^{\star})}{l}\right)l^{\star}n + o(n)\right).$$

Averaging η over $B(w_{\infty}^{\star}, e^{-l^{\star}n})$ deduces the required lower bound (thus upper bound for the dimension) of $\nu_{\pi}(\boldsymbol{B}(\boldsymbol{w}_{\infty}, e^{-l^{\star}n}))$. In this discussion, it is crucial to use the balls with radii $e^{-l^{\star}n}$ rather than e^{-ln} (or other scales) since $\boldsymbol{q}(\boldsymbol{w}_{\infty}, \boldsymbol{w}_n) = e^{-l^{\star}n + o(n)}$, where $l \geq l^{\star}$,

$$q(w_{\infty}, w_n) = e^{-ln + o(n)}$$
 and $q^{\star}(w_{\infty}^{\star}, w_n^{\star}) = e^{-l^{\star}n + o(n)}$.

Concerning the dimension lower bound, a slight strengthened version for the lower bound in Theorem 1.2 enables us to exploit the naive disintegration formula. Roughly, estimating along the following heuristic can be justified:

$$\nu_{\pi} \left(\boldsymbol{B}(\boldsymbol{w}_{\infty}, e^{-ln}) \right) \approx \nu_{\pi}^{\eta} (B(w_{\infty}, e^{-ln}) \times \{\eta\}) \cdot \nu_{\mu^{\star}} \left(B(w_{\infty}^{\star}, e^{-ln}) \right).$$

Since this works only for ν_{π} restricted on a large subset, the argument is merely for the upper bound (thus lower bound for the dimension) of ν_{π} (up to a density lemma which is guaranteed by a weak version of the Lebesgue differentiation theorem Lemma 2.2). Thus Theorem 1.2 and the exact dimensionality of $\nu_{\mu^{\star}}$ with $h(\mu^{\star})/l^{\star}$ conclude the required dimension lower bound on ν_{π} .

The above sketch for hyperbolic groups can be extended to a countable group of isometries acting on a product of two hyperbolic metric spaces in Theorem 4.5. The positive lower bound for $h(\pi) - h(\partial \mathcal{X}^*, \mu^*)$ in Theorem 1.4 uses the pivotal time technique developed by Gouëzel in [Gou22]. We mention possible extensions of Theorems 1.1 and 4.5 and questions in Remark 4.6.

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Organization. Section 2 recalls basics on hyperbolic metric spaces and random walks. Section 3 concerns dimensions of the conditional measures, showing Theorem 1.3 (and thus Theorem 1.2). Section 4 concerns dimensions of the harmonic measures on products of boundaries, showing Theorem 4.5 (and thus Theorem 1.1). Section 5 is about a sufficient condition on a positivity of the dimension for conditional measures, showing Theorem 1.4 in Theorem 5.4.

Notation. We denote by c, C, \ldots , constants whose exact values may vary from line to line, and by C_{δ} a constant which depends on the other constant δ to emphasize its dependency. For a real valued sequence $\{f(n)\}_{n \in \mathbb{Z}_+}$ on non-negative integers \mathbb{Z}_+ , we write f(n) = o(n) if $|f(n)|/n \to 0$ as $n \to \infty$. For a set A, we denote by A^c the complement set, and by #A the cardinality.

2. Preliminaries

2.1. Hyperbolic metric spaces. For background, we refer to the original paper by Gromov [Gro87]. For a metric space (\mathcal{X}, d) , the Gromov product is defined by

$$(x|y)_{z} := \frac{1}{2}(d(x,z) + d(z,y) - d(x,y)) \text{ for } x, y, z \in \mathcal{X}$$

A metric space (\mathcal{X}, d) is δ -hyperbolic for a non-negative real $\delta \in \mathbb{R}_+$ if it holds that

$$(x|y)_w \ge \min\left\{(x|z)_w, (z|y)_w\right\} - \delta \quad \text{for all } x, y, z, w \in \mathcal{X}.$$
(2.1)

It is called **hyperbolic** if it is δ -hyperbolic for some $\delta \in \mathbb{R}_+$. A map $\gamma : I \to \mathcal{X}$ from an interval I in \mathbb{R} to \mathcal{X} is called a C-rough geodesic for $C \in \mathbb{R}_+$ if $|d(\gamma(s), \gamma(t)) - |t-s|| \leq C$ for all $s, t \in I$. Further a map $\gamma : I \to \mathcal{X}$ is called a C-rough geodesic ray in the case when $I = [0, \infty)$. A metric space is called C-roughly geodesic for $C \in \mathbb{R}_+$ if for all pairs of points $x, y \in \mathcal{X}$ there exists a C-rough geodesic $\gamma : [a, b] \to \mathcal{X}$ such that $\gamma(a) = x$ and $\gamma(b) = y$. In this terminology, a metric space is called geodesic if it is 0-roughly geodesic. A graph endowed with a path metric of unit edge length (e.g., a Cayley graph) is also considered as a geodesic metric space by using intervals in the integers \mathbb{Z} in the definition. Let us simply call a metric space roughly geodesic if it is C-roughly geodesic for some $C \in \mathbb{R}_+$. For a hyperbolic group Γ equipped with a left invariant hyperbolic metric dquasi-isometric to a word metric, (Γ, d) is roughly geodesic (cf. [BS00, Proposition 5.6] and [BHM11, Theorem 2.2]). A metric space (\mathcal{X}, d) is proper if for all $x \in \mathcal{X}$ and all $r \in \mathbb{R}_+$, the ball $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ is relatively compact.

For a hyperbolic metric space (\mathcal{X}, d) , the **(Gromov) boundary** $\partial \mathcal{X}$ is defined as the set of equivalence classes of divergent sequences in \mathcal{X} . Let us fix a point $o \in \mathcal{X}$. Further let $q(x, y) := \exp(-(x|y)_o)$ for $x, y \in \mathcal{X}$ and q(x, y) := 0 if x = y. Since the space is δ -hyperbolic for some $\delta \in \mathbb{R}_+$, it holds that

$$q(x,y) \le e^{\delta} \max\left\{q(x,z), q(z,y)\right\} \quad \text{for } x, y, z \in \mathcal{X}.$$
(2.2)

A sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ in \mathcal{X} is called **divergent** if it is a Cauchy sequence with respect to q. Two sequences $\{x_n\}_{n\in\mathbb{Z}_+}$ and $\{y_n\}_{n\in\mathbb{Z}_+}$ are **equivalent** if $q(x_n, y_m) \to 0$ as $n, m \to \infty$. It is indeed an equivalence relation in the set of divergence sequences by (2.2).

For $\xi \in \mathcal{X} \cup \partial \mathcal{X}$, let us write $\xi = [\{x_n\}_{n \in \mathbb{Z}_+}]$ for a divergent sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ which represents ξ if $\xi \in \partial \mathcal{X}$, or for the constant sequence $x_n = \xi$ for all $n \in \mathbb{Z}_+$ if $\xi \in \mathcal{X}$. The Gromov product is extended to $\mathcal{X} \cup \partial \mathcal{X}$ by

$$(\xi|\eta)_o := \inf \left\{ \liminf_{n,m\to\infty} (x_n|y_m)_o : \xi = [\{x_n\}_{n\in\mathbb{Z}_+}], \eta = [\{y_m\}_{m\in\mathbb{Z}_+}] \right\}$$

For a δ -hyperbolic space, the extended Gromov product satisfies (2.1) for $x, y, z \in \mathcal{X} \cup \partial \mathcal{X}$ and w = o. Let us extend q on \mathcal{X} to $\mathcal{X} \cup \partial \mathcal{X}$ and call it the **quasi-metric**:

$$q(\xi,\eta) := \exp(-(\xi|\eta)_o) \quad \text{if } \xi \neq \eta, \quad \text{and} \quad q(\xi,\eta) := 0 \quad \text{if } \xi = \eta, \quad \text{for } \xi, \eta \in \mathcal{X} \cup \partial \mathcal{X}.$$

It is known that there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the power q^{ε} is bi-Lipschitz equivalent to a genuine metric. However, the quasi-metric q is used to define balls and other notions related to metrics without introducing an additional parameter ε . The space $\mathcal{X} \cup \partial \mathcal{X}$ is equipped with the topology defined from the (quasi-)metric. If \mathcal{X} is proper, then $\mathcal{X} \cup \partial \mathcal{X}$ is a compact metrizable space. If in addition \mathcal{X} is C-roughly geodesic for some $C \in \mathbb{R}_+$, then for every $\xi \in \partial \mathcal{X}$ and every $x \in \mathcal{X}$ there exists a C-rough geodesic ray γ from x converging to ξ , i.e., $q(\gamma(t), \xi) \to 0$ as $t \to \infty$ in $\mathcal{X} \cup \partial \mathcal{X}$ (cf. [BS00, Proposition 5.2]). Henceforth it is assumed that \mathcal{X} is a proper roughly geodesic hyperbolic metric space.

Let us denote the open ball of radius $r \in \mathbb{R}_+$ centered at $\xi \in \mathcal{X} \cup \partial \mathcal{X}$ in $\mathcal{X} \cup \partial \mathcal{X}$ by

$$B(\xi, r) := \Big\{ \eta \in \mathcal{X} \cup \partial \mathcal{X} : q(\xi, \eta) < r \Big\}.$$

The shadow (seen from o) at $x \in \mathcal{X}$ with thickness $R \in \mathbb{R}_+$ is defined by

$$\mathcal{O}(x,R) := \Big\{ \eta \in \partial \mathcal{X} : (o|\eta)_x < R \Big\}.$$

The following is used to compare shadows with balls. For each T > 0, there exist constants $R_0, C > 0$ such that for all $R > R_0$, all $\xi \in \partial \mathcal{X}$ and all $x \in \mathcal{X}$ with $(o|\xi)_x \leq T$,

$$B(\xi, C^{-1}e^{-d(o,x)+R}) \cap \partial \mathcal{X} \subset \mathcal{O}(x, R) \subset B(\xi, Ce^{-d(o,x)+R}) \cap \partial \mathcal{X}.$$
(2.3)

For another such hyperbolic metric space (\mathcal{X}^*, d^*) with base point o^* , let q^* denote the quasi-metric in $\partial \mathcal{X}^*$. In the product space, for $\boldsymbol{\xi}_i = (\xi_i, \eta_i) \in (\mathcal{X} \cup \partial \mathcal{X}) \times (\mathcal{X}^* \cup \partial \mathcal{X}^*)$ and i = 1, 2, let

$$q(\xi_1, \xi_2) := \max \{q(\xi_1, \xi_2), q^*(\eta_1, \eta_2)\}.$$

By (2.2) extended to $\mathcal{X} \cup \partial \mathcal{X}$ and $\mathcal{X}^* \cup \partial \mathcal{X}^*$, there exists a constant $C := C_q > 0$ such that

 $\boldsymbol{q}(\boldsymbol{x},\boldsymbol{y}) \leq C \max\{\boldsymbol{q}(\boldsymbol{x},\boldsymbol{z}),\boldsymbol{q}(\boldsymbol{z},\boldsymbol{y})\} \quad \text{for all } \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in (\mathcal{X} \cup \partial \mathcal{X}) \times (\mathcal{X}^{\star} \cup \partial \mathcal{X}^{\star}).$

Let $\boldsymbol{B}(\boldsymbol{\xi}, r)$ denote the ball in $(\mathcal{X} \cup \partial \mathcal{X}) \times (\mathcal{X}^{\star} \cup \partial \mathcal{X}^{\star})$ with respect to \boldsymbol{q} .

2.2. Hausdorff dimensions. Let (\mathcal{M}, q) be a compact metrizable space \mathcal{M} with a quasimetric q. It is basically intended $(\partial \mathcal{X}, q)$ or $(\partial \mathcal{X} \times \partial \mathcal{X}^*, q)$. For a set E in \mathcal{M} , let dim Edenote the Hausdorff dimension of E with respect to the quasi-metric q. The definition is recalled briefly. Let $|E| := \sup\{q(\xi, \eta) : \xi, \eta \in E\}$. For all $\alpha, \Delta \in \mathbb{R}_+$ with $\Delta > 0$, let

$$\mathcal{H}^{\alpha}_{\Delta}(E) := \inf \Big\{ \sum_{i=0}^{\infty} |E_i|^{\alpha} : E \subset \bigcup_{i=0}^{\infty} E_i \text{ and } |E_i| \le \Delta \Big\}.$$

The α -dimensional Hausdorff measure of a set E is defined by

$$\mathcal{H}^{\alpha}(E) := \lim_{\Delta \to 0} \mathcal{H}^{\alpha}_{\Delta}(E) = \sup_{\Delta > 0} \mathcal{H}^{\alpha}_{\Delta}(E).$$

Moreover the **Hausdorff dimension** of a set E is defined by

dim
$$E := \sup \left\{ \alpha \ge 0 : \mathcal{H}^{\alpha}(E) > 0 \right\} = \inf \left\{ \alpha \ge 0 : \mathcal{H}^{\alpha}(E) = 0 \right\}.$$

Let ν be a Borel probability measure on $\partial \mathcal{X}$. The **upper Hausdorff dimension** of ν is

$$\overline{\dim}\nu := \inf \Big\{ \dim E : E \text{ is Borel and } \nu(\mathcal{M} \setminus E) = 0 \Big\},\$$

and the lower Hausdorff dimension of ν is

 $\underline{\dim}\,\nu:=\inf\,\Big\{\,\dim E\ :\ E\ \text{is Borel and}\ \nu(E)>0\Big\}.$

If the upper and lower Hausdorff dimensions of ν coincide, then the common value is called the **Hausdorff dimension** of ν and is denoted by dim ν . The following is a fundamental lemma which relates pointwise behaviors of a measure to Hausdorff dimensions. This is called the Billingsley lemma (in the case of Euclidean spaces).

Lemma 2.1 (cf. Section 8.7 in [Hei01]). For every Borel probability measure ν on \mathcal{M} , if for $\alpha_1, \alpha_2 \in \mathbb{R}_+$,

$$\alpha_1 \leq \liminf_{r \to 0} \frac{\log \nu(B(\xi, r))}{\log r} \leq \alpha_2 \quad \text{for } \nu \text{-almost every } \xi \in \mathcal{M},$$

then $\alpha_1 \leq \overline{\dim} \nu \leq \alpha_2$.

It is deduced that

$$\overline{\dim}\,\nu = \sup_{\nu\text{-a.e. }\xi} \liminf_{r \to 0} \frac{\log \nu(B(\xi, r))}{\log r} \quad \text{and} \quad \underline{\dim}\,\nu = \inf_{\nu\text{-a.e. }\xi} \liminf_{r \to 0} \frac{\log \nu(B(\xi, r))}{\log r}.$$

In the above, $\sup_{\nu-\text{a.e. }\xi}$ and $\inf_{\nu-\text{a.e. }\xi}$ denote the essential supremum and the essential infimum relative to ν respectively. A Borel probability measure ν on \mathcal{M} is **exact dimensional** if the following limit exists and is constant ν -almost everywhere on \mathcal{M} :

$$\lim_{r \to 0} \frac{\log \nu(B(\xi, r))}{\log r}$$

In that case, the Hausdorff dimension of ν exists and equals the constant.

A metric space (\mathcal{X}, d) is called **bounded growth at some scale** if there exist constants $r, R \in \mathbb{R}_+$ with 0 < r < R and $N \in \mathbb{Z}_+$ such that every open ball of radius R is covered by at most N open balls of radius r. The examples include Gromov hyperbolic Riemannian

manifolds whose sectional curvature is uniformly bounded from below and from above and Cayley graphs of hyperbolic groups.

Lemma 2.2. Let (\mathcal{X}, d) and $(\partial \mathcal{X}^*, d^*)$ be hyperbolic metric spaces with bounded growth at some scale. There exists a constant $L \geq 1$ such that the following holds for every Borel probability measure ν on $\partial \mathcal{X} \times \partial \mathcal{X}^*$ and for every Borel set F in $\partial \mathcal{X} \times \partial \mathcal{X}^*$ with $\nu(F) > 0$. For ν -almost every $\boldsymbol{\xi} \in F$, there exists a constant $r(\boldsymbol{\xi}) > 0$ such that for every $r \in (0, r(\boldsymbol{\xi}))$,

$$\nu(F \cap \boldsymbol{B}(\boldsymbol{\xi}, Lr)) \geq \frac{9}{10}\nu(\boldsymbol{B}(\boldsymbol{\xi}, r)).$$

Proof. The assumption on (\mathcal{X}, d) implies that for every $\alpha \in (0, 1)$ there exists a bi-Lipschitz embedding f from $(\partial \mathcal{X}, q^{\alpha})$ to some finite dimensional standard Euclidean space $(\mathbb{E}_0, \|\cdot\|_{\mathbb{E}_0})$ (cf. [BS00, Theorem 9.2] and [Ass83, 2.6. Proposition]). More precisely, there exists a constant $L^0 \geq 1$ such that for all $\xi, \eta \in \partial \mathcal{X}$,

$$(1/L^0)q(\xi,\eta)^{\alpha} \le ||f(\xi) - f(\eta)||_{\mathbf{E}_0} \le L^0 q(\xi,\eta)^{\alpha}$$

Similarly, for (\mathcal{X}^*, d^*) there exists a bi-Lipschitz embedding f^* from $(\partial \mathcal{X}^*, q^{*\alpha})$ into some Euclidean space $(\mathbf{E}^*, \|\cdot\|_{\mathbf{E}^*})$ with a Lipschitz constant $L^* \geq 1$. Let

$$\boldsymbol{f}: \partial \mathcal{X} \times \partial \mathcal{X}^{\star} \to \mathsf{E} := \mathsf{E}^{\mathsf{0}} \times \mathsf{E}^{\star}, \quad (\xi, \eta) \mapsto (f(\xi), f^{\star}(\eta)).$$

The map f is homeomorphic onto the image since $\partial \mathcal{X} \times \partial \mathcal{X}^*$ is compact. The product space E is endowed with the maximum norm $\|\cdot\|_{E}$ of the factors. Let $B_{E}(\boldsymbol{v},r)$ denote the ball in E with respect to the norm. It holds that

$$(1/L)\boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta})^{\alpha} \leq \|\boldsymbol{f}(\boldsymbol{\xi}) - \boldsymbol{f}(\boldsymbol{\eta})\|_{\mathsf{E}} \leq L\boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta})^{\alpha} \quad \text{for } \boldsymbol{\xi}, \boldsymbol{\eta} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star},$$
(2.4)

where $L := \max\{L^0, L^*\}$. The pushforward $f_*\nu$ satisfies that $f_*\nu(B_{\mathsf{E}}(f(\boldsymbol{\xi}), r)) > 0$ for all r > 0 and for ν -almost all $\boldsymbol{\xi} \in F$. This follows since $\nu(F) > 0$ and the intersection of F and the support of ν has a positive ν -measure, it holds that $\nu(\boldsymbol{B}(\boldsymbol{\xi}, r)) > 0$ for all r > 0 and for ν -almost every $\boldsymbol{\xi} \in F$. The Lebesgue differentiation theorem on $f_*\nu$ yields

$$\lim_{r \to 0} \frac{\boldsymbol{f}_* \nu(\boldsymbol{f}(F) \cap B_{\mathsf{E}}(\boldsymbol{f}(\boldsymbol{\xi}), r))}{\boldsymbol{f}_* \nu(B_{\mathsf{E}}(\boldsymbol{f}(\boldsymbol{\xi}), r))} = 1 \quad \text{for } \nu\text{-almost every } \boldsymbol{\xi} \in F.$$

By (2.4), it holds that

$$\liminf_{r \to 0} \frac{\nu(F \cap B(\boldsymbol{\xi}, (Lr)^{1/\alpha}))}{\nu(B(\boldsymbol{\xi}, (r/L)^{1/\alpha}))} \ge 1 \quad \text{for } \nu\text{-almost every } \boldsymbol{\xi} \in F.$$

Hence for ν -almost every $\boldsymbol{\xi} \in F$ there exists some $r(\boldsymbol{\xi}) > 0$ such that

$$\nu(F \cap B(\boldsymbol{\xi}, L^{2/\alpha}r)) \ge \frac{9}{10}\nu(B(\boldsymbol{\xi}, r)) \quad \text{for all } r \in (0, r(\boldsymbol{\xi})).$$

Shifting the constant $L^{2/\alpha}$ to L deduces the claim.

2.3. Random walks. Let Γ be a countable group. Further let $\Omega := \Gamma^{\mathbb{Z}_+}$ be the product space endowed with the σ -algebra \mathcal{F} generated by cylinder sets. For a probability measure π on Γ , let $\pi^{\mathbb{Z}_+}$ be the product measure on $\Gamma^{\mathbb{Z}_+}$. Let us define the map $w : \Gamma^{\mathbb{Z}_+} \to \Omega$, $\{\boldsymbol{x}_n\}_{n \in \mathbb{Z}_+} \mapsto \{\boldsymbol{w}_n\}_{n \in \mathbb{Z}_+}$ where $\boldsymbol{w}_0 := \text{id}$ (the identity element) and

$$\boldsymbol{w}_n := \boldsymbol{x}_1 \cdots \boldsymbol{x}_n \quad \text{for } n = 1, 2, \dots$$

The pushforward of $\pi^{\mathbb{Z}_+}$ by the map w is denoted by \mathbf{P} . The probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is a standard probability space; this is the most basic space in the following discussion. The maps $\Omega \to \Gamma$, $\{w_n\}_{n \in \mathbb{Z}_+} \mapsto w_n$ defines a Markov chain $\{w_n\}_{n \in \mathbb{Z}_+}$ called a π -random walk starting from id.

For a hyperbolic metric space (\mathcal{X}, d) , let Isom \mathcal{X} denote the isometry group. A probability measure μ (with a countable support) on Isom \mathcal{X} is called **non-elementary** if the group generated by the support of μ (as a group) contains a free group of rank 2.

Let (\mathcal{X}, d) and (\mathcal{X}^*, d^*) be hyperbolic metric spaces, and Γ be a countable subgroup of Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$. Further let π be a probability measure on Γ such that the pushforwards μ and μ^* by the projections from Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$ to Isom \mathcal{X} and to Isom \mathcal{X}^* respectively are non-elementary. In this setting, a π -random walk $\{\boldsymbol{w}_n\}_{n \in \mathbb{Z}_+}$ starting from id yields by letting $\boldsymbol{w}_n = (w_n, w_n^*)$, a μ -random walk $\{w_n\}_{n \in \mathbb{Z}_+}$ with $w_0 = \text{id}$ and a μ^* -random walk $\{w_n^*\}_{n \in \mathbb{Z}_+}$ with $w_0^* = \text{id}$. For fixed base points $o \in \mathcal{X}$ and $o^* \in \mathcal{X}^*$, let

$$\boldsymbol{z}_n := (z_n, z_n^{\star}), \quad \text{where } z_n := w_n \cdot o \text{ and } z_n^{\star} := w_n^{\star} \cdot o^{\star} \text{ for } \boldsymbol{w}_n = (w_n, w_n^{\star}).$$

The assumption that μ and μ^* are non-elementary implies that **P**-almost surely there exist $z_{\infty} \in \partial \mathcal{X}$ and $z_{\infty}^* \in \partial \mathcal{X}^*$ such that $z_n \to z_{\infty}$ in $\mathcal{X} \cup \partial \mathcal{X}$ and $z_n^* \to z_{\infty}^*$ in $\mathcal{X}^* \cup \partial \mathcal{X}^*$ as $n \to \infty$ respectively. Let ν_{π} be the distribution of $(z_{\infty}, z_{\infty}^*)$ on $\partial \mathcal{X} \times \partial \mathcal{X}^*$, and ν_{μ} and ν_{μ^*} be the distributions of z_{∞} and of z_{∞}^* respectively. The probability measure ν_{π} is called the **harmonic measure** for the π -random walk. Similarly, ν_{μ} and ν_{μ^*} are called the harmonic measures for the μ -random walk and for the μ^* -random walk respectively. Let us denote the measurable map by

$$\mathbf{bnd} = (\mathbf{bnd}, \mathbf{bnd}^{\star}) : \Omega \to \partial \mathcal{X} \times \partial \mathcal{X}^{\star}, \quad \boldsymbol{w} \mapsto \boldsymbol{z}_{\infty} := (z_{\infty}, z_{\infty}^{\star}).$$

The harmonic measures ν_{π} , ν_{μ} and $\nu_{\mu^{\star}}$ are obtained as the pushforwards of **P** by **bnd**, by bnd and by bnd^{*} respectively.

The group Γ acts on $\partial \mathcal{X} \times \partial \mathcal{X}^*$ through Isom $\mathcal{X} \times$ Isom \mathcal{X}^* . It satisfies that ν_{π} is π -stationary, i.e.,

$$u_{\pi} = \sum_{\boldsymbol{x} \in \boldsymbol{\Gamma}} \pi(\boldsymbol{x}) \boldsymbol{x}
u_{\pi} \quad ext{where } \boldsymbol{x}
u_{\pi} =
u_{\pi} \circ \boldsymbol{x}^{-1}.$$

Similarly, Γ acts on $\partial \mathcal{X}$ and on $\partial \mathcal{X}^*$ through the projection from Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$ to each one of factors, and thus ν_{μ} and ν_{μ^*} are also π -stationary, i.e.,

$$\nu_{\mu} = \sum_{\boldsymbol{x} \in \boldsymbol{\Gamma}} \pi(\boldsymbol{x}) \boldsymbol{x} \nu_{\mu} \quad \text{and} \quad \nu_{\mu^{\star}} = \sum_{\boldsymbol{x} \in \boldsymbol{\Gamma}} \pi(\boldsymbol{x}) \boldsymbol{x} \nu_{\mu^{\star}}.$$
(2.5)

Since μ and μ^* are marginals of π , these further lead to

$$\nu_{\mu} = \sum_{x \in \operatorname{supp} \mu} \mu(x) x \nu_{\mu} \quad \text{and} \quad \nu_{\mu^{\star}} = \sum_{x^{\star} \in \operatorname{supp} \mu^{\star}} \mu^{\star}(x^{\star}) x^{\star} \nu_{\mu^{\star}}.$$

Let us define the metric in $\mathcal{X} \times \mathcal{X}^*$ by

$$d(z_1, z_2) := \max \{ d(z_1, z_2), d^*(z_1^*, z_2^*) \}$$
 for $z_i = (z_i, z_i^*) \in \mathcal{X} \times \mathcal{X}^*$ and $i = 1, 2$.

A probability measure π on $\Gamma < \operatorname{Isom} \mathcal{X} \times \operatorname{Isom} \mathcal{X}^*$ has a finite first moment if

$$\sum_{\boldsymbol{x}\in\boldsymbol{\Gamma}}\boldsymbol{d}(\boldsymbol{o},\boldsymbol{x}\cdot\boldsymbol{o})\pi(\boldsymbol{x})<\infty,\quad\text{where }\boldsymbol{o}=(o,o^{\star}).$$

This condition is independent of the choice of the point $o = (o, o^*)$. Let us assume that π has a finite first moment. The Kingman subadditive ergodic theorem implies that the following limits exist and are constant **P**-almost everywhere:

$$l(\mathcal{X},\mu) := \lim_{n \to \infty} \frac{1}{n} d(o, z_n) \quad \text{and} \quad l(\mathcal{X}^*, \mu^*) := \lim_{n \to \infty} \frac{1}{n} d^*(o^*, z_n^*).$$

The limits $l(\mathcal{X}, \mu)$ and $l(\mathcal{X}^*, \mu^*)$ are called the **drifts** of $\{z_n\}_{n \in \mathbb{Z}_+}$ and $\{z_n^*\}_{n \in \mathbb{Z}_+}$ respectively. In the case when μ and μ^* are non-elementary, then $l(\mathcal{X}, \mu) > 0$ and $l(\mathcal{X}^*, \mu^*) > 0$ ([Kai00, Theorem 7.3], and see also [Gou22, Theorem 1.1] for a more recent account).

2.4. Conditional processes and their entropies. If π has a finite first moment, then the entropy $H(\pi) := -\sum_{\boldsymbol{x} \in \Gamma} \pi(\boldsymbol{x}) \log \pi(\boldsymbol{x})$ is finite [Der86, Section VII, B]. The Shannon theorem for random walks says that for such π , the following limit exists and is constant **P**-almost everywhere:

$$h(\pi) := \lim_{n \to \infty} -\frac{1}{n} \log \pi_n(\boldsymbol{w}_n).$$
(2.6)

See [KV83, Theorem 2.1] and [Der80, Section IV]. The limit $h(\pi)$ is called the **asymptotic** entropy for π -random walk. Let $h(\mu)$ and $h(\mu^*)$ be the asymptotic entropies for μ -random walk and μ^* -random walk respectively; they exist and are defined in the same way. We will also use a conditional version of the notion. First we introduce a conditional process and then define the conditional entropy.

Recall that $(\Omega, \mathcal{F}, \mathbf{P})$ is a standard probability space. Let $\sigma(\mathbf{bnd})$ be the σ -algebra generated by the measurable map $\mathbf{bnd} : \Omega \to \partial \mathcal{X} \times \partial \mathcal{X}^*$. Disintegrating the measure \mathbf{P} with respect to $\sigma(\mathbf{bnd})$ yields the system of conditional probability measures $\{\mathbf{P}^{\mathbf{bnd}(w)}\}_{w\in\Omega}$. More precisely, for every $A \in \mathcal{F}$, the map $\Omega \to \mathbb{R}$, $w \mapsto \mathbf{P}^{\mathbf{bnd}(w)}(A)$ is $\sigma(\mathbf{bnd})$ -measurable, and

$$\mathbf{P} = \int_{\Omega} \mathbf{P}^{\mathbf{bnd}(\boldsymbol{w})} \, d\mathbf{P}(\boldsymbol{w})$$

Noting that $\nu_{\pi} = \mathbf{bnd}_*\mathbf{P}$, let us write $\{\mathbf{P}^{\xi,\eta}\}_{(\xi,\eta)\in\partial\mathcal{X}\times\partial\mathcal{X}^*}$ and

$$\mathbf{P} = \int_{\partial \mathcal{X} \times \partial \mathcal{X}^{\star}} \mathbf{P}^{\xi,\eta} \, d\nu_{\pi}(\xi,\eta)$$

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Similarly, disintegrating \mathbf{P} with respect to the σ -algebra $\sigma(\text{bnd}^*)$ generated by bnd^{*} yields the system of conditional probability measures $\{\mathbf{P}^{\eta}\}_{\eta\in\partial\mathcal{X}^*}$ satisfying that

$$\mathbf{P} = \int_{\partial \mathcal{X}^{\star}} \mathbf{P}^{\eta} \, d\nu_{\mu^{\star}}(\eta). \tag{2.7}$$

Let us disintegrate the harmonic measure ν_{π} . In the present setting, $\partial \mathcal{X} \times \partial \mathcal{X}^*$ is a compact metrizable space and thus $\partial \mathcal{X} \times \partial \mathcal{X}^*$ endowed with the Borel σ -algebra is a standard Borel space. The probability measure ν_{π} is disintegrated with respect to the σ -algebra generated by the projection $\partial \mathcal{X} \times \partial \mathcal{X}^* \to \partial \mathcal{X}^*$. This yields the system of conditional probability measures $\{\nu_{\pi}^{\eta}\}_{\eta\in\partial\mathcal{X}^*}$ such that

$$\nu_{\pi} = \int_{\partial \mathcal{X}^{\star}} \nu_{\pi}^{\eta} \, d\nu_{\mu^{\star}}(\eta).$$

Moreover, it satisfies that $\nu_{\pi}^{\eta}(\partial \mathcal{X} \times \partial \mathcal{X}^{\star}) = \nu_{\pi}^{\eta}(\partial \mathcal{X} \times \{\eta\}) = 1$ for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$. These disintegrations lead to by the Fubini theorem,

$$\mathbf{P}^{\eta} = \int_{\partial \mathcal{X} \times \{\eta\}} \mathbf{P}^{\xi,\eta} \, d\nu_{\pi}^{\eta}(\xi) \quad \text{for } \nu_{\mu^{\star}} \text{-almost every } \eta \in \partial \mathcal{X}^{\star}.$$
(2.8)

For $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$, the conditional probability measure \mathbf{P}^{η} coincides with the distribution of a conditional process on Γ . This is a Markov chain whose transition probability is defined for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$p^{\eta}(\boldsymbol{x},\boldsymbol{y}) := \pi(\boldsymbol{x}^{-1}\boldsymbol{y}) \frac{d\boldsymbol{y}\nu_{\mu^{\star}}}{d\boldsymbol{x}\nu_{\mu^{\star}}}(\eta) \quad \text{if } \pi(\boldsymbol{x}^{-1}\boldsymbol{y}) > 0, \text{ and } 0 \text{ if otherwise, for } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\Gamma}.$$

Note that $\boldsymbol{y}\nu_{\mu^{\star}}$ is absolutely continuous with respect to $\boldsymbol{x}\nu_{\mu^{\star}}$ and $d\boldsymbol{y}\nu_{\mu^{\star}}/d\boldsymbol{x}\nu_{\mu^{\star}}$ is welldefined $\nu_{\mu^{\star}}$ -almost everywhere if $\pi(\boldsymbol{x}^{-1}\boldsymbol{y}) > 0$ by (2.5). Moreover since

$$\frac{d\boldsymbol{y}\nu_{\mu^{\star}}}{d\boldsymbol{x}\nu_{\mu^{\star}}}(\eta) = \frac{d\boldsymbol{x}^{-1}\boldsymbol{y}\nu_{\mu^{\star}}}{d\nu_{\mu^{\star}}}(\boldsymbol{x}^{-1}\eta) \quad \text{for } \nu_{\mu^{\star}}\text{-almost every } \eta \in \partial \mathcal{X}^{\star}$$

the above $p^{\eta}(\boldsymbol{x}, \boldsymbol{y})$ indeed defines a transition probability by the π -stationarity of $\nu_{\mu^{\star}}$. Let $\pi_n^{\eta}(\boldsymbol{x}) := \mathbf{P}^{\eta}(\boldsymbol{w}_n = \boldsymbol{x})$ for $\boldsymbol{x} \in \boldsymbol{\Gamma}$ and $n \in \mathbb{Z}_+$. It holds that for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$, for every cylinder set $[\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n]$ in $(\Omega, \mathcal{F}, \mathbf{P})$,

$$\mathbf{P}^{\eta}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]) = \mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]) \frac{d\boldsymbol{w}_n \nu_{\mu^{\star}}}{d\nu_{\mu^{\star}}}(\eta).$$
(2.9)

In particular, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\pi_n^\eta(oldsymbol{x}) = \pi_n(oldsymbol{x}) rac{doldsymbol{x}
u_{\mu^\star}}{d
u_{\mu^\star}}(\eta) \quad ext{for } oldsymbol{x} \in oldsymbol{\Gamma} ext{ and } n \in \mathbb{Z}_+.$$

For more details, see [Kai00, Sections 3 and 4]. There it is shown (in a more general setting) that the Shannon theorem holds for the conditional process. Namely, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, the following limit exists and is constant \mathbf{P}^{η} -almost everywhere:

$$h(\mathbf{P}^{\eta}) := \lim_{n \to \infty} -\frac{1}{n} \log \pi_n^{\eta}(\boldsymbol{w}_n).$$
(2.10)

Furthermore, the limit is obtained as

$$h(\mathbf{P}^{\eta}) = h(\pi) - \sum_{\boldsymbol{x} \in \boldsymbol{\Gamma}} \pi(\boldsymbol{x}) \int_{\partial \mathcal{X}^{\star}} \log \frac{d\boldsymbol{x}\nu_{\mu^{\star}}}{d\nu_{\mu^{\star}}}(\eta) \, d\boldsymbol{x}\nu_{\mu^{\star}}(\eta).$$

See [Kai00, Theorem 4.5]. The differential entropy for the pair $(\partial \mathcal{X}^*, \mu^*)$ is defined by

$$h(\partial \mathcal{X}^{\star}, \mu^{\star}) := \sum_{x \in \operatorname{supp} \mu^{\star}} \mu^{\star}(x) \int_{\partial \mathcal{X}^{\star}} \log \frac{dx \nu_{\mu^{\star}}}{d\nu_{\mu^{\star}}}(\eta) \, dx \nu_{\mu^{\star}}(\eta).$$

Since μ^{\star} is a marginal of π , it holds that

 $h(\mathbf{P}^{\eta}) = h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star}) \quad \text{for } \nu_{\mu^{\star}}\text{-almost every } \eta \in \partial \mathcal{X}^{\star}.$

Let us mention that the differential entropy arises in the theory of Poisson boundary in the following way: It has been proven that $h(\partial \mathcal{X}^*, \mu^*) \leq h(\mu^*)$ and the equality holds if and only if $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is a Poisson boundary for the μ^* -random walk [Kai00, Theorem 4.6]. In the present setting, since $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is π -stationary (2.5), it holds that $h(\partial \mathcal{X}^*, \mu^*) =$ $h(\partial \mathcal{X}^*, \pi)$, and that $h(\partial \mathcal{X}^*, \pi) \leq h(\pi)$, where the equality holds if and only if $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is a Poisson boundary for (Γ, π) .

3. EXACT DIMENSION OF CONDITIONAL MEASURES

For a proper C-roughly geodesic hyperbolic metric space (\mathcal{X}, d) for some $C \in \mathbb{R}_+$ with a fixed base point $o \in \mathcal{X}$ and for a μ -random walk $\{w_n\}_{n\in\mathbb{Z}_+}$, the following **ray approximation** holds for $z_n = w_n \cdot o$: If μ is non-elementary and has a finite first moment, then **P**-almost surely there exists a C-rough geodesic ray $\gamma_{z_{\infty}}$ such that for $l := l(\mathcal{X}, \mu)$ of $\{z_n\}_{n\in\mathbb{Z}_+}$,

$$d(z_n, \gamma_{z_{\infty}}(ln)) = o(n). \tag{3.1}$$

See [Kai00, Theorem 7.3]. In fact, such an assignment $\xi \mapsto \gamma_{\xi}$ from $\partial \mathcal{X}$ to the space of *C*-rough geodesic rays from o in (\mathcal{X}, d) equipped with the topology of convergence on compact sets is chosen to be Borel measurable by the Borel selection theorem (cf. [Tan19, Section 3.2]). In the same way, there is a Borel measurable map $\eta \mapsto \gamma_{\eta}$ from $\partial \mathcal{X}^*$ to the space of *C*-rough geodesic rays from o^* in (\mathcal{X}^*, d^*) . For the drift $l^* := l(\mathcal{X}^*, \mu^*)$ of $\{z_n^*\}_{n \in \mathbb{Z}_+}$, it holds that **P**-almost surely,

$$d^{\star}(z_n^{\star}, \gamma_{z_{\infty}^{\star}}(l^{\star}n)) = o(n).$$

$$(3.2)$$

In the following subsections, for brevity, let

$$\overline{h} := h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star}), \quad l := l(\mathcal{X}, \mu) \quad \text{and} \quad l^{\star} := l(\mathcal{X}^{\star}, \mu^{\star}).$$

3.1. Upper bounds on dimensions of conditional measures.

Lemma 3.1. Let Γ be a countable subgroup in Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$, and π be a probability measure on Γ with finite first moment and non-elementary marginals μ and μ^* . It holds that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ and for ν_{π}^{η} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^*$,

$$\limsup_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} \leq \frac{h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star})}{l(\mathcal{X}, \mu)}.$$

Proof. Recall that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, the distribution \mathbf{P}^{η} is obtained by a Markov chain whose law at time n is π_n^{η} for $n \in \mathbb{Z}_+$. For every $\varepsilon > 0$ and every interval I in \mathbb{Z}_+ , let

$$A_{\varepsilon,I} := \bigcap_{n \in I} \Big\{ \boldsymbol{w} \in \Omega : (z_n | z_{n+1})_o \ge (l - \varepsilon)n, \ \pi_n^{\mathrm{bnd}^{\star}(\boldsymbol{w})}(\boldsymbol{w}_n) \ge \exp(-n(\overline{h} + \varepsilon)) \Big\}.$$

Note that $(z_n|z_{n+1})_o/n \to l$ as $n \to \infty$ almost surely in **P** since μ has a finite first moment and the μ -random walk has the drift l > 0. By disintegration (2.7), this together with (2.10) implies that for every $\varepsilon \in (0, l)$ and for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\mathbf{P}^{\eta}\Big(\bigcup_{N\in\mathbb{Z}_{+}}A_{\varepsilon,[N,\infty)}\Big)=1.$$

Hence for every $\varepsilon > 0$ and for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, there exists an $N_{\varepsilon,\eta} \in \mathbb{Z}_+$ such that

$$\mathbf{P}^{\eta}\left(A_{\varepsilon,[N_{\varepsilon,\eta},\infty)}\right) \geq 1 - \varepsilon.$$

Let $N := N_{\varepsilon,\eta}$ and $A := A_{\varepsilon,N_{\varepsilon,\eta}}$. Further for all n > N, let

$$A_{[N,n)} := A_{\varepsilon,[N,n)}$$
 and $A_{[n,\infty)} := A_{\varepsilon,[n,\infty)}$.

Note that $A = A_{[N,n)} \cap A_{[n,\infty)}$. For $n \in \mathbb{Z}_+$ and $\boldsymbol{w} \in \Omega$, let

$$C_n(\boldsymbol{w}) := \left\{ \boldsymbol{w}' \in \Omega : \boldsymbol{w}'_n = \boldsymbol{w}_n
ight\}$$

This is the event where the position of the chain is \boldsymbol{w}_n at time n. Since the conditional process is a Markov chain, for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$, for all $\boldsymbol{w} \in \Omega$ and all n > N,

$$\mathbf{P}^{\eta}(A \mid C_n(\boldsymbol{w})) = \mathbf{P}^{\eta}(A_{[N,n)} \mid C_n(\boldsymbol{w})) \cdot \mathbf{P}^{\eta}(A_{[n,\infty)} \mid C_n(\boldsymbol{w})).$$
(3.3)

Furthermore for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$\mathbf{P}^{\eta}(A_{[N,n)} \mid C_n(\boldsymbol{w})) = \mathbf{P}^{\eta}(A_{[N,n)} \mid \sigma(\boldsymbol{w}_n, \boldsymbol{w}_{n+1}, \dots)) \quad \text{almost everywhere in } \mathbf{P}^{\eta}, \quad (3.4)$$

where $\sigma(\boldsymbol{w}_n, \boldsymbol{w}_{n+1}, \dots)$ is the σ -algebra generated by $\boldsymbol{w}_n, \boldsymbol{w}_{n+1}, \dots$ Similarly, one has

$$\mathbf{P}^{\eta}(A_{[n,\infty)} \mid C_n(\boldsymbol{w})) = \mathbf{P}^{\eta}(A_{[n,\infty)} \mid \sigma(\boldsymbol{w}_0, \dots, \boldsymbol{w}_n)) \quad \text{almost everywhere in } \mathbf{P}^{\eta}, \quad (3.5)$$

where $\sigma(\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n)$ is the σ -algebra generated by $\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n$. Let us denote the tail σ -algebra by

$$\mathcal{T} := igcap_{n \in \mathbb{Z}_+} \sigma(oldsymbol{w}_n, oldsymbol{w}_{n+1}, \dots).$$

Note that \mathbf{P}^{η} -almost everywhere on $A = A_{[N,n)} \cap A_{[n,\infty)}$,

$$\mathbf{P}^{\eta}(A_{[N,n)} \mid \sigma(\boldsymbol{w}_n, \boldsymbol{w}_{n+1}, \dots)) = \mathbf{P}^{\eta}(A \mid \sigma(\boldsymbol{w}_n, \boldsymbol{w}_{n+1}, \dots)).$$

By (3.4), the bounded martingale convergence theorem shows that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\lim_{n \to \infty} \mathbf{P}^{\eta}(A_{[N,n)} \mid C_n(\boldsymbol{w})) = \mathbf{P}^{\eta}(A \mid \mathcal{T}) \quad \text{almost everywhere in } \mathbf{P}^{\eta} \text{ on } A.$$
(3.6)

Analogously, note that \mathbf{P}^{η} -almost everywhere on $A = A_{[N,n)} \cap A_{[n,\infty)}$,

$$\mathbf{P}^{\eta}(A_{[n,\infty)} \mid \sigma(\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n)) = \mathbf{P}^{\eta}(A \mid \sigma(\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n)).$$

By (3.5), the bounded martingale convergence theorem shows that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\lim_{n \to \infty} \mathbf{P}^{\eta}(A_{[n,\infty)} \mid C_n(\boldsymbol{w})) = \mathbf{1}_A \quad \text{almost everywhere in } \mathbf{P}^{\eta} \text{ on } A.$$
(3.7)

For $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$, it holds that

$$\mathbf{P}^{\eta}(A \mid \mathcal{T}) > 0$$
 almost everywhere in \mathbf{P}^{η} on A . (3.8)

Indeed, if we define $\mathcal{N} := \{ \boldsymbol{w} \in \Omega : \mathbf{P}^{\eta}(A \mid \mathcal{T}) = 0 \}$, then \mathcal{N} is \mathcal{T} -measurable and $\mathbf{P}^{\eta}(A \cap \mathcal{N} \mid \mathcal{T}) = 0$, implying that $\mathbf{P}^{\eta}(A \cap \mathcal{N}) = 0$. Thus we have (3.8). Therefore by (3.3), (3.6), (3.7) and (3.8), for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$\lim_{n \to \infty} \mathbf{P}^{\eta}(A \mid C_n(\boldsymbol{w})) = \mathbf{P}^{\eta}(A \mid \mathcal{T}) > 0 \quad \text{almost everywhere in } \mathbf{P}^{\eta} \text{ on } A.$$
(3.9)

For every $\boldsymbol{w} \in A$, it holds that

$$q(z_n, z_\infty) = \exp(-(z_n | z_\infty)_o) \le C e^{-(l-\varepsilon)n}$$
 for all $n > N$,

where C is independent of \boldsymbol{w} . This follows since q with a power is bi-Lipschitz to a genuine metric (cf. Section 2.1). Further for n > N and for \mathbf{P}^{η} -almost every $\boldsymbol{w}' \in A \cap C_n(\boldsymbol{w})$,

$$\boldsymbol{w}_n' = \boldsymbol{w}_n \quad \text{and} \quad q(z_n', z_\infty') \leq C e^{-(l-\varepsilon)n}$$

where $z'_n := w'_n \cdot o$ for $\boldsymbol{w}_n = (w'_n, w^{\star'}_n)$ and $z'_n \to z'_\infty$ as $n \to \infty$ in $\mathcal{X} \cup \partial \mathcal{X}$. Since $\boldsymbol{w}'_n = \boldsymbol{w}_n$ and thus $z'_n = z_n$, by (2.1) extended on $\mathcal{X} \cup \partial \mathcal{X}$,

$$q(z_{\infty}, z'_{\infty}) \le Ce^{\delta - (l-\varepsilon)n}$$
, i.e., $z'_{\infty} \in B(z_{\infty}, Ce^{\delta - (l-\varepsilon)n})$ for all $n > N$.

Hence for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\mathbf{P}^{\eta}(A \cap C_n(\boldsymbol{w})) \le \nu_{\pi}^{\eta}(B(z_{\infty}, Ce^{\delta - (l-\varepsilon)n}) \times \partial \mathcal{X}^{\star}) \quad \text{for all } n > N.$$

The right hand side coincides with $\nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{z}_{\infty}, Ce^{\delta-(l-\varepsilon)n}))$ where $\boldsymbol{z}_{\infty} = (z_{\infty}, z_{\infty}^{\star})$ since ν_{π}^{η} is supported in $\partial \mathcal{X} \times \{\eta\}$ and $\eta = z_{\infty}^{\star}$ for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$. For \mathbf{P}^{η} -almost every $\boldsymbol{w} \in A$, it holds that for all n > N,

$$\mathbf{P}^{\eta}(A \cap C_n(\boldsymbol{w})) = \mathbf{P}^{\eta}(A \mid C_n(\boldsymbol{w})) \cdot \mathbf{P}^{\eta}(C_n(\boldsymbol{w})) \ge \mathbf{P}^{\eta}(A \mid C_n(\boldsymbol{w})) \cdot e^{-n(h+\varepsilon)}.$$

This shows that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for \mathbf{P}^{η} -almost every $\boldsymbol{w} \in A$ and for all n > N,

$$\nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{z}_{\infty}, Ce^{\delta - (l-\varepsilon)n})) \geq \mathbf{P}^{\eta}(A \mid C_{n}(\boldsymbol{w})) \cdot e^{-n(\overline{h}+\varepsilon)}$$

By (3.9), for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$ and for \mathbf{P}^{η} -almost every $\boldsymbol{w} \in A$,

$$\limsup_{n \to \infty} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{z}_{\infty}, r))}{\log r} \le \frac{\overline{h} + \varepsilon}{l - \varepsilon}.$$
(3.10)

This follows first for the sequence $r_n \to 0$ as $n \to \infty$ where $r_n := Ce^{\delta - (l-\varepsilon)n}$ and then for r > 0 and $r \to 0$ by noting that $r_{n+1} = e^{-(l-\varepsilon)}r_n$. Recall that $\mathbf{P}^{\eta}(A) \ge 1 - \varepsilon$ for the event $A = A_{\varepsilon,[N_{\varepsilon,\eta,\infty})}$ for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$. For an arbitrary decreasing sequence $\varepsilon_n \to 0$ as $n \to \infty$, one has $\mathbf{P}^{\eta}(\bigcap_{m \in \mathbb{Z}_+} \bigcup_{n \ge m} A_{\varepsilon_n, [N_{\varepsilon_n, \eta, \infty})}) = 1$ for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$. Therefore by (3.10) for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\limsup_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{z}_{\infty}, r))}{\log r} \leq \frac{\overline{h}}{l} \quad \text{almost everywhere in } \mathbf{P}^{\eta}.$$

Noting that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, the distribution of \boldsymbol{z}_{∞} is ν_{π}^{η} , we obtain for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\limsup_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} \leq \frac{\overline{h}}{l} \quad \text{for } \nu_{\pi}^{\eta} \text{-almost every } \boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$$

This concludes the claim.

We use the following version of Lemma 3.1 in Section 4.

Lemma 3.2. In the same setting as in Lemma 3.1, if for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ there exists a Borel set F_η in $\partial \mathcal{X} \times \partial \mathcal{X}^*$ such that $\nu_{\pi}^{\eta}(F_\eta) > 0$, then for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ and for ν_{π}^{η} -almost every $\boldsymbol{\xi} \in F_\eta$,

$$\limsup_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r) \cap F_{\eta})}{\log r} \le \frac{h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star})}{l(\mathcal{X}, \mu)}$$

Proof. This follows from Lemmas 2.2 and 3.1.

3.2. Lower bounds on dimensions of conditional measures.

Lemma 3.3. Let Γ be a countable subgroup in Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$ with finite exponential growth relative to $(\mathcal{X} \times \mathcal{X}^*, \mathbf{d})$, and π be a probability measure on Γ with finite first moment and non-elementary marginals μ and μ^* . For every $\varepsilon > 0$, there exist

- (1) an $N \in \mathbb{Z}_+$,
- (2) a Borel set D in $\partial \mathcal{X}^*$ with $\nu_{\mu^*}(D) \geq 1 \varepsilon$, and
- (3) a Borel set F in $\partial \mathcal{X} \times \partial \mathcal{X}^*$ with $\nu_{\pi}^{\eta}(F) \geq 1 \varepsilon$ for ν_{μ^*} -almost every $\eta \in D$ and $\nu_{\pi}(F) \geq 1 \varepsilon$,

such that the following holds: For ν_{μ^*} -almost every $\eta \in D$, for all $\xi \in \partial \mathcal{X}$ and all $n \geq N$,

$$\nu_{\pi}^{\eta}\left(B(\xi, e^{-ln}) \times \partial \mathcal{X}^{\star} \cap F\right) \leq C_{\varepsilon} e^{-n(\overline{h} - \varepsilon)},$$

where C_{ε} is a constant depending only on ε .

Proof. For every $\varepsilon > 0$ and every $N \in \mathbb{Z}_+$, let

$$A_{\varepsilon,N} := \bigcap_{n \ge N} \Big\{ \boldsymbol{w} \in \Omega : \boldsymbol{d}(\boldsymbol{z}_n, (\gamma_{\boldsymbol{z}_{\infty}}(ln), \gamma_{\boldsymbol{z}_{\infty}^{\star}}(l^{\star}n))) \le \varepsilon n, \ \pi_n^{\mathrm{bnd}^{\star}(\boldsymbol{w})}(\boldsymbol{w}_n) \le \exp(-n(\overline{h} - \varepsilon)) \Big\}.$$

The disintegration formula (2.7) implies that for all event $A \subset \Omega$ if $\mathbf{P}(A) = 1$, then $\mathbf{P}^{\eta}(A) = 1$ for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$. This together with (3.1) and (3.2), and further (2.10) imply that for every $\varepsilon > 0$,

$$\mathbf{P}^{\eta} \Big(\bigcup_{N \in \mathbb{Z}_{+}} A_{\varepsilon,N}\Big) = 1 \quad \text{for } \nu_{\mu^{\star}}\text{-almost every } \eta \in \partial \mathcal{X}^{\star}$$

For $N \in \mathbb{Z}_+$, let

$$D_{\varepsilon,N} := \Big\{ \eta \in \partial \mathcal{X}^{\star} : \mathbf{P}^{\eta}(A_{\varepsilon,N}) \ge 1 - \varepsilon \Big\}.$$

Since $A_{\varepsilon,N}$ is increasing and $\mathbf{P}^{\eta}(A_{\varepsilon,N}) \to 1$ monotonically as $N \to \infty$ for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$, there exists an $N_{\varepsilon} \in \mathbb{Z}_{+}$ such that

$$\nu_{\mu^{\star}}\left(D_{\varepsilon,N_{\varepsilon}}\right) \ge 1 - \varepsilon. \tag{3.11}$$

Let $N := N_{\varepsilon}, D := D_{\varepsilon,N_{\varepsilon}}$ and $A := A_{\varepsilon,N_{\varepsilon}}$. It holds that

$$\mathbf{P}^{\eta}(A) \ge 1 - \varepsilon \quad \text{for } \nu_{\mu^{\star}} \text{-almost every } \eta \in D.$$
 (3.12)

Let us define

$$F_{\varepsilon} := \Big\{ (\xi, \eta) \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star} : \mathbf{P}^{\xi, \eta}(A) \ge \varepsilon \Big\},\$$

and $F := F_{\varepsilon}$. We claim that

$$\nu_{\pi}^{\eta}(F) \ge 1 - 2\varepsilon \quad \text{for } \nu_{\mu^{\star}} \text{-almost every } \eta \in D.$$
 (3.13)

Indeed, by (3.12) and by (2.8), for ν_{μ^*} -almost every $\eta \in D$,

$$1 - \varepsilon \leq \mathbf{P}^{\eta}(A) = \int_{\partial \mathcal{X} \times \{\eta\}} \mathbf{P}^{\xi,\eta}(A) \, d\nu_{\pi}^{\eta}(\xi) = \int_{F} \mathbf{P}^{\xi,\eta}(A) \, d\nu_{\pi}^{\eta}(\xi) + \int_{\partial \mathcal{X} \times \partial \mathcal{X}^{\star} \setminus F} \mathbf{P}^{\xi,\eta}(A) \, d\nu_{\pi}^{\eta}(\xi).$$

This implies that

$$1 - \varepsilon \le \nu_{\pi}^{\eta}(F) + \varepsilon \cdot \nu_{\pi}^{\eta}(\partial \mathcal{X} \times \partial \mathcal{X}^{\star} \setminus F) \le \nu_{\pi}^{\eta}(F) + \varepsilon,$$

showing (3.13).

Furthermore it holds that

$$\nu_{\pi}(F) \ge 1 - 3\varepsilon. \tag{3.14}$$

This follows since $\nu_{\pi}^{\eta}(F) \geq (1-2\varepsilon)\mathbf{1}_D$ by (3.13), integration with respect to $\nu_{\mu^{\star}}$ yields

$$\nu_{\pi}(F) = \int_{\partial \mathcal{X}^{\star}} \nu_{\pi}^{\eta}(F) \, d\nu_{\mu^{\star}}(\eta) \ge (1 - 2\varepsilon)\nu_{\mu^{\star}}(D) \ge (1 - 2\varepsilon)(1 - \varepsilon) \ge 1 - 3\varepsilon,$$

where the second inequality uses (3.11). It holds that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for every $\xi \in \partial \mathcal{X}$ and R > 0,

$$\nu_{\pi}^{\eta}(\mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F) = \mathbf{P}^{\eta}(\{\boldsymbol{z}_{\infty} \in \mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F\})$$
$$= \mathbf{P}^{\eta}(\{\boldsymbol{z}_{\infty} \in \mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F\} \cap A)$$
$$+ \mathbf{P}^{\eta}(\{\boldsymbol{z}_{\infty} \in \mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F\} \cap A^{\mathsf{c}}).$$
(3.15)

In the above, A^{c} denote the complement event of A, and $\mathbf{z}_{\infty} = (z_{\infty}, z_{\infty}^{\star})$. First let us bound the first term in (3.15). On the event A, if $z_{\infty} \in \mathcal{O}(\gamma_{\xi}(ln), R)$ and $n \geq N$, then

$$d(z_n, \gamma_{\xi}(ln)) \le d(z_n, \gamma_{z_{\infty}}(ln)) + d(\gamma_{z_{\infty}}(ln), \gamma_{\xi}(ln)) \le \varepsilon n + C_R,$$

where $C_R := 2R + 2C$ since γ_{ξ} is a *C*-rough geodesic ray. Moreover, since ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ it holds that $z_{\infty}^* = \eta$ almost surely in \mathbf{P}^{η} , it holds that \mathbf{P}^{η} -almost everywhere on A for all $n \geq N$,

$$d^{\star}(z_n^{\star}, \gamma_{\eta}(l^{\star}n)) \le \varepsilon n.$$

Hence \mathbf{P}^{η} -almost everywhere on A for all $n \geq N$,

$$\boldsymbol{d}(\boldsymbol{z}_n, \gamma_{\xi,\eta,n}) \leq \varepsilon n + C_R \quad \text{where } \gamma_{\xi,\eta,n} := (\gamma_{\xi}(ln), \gamma_{\eta}(l^*n)).$$

This shows that letting $\boldsymbol{B}(\boldsymbol{x},r)$ denote the ball in $(\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}}^{\star}, \boldsymbol{d})$, we have for all $n \geq N$,

$$\mathbf{P}^{\eta}(\{\boldsymbol{z}_{\infty} \in \mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F\} \cap A) \\ \leq \mathbf{P}^{\eta}(\{\boldsymbol{z}_{n} \in \boldsymbol{B}(\gamma_{\xi,\eta,n}, \varepsilon n + C_{R})\} \cap \{\pi_{n}^{\eta}(\boldsymbol{w}_{n}) \leq \exp(-n(\overline{h} - \varepsilon))\}).$$

The right hand side is at most $\sum \pi_n^{\eta}(\boldsymbol{x})$ where the summation runs over all $\boldsymbol{x} \in \boldsymbol{\Gamma}$ such that $\boldsymbol{x} \cdot \boldsymbol{o} \in \boldsymbol{B}(\gamma_{\xi,\eta,n}, \varepsilon n + C_R)$ and $\pi_n^{\eta}(\boldsymbol{x}) \leq \exp(-n(\overline{h} - \varepsilon))$. This is at most

$$\# \{ \boldsymbol{x} \in \boldsymbol{\Gamma} : \boldsymbol{x} \cdot \boldsymbol{o} \in \boldsymbol{B}(\gamma_{\xi,\eta,n}, \varepsilon n + C_R) \} \cdot e^{-n(\overline{h} - \varepsilon)} \leq c e^{c(\varepsilon n + C_R)} \cdot e^{-n(\overline{h} - \varepsilon)},$$

for a constant c > 0 since Γ has a finite exponential growth relative to $(\mathcal{X} \times \mathcal{X}^*, \mathbf{d})$. Thus for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ and for every $\xi \in \partial \mathcal{X}$, for all $n \ge N$,

$$\mathbf{P}^{\eta}(\{\boldsymbol{z}_{\infty} \in \mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F\} \cap A) \leq c e^{c(\varepsilon n + C_R)} \cdot e^{-n(\overline{h} - \varepsilon)}.$$
(3.16)

Next let us bound the second term in (3.15). By (2.8), it holds that for ν_{μ^*} -almost every $\eta \in D$,

$$\mathbf{P}^{\eta}(\{\boldsymbol{z}_{\infty} \in \mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F\} \cap A^{\mathsf{c}}) = \int_{\mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F} \mathbf{P}^{\zeta, \eta}(A^{\mathsf{c}}) \, d\nu_{\pi}^{\eta}(\zeta) \leq (1 - \varepsilon) \cdot \nu_{\pi}^{\eta}(\mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F).$$
(3.17)

In the above, the inequality holds since $\mathbf{P}^{\zeta,\eta}(A^{\mathsf{c}}) \leq 1 - \varepsilon$ for ν_{π}^{η} -almost every $(\zeta,\eta) \in F$ for $\nu_{\mu^{\star}}$ -almost every $\eta \in D$ by the definition of F.

Finally, combining (3.15), (3.16) and (3.17) yields for ν_{μ^*} -almost every $\eta \in D$, for every $\xi \in \partial \mathcal{X}$ and for all $n \geq N$,

$$\nu_{\pi}^{\eta}(\mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F) \leq ce^{c(\varepsilon n + C_R)} \cdot e^{-n(\overline{h} - \varepsilon)} + (1 - \varepsilon) \cdot \nu_{\pi}^{\eta}(\mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F).$$

Therefore for ν_{μ^*} -almost every $\eta \in D$ and for every $\xi \in \partial \mathcal{X}$, for all $n \geq N$,

$$\varepsilon \cdot \nu_{\pi}^{\eta} \left(\mathcal{O}(\gamma_{\xi}(ln), R) \times \partial \mathcal{X}^{\star} \cap F \right) \leq c e^{c(\varepsilon n + C_R)} \cdot e^{-n(h-\varepsilon)}.$$
 (3.18)

Note that $B(\xi, C^{-1}e^{-ln+R}) \cap \partial \mathcal{X} \subset \mathcal{O}(\gamma_{\xi}(ln), R)$ by (2.3), where we choose a large enough constant R so that $e^R/C \geq 1$ and C depends only on the hyperbolicity constant. In (3.18), the constant c depends only on Γ and $(\mathcal{X} \times \mathcal{X}^*, \mathbf{d})$. For every $\varepsilon > 0$, we argue with $\varepsilon' = \varepsilon/(3+c)$. By (3.11), (3.13) and (3.14), we obtain

$$\nu_{\mu^{\star}}(D) \ge 1 - \varepsilon' \ge 1 - \varepsilon, \quad \nu_{\pi}^{\eta}(F) \ge 1 - 2\varepsilon' \ge 1 - \varepsilon \quad \text{for } \nu_{\mu^{\star}}\text{-almost every } \eta \in D,$$

and $\nu_{\pi}(F) \geq 1 - 3\varepsilon' \geq 1 - \varepsilon$. Further by (3.18), for $\nu_{\mu^{\star}}$ -almost every $\eta \in D$, for every $\xi \in \partial \mathcal{X}$ and for all $n \geq N$,

$$\nu_{\pi}^{\eta} \left(B(\xi, e^{-ln}) \cap F \right) \le (c/\varepsilon') e^{cC_R} \cdot e^{-n(\overline{h}-\varepsilon)}$$

Defining the constant $C_{\varepsilon} := (c/\varepsilon')e^{cC_R}$ yields the claim.

Lemma 3.4. In the same setting as in Lemma 3.3, it holds that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ and for ν_{π}^{η} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^*$,

$$\liminf_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} \geq \frac{h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star})}{l(\mathcal{X}, \mu)}.$$

Proof. By Lemma 3.3, for every $\varepsilon > 0$ there exist a Borel set D in $\partial \mathcal{X}^*$ with $\nu_{\mu^*}(D) \ge 1 - \varepsilon$ and a Borel set F in $\partial \mathcal{X} \times \partial \mathcal{X}^*$ with $\nu_{\pi}(F) \ge 1 - \varepsilon$ such that the following holds: For ν_{μ^*} -almost every $\eta \in D$ and for every $\xi \in \partial \mathcal{X}$,

$$\liminf_{n \to \infty} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}((\xi, \eta), e^{-ln}) \cap F)}{-ln} \geq \frac{\overline{h} - \varepsilon}{l}$$

In fact, in the above the sequence $r_n := e^{-ln}$ for $n \in \mathbb{Z}_+$ is replaced by positive reals r tending to 0 since $r_{n+1} = e^{-l}r_n$ for all $n \in \mathbb{Z}_+$. Applying to Lemma 2.2 the measures ν_{π}^{η} and F implies the following: There exists a constant $L \ge 1$ such that for ν_{μ^*} -almost every $\eta \in D$, for ν_{π}^{η} -almost every $(\xi, \eta) \in F$ and for a constant $r(\xi, \eta) > 0$,

$$\nu_{\pi}^{\eta}(\boldsymbol{B}((\xi,\eta),Lr)\cap F) \geq \frac{9}{10}\nu_{\pi}^{\eta}(\boldsymbol{B}((\xi,\eta),r)) \quad \text{for all } r \in (0,r(\xi,\eta)).$$

Hence for ν_{μ^*} -almost every $\eta \in D$ and for ν_{π}^{η} -almost every $\boldsymbol{\xi} = (\xi, \eta) \in F$,

$$\liminf_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} \geq \frac{\overline{h} - \varepsilon}{l}$$

Since for every $\varepsilon > 0$ there exists such an F denoted by F_{ε} with $\nu_{\pi}^{\eta}(F_{\varepsilon}) \ge 1 - \varepsilon$, for $\nu_{\mu^{\star}}$ almost every $\eta \in D$, it holds that $\nu_{\pi}^{\eta}(\bigcap_{m \in \mathbb{Z}_{+}} \bigcup_{n \ge m} F_{\varepsilon_{n}}) = 1$ for an arbitrary decreasing sequence $\varepsilon_{n} \to 0$ as $n \to \infty$. Therefore it follows that for $\nu_{\mu^{\star}}$ -almost every $\eta \in D$ and for ν_{π}^{η} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$,

$$\liminf_{r \to 0} \frac{\log \nu_{\pi}^{\eta}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} \geq \frac{\overline{h}}{l}.$$

For every $\varepsilon > 0$ there exists such a D with $\nu_{\mu^*}(D) \ge 1 - \varepsilon$, the above holds for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$. This concludes the claim.

3.3. Proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.3. Since μ and μ^* are non-elementary and have finite first moments, the drift $l(\mathcal{X}, \mu)$ is finite and positive, further the asymptotic entropy $h(\pi)$ and the differential entropy $h(\partial \mathcal{X}^*, \mu^*)$ are finite. The first claim follows from Lemmas 3.1 and 3.4. The second claim follows from Lemma 2.1.

Proof of Theorem 1.2. Since a probability measure μ^* on Γ^* is non-elementary with finite first moment, it holds that $h(\mu^*) = h(\partial\Gamma^*, \mu^*)$ since $(\partial\Gamma^*, \nu_{\mu^*})$ is a Poisson boundary for (Γ^*, μ^*) [Kai00, Theorem 7.4]. The claim follows from Theorem 1.3 by applying Γ and Γ^* endowed with left invariant hyperbolic metrics quasi-isometric to word metrics respectively to (\mathcal{X}, d) and (\mathcal{X}^*, d^*) .

4. EXACT DIMENSION OF HARMONIC MEASURES IN PRODUCTS SPACES

As in Section 3, for brevity, let

$$l := l(\mathcal{X}, \mu), \quad l^{\star} := l(\mathcal{X}^{\star}, \mu^{\star}), \quad h^{\star} := h(\partial \mathcal{X}^{\star}, \mu^{\star}) \quad \text{and} \quad \overline{h} := h(\pi) - h^{\star}.$$

4.1. Upper bounds on dimensions of harmonic measures in product spaces. The proof of the following proposition is inspired by [LL23, Section 8].

Proposition 4.1. Let Γ be a countable subgroup in Isom $\mathcal{X} \times \text{Isom } \mathcal{X}^*$, and π be a probability measure on Γ with finite first moment and non-elementary marginals μ and μ^* . If $l(\mathcal{X}, \mu) \geq l(\mathcal{X}^*, \mu^*)$, then it holds that for ν_{π} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^*$,

$$\limsup_{r \to 0} \frac{\log \nu_{\pi} \left(\boldsymbol{B}(\boldsymbol{\xi}, r) \right)}{\log r} \leq \frac{h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star})}{l(\mathcal{X}, \mu)} + \frac{h(\partial \mathcal{X}^{\star}, \mu^{\star})}{l(\mathcal{X}^{\star}, \mu^{\star})}$$

We assume that $l \ge l^*$: if otherwise we argue after exchanging the notations l and l^* . Fix an arbitrary $\varepsilon \in (0, l^*)$. Let

$$r_n := e^{-(l^\star - \varepsilon)n} \quad \text{for } n \in \mathbb{Z}_+.$$

Let us define (recalling that $\boldsymbol{z}_t = \boldsymbol{w}_t \cdot \boldsymbol{o}$)

$$A_{\varepsilon,n} := \bigcap_{t \ge n} \Big\{ \boldsymbol{w} \in \Omega : \boldsymbol{z}_{\infty} = (z_{\infty}, z_{\infty}^{\star}) \text{ exists and } \boldsymbol{q}(\boldsymbol{z}_t, \boldsymbol{z}_{\infty}) \le r_n \Big\}.$$

Lemma 4.2. The events $A_{\varepsilon,n}$ are increasing in $n \in \mathbb{Z}_+$, and it holds that

$$\mathbf{P}\Big(\bigcup_{n\in\mathbb{Z}_+}A_{\varepsilon,n}\Big)=1.$$

Proof. By definition $A_{\varepsilon,n}$ are increasing in $n \in \mathbb{Z}_+$. For **P**-almost every $\boldsymbol{w} \in \Omega$, for all large enough $t \in \mathbb{Z}_+$ (recalling that $z_t = w_t \cdot o$ and $z_t^* = w_t^* \cdot o^*$),

$$(z_t|z_{\infty})_o \ge (l-\varepsilon)t$$
 and $(z_t^{\star}|z_{\infty}^{\star})_{o^{\star}} \ge (l^{\star}-\varepsilon)t.$

In which case, $\max\{e^{-(z_t|z_{\infty})_o}, e^{-(z_t^*|z_{\infty}^*)_{o^*}}\} \le e^{-(l^*-\varepsilon)t}$ since $l \ge l^*$, showing the claim. \Box

For each $\eta \in \partial \mathcal{X}^*$ and $n \in \mathbb{Z}_+$, let

$$E_{\varepsilon,n}(\eta) := \Big\{ \boldsymbol{w} \in \Omega : \mathbf{P}^{\eta}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n] \cap A_{\varepsilon,n}) \le e^{n(h^{\star} + \varepsilon)} \mathbf{P}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n] \cap A_{\varepsilon,n}) \Big\},$$

and $E_{\varepsilon,[n,\infty)}(\eta) := \bigcap_{t \ge n} E_{\varepsilon,t}(\eta).$

Lemma 4.3. For each $\eta \in \partial \mathcal{X}^*$, the events $E_{\varepsilon,[n,\infty)}(\eta)$ are increasing in $n \in \mathbb{Z}_+$, and for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\mathbf{P}^{\eta} \Big(\bigcup_{n \in \mathbb{Z}_+} E_{\varepsilon, [n, \infty)}(\eta) \Big) = 1.$$

Proof. In the following, let $A_n := A_{\varepsilon,n}$ for $n \in \mathbb{Z}_+$. By (2.9), for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for every cylinder set $[\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n]$ in $(\Omega, \mathcal{F}, \mathbf{P})$,

$$\mathbf{P}^{\eta}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]) = \mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]) \frac{d\boldsymbol{w}_n \nu_{\mu^\star}}{d\nu_{\mu^\star}}(\eta)$$

Note that for **P**-almost every $\boldsymbol{w} \in \Omega$, for all $n \in \mathbb{Z}_+$,

$$\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n])>0 \quad ext{and} \quad \mathbf{P}^{ ext{bnd}^\star(\boldsymbol{w})}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n])>0.$$

The Birkhoff ergodic theorem implies that **P**-almost every $\boldsymbol{w} \in \Omega$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{d\boldsymbol{w}_n \nu_{\mu^\star}}{d\nu_{\mu^\star}} (\text{bnd}^\star(\boldsymbol{w})) = h^\star.$$

See [Kai00, the proof of Theorem 4.5]. This implies that by disintegration of \mathbf{P} into \mathbf{P}^{η} for $\eta \in \partial \mathcal{X}^{\star}$, for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$, for \mathbf{P}^{η} -almost every $\boldsymbol{w} \in \Omega$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{d\boldsymbol{w}_n \nu_{\mu^\star}}{d\nu_{\mu^\star}}(\eta) = h^\star$$

Therefore for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for \mathbf{P}^{η} -almost every $\boldsymbol{w} \in \Omega$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\mathbf{P}^{\eta}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n])}{\mathbf{P}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n])} = h^{\star}.$$
(4.1)

Fix an arbitrary $N \in \mathbb{Z}_+$. For all $n \in [N, \infty) \cap \mathbb{Z}_+$, for all cylinder set $[\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n]$ in $(\Omega, \mathcal{F}, \mathbf{P})$ of positive **P**-measure, it holds that

$$\frac{\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]\cap A_N)}{\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n])} \leq \frac{\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]\cap A_n)}{\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n])} \leq 1.$$

The left most side equals $\mathbf{P}(A_N \mid \sigma(\boldsymbol{w}_0, \dots, \boldsymbol{w}_n))$ almost everywhere in **P**. The martingale convergence theorem yields

$$\lim_{n\to\infty} \mathbf{P}(A_N \mid \sigma(\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n)) = \mathbf{1}_{A_N} \quad \text{for } \mathbf{P}\text{-almost every } \boldsymbol{w} \in \Omega.$$

Since N is arbitrary and $\mathbf{P}(\bigcup_{N \in \mathbb{Z}_+} A_N) = 1$, it holds that

$$\lim_{n \to \infty} \frac{\mathbf{P}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n] \cap A_n)}{\mathbf{P}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n])} = \mathbf{1} \quad \text{for } \mathbf{P}\text{-almost every } \boldsymbol{w} \in \Omega$$

By disintegration of **P** into \mathbf{P}^{η} for $\eta \in \partial \mathcal{X}^{\star}$, for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$\lim_{n \to \infty} \frac{\mathbf{P}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n] \cap A_n)}{\mathbf{P}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n])} = \mathbf{1} \quad \text{for } \mathbf{P}^{\eta}\text{-almost every } \boldsymbol{w} \in \Omega.$$
(4.2)

Applying the same discussion to \mathbf{P}^{η} as for \mathbf{P} , we obtain for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$\lim_{n \to \infty} \frac{\mathbf{P}^{\eta}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n] \cap A_n)}{\mathbf{P}^{\eta}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n])} = \mathbf{1} \quad \text{for } \mathbf{P}^{\eta}\text{-almost every } \boldsymbol{w} \in \Omega.$$
(4.3)

Combining (4.1), (4.2) and (4.3) yields for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\mathbf{P}^{\eta}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n] \cap A_n)}{\mathbf{P}([\boldsymbol{w}_0, \dots, \boldsymbol{w}_n] \cap A_n)} = h^{\star} \quad \text{for } \mathbf{P}^{\eta}\text{-almost every } \boldsymbol{w} \in \Omega.$$
(4.4)

By definition for each $\eta \in \partial \mathcal{X}^*$ the events $E_{\varepsilon,[n,\infty)}(\eta)$ are increasing in $n \in \mathbb{Z}_+$ respectively, and by (4.4) for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\mathbf{P}^{\eta}\Big(\bigcup_{n\in\mathbb{Z}_{+}}E_{\varepsilon,[n,\infty)}(\eta)\Big)=1,$$

as claimed.

Proof of Proposition 4.1. Fix an arbitrary $\varepsilon \in (0, l^*)$ and recall that $r_n := e^{-(l^*-\varepsilon)n}$ for $n \in \mathbb{Z}_+$. Let A_n , $E_n(\eta)$ and $E_{[n,\infty)}(\eta)$ denote $A_{\varepsilon,n}$, $E_{\varepsilon,n}(\eta)$ and $E_{\varepsilon,[n,\infty)}(\eta)$ respectively for brevity. Lemma 4.2 implies that by disintegration of **P** into \mathbf{P}^{η} for $\eta \in \partial \mathcal{X}^*$, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$,

$$\mathbf{P}^{\eta}\Big(\bigcup_{n\in\mathbb{Z}_+}A_n\Big)=1.$$

By this together with Lemma 4.3, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, there exists an $N_{\varepsilon,\eta} \in \mathbb{Z}_+$ such that

$$\mathbf{P}^{\eta}\left(E_{[N_{\varepsilon,\eta},\infty)}(\eta)\cap A_{N_{\varepsilon,\eta}}\right)\geq 1-\varepsilon.$$
(4.5)

Let $N := N_{\varepsilon,\eta}, E_{[N,\infty)}(\eta) := E_{[N_{\varepsilon,\eta},\infty)}(\eta)$ and $A_N := A_{N_{\varepsilon,\eta}}$. Further let

$$F_{\eta} := \Big\{ \boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star} : \mathbf{P}^{\boldsymbol{\xi}} \big(E_{[N,\infty)}(\eta) \cap A_N \big) \ge \varepsilon \Big\}.$$

Note that F_{η} is a Borel measurable set in $\partial \mathcal{X} \times \partial \mathcal{X}^{\star}$ since for each $B \in \mathcal{F}$ in $(\Omega, \mathcal{F}, \mathbf{P})$, the map $\boldsymbol{\xi} \mapsto \mathbf{P}^{\boldsymbol{\xi}}(B)$ is Borel measurable. By (4.5) and by disintegration of \mathbf{P}^{η} into $\mathbf{P}^{\boldsymbol{\xi}}$ for $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$, for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$1 - \varepsilon \leq \mathbf{P}^{\eta} \left(E_{[N,\infty)}(\eta) \cap A_N \right) = \int_{\partial \mathcal{X} \times \partial \mathcal{X}^{\star}} \mathbf{P}^{\boldsymbol{\xi}} \left(E_{[N,\infty)}(\eta) \cap A_N \right) \, d\nu_{\pi}^{\eta}(\boldsymbol{\xi}) \leq \nu_{\pi}^{\eta}(F_{\eta}) + \varepsilon \nu_{\pi}^{\eta}(F_{\eta}^{\mathsf{c}}).$$

Therefore for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$,

$$\nu_{\pi}^{\eta}(F_{\eta}) \ge 1 - 2\varepsilon. \tag{4.6}$$

Furthermore, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$ and for every $\boldsymbol{\xi}_0 \in \partial \mathcal{X} \times \partial \mathcal{X}^*$,

$$\nu_{\pi}^{\eta} \left(\boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta} \right) = \mathbf{P}^{\eta} \left(\left\{ \boldsymbol{z}_{\infty} \in \boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta} \right\} \cap E_{[N,\infty)}(\eta) \cap A_{N} \right) \\ + \mathbf{P}^{\eta} \left(\left\{ \boldsymbol{z}_{\infty} \in \boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta} \right\} \cap \left(E_{[N,\infty)}(\eta) \cap A_{N} \right)^{\mathsf{c}} \right) \right\}$$

By definition of F_{η} , it holds that

$$\mathbf{P}^{\eta}\left(\left\{\boldsymbol{z}_{\infty} \in \boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta}\right\} \cap \left(E_{[N,\infty)}(\eta) \cap A_{N}\right)^{\mathsf{c}}\right)$$
$$= \int_{\boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta}} \mathbf{P}^{\boldsymbol{\xi}}\left(\left(E_{[N,\infty)}(\eta) \cap A_{N}\right)^{\mathsf{c}}\right) \, d\nu_{\pi}^{\eta}(\boldsymbol{\xi}) \leq (1-\varepsilon)\nu_{\pi}^{\eta}\left(\boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta}\right).$$

In summary, for all $n \ge N$,

$$\varepsilon \nu_{\pi}^{\eta} \left(\boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta} \right) \leq \mathbf{P}^{\eta} \left(\left\{ \boldsymbol{z}_{\infty} \in \boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta} \right\} \cap E_{[N, \infty)}(\eta) \cap A_{N} \right).$$

By Lemmas 4.2 and 4.3, the events A_n and $E_{[n,\infty)}(\eta)$ are increasing in $n \in \mathbb{Z}_+$, and $E_{[n,\infty)}(\eta) \subset E_n(\eta)$ by the definition. Thus, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for every $\boldsymbol{\xi}_0 \in \partial \mathcal{X} \times \partial \mathcal{X}^*$, and for all $n \geq N = N_{\varepsilon,\eta}$,

$$\varepsilon \nu_{\pi}^{\eta} \left(\boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta} \right) \leq \mathbf{P}^{\eta} \left(\left\{ \boldsymbol{z}_{\infty} \in \boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \right\} \cap E_{n}(\eta) \cap A_{n} \right).$$
(4.7)

By definition of A_n , if A_n holds, then $\boldsymbol{z}_n \in \boldsymbol{B}(\boldsymbol{z}_{\infty}, r_n)$, whence for $C := C_q > 0$,

$$\mathbf{P}^{\eta}\left(\left\{\boldsymbol{z}_{\infty}\in\boldsymbol{B}(\boldsymbol{\xi}_{0},r_{n})\right\}\cap E_{n}(\eta)\cap A_{n}\right)\leq\mathbf{P}^{\eta}\left(\left\{\boldsymbol{z}_{n}\in\boldsymbol{B}(\boldsymbol{\xi}_{0},Cr_{n})\right\}\cap E_{n}(\eta)\cap A_{n}\right).$$
 (4.8)

For every $\boldsymbol{\xi}_0 \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$ and for every $n \in \mathbb{Z}_+$, let

$$B_n(\boldsymbol{\xi}_0) := \big\{ \boldsymbol{z}_n \in \boldsymbol{B}(\boldsymbol{\xi}_0, Cr_n) \big\}.$$

For each such fixed $\boldsymbol{\xi}_0$ and n, the event $B_n(\boldsymbol{\xi}_0)$ is $\sigma(\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n)$ -measurable. Further each fixed η and n, the event $E_n(\eta)$ is $\sigma(\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n)$ -measurable. Hence for each fixed $\boldsymbol{\xi}_0, \eta$ and n, the event $B_n(\boldsymbol{\xi}_0) \cap E_n(\eta)$ is $\sigma(\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n)$ -measurable and is obtained as a

(countable) sum of cylinder sets $[\boldsymbol{w}_0, \ldots, \boldsymbol{w}_n]$ in $(\Omega, \mathcal{F}, \mathbf{P})$. Decomposing the event into a sum of cylinder sets yields

$$\mathbf{P}^{\eta}\left(B_{n}(\boldsymbol{\xi}_{0})\cap E_{n}(\eta)\cap A_{n}\right)=\sum_{\left[\boldsymbol{w}_{0},\ldots,\boldsymbol{w}_{n}\right]\subset B_{n}(\boldsymbol{\xi}_{0})\cap E_{n}(\eta)}\mathbf{P}^{\eta}\left(\left[\boldsymbol{w}_{0},\ldots,\boldsymbol{w}_{n}\right]\cap A_{n}\right)\right)$$

In the right hand side, each summand is at most $e^{n(h^*+\varepsilon)}\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]\cap A_n)$ since $[\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n] \subset E_n(\eta)$. Furthermore since $\mathbf{P}([\boldsymbol{w}_0,\ldots,\boldsymbol{w}_n]\cap A_n)$ over those cylinder sets add up to $\mathbf{P}(B_n(\boldsymbol{\xi}_0)\cap E_n(\eta)\cap A_n)$, it holds that

$$\mathbf{P}^{\eta}\left(B_{n}(\boldsymbol{\xi}_{0})\cap E_{n}(\eta)\cap A_{n}\right)\leq e^{n(h^{\star}+\varepsilon)}\mathbf{P}\left(B_{n}(\boldsymbol{\xi}_{0})\cap E_{n}(\eta)\cap A_{n}\right).$$
(4.9)

By the definitions of $B_n(\boldsymbol{\xi}_0)$ and A_n , if $B_n(\boldsymbol{\xi}_0) \cap A_n$ holds, then $\boldsymbol{z}_{\infty} \in \boldsymbol{B}(\boldsymbol{\xi}_0, C^2 r_n)$. This in particular implies that

$$\mathbf{P}\left(B_n(\boldsymbol{\xi}_0) \cap E_n(\eta) \cap A_n\right) \le \mathbf{P}\left(\left\{\boldsymbol{z}_{\infty} \in \boldsymbol{B}(\boldsymbol{\xi}_0, C^2 r_n)\right\}\right) = \nu_{\pi}\left(\boldsymbol{B}(\boldsymbol{\xi}_0, C^2 r_n)\right)$$
(4.10)

Combining (4.7), (4.8), (4.9) and (4.10) implies that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for every $\boldsymbol{\xi}_0 \in \partial \mathcal{X} \times \partial \mathcal{X}^*$, and for all $n \geq N = N_{\varepsilon,\eta}$,

$$\varepsilon \nu_{\pi}^{\eta} \left(\boldsymbol{B}(\boldsymbol{\xi}_{0}, r_{n}) \cap F_{\eta} \right) \leq e^{n(h^{\star} + \varepsilon)} \nu_{\pi} \left(\boldsymbol{B}(\boldsymbol{\xi}_{0}, C^{2} r_{n}) \right).$$
(4.11)

Recall that $\overline{h} = h(\pi) - h^*$. By Lemma 3.2, for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for ν_{π}^{η} -almost every $\boldsymbol{\xi}_0 \in F_{\eta}$, there exists an $N_{\eta,\boldsymbol{\xi}_0} \in \mathbb{Z}_+$ such that for all $n \geq N_{\eta,\boldsymbol{\xi}_0}$,

$$\left(\frac{\overline{h}}{l}+\varepsilon\right)\log r_n\leq \log \nu_{\pi}^{\eta}\left(\boldsymbol{B}(\boldsymbol{\xi}_0,r_n)\cap F_{\eta}\right).$$

This together with (4.11) shows that for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, for ν_{π}^{η} -almost every $\boldsymbol{\xi}_0 \in F_{\eta}$ (recalling that $r_n = e^{-(l^* - \varepsilon)n}$),

$$\limsup_{n \to \infty} \frac{\log \nu_{\pi} \left(\boldsymbol{B}(\boldsymbol{\xi}_0, C^2 r_n) \right)}{\log r_n} \leq \frac{\overline{h}}{l} + \varepsilon + \frac{h^* + \varepsilon}{l^* - \varepsilon}.$$

Recall that $\nu_{\pi}^{\eta}(F_{\eta}) \geq 1 - 2\varepsilon$ by (4.6) and that this holds for arbitrary $\varepsilon \in (0, l^{\star})$. Therefore after replacing the sequence r_n by reals r tending to 0, for $\nu_{\mu^{\star}}$ -almost every $\eta \in \partial \mathcal{X}^{\star}$, for ν_{π}^{η} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$,

$$\limsup_{r \to 0} \frac{\log \nu_{\pi} \left(\boldsymbol{B} \left(\boldsymbol{\xi}, r \right) \right)}{\log r} \leq \frac{\overline{h}}{l} + \frac{h^{\star}}{l^{\star}}.$$

The disintegration of ν_{π} into ν_{π}^{η} shows that the above holds for ν_{π} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$, concluding the claim.

4.2. Lower bounds on dimensions of harmonic measures in product spaces.

Proposition 4.4. Let $\Gamma = \Gamma \times \Gamma^*$ where Γ and Γ^* are countable subgroups in Isom \mathcal{X} and in Isom \mathcal{X}^* with finite exponential growth relative to (\mathcal{X}, d) and to (\mathcal{X}^*, d^*) respectively. For every probability measure π on Γ with finite first moment and non-elementary marginals μ and μ^* , it holds that for ν_{π} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \mathcal{X}^*$,

$$\liminf_{r \to 0} \frac{\log \nu_{\pi} \left(\boldsymbol{B}(\boldsymbol{\xi}, r) \right)}{\log r} \geq \frac{h(\pi) - h(\mu^{\star})}{l(\mathcal{X}, \mu)} + \frac{h(\mu^{\star})}{l(\mathcal{X}^{\star}, \mu^{\star})}.$$

Proof. Note that Γ has a finite exponential growth relative to $(\mathcal{X} \times \mathcal{X}^*, d)$ by the assumption. By Lemma 3.3, for every $\varepsilon > 0$ there exists an $N \in \mathbb{Z}_+$, a Borel set D in $\partial \mathcal{X}^*$ with $\nu_{\mu^*}(D) \ge 1 - \varepsilon$ and a Borel set F in $\partial \mathcal{X} \times \partial \mathcal{X}^*$ with $\nu_{\pi}(F) \ge 1 - \varepsilon$ as stated. If we define $\hat{F} := (\partial \mathcal{X} \times D) \cap F$, then

$$\nu_{\pi}(\tilde{F}) \ge 1 - 2\varepsilon. \tag{4.12}$$

This follows since $\nu_{\pi}(\partial \mathcal{X} \times D) = \nu_{\mu^{\star}}(D) \ge 1 - \varepsilon$ and $\nu_{\pi}(F) \ge 1 - \varepsilon$.

Let $r_n := e^{-ln}$ for $n \in \mathbb{Z}_+$. For every $\boldsymbol{\xi} = (\xi, \xi^*) \in \partial \mathcal{X} \times \partial \mathcal{X}^*$, by disintegration of ν_{π} into ν_{π}^{η} for $\eta \in \partial \mathcal{X}^*$,

$$\nu_{\pi} \big(B(\xi, r_n) \times B(\xi^{\star}, r_n) \cap \hat{F} \big) = \int_{B(\xi^{\star}, r_n) \cap D} \nu_{\pi}^{\eta} \left(B(\xi, r_n) \times \partial \mathcal{X}^{\star} \cap F \right) \, d\nu_{\mu^{\star}}(\eta).$$

By Lemma 3.3, for ν_{μ^*} -almost every $\eta \in D$, for every $\xi \in \partial \mathcal{X}$ and for all $n \geq N$,

$$\nu_{\pi}^{\eta}\left(B(\xi, r_n) \times \partial \mathcal{X}^{\star} \cap F\right) \le C_{\varepsilon} e^{-n(\overline{h} - \varepsilon)}.$$

Therefore for every $\boldsymbol{\xi} = (\xi, \xi^{\star}) \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$ and for all $n \geq N$,

$$\nu_{\pi} \left(B(\xi, r_n) \times B(\xi^{\star}, r_n) \cap \hat{F} \right) \le C_{\varepsilon} e^{-n(\overline{h} - \varepsilon)} \nu_{\mu^{\star}} \left(B(\xi^{\star}, r_n) \cap D \right).$$
(4.13)

By the dimension formula for (Γ^*, μ^*) in [Tan19, Theorem 1.2], for ν_{μ^*} -almost every $\xi^* \in \partial \mathcal{X}^*$,

$$\lim_{n \to \infty} \frac{\log \nu_{\pi} \left(B(\xi^{\star}, r_n) \right)}{\log r_n} = \frac{h(\mu^{\star})}{l^{\star}}$$

(The proof presented there is for geodesic spaces, but it is adapted to roughly geodesic spaces \mathcal{X} .) Since $\nu_{\pi} (B(\xi^{\star}, r_n) \cap D) \leq \nu_{\pi} (B(\xi^{\star}, r_n))$ and $\log r_n < 0$, for $\nu_{\mu^{\star}}$ -almost every $\xi^{\star} \in \partial \mathcal{X}^{\star}$,

$$\liminf_{n \to \infty} \frac{\log \nu_{\pi} \left(B(\xi^{\star}, r_n) \cap D \right)}{\log r_n} \ge \frac{h(\mu^{\star})}{l^{\star}}.$$

This together with (4.13) implies that since $\nu_{\pi}(\partial \mathcal{X} \times D) = \nu_{\mu^{\star}}(D)$ and $B(\boldsymbol{\xi}, r_n) = B(\boldsymbol{\xi}, r_n) \times B(\boldsymbol{\xi}^{\star}, r_n)$, for ν_{π} -almost every $\boldsymbol{\xi} = (\boldsymbol{\xi}, \boldsymbol{\xi}^{\star}) \in \partial \mathcal{X} \times D$ (recalling that $r_n = e^{-ln}$),

$$\liminf_{n \to \infty} \frac{\log \nu_{\pi} \left(\boldsymbol{B}(\boldsymbol{\xi}, r_n) \cap \tilde{F} \right)}{\log r_n} \ge \frac{\overline{h} - \varepsilon}{l} + \frac{h(\mu^{\star})}{l^{\star}}.$$
(4.14)

By Lemma 2.2, for ν_{π} -almost every $\boldsymbol{\xi} \in \hat{F}$ (where $\hat{F} \subset \partial \mathcal{X} \times D$),

$$\liminf_{n \to \infty} \frac{\log \nu_{\pi} (\boldsymbol{B}(\boldsymbol{\xi}, r_n))}{\log r_n} \ge \frac{\overline{h} - \varepsilon}{l} + \frac{h(\mu^{\star})}{l^{\star}}.$$
(4.15)

By (4.12), one has $\nu_{\pi}(\hat{F}) \geq 1-2\varepsilon$, and for every $\varepsilon > 0$ there exists such an \hat{F} in $\partial \mathcal{X} \times \partial \mathcal{X}^{\star}$. Thus after replacing the sequence r_n for $n \in \mathbb{Z}_+$ by positive reals r tending to 0, we obtain for ν_{π} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^{\star}$,

$$\liminf_{r \to 0} \frac{\log \nu_{\pi} \big(\boldsymbol{B}(\boldsymbol{\xi}, r) \big)}{\log r} \ge \frac{\overline{h}}{l} + \frac{h(\mu^{\star})}{l^{\star}}.$$

This concludes the claim.

4.3. Exact dimension and the proof of Theorem 1.1.

Theorem 4.5. Let (\mathcal{X}, d) and (\mathcal{X}^*, d^*) be roughly geodesic hyperbolic metric spaces with bounded growth at some scale. Let $\Gamma = \Gamma \times \Gamma^*$ where Γ and Γ^* are countable subgroups in Isom \mathcal{X} and in Isom \mathcal{X}^* with finite exponential growth relative to (\mathcal{X}, d) and to (\mathcal{X}^*, d^*) respectively. For every probability measure π on Γ with finite first moment and nonelementary marginals μ and μ^* , the harmonic measure ν_{π} on $\partial \mathcal{X} \times \partial \mathcal{X}^*$ is exact dimensional. Moreover, if $l(\mathcal{X}, \mu) \geq l(\mathcal{X}^*, \mu^*)$, then it holds that

$$\dim \nu_{\pi} = \frac{h(\pi) - h(\mu^{\star})}{l(\mathcal{X}, \mu)} + \frac{h(\mu^{\star})}{l(\mathcal{X}^{\star}, \mu^{\star})}$$

Proof. If Γ^* has finite exponential growth relative to (\mathcal{X}^*, d^*) and μ^* is non-elementary and of finite first moment, then $(\partial \mathcal{X}^*, \nu_{\mu^*})$ is a Poisson boundary for (Γ^*, μ^*) [Kai00, Theorem 7.4]. In particular, $h(\mu^*) = h(\partial \mathcal{X}^*, \mu^*)$. Therefore by Propositions 4.1 and 4.4, if $l(\mathcal{X}, \mu) \geq l(\mathcal{X}^*, \mu^*)$, then for ν_{π} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X} \times \partial \mathcal{X}^*$,

$$\lim_{r \to 0} \frac{\log \nu_{\pi}(\boldsymbol{B}(\boldsymbol{\xi}, r))}{\log r} = \frac{h(\pi) - h(\mu^{\star})}{l(\mathcal{X}, \mu)} + \frac{h(\mu^{\star})}{l(\mathcal{X}^{\star}, \mu^{\star})}$$

This shows that ν_{π} is exact dimensional. The second claim follows from Lemma 2.1. *Proof of Theorem 1.1.* The claim follows from Theorem 4.5 as a special case.

Remark 4.6. Let us mention possible extensions and related questions.

(1) The proof of Theorem 4.5 can be extended to a product of more than two hyperbolic metric spaces. For a positive $N \in \mathbb{Z}_+$, let $(\mathcal{X}^{(i)}, d^{(i)})$ for $i = 1, \ldots, N$ be proper roughly geodesic hyperbolic metric spaces with bounded growth at some scale. Further $\Gamma^{(i)}$ are countable subgroups in Isom $\mathcal{X}^{(i)}$ with finite exponential growth for each $i = 1, \ldots, N$. Let $\Gamma := \Gamma^{(1)} \times \cdots \times \Gamma^{(N)}$. For a probability measure π on Γ with non-elementary marginals $\mu^{(i)}$ of π in Isom $\mathcal{X}^{(i)}$ of finite first moment, the harmonic measure ν_{π} on $\partial \mathcal{X}^{(1)} \times \cdots \times \partial \mathcal{X}^{(N)}$ is exact dimensional: Let $\pi^{(i)}$ be the pushforward of π to Isom $\mathcal{X}^{(i)} \times \cdots \times \text{Isom } \mathcal{X}^{(N)}$ for $i = 1, \ldots, N$. If $l(\mathcal{X}^{(1)}, \mu^{(1)}) \geq \cdots \geq l(\mathcal{X}^{(N)}, \mu^{(N)})$, then for ν_{π} -almost every $\boldsymbol{\xi} \in \partial \mathcal{X}^{(1)} \times \cdots \times \partial \mathcal{X}^{(N)}$,

$$\dim \nu_{\pi} = \sum_{i=1}^{N-1} \frac{h(\pi^{(i)}) - h(\pi^{(i+1)})}{l(\mathcal{X}^{(i)}, \mu^{(i)})} + \frac{h(\pi^{(N)})}{l(\mathcal{X}^{(N)}, \mu^{(N)})}$$

In the above, $h(\pi^{(i)})$ denotes the asymptotic entropy for a $\pi^{(i)}$ -random walk and $l(\mathcal{X}^{(i)}, \mu^{(i)})$ denotes the drift associated with a $\mu^{(i)}$ -random walk for each $i = 1, \ldots, N$. Further the Hausdorff dimension is computed by the quasi-metric defined as maximum of the ones in $\partial \mathcal{X}^{(i)}$. The proof proceeds by the reverse induction in i from N to 1 upon extending Propositions 4.1 and 4.4 and Theorem 4.5 to the spaces $\mathcal{X}^{(i)} \times \cdots \times \mathcal{X}^{(N)}$ for $i = 1, \ldots, N$. Since writing out all the details in this generality would hurt readability, we refrain from producing the whole argument.

(2) In [Tan19], the exact dimensionality of the harmonic measures for a single hyperbolic metric space \mathcal{X} has been extended to several directions. For example, \mathcal{X} can be replaced by a proper hyperbolic \mathcal{X} without assuming bounded growth at some scale, and by a non-proper, separable and geodesic hyperbolic \mathcal{X} with acylindrical action of a group. In those cases, probability measures μ for μ -random walks are assumed to satisfy that the support generates a non-elementary subgroup of isometries as a *semigroup* rather than a *group*. It is expected that results in the present paper are extended to products of such hyperbolic spaces (with right assumption on random walks). However, we need that the boundary be Polish (at least the space endowed with Borel structure be a standard Borel space) so that the conditional measures are well-defined. Thus it is not clear as to whether the separability of \mathcal{X} could be dropped.

(3) It is not clear as to whether one can remove the condition on finite exponential growth relative to each factor in Theorem 4.5. It is expected that the harmonic measure is exact dimensional without the assumption in regards of results in [HS17] and [LL23]. The issue in the present setting lies in a lack of Lebesgue differentiation theorem on boundaries. This is available, for example, if the boundaries are Euclidean spaces, more generally, Riemannian manifolds, or if the harmonic measures are doubling (which is stringent). In this paper, we have used a weaker version of Lebesgue differentiation theorem (Lemma 2.2). However, we do not know if this would suffice to remove the condition on growth. See a related question in [Tan19, Quesion 4.3].

5. A Positive lower bound for dimension

5.1. **Pivotal times.** Let us recall the terminology and methods from [Gou22]. For $\delta \in \mathbb{R}_+$, let (\mathcal{X}, d) be a δ -hyperbolic space with a base point o. A sequence of points x_0, \ldots, x_n is called a (C, D)-chain for some $C, D \in \mathbb{R}_+$ if

 $(x_{i-1}|x_{i+1})_{x_i} \le C$ for all $i = 1, \dots, n-1$, and $d(x_{i-1}, x_i) \ge D$ for all $i = 1, \dots, n$.

If a sequence x_0, \ldots, x_n is a (C, D)-chain with $C \in \mathbb{R}_+$ and $D \ge 2C + 2\delta + 1$, then

$$(x_0|x_n)_{x_1} \le C + \delta$$
 and $d(x_0, x_n) \ge \sum_{i=0}^{n-1} (d(x_i, x_{i+1}) - (2C + 2\delta)) \ge n.$ (5.1)

See [Gou22, Lemma 3.7]. For $C \in \mathbb{R}_+$, $D = 2C + 2\delta + 1$ and for $x, y \in \mathcal{X}$, the chain shadow $\mathcal{CS}_x(y, C)$ of y seen from x is the set

$$\left\{z \in \mathcal{X} : \text{ there exists a } (C, D)\text{-chain } x_0, \dots, x_n; x_0 = x, x_n = z \text{ and } (x_0|x_1)_y \le C\right\}.$$

Definition 5.1. For $\varepsilon, C, D \in \mathbb{R}_+$, a set of isometries \mathcal{S} is called an (ε, C, D) -Schottky set if the following three conditions are satisfied:

- (1) $\#\{s \in \mathcal{S} : (x|s \cdot y)_o \leq C\} \geq (1-\varepsilon)\#\mathcal{S} \text{ for all } x, y \in \mathcal{X},$
- (2) $\#\{s \in \mathcal{S} : (x|s^{-1} \cdot y)_o \le C\} \ge (1-\varepsilon)\#\mathcal{S}$ for all $x, y \in \mathcal{X}$, and
- (3) $d(o, s \cdot o) \ge D$ for all $s \in \mathcal{S}$.

Let μ be a non-elementary probability measure on Isom \mathcal{X} with a countable support. It is shown basically by a classical ping-pong argument that for every $\varepsilon > 0$ there exists a $C_0 \in \mathbb{R}_+$ satisfying the following: For all $D \in \mathbb{R}_+$ there exists an $M \in \mathbb{Z}_+$ such that the

support of the *M*-fold convolution μ^{*M} of μ contains an (ε, C_0, D) -Schottky set \mathcal{S} [Gou22, Corollary 3.13]. Let us fix the constants

$$\varepsilon = 1/100, \quad C_0 \in \mathbb{R}_+ \text{ and } D \ge 20C_0 + 100\delta + 1.$$

Let $\lambda_{\mathcal{S}}$ denote the uniform distribution on \mathcal{S} .

Given a sequence of isometries u_0, u_1, \ldots on \mathcal{X} and a sequence of independent random isometries s_1, s_2, \ldots with the identical distribution $\lambda_{\mathcal{S}}^{*2}$, let

$$y_n^- := u_0 s_1 u_1 \cdots s_{n-1} u_{n-1} \cdot o_n$$

Letting $s_i = a_i b_i$ where a_i and b_i are independent and distributed as λ_S , we define

$$y_n := u_0 s_1 u_1 \cdots s_{n-1} u_{n-1} a_n \cdot o$$
 and $y_n^+ := u_0 s_1 u_1 \cdots s_{n-1} u_{n-1} a_n b_n \cdot o$.

A sequence of **pivotal times** $P_n \subseteq \{1, \ldots, n\}$ is defined inductively as in the following: Let $P_0 := \emptyset$ (the empty set). Given P_{n-1} , let k = 0 and $y_k := o$ if $P_{n-1} = \emptyset$. Suppose that $P_{n-1} \neq \emptyset$. Let us say that the **local geodesic condition** is satisfied at time n if

$$(y_k|y_n)_{y_n^-} \le C_0, \quad (y_n^-|y_n^+)_{y_n} \le C_0 \quad \text{and} \quad (y_n|y_{n+1}^-)_{y_n^+} \le C_0.$$
 (5.2)

If the local geodesic condition is satisfied at time n, then we define

$$P_n := P_{n-1} \cup \{n\}.$$

If otherwise, then letting m be the largest pivotal time in P_{n-1} such that

$$y_{n+1}^- \in \mathcal{CS}_{y_m}(y_m^+, C_0 + \delta)$$

we define $P_n := P_{n-1} \cap \{1, \ldots, m\}$, and $P_n := \emptyset$ in the case when there is no such m. Note that the set P_n depends only on the sequence s_1, \ldots, s_n for fixed u_0, \ldots, u_n .

Lemma 5.2. If $P_n := \{k_1, \ldots, k_p\}$ where $k_1 < \cdots < k_p$, then the sequence

 $o, y_{k_1}, y_{k_2}^-, y_{k_2}, \dots, y_{k_p}^-, y_{k_p}, y_{n+1}^-,$

forms a $(2C_0 + 4\delta, D - 2C_0 - 3\delta)$ -chain. Moreover, if $D \ge 6C_0 + 13\delta + 1$, then for every $i = 2, \ldots, p$, the sequence $y_{k_i}^-, y_{k_i}, y_{n+1}^-$ is a $(2C_0 + 5\delta, D - 6C_0 - 13\delta)$ -chain.

Proof. The first claim is [Gou22, Lemma 4.4]. The second claim follows from the first claim and (5.1). Indeed, applying to them the sequence $y_{k_i}, y_{k_i}, \ldots, y_{k_p}, y_{n+1}^-$ for each $i = 2, \ldots, p$ shows that $(y_{k_i}^-|y_{n+1}^-)_{y_{k_i}} \leq 2C_0 + 5\delta$, and further,

$$d(y_{k_i}, y_{n+1}) \ge d(y_{k_i}, y_{k_{i+1}}) - 2(2C_0 + 4\delta) - 2\delta \ge D - 6C_0 - 13\delta.$$

The claim follows.

Let $\overline{s} = (s_1, \ldots, s_n)$. Let us say that a sequence $\overline{s}' = (s'_1, \ldots, s'_n)$ where $s'_i = a'_i b'_i$ is **pivoted from** \overline{s} if \overline{s}' and \overline{s} have the same pivotal times, $b'_i = b_i$ for all $i = 1, \ldots, n$, and $a'_i = a_i$ for all i which is not a pivotal time. The relation that \overline{s}' is pivoted from \overline{s} defines an equivalence relation among sequences. In this notation, we understand that b_i for all $i = 1, \ldots, n$ and a_i for i which is not a pivotal time are determined in \overline{s} .

Let $\mathcal{E}_n(\overline{s})$ be the set of sequences which are pivoted from \overline{s} . Note that if u_0, u_1, \ldots are fixed, then conditioned on $\mathcal{E}_n(\overline{s})$, all a_i are independent. However, their distributions may depend on i. For each $i = 1, \ldots, n$, let

$$A_i(\overline{s}) := \left\{ a \in \mathcal{S} : s'_i = ab_i \text{ for some } \overline{s}' = (s'_1, \dots, s'_i, \dots, s'_n) \in \mathcal{E}_n(\overline{s}) \right\}.$$

It holds that for each pivotal time i of \overline{s} ,

$$\mathbf{P}(a_i = a \mid \mathcal{E}_n(\overline{s})) = \lambda_{\mathcal{S}}(a \mid A_i(\overline{s})) = \frac{\lambda_{\mathcal{S}}(a)}{\lambda_{\mathcal{S}}(A_i(\overline{s}))} \quad \text{for } a \in A_i(\overline{s}).$$

If *i* is a pivotal time of \overline{s} and $\overline{s}' = (s_1, \ldots, s'_i, \ldots, s_n)$ in which $s_i = a_i b_i$ is replaced by $s'_i = a'_i b_i$ satisfies the local geodesic condition at time *i*, then \overline{s}' is pivoted from \overline{s} [Gou22, Lemma 4.7]. By the definition of (ε, C_0, D) -Schottky set (Definition 5.1), there are at most $2\varepsilon \# S$ elements for which the local geodesic condition (5.2) does not hold at *i*. Therefore for each pivotal time *i* in \overline{s} ,

$$#A_i(\overline{s}) \ge (1 - 2\varepsilon) # \mathcal{S}.$$
(5.3)

Lemma 5.3. Let $D \ge 10C_0 + 25\delta + 1$. For $\overline{s}' \in \mathcal{E}_n(\overline{s})$, if $y_{n+1}^- = y_{n+1}'^-$, then $\overline{s} = \overline{s}'$.

Proof. Let $P_n = \{k_1, \ldots, k_p\}$ with $k_1 < \cdots < k_p$ be the set of pivotal times in \overline{s} . By Lemma 5.2, the sequences $y_{k_i}^-, y_{k_i}, y_{n+1}^-$ and $y_{k_i}^{\prime-}, y_{k_i}^{\prime}, y_{n+1}^{\prime-}$ are $(2C_0 + 5\delta, D - 6C_0 - 13\delta)$ chains respectively. For $\overline{s}' \in \mathcal{E}_n(\overline{s})$ with $\overline{s}' \neq \overline{s}$, let *i* be the first *i* for which $s_{k_i} \neq s_{k_i}'$. For $s_{k_i} = a_{k_i}b_{k_i}$ and $s_{k_i}' = a_{k_i}'b_{k_i}'$, it holds that $a_{k_i} \neq a_{k_i'}$ and these a_{k_i} and a_{k_i}' are in the Schottky set \mathcal{S} , whence $(a_{k_i} \cdot o|a_{k_i}' \cdot o)_o \leq C_0$. This shows that the sequence

$$y'_{n+1}, y'_{k_i}, y_{k_i}, y_{k_i}, y_{n+1}, \text{ where } y_{k_i}^- = y'_{k_i},$$

forms a $(2C_0 + 5\delta, D - 6C_0 - 13\delta)$ -chain. For such D, one has $d(y'_{n+1}, y_{n+1}) > 0$ by (5.1), and thus $y'_{n+1} \neq y_{n+1}$, as required.

5.2. A lower bound for entropy.

Theorem 5.4. Let Γ and Γ^* be countable subgroups in Isom \mathcal{X} and in Isom \mathcal{X}^* respectively, and $\Gamma := \Gamma \times \Gamma^*$. Further let us consider a probability measure π on Γ of the following form:

$$\pi = \alpha \lambda \times \lambda^{\star} + (1 - \alpha) \pi_0$$

for some $\alpha \in (0, 1]$, non-elementary probability measures λ and λ^* on Γ and on Γ^* respectively, and a probability measure π_0 on Γ . For the marginal μ^* of π on Γ^* , it holds that $h(\pi) - h(\partial \mathcal{X}^*, \mu^*) > 0$. Moreover, if in addition Γ has a finite exponential growth relative to $(\mathcal{X}, \mathbf{d})$, then for ν_{μ^*} -almost every $\eta \in \partial \mathcal{X}^*$, the Hausdorff dimension of the conditional measure ν_{π}^{η} is positive.

Fix constants $\varepsilon = 1/100$, $C_0 \ge 0$ and $D \ge 20C_0 + 100\delta + 1$, and a $(1/100, C_0, D)$ -Schottky set S in Γ , contained in the support of λ^{*M} for some $M \in \mathbb{Z}_+$. For N := 2M, let us write for some $\beta \in (0, 1]$ and for a probability measure λ_0 on Γ ,

$$\lambda^{*N} = \beta \lambda_{\mathcal{S}}^{*2} + (1 - \beta) \lambda_0.$$

Let us also write for a probability measure $\overline{\pi}_0$ on $\Gamma = \Gamma \times \Gamma^*$,

$$\pi^{*N} = \alpha^N \beta \lambda_{\mathcal{S}}^{*2} \times \lambda^{\star*N} + (1 - \alpha^N \beta) \overline{\pi}_0.$$

For a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of independent Bernoulli random variables with the common parameter $\alpha^N \beta$, let us define a sequence of independent random group elements $\gamma_1, \gamma_2, \ldots$ where $\gamma_i = (\gamma_i, \gamma_i^*) \in \Gamma$ is distributed as $\lambda_S^{*2} \times \lambda^{**N}$ if $\varepsilon_i = 1$ and as $\overline{\pi}_0$ if $\varepsilon_i = 0$. Note in particular that for $i = 1, 2, \ldots$, conditioned on the event $\{\varepsilon_i = 1\}$, random group elements γ_i and γ_i^* are independent.

Further we realize a π -random walk \boldsymbol{w}_{nN} at time nN by a product $\boldsymbol{\gamma}_1 \cdots \boldsymbol{\gamma}_n$ through a coupling on an enlarged probability space. Let us define a sequence of pivotal times for $z_n = w_n \cdot o$ on \mathcal{X} , where $\{w_n\}_{n \in \mathbb{Z}_+}$ is a μ -random walk and μ is the marginal of π on Γ .

Let t_1, t_2, \ldots be the sequence of *i* such that $\varepsilon_i = 1$. For every positive $n \in \mathbb{Z}_+$, let $\tau = \tau(n)$ be the maximum of *j* with $Nt_j \leq n$. It holds that

$$(N(t_j-1), Nt_j] \subset (0, n]$$
 for all $j = 1, \ldots, \tau$.

For each $j = 1, ..., \tau$, let s_{t_j} be γ_j , which is realized as the product of elements x_i over $i \in (N(t_j - 1), Nt_j]$ in the natural order from \mathbb{Z}_+ . Let us write $s'_j := s_{t_j}$ for brevity, and

$$u_0 := x_1 \cdots x_{N(t_1-1)}, \quad u_j := x_{Nt_j+1} \cdots x_{N(t_{j+1}-1)} \text{ and } u(n) := x_{Nt_{\tau}+1} \cdots x_n.$$

In the above, u_0 , u_j and u(n) are defined as the identity if they are empty words. For a μ -random walk w_n at time n, the orbit z_n is realized as

$$z_n = w_n \cdot o = u_0 s'_1 u_1 \cdots s'_\tau u(n) \cdot o$$

Let $P_1, \ldots, P_{\tau(n)}$ be the sequence of pivotal times of z_n given $u_0, u_1, \ldots, u(n)$. Note that $P_{\tau(n)}$ depends not only on $\tau(n)$ but also on n. It is shown that there exists a constant $\kappa > 0$ such that

$$\mathbf{P}(\#P_{\tau(n)} \le \kappa n) \le e^{-\kappa n} \quad \text{for all } n \in \mathbb{Z}_+,$$
(5.4)

[Gou22, Proposition 4.11].

Note that the sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ determines $\{t_i\}_{i=1}^{\infty}$ and $\tau = \tau(n)$ for each n. Let us define the σ -algebra

$$\mathcal{G} := \sigma\Big(\varepsilon_i, t_i, x_i^{\star} \text{ for } i = 1, 2, \dots \text{ and } x_i \text{ for } i \notin \bigcup_{j=1}^{\tau(n)} (N(t_j - 1), Nt_j)\Big).$$

Conditioning on \mathcal{G} amounts to fix a typical sequence $\{\varepsilon_i\}_{i=1}^{\infty}$, $\{t_i\}_{i=1}^{\infty}$ and a trajectory of μ^* -random walk $\{w_n^*\}_{n\in\mathbb{Z}_+}$, increments x_i of μ -random walk outside the time intervals $(N(t_j - 1), Nt_j)$ for $j = 1, \ldots, \tau(n)$. Under this conditioning, s'_1, \ldots, s'_{τ} is a sequence of independent random elements in Γ with the common distribution λ_S^{*2} .

Let $\overline{s} := (s'_1, \ldots, s'_{\tau})$, and $\mathcal{E}_{\tau}(\overline{s})$ be the set of sequences pivoted from \overline{s} . Conditioned on $\mathcal{E}_{\tau}(\overline{s})$ and \mathcal{G} , the random group elements a_i at pivotal times where $s'_i = a_i b_i$ are independent and each a_i is distributed as $\lambda_{\mathcal{S}}(\cdot \mid A_i(\overline{s}))$. Let $\sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G})$ denote the σ algebra generated by $\mathcal{E}_{\tau}(\overline{s})$ and \mathcal{G} . Given $\mathcal{E}_{\tau}(\overline{s})$ and $u_0, \ldots, u(n)$, let us define the map

$$\prod_{i \in P_{\tau(n)}} A_i(\overline{s}) \to \mathcal{X}, \quad (a_i)_{i \in P_{\tau(n)}} \mapsto w_n \cdot o = u_0 a_1 b_1 u_1 \cdots u_{\tau-1} a_\tau b_\tau u(n) \cdot o_1 u_1 \cdots u_{\tau-1} u_\tau u(n) \cdot o_1 u_1 \cdots u_{\tau-1} u_\tau u(n) \cdot o_1 u_1 \cdots u_{\tau-1} u_\tau u(n) \cdot o_1 u_1 \cdots u_{\tau-1} u_1 \dots u_{\tau-1}$$

In the above, we understand that $\{b_i\}_{i=1,...,\tau(n)}$ and $\{a_i\}_{i\notin P_{\tau(n)}}$ are determined by $\mathcal{E}_{\tau}(\overline{s})$. Under the conditioning, the map is injective by Lemma 5.3. Furthermore the conditional distribution $\mathbf{P}(w_n \cdot o \in \cdot \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G}))$ is the pushforward by the injective map of the product measure $\lambda_{\mathcal{S}}(\cdot \mid A_i(\overline{s}))$ over $i \in P_{\tau(n)}$ almost everywhere in \mathbf{P} .

Proof of Theorem 5.4. Let us denote by H(w) the entropy $H(\mu)$ for a random variable w with the distribution μ . If \mathcal{F} is a sub σ -algebra of $\sigma(\{(w_n, w_n^{\star})\}_{n \in \mathbb{Z}_+})$, then the conditional entropy of w_n with respect to \mathcal{F} is defined by

$$H(w_n \mid \mathcal{F}) := \mathbf{E} \left[-\sum_{x \in \operatorname{supp} \mu_n} \mathbf{P}(w_n = x \mid \mathcal{F}) \log \mathbf{P}(w_n = x \mid \mathcal{F}) \right].$$

It holds that $H(w_n) \geq H(w_n | \mathcal{F})$, further that if \mathcal{F}_1 and \mathcal{F}_2 are sub σ -algebras of $\sigma(\{(w_n, w_n^{\star})\}_{n \in \mathbb{Z}_+})$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $H(w_n | \mathcal{F}_1) \geq H(w_n | \mathcal{F}_2)$. Since the σ -algebra $\sigma(w_n^{\star})$ generated by w_n^{\star} is included in \mathcal{G} , it follows that

$$H(w_n \mid \sigma(w_n^{\star})) \ge H(w_n \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G})).$$

Let us find a lower bound on the right hand side of the following:

$$H(w_n \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G})) = \mathbf{E}\left[-\sum_{x \in \text{supp } \mu_n} \mathbf{P}\left(w_n = x \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G})\right) \log \mathbf{P}\left(w_n = x \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G})\right)\right].$$

Since $\mathbf{P}(w_n \cdot o \in \cdot \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G}))$ is the pushforward by an injective map of the product measure of $\lambda_{\mathcal{S}}(\cdot \mid A_i(\overline{s}))$ over $i \in P_{\tau(n)}$, one has **P**-almost everywhere,

$$-\sum_{x\in\operatorname{supp}\mu_n} \mathbf{P}(w_n = x \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G})) \log \mathbf{P}(w_n = x \mid \sigma(\mathcal{E}_{\tau}(\overline{s}), \mathcal{G})) = -\sum_{i\in P_{\tau(n)}} \log \frac{1}{\#A_i(\overline{s})}.$$

This shows that for each $n \in \mathbb{Z}_+$ and for $\kappa > 0$ in (5.4),

$$H(w_n \mid \sigma(w_n^*)) \ge \mathbf{E} \left[\left(\sum_{i \in P_{\tau(n)}} \log \# A_i(\overline{s}) \right) \cdot \mathbf{1}_{\{\# P_{\tau(n)} \ge \kappa n\}} \right].$$

If $i \in P_{\tau(n)}$, then $\#A_i(\overline{s}) \ge (1-2\varepsilon)\#S$ by (5.3) and $\mathbf{P}(\#P_{\tau(n)} \le \kappa n) \le e^{-\kappa n}$ by (5.4), the right hand side of the above inequality is at least

$$\kappa n \log((1-2\varepsilon)\#\mathcal{S}) \cdot \mathbf{P}(\#P_{\tau(n)} \ge \kappa n) \ge \kappa n \log((1-2\varepsilon)\#\mathcal{S}) \cdot (1-e^{-\kappa n}).$$

Therefore it holds that

$$\liminf_{n \to \infty} \frac{1}{n} H(w_n \mid \sigma(w_n^*)) \ge \kappa \log((1 - 2\varepsilon) \# \mathcal{S}) > 0$$

Noting that $H(w_n, w_n^{\star}) = H(w_n^{\star}) + H(w_n \mid \sigma(w_n^{\star}))$ and $h(\partial \mathcal{X}^{\star}, \mu^{\star}) \leq h(\mu^{\star})$, we obtain

$$h(\pi) - h(\partial \mathcal{X}^{\star}, \mu^{\star}) \ge h(\pi) - h(\mu^{\star}) \ge \liminf_{n \to \infty} \frac{1}{n} H(w_n \mid \sigma(w_n^{\star})) > 0.$$

(In the above, the second inequality is in fact the equality and the limit is the limit.) Thus the first claim follows. The second claim follows from the first claim together with Theorem 1.3.

Proof of Theorem 1.4. This is contained in Theorem 5.4.

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