Time reversal invariant topological 1D and 2D superconductors: doubling the Sau-Luchtin-Tewari-Sarma [1] and Oreg-Refael-von Oppen [2, 3] proposals

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In this work we present doubled versions of the Sau-Luchtin-Tewari-Sarma [1] and Oreg-Refael-von Oppen [2, 3] proposals thereby obtaining time reversal invariant p-wave superconductivity in both 1D and 2D. This construction is much like the Kane-Mele spin Hall model [4], which is a time reversal invariant doubling of the Haldane model [5]. We show that the low energy effective action for these doubled versions of the Sau-Luchtin-Tewari-Sarma [1] and Oreg-Refael-von Oppen [2, 3] models correspond to single band p-wave time reversal invariant superconductors with pseudospin degree of freedom instead of spin degree of freedom. There are Majorana fermions at the ends of wires or in vortex cores of these superconductors. Furthermore these Majorana fermions are shown to be stable to small perturbations. In the supplement we present a physical realization of the system with cold atoms [6] and show that a related "no-go" theorem given in Ref. [7, 8] has too restrictive assumptions to apply to this proposal.

I. INTRODUCTION

The scientific community has been studying topological superconductivity and superfluidity for a relatively long time. It is now known that the ^3He -A and -B phases are topological superfluids. The ^3He -A and -B phases have been characterized by topological bulk invariants [9]. $p+ip$ superconductors in 2D also posses two topologically invariant phases and exhibit Majorana fermions on edges between them [10]. The smoking gun for topological superconductors and superfluids is the existence of zero energy Majorana modes in the vortices of their order parameters [1–3, 8, 9, 11–17, 19]. Majorana fermions are real fermions which are their own antiparticles [20]. Sau et. al. [1] suggested creating Majorana fermions in vortices of ferromagnetic insulator/semiconductor/s-wave superconductor superstructures. The authors of [1] showed that Majorana fermions exist in this setup by solving the single vortex core problem for the effective Hamiltonian for the superstructure [1]. Alicea [11] extended the work of Sau et al. [1] to replace the external magnetic field with a ferromagnet. Shen presented [17] an equivalence between the model in Sau et. al. [1] and spinless $p+ip$ superconductors. In this work for clarity in the supplement we extend these results and show that the low energy effective action for large magnetic fields is that of a spinless $p+ip$ superconductor [17] (however this is not the main thrust of this work). These ideas about 2D s-wave superconductor proximitized semiconductors were extended to 1D by [2, 3] where it was shown that such ferromagnetic insulator/semiconductor/s-wave superconductor heterostructures but with 1D semiconductor wires have Majorana modes at the ends of the wires much like 1D spinless p-wave superconductors. In this work, in the supplement [21], we show that the low energy effective action is that of a 1D spinless p-wave superconductor as well.

The main thrust of this this work is to extend the ideas of Sau et al. [1] as well as the 1D case of Refs. [2, 3] to the case of two band s-wave superconductor proximitized semiconductors with both Kane Mele like [4] and Rashba [22] spin orbit coupling. Furthermore no large magnetic fields (or ferromagnets) are involved as in the Sau-Luchtin-Tewari-Sarma [1] and Oreg-Refael-von Oppen [2, 3] proposals - which is highly advantageous to superconductivity [23]. We replace the time reversal symmetry breaking effects of the magnetic field with spin orbit coupling (which is time reversal (TR) invariant) thereby doubling the proposals in Refs. [1–3] in a TR invariant way. This is much like the Kane-Mele spin Hall proposal [4] doubled the Haldane model [5] in a TR invariant way. We show that to leading order the low energy effective action in 2D is time reversal invariant p-wave superconductivity with two copies of $p+ip$ superconductors with opposite chirality, the two copies are in pseudospin space rather then in spin space as is the usual case with TR invariant superconductors [17, 19]. Furthermore in the supplement we show that "no-go" theorems derived previously [7, 8] make assumptions that are not general enough to cover proximity effects from multi-band superconductors considered here and as such do not apply to the setup presented in this work. Majorana fermions are found in vortex cores for the superconductors in this work and their stability is proved for small perturbations. In 1D, for the superconductors in this work, we find Majorana fermions at the end of wires (these are also stable to small perturbations) with the effective low energy theory of the doubled Oreg-Refael-von Oppen [2, 3] proposal is that of a single band time reversal invariant p-wave superconductor with pseudospin rather then spin degree of freedom.

II. GENERAL CONSIDERATIONS ABOUT TIME REVERSAL AND THE TYPE OF ORBITALS NEEDED FOR OUR CONSTRUCTION

In order to obtain the doubled Sau-Luchtin-Tewari-Sarma [1] and Oreg-Refael-von Oppen [2, 3] proposals we require two spinful orbitals with specific time reversal properties given by Eq. (5) below. Any two orbitals with this transform property under time reversal will do. Here we will show that these time reversal properties are not exotic by presenting an explicit physical realization of such orbitals with $\left|Y_1^1\right\rangle, \left|Y_{-1}^1\right\rangle$ (which transform under TR as in Eq. (5)). Here Y_m^l are spherical harmonics. For the rest of the paper, for simplicity, the reader may focus on this realization if they so please, though many others are possible.

A. Physical setup

We now present explicit orbitals with time reversal properties given by Eq. (5) below. For concreteness we will consider the case of p_x and p_y orbitals which are a linear combination of $\left|Y_1^1\right\rangle, \left|Y_{-1}^1\right\rangle$. We will work in the basis:

$$|+\rangle = \left|Y_1^1\right\rangle \sim \exp\left(i\theta\right),\ |-\rangle = \left|Y_{-1}^1\right\rangle \sim -\exp\left(-i\theta\right). \tag{1}$$

Here θ is the azimuthal angle. For each lattice site for the systems we will consider there will be four relevant basis states:

$$|+,\uparrow\rangle,\ |+,\downarrow\rangle,\ |-,\uparrow\rangle,\ |-,\downarrow\rangle \tag{2}$$

We will use τ as the Pauli matrices within the orbital space and σ to be the Pauli matrices within the spin space, below we shall also introduce μ which are Pauli matrices for particle-hole space. We note that $\left|Y_1^2\right\rangle, \left|Y_{-1}^2\right\rangle$ are just as good for our purposes of obtaining Eq. (5) as well as many other orbitals.

B. Time reversal symmetry

Under time reversal symmetry we have that $\vec{\sigma} \to -\vec{\sigma}$. Furthermore under time reversal $Y_m^l \to (-1)^m Y_{-m}^l$ (as $i \to -i$, see Eq. (1)) which means that we have that in the model we propose below the time reversal operator is given by

$$T : -\tau_x i\sigma_y K \tag{3}$$

Where K is complex conjugation. This means that we have

$$T : \tau_z \to -\tau_z,\ \tau_x \to \tau_x,\ \tau_y \to \tau_y \tag{4}$$

Furthermore under time reversal $\mathbf{k} \to -\mathbf{k}$ (here $\mathbf{k}$ is the pseudo-momentum in the first Brillouin zone). As such we have that:

$$\begin{aligned} T : & c^\dagger_{\mathbf{k}+\uparrow} \to -c^\dagger_{-\mathbf{k}-\downarrow},\ c^\dagger_{\mathbf{k}+\downarrow} \to +c^\dagger_{-\mathbf{k}-\uparrow}, \\ & c^\dagger_{\mathbf{k}-\uparrow} \to -c^\dagger_{-\mathbf{k}+\downarrow},\ c^\dagger_{\mathbf{k}-\downarrow} \to +c^\dagger_{-\mathbf{k}+\uparrow} \end{aligned} \tag{5}$$

III. DOUBLING THE SAU-LUCHTIN-TEWARI-SARMA PROPOSAL (P-WAVE TIME REVERSAL PRESERVING HAMILTONIAN)

A. Global analysis

We consider a two band spinful BDG Hamiltonian given by:

$$\begin{aligned} H\left(\mathbf{k}\right) = \mu_z & \left[\left(\frac{\mathbf{k}^2}{2m} - \mu\right) + \lambda_{SO}\sigma_z\tau_z + \lambda_R\left[\mathbf{k}_y\sigma_x - \mathbf{k}_x\sigma_y\right]\right] \\ & + \Delta_s\mu_x \end{aligned} \tag{6}$$

We note that $\left[H\left(\mathbf{k}\right), \tau_z\right] = 0$ so the two orbitals decouple and the Hamiltonian is doubled and $\lambda_{SO}\sigma_z\tau_z$ is similar to the spin orbit coupling used by Kane and Mele 4 [4] except τ_z is not in the valley space for graphene 4 [4]. We now use the basis:

$$\begin{aligned} \Psi_{\mathbf{k}}^{\tau_z} = \Big(& c_{\mathbf{k}+\uparrow}, c_{\mathbf{k}+\downarrow}, c^\dagger_{-\mathbf{k}+\downarrow}, -c^\dagger_{-\mathbf{k}+\uparrow} \\ & c_{\mathbf{k}-\uparrow}, c_{\mathbf{k}-\downarrow}, c^\dagger_{-\mathbf{k}-\downarrow}, -c^\dagger_{-\mathbf{k}-\uparrow} \Big)^T \end{aligned} \tag{7}$$

As such:

$$H\left(\mathbf{k}\right) = \begin{pmatrix} H_+\left(\mathbf{k}\right) & 0 \\ 0 & H_-\left(\mathbf{k}\right) \end{pmatrix} \tag{8}$$

with

$$\begin{aligned} H_\pm\left(\mathbf{k}\right) = & \mu_z\left[\left(\frac{\mathbf{k}^2}{2m} - \mu\right) \pm \lambda_{SO}\sigma_z + \lambda_R\left[\mathbf{k}_y\sigma_x - \mathbf{k}_x\sigma_y\right]\right] \\ & + \Delta_s\mu_x \end{aligned} \tag{9}$$

being two time reversed copies of the Sau-Luchtin-Tewari-Sarma Hamiltonians (the total Hamiltonian is TR invariant see Eq. (5)). Furthermore since this is an exact doubling the phase boundaries (where the Hamiltonian become gapless) are identical to those in the proposal by Sau et. al. [1] with the critical boundary given by [1]:

$$\mu = \sqrt{\lambda_{SO}^2 - \Delta_s^2} \tag{10}$$

B. Effective low energy theory

1. Hamiltonian ignoring the effects of pairing

We now introduce the spinor $\Psi_{\mathbf{k}} = \left(c_{\mathbf{k}+\uparrow}, c_{\mathbf{k}+\downarrow}, c_{\mathbf{k}-\uparrow}, c_{\mathbf{k}-\downarrow}\right)^T$ then we consider the Hamiltonian given by:

$$H\left(\mathbf{k}\right) = \Psi_{\mathbf{k}}^\dagger \mathcal{H}\left(\mathbf{k}\right)\Psi_{\mathbf{k}} \tag{11}$$

with:

$$\mathcal{H}\left(\mathbf{k}\right) = \left(\frac{\mathbf{k}^2}{2m} - \mu\right) + \lambda_{SO}\tau_z\sigma_z + \lambda_R\left(\mathbf{k}_x\sigma_y - \mathbf{k}_y\sigma_x\right) \tag{12}$$

It is straightforward to check that $\left[\mathcal{H}\left(\mathbf{k}\right),\tau_z\right]=0$ and that the Hamiltonian in Eq. (12) is TR invariant and corresponds to the non-pairing piece of the Hamiltonian in Eq. (6). Now we will assume that $\lambda_{SO}\gg\lambda_R$ so that the low energy subspace of the Hamiltonian has a basis given by [21]:

$$
\begin{aligned}
\left|\Uparrow\right\rangle &= \left|+,\downarrow\right\rangle - \frac{\lambda_R}{2\lambda_{SO}}\left(i\mathbf{k}_x-\mathbf{k}_y\right)\left|+,\uparrow\right\rangle + ... \\
\left|\Downarrow\right\rangle &= \left|-,\uparrow\right\rangle - \frac{\lambda_R}{2\lambda_{SO}}\left(-i\mathbf{k}_x-\mathbf{k}_y\right)\left|-,\downarrow\right\rangle + ...
\end{aligned} \tag{13}
$$

this being a Kramers doublet, with the Hamiltonian is this basis being given by [21]:

$$
H\left(\mathbf{k}\right)=\left[\left(\frac{\mathbf{k}^2}{2m}-\mu\right)-\lambda_{SO}\right]\left[c^\dagger_{\mathbf{k},\Uparrow}c_{\mathbf{k},\Uparrow}+c^\dagger_{\mathbf{k},\Downarrow}c_{\mathbf{k},\Downarrow}\right]+... \tag{14}
$$

2. Adding a small pairing

We consider adding

$$
H_\Delta\left(\mathbf{k}\right)=\Delta c^\dagger_{\mathbf{k}+\uparrow}c^\dagger_{-\mathbf{k}+\downarrow}+\Delta^* c^\dagger_{\mathbf{k}-\uparrow}c^\dagger_{-\mathbf{k}-\downarrow}+h.c. \tag{15}
$$

to the Hamiltonian in Eq. (11) which corresponds to the pairing piece of the Hamiltonian in Eq. (6).. Then we have that:

$$
\begin{aligned}
&T\sum_{\mathbf{k}}H_\Delta\left(\mathbf{k}\right)T= \\
&=\sum_{\mathbf{k}}\left[-\Delta^* c^\dagger_{-\mathbf{k}-\downarrow}c^\dagger_{\mathbf{k}-\uparrow}-\Delta c^\dagger_{-\mathbf{k}+\downarrow}c_{\mathbf{k}-\uparrow}+h.c.\right] \\
&=\sum_{\mathbf{k}}H_\Delta\left(\mathbf{k}\right),
\end{aligned} \tag{16}
$$

which means the pairing term in Eq. (15) is time reversal invariant. Now we consider the case where Δ is the smallest energy scale to the Hamiltonian in Eq. (11). As such it is sufficient to do zeroth order perturbation theory in Δ. Now we write [21]:

$$
\begin{aligned}
c^\dagger_{\mathbf{k}+\uparrow} &= -\frac{\lambda_R}{2\lambda_{SO}}\left(-i\mathbf{k}_x-\mathbf{k}_y\right)c^\dagger_{\mathbf{k}\Uparrow}+... \\
c^\dagger_{\mathbf{k}+\downarrow} &= c^\dagger_{\mathbf{k}\Uparrow}+... \\
c^\dagger_{\mathbf{k}-\uparrow} &= c^\dagger_{\mathbf{k}\Downarrow}+... \\
c^\dagger_{\mathbf{k}-\downarrow} &= -\frac{\lambda_R}{2\lambda_{SO}}\left(+i\mathbf{k}_x-\mathbf{k}_y\right)c^\dagger_{\mathbf{k}\Downarrow}+...
\end{aligned} \tag{17}
$$

As such computing matrix elements of the Hamiltonian in Eq. (15) we see that within the low energy subspace:

$$
\begin{aligned}
H_\Delta\left(\mathbf{k}\right)=&\frac{\lambda_R\Delta}{2\lambda_{SO}}\left(i\mathbf{k}_x-\mathbf{k}_y\right)c^\dagger_{\mathbf{k}\Uparrow}c^\dagger_{-\mathbf{k}\Uparrow}+h.c. \\
&-\frac{\lambda_R\Delta^*}{2\lambda_{SO}}\left(-i\mathbf{k}_x-\mathbf{k}_y\right)c^\dagger_{\mathbf{k}\Downarrow}c^\dagger_{-\mathbf{k}\Downarrow}+h.c.
\end{aligned} \tag{18}
$$

As such the total low energy effective Hamiltonian is given by:

$$
\begin{aligned}
H\left(\mathbf{k}\right)=&\left[\left(\frac{\mathbf{k}^2}{2m}-\mu\right)-\lambda_{SO}\right]\left[c^\dagger_{\mathbf{k},\Uparrow}c_{\mathbf{k},\Uparrow}+c^\dagger_{\mathbf{k},\Downarrow}c_{\mathbf{k},\Downarrow}\right] \\
&+\left[\frac{\lambda_R\Delta}{2\lambda_{SO}}\left(-i\mathbf{k}_x-\mathbf{k}_y\right)c^\dagger_{\mathbf{k}\Uparrow}c^\dagger_{-\mathbf{k}\Uparrow}\right. \\
&\left.-\frac{\lambda_R\Delta^*}{2\lambda_{SO}}\left(i\mathbf{k}_x-\mathbf{k}_y\right)c^\dagger_{\mathbf{k}\Downarrow}c^\dagger_{-\mathbf{k}\Downarrow}+h.c.\right]
\end{aligned} \tag{19}
$$

Which is a 2D TR invariant Hamiltonian with pseudospin degree of freedom: $\Uparrow$, $\Downarrow$.

C. Stability analysis

It is known that superconductors whose effective theory is given by Eq. (19) have a pair of Majorana Fermions inside vortex cores [17, 19], denoted by γ_1 and γ_2. We now show that these modes are stable until a gap closing transition. Indeed from Brillouin-Wigner perturbation theory we know that the zero energy manifold for a vortex has an effective Hamiltonian which converges until there is a gap closing phase transition [18]. By requiring the Hamiltonian be Hermitian and using fermion parity conservation we must have that the effective Hamiltonian for the zero energy subspace is given by:

$$
H_{eff}=i\Gamma\gamma_1\gamma_2 \tag{20}
$$

For some Γ. If we further assume the perturbation preserves time reversal T we must have that:

$$
Ti\Gamma\gamma_1\gamma_2T^{-1}=-i\Gamma\gamma_1\gamma_2=i\Gamma\gamma_1\gamma_2\Rightarrow H_{eff}=0 \tag{21}
$$

so the Majorana modes are stable (cannot open a gap) and hence the band topology is stable under small TR preserving perturbations as well.

IV. 1-D CASE: DOUBLING THE OREG-REFAEL-VON OPPEN PROPOSAL

A. Global analysis

We consider a total (doubled) Hamiltonian given by:

$$
\begin{aligned}
H\left(k\right)=&\mu_z\left[\left(\frac{k^2}{2m}-\mu\right)+\lambda_{SO}\sigma_z\tau_z+\lambda_R k\sigma_x\right] \\
&+\Delta_s\mu_x
\end{aligned} \tag{22}
$$

We note that $\left[H\left(k\right),\tau_z\right]=0$ so the two bands decouple and the Hamiltonian is doubled. We now use the basis:

$$
\begin{aligned}
\Psi^{\tau_z}_k=&\left(c_{k+\uparrow},c_{k+\downarrow},c^\dagger_{-k+\downarrow},-c^\dagger_{-k+\uparrow}\right. \\
&\left.c_{k-\uparrow},c_{k-\downarrow},c^\dagger_{-k-\downarrow},-c^\dagger_{-k-\uparrow}\right)^T
\end{aligned} \tag{23}
$$

So that the Hamiltonian is explicitly doubled:

$$H(k)=\begin{pmatrix} H_{+}(k) & 0 \\ 0 & H_{-}(k) \end{pmatrix} \tag{24}$$

with

$$H_{\pm}(k)=\mu_z\left[\left(\frac{k^2}{2m}-\mu\right)\pm\lambda_{SO}\sigma_z+\lambda_R k\sigma_x\right]+\Delta_s\mu_x \tag{25}$$

being two time reversed copies of the Oreg-Refael-von Oppen Hamiltonians [2, 3]. Therefore the phase boundaries are identical to those in that proposal with the critical boundary be given by the gap closing transition [2] given by Eq. (10) above.

B. Effective low energy theory

1. Hamiltonian no pairing

We now introduce the spinor $\Psi_k = (c_{k+\uparrow}, c_{k+\downarrow}, c_{k-\uparrow}, c_{k-\downarrow})^T$ then we consider the Hamiltonian given by:

$$H(k)=\Psi_k^{\dagger}\mathcal{H}(k)\Psi_k \tag{26}$$

with:

$$\mathcal{H}(k)=\left(\frac{k^2}{2m}-\mu\right)+\lambda_{SO}\tau_z\sigma_z+\lambda_R k\sigma_x \tag{27}$$

It is straightforward to check that $[\mathcal{H}(k),\tau_z]=0$ and that the Hamiltonian in Eq. (27) is time reversal invariant and corresponds to the non-pairing piece of the Hamiltonian in Eq. (22). Now we will assume that $\lambda_{SO}\gg\lambda_R$ so that the low energy Hamiltonian is given by [21]:

$$\begin{aligned} |\Uparrow\rangle &= |+,\downarrow\rangle-\frac{\lambda_R}{2\lambda_{SO}}k\,|+,\uparrow\rangle+... \\ |\Downarrow\rangle &= |-,\uparrow\rangle-\frac{\lambda_R}{2\lambda_{SO}}k\,|-,\downarrow\rangle+... \end{aligned} \tag{28}$$

being a Kramers doublet, with

$$H(\mathbf{k})=\left[\left(\frac{k^2}{2m}-\mu\right)-\lambda_{SO}\right]\left[c_{k,\Uparrow}^{\dagger}c_{k,\Uparrow}+c_{k,\Downarrow}^{\dagger}c_{k,\Downarrow}\right]+... \tag{29}$$

where we have focused just on the low energy subspace.

2. Adding pairing

We consider adding

$$H_{\Delta}(k)=\Delta c_{k+\uparrow}^{\dagger}c_{-k+\downarrow}^{\dagger}+\Delta^{*}c_{k-\uparrow}^{\dagger}c_{-k-\downarrow}^{\dagger}+h.c. \tag{30}$$

to the Hamiltonian in Eq. (26) which corresponds to the pairing piece of the Hamiltonian in Eq. (22) and is TR invariant see Eq. (16). Now we write [21]:

$$\begin{aligned} c_{k+\uparrow}^{\dagger} &= -\frac{\lambda_R}{2\lambda_{SO}}kc_{k\Uparrow}^{\dagger}+... \\ c_{k+\downarrow}^{\dagger} &= c_{k\Uparrow}^{\dagger}+... \\ c_{k-\uparrow}^{\dagger} &= c_{k\Downarrow}^{\dagger}+... \\ c_{k-\downarrow}^{\dagger} &= -\frac{\lambda_R}{2\lambda_{SO}}kc_{k\Downarrow}^{\dagger}+... \end{aligned} \tag{31}$$

As such we have that within the low energy subspace [21]:

$$H_{\Delta}(k)=\frac{\lambda_R\Delta}{2\lambda_{SO}}kc_{k\Uparrow}^{\dagger}c_{-k\Uparrow}^{\dagger}-\frac{\lambda_R\Delta^{*}}{2\lambda_{SO}}kc_{k\Downarrow}^{\dagger}c_{-k\Downarrow}^{\dagger}+h.c. \tag{32}$$

As such the total low energy effective Hamiltonian is given by [21]:

$$\begin{aligned} H(k) &= \left[\left(\frac{k^2}{2m}-\mu\right)-\lambda_{SO}\right]\left[c_{k,\Uparrow}^{\dagger}c_{k,\Uparrow}+c_{k,\Downarrow}^{\dagger}c_{k,\Downarrow}\right] \\ &+\left[\frac{\lambda_R\Delta}{2\lambda_{SO}}kc_{k\Uparrow}^{\dagger}c_{-k\Uparrow}^{\dagger}-\frac{\lambda_R\Delta^{*}}{2\lambda_{SO}}kc_{k\Downarrow}^{\dagger}c_{-k\Downarrow}^{\dagger}+h.c.\right] \end{aligned} \tag{33}$$

Which is a p-wave 1D time reversal invariant Hamiltonian with pseudospin degree of freedom.

C. Stability analysis

The stability analysis is verbatim that of Section III C.

V. CONCLUSIONS & OUTLOOK

In this work we have doubled in a TR preserving way the Sau-Luchtin-Tewari-Sarma [1] and Oreg-Refael-von Oppen [2, 3] proposals much like the Kane-Mele proposal (spin Hall insulator) [4] doubles the Haldane model [5] in a TR preserving way. We have shown that the low energy effective action for these models in 1D and 2D are the time reversal invariant single band p-wave superconductors with pseudospin degree of freedom. Similarly the effective action for the Sau-Luchtin-Tewari-Sarma [1] and Oreg-Refael-von Oppen [2, 3] are the single band $p+ip$ and single band 1D p-wave superconductors with spin degree of freedom. In the supplement [21] we have presented physical realization of the system within the cold atoms setup [7, 8]. In future works it would be of interest to study real solid state systems with spin orbit coupling and proximity induces s-wave superconductivity for a completely realistic realization of the models presented in this work in solid state heterostructures which would open this proposal to many applications in quantum computing [24].

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Supplementary Online Information

Appendix A: Background (lightening review)

1. Topological superconductivity

a. *Time reversal symmetry breaking p-wave 2D superconductors*

The simplest time reversal symmetry breaking p-wave 2D superconductors have the following Hamiltonian:

$$\begin{aligned} H(\mathbf{k}) = & \left(\frac{\mathbf{k}^2}{2m} - \mu\right) c^\dagger_{\mathbf{k}} c_{\mathbf{k}} \\ & + \Delta\left(\mathbf{k}_x \pm i\mathbf{k}_y\right) c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} + h.c. \end{aligned} \tag{A1}$$

Where $\pm$ corresponds to $p_x + ip_y$ and $p_x - ip_y$ Hamiltonians. It is known to have Majorana fermions in the vortices of its order parameter [17, 19].

b. *Time reversal symmetry invariant p-wave 2D Hamiltonians*

The simplest time reversal symmetry preserving p-wave 2D superconductors have the following Hamiltonian [7]:

$$\begin{aligned} H(\mathbf{k}) = & \sum_{\sigma=\pm}\left(\frac{\mathbf{k}^2}{2m} - \mu\right) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \\ & + \Delta\left(\mathbf{k}_x \pm i\mathbf{k}_y\right) c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\uparrow} + h.c. \\ & - \Delta^*\left(\mathbf{k}_x \mp i\mathbf{k}_y\right) c^\dagger_{\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow} + h.c. \end{aligned} \tag{A2}$$

Which is just two time reversal invariant copies of the Hamiltonian in Eq. (A1).

c. *Time reversal symmetry breaking p-wave 1D superconductors*

The simplest time reversal symmetry breaking p-wave 1D superconductors have the following Hamiltonian:

$$H(k) = \left(\frac{k^2}{2m} - \mu\right) c^\dagger_k c_k + \left[\Delta k c^\dagger_k c^\dagger_{-k} + h.c.\right] \tag{A3}$$

It is known to have Majorana fermions at the ends of 1D wires [17, 19].

d. *Time reversal symmetry invariant p-wave 1D Hamiltonians*

The simplest time reversal symmetry preserving p-wave 2D superconductors have the following Hamiltonian [7]:

$$\begin{aligned} H(k) = & \sum_{\sigma=\pm}\left(\frac{k^2}{2m} - \mu\right) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \\ & + \left[\Delta k c^\dagger_{k\uparrow} c^\dagger_{-k\uparrow} - \Delta^* k c^\dagger_{k\downarrow} c^\dagger_{-k\downarrow} + h.c.\right] \end{aligned} \tag{A4}$$

Which is just two time reversal invariant copies of the Hamiltonian in Eq. (A3).

2. Quantum mechanics

a. *First order perturbation theory*

We note that if

$$H_0 |E_0\rangle = E_0 |E_0\rangle \tag{A5}$$

and $H = H_0 + V$ with $V \ll H_0$ then:

$$\begin{aligned} H|E\rangle &= E|E\rangle \\ E &= E_0 + \langle E_0| V |E_0\rangle + ... \\ |E\rangle &= |E_0\rangle - \sum_{E'_0 \neq E_0} \frac{\langle E'_0| V |E_0\rangle}{E'_0 - E_0} |E'_0\rangle + ... \end{aligned} \tag{A6}$$

b. *Change of basis*

Suppose you have an orthonormal basis $\left\{|U_i\rangle_{i=1,..N}\right\}$ and a second orthonormal basis $\left\{|V_i\rangle_{i=1,...N}\right\}$ then we have that:

$$|U_j\rangle = \sum_{i=1}^{N} |V_i\rangle \langle V_i \mid U_j\rangle \tag{A7}$$

Now suppose there is some relevant Hamiltonian such that $|V_i\rangle_{i=1,..M}$ is much lower in energy then $|V_i\rangle_{i=M+1,...N}$ then in many cases we may write that:

$$|U_j\rangle = \sum_{i=1}^{M} |V_i\rangle \langle V_i \mid U_j\rangle + ... \tag{A8}$$

Appendix B: Proximity induced superconductivity

1. Sau-Luchtin-Tewari-Sarma proposal [1] (simplified treatment)

Consider the following Hamiltonian:

$$H(\mathbf{k}) = \left(c^\dagger_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\downarrow}\right) h(\mathbf{k}) \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{\mathbf{k}\downarrow} \end{pmatrix} \tag{B1}$$

with

$$h(\mathbf{k}) = \left(\frac{\mathbf{k}^2}{2m} - \mu\right) + \lambda_R(\mathbf{k}_x\sigma_y - \mathbf{k}_y\sigma_x) + B\sigma_z$$
$$h_0(\mathbf{k}) = \left(\frac{\mathbf{k}^2}{2m} - \mu\right) + B\sigma_z$$
$$V(\mathbf{k}) = \lambda_R(\mathbf{k}_x\sigma_y - \mathbf{k}_y\sigma_x) \tag{B2}$$

Then using Eq. (A6) we have that

$$|\Omega\rangle = |\downarrow\rangle - \frac{\lambda_R}{2B}(i\mathbf{k}_x - \mathbf{k}_y)|\uparrow\rangle + ...$$
$$E_\Omega = \left(\frac{\mathbf{k}^2}{2m} - \mu\right) - B + ... \tag{B3}$$

Where we have assumed that $B \gg \lambda_R$. Now we write:

$$c^\dagger_{\mathbf{k}\downarrow} = c^\dagger_{\mathbf{k},\Omega} + ...$$
$$c^\dagger_{\mathbf{k}\uparrow} = -\frac{\lambda_R}{2B}(-i\mathbf{k}_x - \mathbf{k}_y)c^\dagger_{\mathbf{k},\Omega} + ... \tag{B4}$$

Now we add s-wave paring to the Hamiltonian in Eq. (B1) with the new Hamiltonian being given by:

$$H(\mathbf{k}) = \left(c^\dagger_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\downarrow}\right) h(\mathbf{k}) \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{\mathbf{k}\downarrow} \end{pmatrix} + \left[\Delta c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + h.c.\right] \tag{B5}$$

Now within the low energy subspace spanned by $|\Omega\rangle$ we have that:

$$H_\Omega(\mathbf{k}) = \left[\left(\frac{\mathbf{k}^2}{2m} - \mu\right) - B\right] c^\dagger_{\mathbf{k},\Omega} c_{\mathbf{k},\Omega}$$
$$+ \left[\frac{\lambda_R\Delta}{2B} c^\dagger_{\mathbf{k}\Omega} c^\dagger_{-\mathbf{k}\Omega}(\mathbf{k}_y + i\mathbf{k}_x) + h.c.\right] \tag{B6}$$

or a 2D p-wave time reversal breaking Hamiltonian.

2. Oreg-Refael-von Oppen proposal [2, 3] (simplified treatment)

Consider the following Hamiltonian:

$$H(k) = \left(c^\dagger_{k\uparrow}, c^\dagger_{k\downarrow}\right) h(k) \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} \tag{B7}$$

with

$$h(k) = \left(\frac{k^2}{2m} - \mu\right) + \lambda_R k\sigma_x + B\sigma_z$$
$$h_0(k) = \left(\frac{k^2}{2m} - \mu\right) + B\sigma_z$$
$$V(k) = \lambda_R k\sigma_x \tag{B8}$$

Then using Eq. (A6) we have that

$$|\Omega\rangle = |\downarrow\rangle - \frac{\lambda_R}{2B} k|\uparrow\rangle + ...$$
$$E_\Omega = \left(\frac{k^2}{2m} - \mu\right) - B + ... \tag{B9}$$

Where we have assumed that $B \gg \lambda_R$. Now we write:

$$c^\dagger_{k\downarrow} = c^\dagger_{k\Omega} + ...$$
$$c^\dagger_{k\uparrow} = -\frac{\lambda_R}{2B} k c^\dagger_{k,\Omega} + ... \tag{B10}$$

Now we add s-wave paring to the Hamiltonian in Eq. (B7) with the new Hamiltonian being given by:

$$H(k) = \left(c^\dagger_{k\uparrow}, c^\dagger_{k\downarrow}\right) h(k) \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} + \left[\Delta c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + h.c.\right] \tag{B11}$$

Now within the low energy subspace spanned by $|\Omega\rangle$ we have that:

$$H_\Omega(k) = \left[\left(\frac{k^2}{2m} - \mu\right) - B\right] c^\dagger_{k,\Omega} c_{k,\Omega}$$
$$+ \frac{\lambda_R\Delta}{2B} c^\dagger_{k\Omega} c^\dagger_{-k\Omega} k + h.c. \tag{B12}$$

or a 1D p-wave time reversal breaking Hamiltonian.

Appendix C: Cold atoms realizations of the proposals in the main text

We would like to emulate the Hamiltonian given in Ref. [16] which is similar to ours but has different time reversal properties using cold atoms. The Hamiltonian we would like to emulate is given by:

$$H(k) = \mu_z\left[\left(\frac{k^2}{2m} - \mu\right) + \lambda_{SO}\sigma_x + \lambda_R k\sigma_z\tau_z\right]$$
$$+ \Delta_s\mu_x\tau_z \tag{C1}$$

We will closely be following [6]. Now we consider the setup with four lasers see Fig. (1). We have that the Hamiltonian is given by:

$$H = \sum_k \left(\frac{k^2}{2m} - V\right) n_k +$$
$$B\sum_k \left(c^\dagger_{k+p+\uparrow} c_{k-p+\downarrow} + c^\dagger_{k+p-\uparrow} c_{k-p-\downarrow} + h.c.\right) \tag{C2}$$

Where

$$n_k = c^\dagger_{k+\uparrow} c_{k+\uparrow} + c^\dagger_{k+\downarrow} c_{k+\downarrow} + c^\dagger_{k-\uparrow} c_{k-\uparrow} + c^\dagger_{k-\downarrow} c_{k-\downarrow}$$
$$B = \frac{\Omega_1\Omega_2^*}{\Delta} \tag{C3}$$

and p is the momentum of the lasers. We will now assume two species of molecules with the Hamiltonian:

$$H_{Fes} = g\int dx b_+(x) c^\dagger_{+\uparrow}(x) c^\dagger_{+\downarrow}(x) + h.c.$$
$$+ g\int dx b_-(x) c^\dagger_{-\uparrow}(x) c^\dagger_{-\downarrow}(x) + h.c. \tag{C4}$$

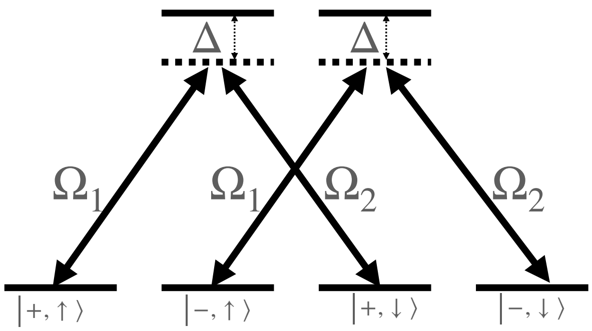

Figure 1. Thelase setup for the could atom realization.

Now we assume that due to molecule condensation and an order from disorder effect

$$g\left\langle b_{+}\left(x\right)\right\rangle =-g\left\langle b_{-}\left(x\right)\right\rangle =\Xi \tag{C5}$$

This means that:

$$\begin{aligned}H=&\sum_{k}\left(\frac{k^{2}}{2m}-V\right)n_{k}\\&+\sum_{k}B\left(c_{k+p+\uparrow}^{\dagger}c_{k-p+\downarrow}+c_{k+p-\uparrow}^{\dagger}c_{k-p-\downarrow}+h.c.\right)\\&+\sum_{k}\Xi\left(c_{k+\uparrow}^{\dagger}c_{-k+\downarrow}^{\dagger}-c_{k-\uparrow}^{\dagger}c_{-k-\downarrow}^{\dagger}+h.c.\right)\end{aligned} \tag{C6}$$

We now perform the transform

$$\exp\left(ip\int x\left(n_{+\uparrow}\left(x\right)-n_{+\downarrow}\left(x\right)-n_{-\uparrow}\left(x\right)+n_{-\downarrow}\left(x\right)\right)\right) \tag{C7}$$

This transforms the Hamiltonian to the form:

$$\begin{aligned}H=&\sum_{k}\sum_{\sigma_z=\pm;\tau_z=\pm}\left(\frac{\left(k-p\sigma_z\tau_z\right)^{2}}{2m}-V\right)n_{k\sigma_z\tau_z}\\&+\sum_{k}B\left(c_{k+\uparrow}^{\dagger}c_{k+\downarrow}+c_{k-\uparrow}^{\dagger}c_{k-\downarrow}+h.c.\right)\\&+\sum_{k}\Xi\left(c_{k-p+\uparrow}^{\dagger}c_{-k+p+\downarrow}^{\dagger}-c_{k-p-\uparrow}^{\dagger}c_{-k+p-\downarrow}^{\dagger}+h.c.\right)\\=&\sum_{k}\left(\frac{k^{2}}{2m}-V+\frac{p^{2}}{2m}\right)n_{k}\\&-\sum_{k}\sum_{\sigma_z=\pm;\tau_z=\pm}\left(\frac{kp\sigma_z\tau_z}{m}\right)n_{k\sigma_z\tau_z}\\&+\sum_{k}B\left(c_{k+\uparrow}^{\dagger}c_{k+\downarrow}+c_{k-\uparrow}^{\dagger}c_{k-\downarrow}+h.c.\right)\\&+\sum_{k}\Xi\left(c_{k-p+\uparrow}^{\dagger}c_{-k+p+\downarrow}^{\dagger}-c_{k-p-\uparrow}^{\dagger}c_{-k+p-\downarrow}^{\dagger}+h.c.\right)\end{aligned} \tag{C8}$$

Appendix D: Comparison with Ref. [7]

In Appendix A of Ref. [7] the authors present results that the constructions presented in the main text are impossible. We make no quarrel with the math presented in Ref. [7] however their main assumption in Appendix A is that the paring Hamiltonian multiplied by the time reversal matrix is positive semidefinite (in which case, to summarize their results, the order parameter cannot wind in $\mathbf{k}$ space, see e.g. Eq. (A1) so the superconductor is trivial) is too restrictive. However if we consider the form of Eq. (5) we see their main assumption about positive semi definiteness does not apply to our setup. Indeed we see that:

$$\begin{aligned}&T\Delta_{\mathbf{k}}c_{\mathbf{k}+\uparrow}^{\dagger}\rightarrow-\Delta Tc_{-\mathbf{k}+\downarrow}^{\dagger}\rightarrow-\Delta c_{\mathbf{k}-\uparrow}^{\dagger},\\&T\Delta_{\mathbf{k}}c_{\mathbf{k}+\downarrow}^{\dagger}\rightarrow\Delta Tc_{-\mathbf{k}+\downarrow}^{\dagger}\rightarrow-\Delta c_{\mathbf{k}-\downarrow}^{\dagger},\\&T\Delta_{\mathbf{k}}c_{\mathbf{k}-\uparrow}^{\dagger}\rightarrow-\Delta^{*}Tc_{-\mathbf{k}-\downarrow}^{\dagger}\rightarrow-\Delta^{*}c_{\mathbf{k}+\uparrow}^{\dagger},\\&T\Delta_{\mathbf{k}}c_{\mathbf{k}-\downarrow}^{\dagger}\rightarrow\Delta^{*}Tc_{-\mathbf{k}-\downarrow}^{\dagger}\rightarrow-\Delta^{*}c_{\mathbf{k}+\downarrow}^{\dagger}\end{aligned} \tag{D1}$$

Where $\Delta_{\mathbf{k}}$ is the restriction of the pairing matrix to the wavevector $\mathbf{k}$. As such we have that:

$$T\Delta_{\mathbf{k}}=-\left(\begin{array}{cccc}0&0&\Delta^{*}&0\\0&0&0&\Delta^{*}\\\Delta&0&0&0\\0&\Delta&0&0\end{array}\right) \tag{D2}$$

is not positive or negative semidefinite as it has eigenvalues $\pm\left|\Delta\right|$, as such the main assumption of Appendix A of Ref. [7] does not apply to our setup.

[1] J. D. Sau, R. M. Luchtyn, S. Tewari, and S. Das Sarma, Phys. Rev. Lett. **104**, 040502 (2010).

[2] Y. Oreg, G, Refael, F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).

[3] R. M. Luchtyn, J. D. Sau, S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
[4] C. L. Kane and E. G. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[5] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[6] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, P. Zoller, Phys. Rev. Lett. **106**, 220402 (2011).
[7] A. Haim, Y. Oreg, Phys. Rep. **825**, 1 (2019).
[8] A. Haim, E. Berg, K. Flensberg and Y. Oreg, Phys. Rev. B **94**, 161110 (2016).
[9] G. E. Volovik, *The Universe in a Helium Droplet* (Claredon Press, Oxford, 2003).
[10] N. Read, and D. Green, Phys. Rev. B **61**, 10267 (2000).
[11] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Nature Physics **7**, 412 (2011).
[12] H. Pan, and S. Das Sarma, Phys. Rev. Res. **2**, 013377 (2020).
[13] S. Das Sarma and H. Pan, Phys. Rev. B **103**, 195158 (2021).
[14] N. H. Lindler, G. Refael, V. Galitski, Nature Physics **7**, 490 (2011).
[15] X. L. Qi, T. L. Hughes and S.-C. Zhang, Phys. Rev. B **81**, 134508 (2010).
[16] A. Keselman, L. Fu, A. Stern and E. Berg, Phys. Rev. Lett. **111**, 116402 (2013).
[17] S.-Q. Shen, *Topological Insulators: Dirac Equation in Condensed Matter* (Springer, Singapore, 2017).
[18] I. Hubac and S. Wilson, *Brillouin-Wigner methods for Many body systems* (Springer, New York, 2010).
[19] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, 2013).
[20] E. Majorana, Nuovo Cimento **5**, 171 (1937).
[21] See supplementary online information.
[22] A. Manchon, H. C. Koo, J. Nitta, S. M. Frolov and R. A. Duine, nature materials **14**, 871 (2015).
[23] P. Coleman, *Introduction to Many-Body Physics* (Cambridge University Press, Cambridge, 2016).
[24] M. A. Nielsen, and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2011).