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COHOMOLOGICAL FLATNESS OVER DISCRETE VALUATION RINGS: NUMERICAL AND LOGARITHMIC CRITERIA

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Abstract. We give sufficient conditions for cohomological flatness (in dimension 0) over discrete valuation rings, generalising a classical result of Raynaud in two different ways. The first is an extension of Raynaud's numerical criterion to higher dimensions. The second is a logarithmic criterion: we show that a proper, fs log smooth morphism to a log regular discrete valuation ring is cohomologically flat in dimension 0. We apply this latter result to curves and torsors under abelian varieties with good reduction, providing necessary and sufficient conditions for the log smoothness of their regular models over arbitrary discrete valuation rings.

1. Introduction

As part of his groundbreaking work on algebraic spaces, Michael Artin proved a theorem that effectively settled the question of the existence of the Picard scheme. Recall that a morphism of schemes $f : X \to T$ is *cohomologically flat in dimension 0* if the formation of $f_*\mathcal{O}_X$ commutes with base change.

Theorem 1 (Artin [1, 2])**.** *Assume f is a proper, finitely presented flat morphism of schemes. If f is cohomologically flat in dimension 0, then the Picard functor $\mathbf{Pic}_{X/T}$ is representable by an algebraic space.*

This fundamental result was the crowning achievement of the work of a number of mathematicians on the Picard functor; see [16] for the history of this problem. Notably, Grothendieck had studied the problem in detail using the arsenal of new ideas and techniques he was introducing into algebraic geometry, reporting on his insights in the Séminaire Bourbaki [8]. In particular, he had identified the condition of cohomological flatness in dimension 0 as one of central importance to the problem and suggested that it might be sufficient to ensure the representability of the Picard functor as a scheme, cf. [8, V, 5.2].

As Artin's theorem suggests, this turned out to be too optimistic, and Mumford soon furnished a counterexample [8, VI, §0].[1] But the importance of cohomological flatness was noted by Michel Raynaud, who studied it in relation to questions of representability of functors related to **Pic** in the case T is the spectrum of a discrete valuation ring. The fruits of his efforts were published in [28], which, among other results, established the following 'numerical criterion' for cohomological flatness.

Theorem 2 (Raynaud [28])**.** *Let T be the spectrum of a discrete valuation ring and let $f : X \to T$ be a proper morphism. Assume the following conditions hold:*

(a) *X is integrally closed in the generic fibre of f*
(b) $f_*\mathcal{O}_X = \mathcal{O}_T$

[1] Mumford's example seems to have been an important step towards Theorem 1, cf. [5, p. 210]

(c) *either* $\dim X \leq 2$ *or* T *has residue characteristic 0.*

If the gcd of the geometric multiplicities of the components of the special fibre of f *is invertible on* T*, then* f *is cohomologically flat in dimension 0.*

Raynaud's theorem raises two immediate questions:

(1) Does the result hold without restriction on $\dim X$?
(2) If the gcd of the multiplicities is divisible by the residue characteristic of $\mathcal{O}(T)$, is there a condition of local nature that one can substitute for (a) to guarantee cohomological flatness in dimension 0?

In this paper we answer both of these questions in the affirmative. First of all, we show that the answer to (1) is a straightforward 'yes'. More precisely, we show

Theorem 3. *Let* T *be the spectrum of a discrete valuation ring and* $f : X \to T$ *a proper morphism. Denote by* $k(t)$ *the residue field of* $\mathcal{O}(T)$*. Assume the following conditions hold:*

(a) X *is integrally closed in the generic fibre of* f
(b) $H^0(X_0, \mathcal{O}_{X_0}) = k(t)$*, where* X_0 *is the reduction of the special fibre of* f*.*

If the gcd of the (apparent) multiplicities of the components of the special fibre of f *is invertible on* T*, then* f *is cohomologically flat in dimension 0.*

Even for $\dim X = 2$, this is more general Raynaud's Theorem. Indeed, in Theorem 3 the restriction on the multiplicities is weaker if $k(t)$ is imperfect. It implies Raynaud's theorem, without limitation on the dimension, see Corollary 6.

For question (2), Raynaud's examples in [28] show that the question is not as straightforward. There was essentially no progress until very recently, when we realised that cohomological flatness is a necessary condition for the log smoothness of regular curves over discrete valuation rings [20] (see also [21]). In fact, as the examples given in that paper show, log smoothness provides a satisfactory answer to question (2) for curves.

This recent result for curves begged the question of whether it held in greater generality, and indeed this is the case. To state the theorem, let T be the spectrum of a discrete valuation ring and M_T a fs log structure on T such that (T, M_T) is log regular.

Theorem 4. *Let* $f_{\log} : (X, M_X) \to (T, M_T)$ *be a morphism of fs log schemes whose underlying morphim of schemes* $f : X \to T$ *is proper. If* $f_{\log}$ *is log smooth, then* f *is cohomologically flat in dimension 0.*

The proof is surprisingly simple, the idea being to pass to a finite extension of T where $f_{\log}$ acquires reduced fibres (which exists thanks to Tsuji [32]), and then to descend by taking the quotient under the action of roots of unity.

Theorem 4 answers question (2), in the sense that if we fix a log structure M_X, then we have a local criterion for cohomological flatness in dimension 0 that applies to situations where the gcd of the multiplicities of the special fibre may be divisible by the residue characteristic. This, however, depends on the choice of M_X. Nevertheless, in many situations there is a natural choice of M_X, and the converse to Theorem 4 may even be true. Namely, for torsors under abelian varieties with good reduction, there is a canonical model (X, M_X), and we show that this model is log smooth if and only if the underlying morphism $X \to T$ is cohomologically flat in dimension 0. This generalises a result for genus 1 curves shown in [21] and [20].

For curves of higher genus, regular normal crossings models have natural log structures, but the resulting morphism of log schemes might not be log smooth, even if the underlying scheme is cohomologically flat over T. So we provide necessary and sufficient conditions for the log smoothness of such models, extending the main result of [20], notably to the case $k(t)$ is imperfect. This was an early motivation for Theorem 4.

To finish this introduction, we give a short overview of the paper and its main results.

We begin §2 by recalling the basic properties of cohomological flatness in dimension 0, before proving a converse to Artin's Theorem 1 (Prop. 2).

In §3, we recall the definition of multiplicities, Raynaud's condition (N), and show some basic properties. Then we prove Theorem 3 by studying 1-cycles with rational coefficients and their associated sheaves.

In §4, after some preliminaries we give the proof of Theorem 4, as described above. The rest of the section is devoted to the study of torsors under abelian schemes and curves.

In §4.3, we study models of torsors under the generic fibre of an abelian scheme over T. We show that the Raynaud regular model of such a torsor is log smooth if and only if it is cohomologically flat in dimension 0 over T, providing a necessary and sufficient condition for the existence of a proper log smooth model (Cor. 7).

Finally, in §4.4 we give necessary and sufficient conditions for log smoothness of proper regular curves over T, extending the main theorem of [20] to imperfect $k(t)$ and allowing horizontal components in the log structure (Thm. 6). To show this, we establish some properties of regular log smooth T-schemes of arbitrary dimension (Prop. 4).

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1.1. **Conventions.** Except in §2, T denotes the spectrum of a discrete valuation ring, $t \in T$ its closed point with residue field $k(t)$, and $u \in T$ its generic point with residue field $k(u)$. We let p denote the characteristic of $k(t)$.

An *extension* of (spectra of) discrete valuation rings is (the spectrum of) an injective local homomorphism.

Irreducible components are endowed with the reduced induced subscheme structure.

We only consider fs log schemes. In particular, fibre products are taken in the category of fs log schemes. Schemes can be viewed as fs log schemes with the trivial log structure; in that case we will often simply omit the log structure from notation. For a log structure M on a scheme X, we write $\overline{M} = M/\mathcal{O}_X^*$.

For an abelian group A and integer n, we write $A[n] \subset A$ for the subset of elements annhilated by n, and $A[\mathrm{tor}] = \varinjlim_n A[n]$ for the torsion subgroup of A. If A is finite we denote its order $|A|$.

2. Cohomological flatness

Let T be a locally noetherian scheme and $f : X \to T$ be a proper and flat morphism. Following Raynaud [28, 1.4], we will say that f is *cohomologically flat* if f is cohomologically flat in dimension 0.[2]

[2] Warning: this conflicts with the terminology of [9, III_2, 7.8.1]!

We gather some basic properties of cohomological flatness.

Proposition 1. (i) *f is cohomologically flat if and only if there is a faithfully flat morphism $S \to T$ such that $f \times_T S$ is cohomologically flat.*
(ii) *If f has geometrically reduced fibres, then f is cohomologically flat.*
(iii) *If T is reduced, then f is cohomologically flat if and only if $\dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t})$ is a locally constant function on T.*
(iv) *If T is the spectrum of a discrete valuation ring, then f is cohomologically flat if and only if $H^1(X, \mathcal{O}_X)$ is a torsion-free $\mathcal{O}(T)$-module.*
(v) *If $g : T \to S$ a finite locally free morphism and f is cohomologically flat, then so is $g \circ f$.*

Proof. (i) follows from [9, III$_2$, 7.7.11], (ii) from [9, III$_2$, 7.8.6], (iii) from [9, III$_2$, 7.8.4], and (iv) follows easily from (iii).

For (v), note that cohomological flatness implies that $f_*\mathcal{O}_X$ is a vector bundle on T (cf. [9, III$_2$, §7]). Hence $g_*f_*\mathcal{O}_X$ is a vector bundle on S. Now it follows from the general (derived) base change formula ([4, Exp. IV, 3.1.0]) that formation of $g_*f_*\mathcal{O}_X$ commutes with base change, since $f_*\mathcal{O}_X$ does. □

2.1. **Converse to Artin's theorem.** We show that Artin's Theorem 1 admits a converse, providing a criterion for cohomological flatness (cf. [5, 8.3, Rmk. 2] for the case T reduced).

Proposition 2. *If $\mathbf{Pic}_{X/T}$ is representable by an algebraic space, then f is cohomologically flat.*

Proof. By [9, III$_2$, 7.7.10 and 7.8.4], it suffices to show that the canonical map $H^0(X, \mathcal{O}_X) \to H^0(X_k, \mathcal{O}_{X_k})$ is surjective when $T = \operatorname{Spec}(A)$ is the spectrum of an artinian local ring with residue field k. Moreover, by Proposition 1 (i), we may assume A to be strictly henselian.

Define
$$C := \operatorname{cok}(H^0(X, \mathcal{O}_X^*) \to H^0(X_k, \mathcal{O}_{X_k}^*))$$

Lemma 1. *If $C = 0$, then $H^0(X, \mathcal{O}_X) \to H^0(X_k, \mathcal{O}_{X_k})$ is surjective.*

Proof. Exercise (use Nakayama's lemma and the fact k is infinite). □

So it suffices to show $C = 0$. Consider the fibre product $A \times_k A$. It is an A-algebra via the diagonal map.

Lemma 2. *The canonical map $\mathcal{O}_X \otimes_A (A \times_k A) \to \mathcal{O}_X \times_{\mathcal{O}_{X_k}} \mathcal{O}_X$ is an isomorphism.*

Proof. Indeed, since f is flat, the functor $\mathcal{O}_X \otimes_A (-)$ is exact, hence commutes with finite limits. □

Write $P = \mathbf{Pic}_{X/T}$.

Lemma 3. $\ker(P(A \times_k A) \rightrightarrows P(A)) \cong P(A) \oplus C$

Proof. Write $B := A \times_k A$. The two projections $B \rightrightarrows A$ induce canonical maps $\mathcal{O}_{X_B}^* \rightrightarrows \mathcal{O}_X^*$. Using Lemma 2, one computes the kernel to be the image of the diagonal map $\mathcal{O}_X^* \to \mathcal{O}_{X_B}^*$, and the cokernel to be $\mathcal{O}_{X_k}^*$. So, if $\mathcal{U} := \ker(\mathcal{O}_X^* \to \mathcal{O}_{X_k}^*)$, we have a split exact sequence
$$1 \to \mathcal{O}_X^* \to \mathcal{O}_{X_B}^* \to \mathcal{U} \to 1$$

hence

$$P(B) \cong P(A) \oplus H^1(X, \mathcal{U})$$

We also have an exact sequence

$$1 \to \mathcal{U} \to \mathcal{O}_X^* \to \mathcal{O}_{X_k}^* \to 1$$

which gives $C = \ker(H^1(X,\mathcal{U}) \to P(A))$. Hence $\ker(P(B) \rightrightarrows P(A))/P(A) \cong C$. □

To complete the proof of Proposition 2, note that since P is representable we have $P(A \times_k A) = P(A) \times_{P(k)} P(A)$ ([8, II]). Thus, $\ker(P(A \times_k A) \rightrightarrows P(A)) = P(A)$ and the lemma implies $C = 0$. □

3. Numerical criterion

Let T be as in §1.1 and $f : X \to T$ be a morphism of noetherian schemes.

3.1. **Multiplicities.** Assume f is of finite type and let $\eta \in X_t$ be a maximal point. The *(apparent) multiplicity* d_η *of* η *in* X_t is, by definition, the length of the artinian local ring $\mathcal{O}_{X_t,\eta}$.

Let $\bar{t} := \mathrm{Spec}(\overline{k(t)})$ be the spectrum of an algebraic closure of $k(t)$ and $\bar{\eta} \in X_t \times_t \bar{t}$ a point lying over η. The *geometric multiplicity* δ_η *of* η *in* X_t is defined $\delta_\eta := d_{\bar{\eta}}$ [5, §9.1, Def. 3].

The *(apparent) multiplicity* d_f (resp. *geometric multiplicity* δ_f) of X_t is the gcd of the integers d_η (resp. δ_η) as η ranges over the finite set of maximal points of X_t.

Lemma 4.
- (i) *δ_η is independent of the choice of algebraic closure of $k(t)$.*
- (ii) *$(\delta_f, p) = 1$ if and only if there is an irreducible component of X_t whose apparent multiplicity is prime to p and whose reduction is geometrically reduced.*
- (iii) *If f is flat and X is regular at a maximal point $\eta \in X_t$, then d_η is equal to the valuation at η of a uniformiser of $\mathcal{O}(T)$.*
- (iv) *Assume f is flat and X is regular at the maximal points of its special fibre. If $g : T \to S$ is a finite flat morphism of spectra of discrete valuation rings, then $d_{g\circ f} = e d_f$ and $\delta_{g \circ f} = e\delta_f$, where e is the ramification index of g.*
- (v) *Let $h : X' \to X$ be a proper birational morphism of T-schemes which are regular at the maximal points of their special fibres. Then $\delta_{f\circ h} | \delta_f$.*
- (vi) *If $S \to T$ is a morphism of spectra of discrete valuation rings such that S_t is the spectrum of a separable extension of $k(t)$, then $d_f = d_{f\times_T S}$ and $\delta_f = \delta_{f \times_T S}$.*

Proof. (i) follows from [9, IV$_2$, 4.7.8], (ii) from [9, IV$_2$, 4.7.10], and (iii) is a straighforward check.

For (iv), let $s \in S$ denote the closed point. Note that $X_t \to X_s$ is a closed immersion which is a homeomorphism. If $\eta \in X_t$ is a maximal point and $\zeta \in X_s$ its image, then $\mathcal{O}_{X,\eta} = \mathcal{O}_{X,\zeta}$. Computing valuations and using (iii) one sees that $d_\zeta = e d_\eta$, where e is the ramification index of $T \to S$. Since X_t and X_s have the same reduction, by [9, IV$_2$, 4.7.3.1] we have $\delta_\zeta / d_\zeta = \delta_\eta / d_\eta$. Thus, $\delta_\zeta = e\delta_\eta$, hence $\delta_{g\circ f} = e\delta_f$ and the claim follows.

Finally, (v) holds because every maximal point of X_t lifts uniquely to a maximal point of X'_t by the valuative criterion for properness, and (vi) follows from the definitions. □

3.2. **Raynaud's condition (N).** Raynaud stated Theorem 2 under a condition he called (N). Recall that this condition states:

(N) X_t is S_1 and X is regular at the maximal points of X_t

For instance, this holds if X is normal.

The next result, due to Kollár, provides a useful characterisation of condition (N). For its statement, let $u \in T$ be the generic point and $j : X_u \to X$ be the canonical map. We say that X *integrally closed in* X_u if X is integrally closed in $j_*\mathcal{O}_{X_u}$ (in the sense of [9, II, 6.3]).

Proposition 3 ([17]). *The following are equivalent*

(i) *f is flat and satisfies (N)*
(ii) *X is integrally closed in X_u*
(iii) *$X_t \subset X$ is a normal pair (cf. [17]).*

Proof. Condition (ii) means that X is the *relative normalization* of the pair $X_t \subset X$ in the sense of [17, Def. 1], which is equivalent to (iii) by definition (cf. [17, Def. 2]). The equivalence of (i) and (iii) follows easily from [17, Cor. 7]. □

We gather some facts about (N) in the next couple of lemmata.

Lemma 5. *Assume $f : X \to T$ is a morphism of finite type. Let $S \to T$ be a morphism of spectra of discrete valuation rings such that S_t is the spectrum of a separable extension of $k(t)$. Then f satisfies (N) if and only if $f \times_T S$ does.*

Proof. See [28, 6.1.7]. □

Lemma 6. *Assume f is proper, flat, and satisfies (N). Let $X \xrightarrow{f'} T' \to T$ be the Stein factorisation of f. Then*

(i) *T' is the spectrum of a one-dimensional semilocal normal ring*
(ii) *f' is proper and flat*
(iii) *for each closed point $t' \in T'$, $f' \times_{T'} \operatorname{Spec}(\mathcal{O}_{T',t'})$ satisfies (N)*
(iv) *f is cohomologically flat if and only if f' is.*

Proof. For (i), see [28, 6.1.8] (and use that f is flat). Note that it implies (ii). For (iii), first note that since f is flat and X_t is S_1, the inequality $\operatorname{depth} \mathcal{O}_{X,x} \geq \min\{2, \dim \mathcal{O}_{X,x}\}$ is satisfied for all $x \in X_t$. Since f' is flat, it follows that its special fibre is S_1. Moreover, X is clearly regular at the maximal points of the special fibre of f' (since this holds for f), so (iii) follows. Finally, (iv) follows from Proposition 1 (iv). □

3.3. **1-cycles.** From now on we assume f is flat and satisfies (N).

Let $X_1, ..., X_c$ denote the irreducible components of X_t and, for $1 \leq i \leq c$, $\eta_i \in X_i$ the generic point. Then $\mathcal{O}_{X,\eta_i}$ is regular of dimension 1; let v_i the discrete valuation of $\mathcal{O}_{X,\eta_i}$, with the usual convention $v_i(0) = +\infty$. Let $j : X_u \to X$ be the inclusion of the generic fibre of f. Note that v_i extends to a function on $j_*\mathcal{O}_{X_u}$.

Let $\mathbb{N}_\infty = \mathbb{N} \cup \{+\infty\}$ and $\mathbb{Z}_\infty = \mathbb{Z} \cup \{+\infty\}$, with the obvious additive monoid structures. Define a map of (Zariski) sheaves

$$\operatorname{div}_X : j_*\mathcal{O}_{X_u} \to \oplus_{1 \leq i \leq c} \eta_{i,*}\mathbb{Z}_\infty$$
$$s \mapsto \sum_{i=1}^{c} v_i(s)[X_i]$$

where $[X_i] = 1 \in \eta_{i,*}\mathbb{Z}$.

Lemma 7. $\mathcal{O}_X(U) = \{s \in j_*\mathcal{O}_{X_u}(U) : \mathrm{div}_U(s) \geq 0\}$

Proof. Let $\mathcal{M}$ be the presheaf $\mathcal{M}(U) = \{s \in j_*\mathcal{O}_{X_u}(U) : \mathrm{div}_U(s) \geq 0\}$. Then

$$\mathcal{M} = (j_*\mathcal{O}_{X_u}) \times_{(\oplus_i \eta_{i,*}\mathbb{Z}_\infty)} (\oplus_i \eta_{i,*}\mathbb{N}_\infty)$$

so $\mathcal{M}$ is a sheaf and clearly $\mathcal{O}_X \subset \mathcal{M}$. We prove that this inclusion is an isomorphism by induction on $\dim X$, the case $\dim X = 1$ being straightforward. So assume the result for noetherian flat T-schemes of dimension less than $\dim X$ satisfying (N).

The claim is local at $x \in X_t$ so we may assume $X = \mathrm{Spec}(\mathcal{O}_{X,x})$ and $\dim \mathcal{O}_{X,x} > 1$. Let $U = X \setminus \{x\}$. Then $\dim U < \dim X$, so by induction the result holds for U. By (N) and flatness over T, we have depth $\mathcal{O}_{X,x} \geq 2$, hence $\mathcal{O}_X(U) = \mathcal{O}_X(X)$. Since U contains all of the maximal points of X_t and $U_u = X_u$, we have $\mathcal{M}(U) = \mathcal{M}(X)$ and the claim follows. □

A *1-cycle supported on* X_t is, by definition, a global section of the sheaf $\oplus_{1\leq i\leq c}\eta_{i,*}\mathbb{Z}$. The *class group* of such cycles is defined

$$C(X) := \mathrm{cok}\left(H^0(X_u, \mathcal{O}^*_{X_u}) \overset{\mathrm{div}_X}{\to} H^0(X, \oplus_{1\leq i\leq c}\eta_{i,*}\mathbb{Z})\right)$$

Define

$$d'_f := |C(X)[\mathrm{tor}]|$$

where $C(X)[\mathrm{tor}] \subset C(X)$ is the torsion subgroup.

Lemma 8. *In addition to our standing hypotheses, assume f is proper and $f_*\mathcal{O}_X = \mathcal{O}_T$.*

(i) *$d'_f = d_f$ is the (apparent) multiplicity of X_t (§3.1).*
(ii) *Assume X is locally factorial. If $g : Y \to T$ is another locally factorial, proper and flat T-scheme such that $g_*\mathcal{O}_Y = \mathcal{O}_T$ and $Y_u \simeq X_u$, then $d_f = d_g$.*

Proof. Note that $H^0(X_u, \mathcal{O}^*_{X_u}) = k(u)^*$, hence $C(X) = (\oplus_{1\leq i\leq c}\mathbb{Z})/\langle \mathrm{div}_X(\pi)\rangle$. So $C(X)[\mathrm{tor}]$ is a cyclic group; let $D = \sum_{i=1}^c a_i[X_i]$ be a generator. Consider the exact sequence

$$0 \to \mathbb{Z} \overset{\cdot D}{\to} H^0(X, \oplus_{1\leq i\leq c}\eta_{i,*}\mathbb{Z}) \to B \to 0$$

Taking the tensor product with $\mathbb{Z}/p$ for a prime p we obtain an exact sequence

$$0 \to B[p] \to \mathbb{Z}/p \to \oplus_{1\leq i\leq c}\mathbb{Z}/p$$

Since D is a generator of $C(X)[\mathrm{tor}]$, we see that $B[p] = 0$ if $p \mid d'_f$. Thus, $B[\mathrm{tor}]$ is annihilated by an integer prime to d'_f, hence $B[\mathrm{tor}]$ is a cyclic group of order m with $(m, d'_f) = 1$. Now a diagram chase shows that $B[\mathrm{tor}] \cong \langle \mathrm{div}_X(\pi)\rangle/\langle d'_f D\rangle$, so $m\,\mathrm{div}_X(\pi) = nd'_f D = \sum_i nd'_f a_i[X_i]$ for some integer n. By Lemma 4 (iii), we see that d'_f divides all of the multiplicities of X_t, so $d'_f \mid d_f$. Conversely, by Lemma 4 (iii), we can write $\mathrm{div}_X(\pi) = d_f(\sum_i b_i[X_i])$ for some $b_i \in \mathbb{N}$, so $C(X)$ has an element of order d_f. Thus, $d_f \mid d'_f$ and this proves (i).

For (ii), first recall ([10, §1]) that, by local factoriality, the map div_X induces an isomorphism

$$j_*\mathcal{O}^*_{X_u}/\mathcal{O}^*_X \cong \oplus_{1\leq i\leq c}\eta_{i,*}\mathbb{Z}$$

In particular, the right hand side is functorial in (locally factorial) X (without any further assumptions on X). Now, since X is normal and g proper, there is an open subset $U \subset X$ with $\mathrm{codim}_X(X \setminus U) \geq 2$ and a unique morphism $U \to Y$ extending the isomorphism on the generic fibres. Note that $H^0(U,\mathcal{O}_U)=H^0(X,\mathcal{O}_X)=\mathcal{O}(T)$ and $C(U)=C(X)$.

Let $Y_1,...,Y_d$ be the irreducible components of Y_t and $\xi_i\in Y_i$ the generic points. By the above, the morphism $U\to Y$ yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(Y_u,\mathcal{O}^*_{Y_u})/H^0(Y,\mathcal{O}^*_Y) & \longrightarrow & H^0(Y,\oplus_{1\le i\le d}\xi_{i,*}\mathbb{Z}) & \longrightarrow & C(Y) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X_u,\mathcal{O}^*_{X_u})/H^0(U,\mathcal{O}^*_U) & \longrightarrow & H^0(U,\oplus_{1\le i\le c}\eta_{i,*}\mathbb{Z}) & \longrightarrow & C(U) & \longrightarrow & 0 \end{array}$$

thus $\ker(C(Y)\to C(U)=C(X))$ is torsion free. So, $C(Y)[\mathrm{tor}]\subset C(X)[\mathrm{tor}]$ and $d'_g\mid d'_f$. Inverting the roles of X and Y shows $d'_f\mid d'_g$, hence $d'_f=d'_g$ and (ii) follows from (i). □

Let D be a 1-cycle supported on X_t. Define a presheaf $\mathcal{O}(D)$ by

$$\mathcal{O}(D)(U)=\{s\in j_*\mathcal{O}_{X_u}(U):\mathrm{div}_U(s)+D|_U\ge 0\}$$

This definition also makes sense for 1-cycles with rational coefficients (briefly, *rational 1-cycles*). Moreover, by Lemma 7, we have $\mathcal{O}(0)=\mathcal{O}_X$.

Lemma 9. *Let D and E be rational 1-cycles supported on X_t.*

(i) *$\mathcal{O}(D)$ is a coherent sheaf of $\mathcal{O}_X$-modules.*
(ii) *$\mathcal{O}(D)=\mathcal{O}(\lfloor D\rfloor)$, where $\lfloor\cdot\rfloor$ denotes the floor function, applied componentwise to D.*
(iii) *If $E\le D$, then $\mathcal{O}(E)\subset\mathcal{O}(D)$. In particular, if $E\le 0$, then $\mathcal{O}(E)\subset\mathcal{O}_X$.*
(iv) *There is a canonical map*
$$\mathcal{O}(D)\otimes_{\mathcal{O}_X}\mathcal{O}(E)\to\mathcal{O}(D+E)$$
(v) *If $r\in j_*\mathcal{O}_{X_u}(X)^*$ and $D=\mathrm{div}_X(r)$, then $\mathcal{O}(-D)=r\mathcal{O}_X$.*

Proof. One proves that $\mathcal{O}(D)$ is a sheaf as in the proof of Lemma 7. Clearly, it is an $\mathcal{O}_X$-module. To check that it is coherent is local so we may assume $X=\mathrm{Spec}(A)$ is affine. Let $a\in A$, $B=A[z]/(az-1)$, $U=\mathrm{Spec}(B)$, $M=\mathcal{O}(D)(X)$.

We first show that $\mathcal{O}(D)$ is quasi-coherent. It suffices to show that the canonical map $M\otimes_A B\to\mathcal{O}(D)(U)$ is an isomorphism (cf. [9, I, 1.4.1]). Note that the map is injective, since $j_*\mathcal{O}_{X_u}$ is quasi-coherent and contains $\mathcal{O}(D)$. For the surjectivity, we need to show that for any $b\in\mathcal{O}(D)(U)$ there is $N\ge 0$ such that a^Nb is the image an element of M. By the quasi-coherence of $j_*\mathcal{O}_{X_u}$, a^Nb is the image of $s\in j_*\mathcal{O}_{X_u}(X)$ for $N\gg 0$. Write $D=\sum n_i[X_i]$. If $v_i(a)=0$, then $\eta_i\in U$ and $v_i(b)\ge -n_i$. If $v_i(a)>0$, then for $N\gg 0$ we have $v_i(a^Nb)=Nv_i(a)+v_i(b)\ge -n_i$. Thus, $v_i(s)\ge -n_i$ for all $1\le i\le c$, so $\mathrm{div}_X(s)\ge -D$ and $s\in M$, as required.

For coherence, let $\pi\in\mathcal{O}(T)$ be a uniformiser and choose $N\ge 0$ such that $\mathrm{div}_X(\pi^N)\ge D$. Then $\pi^N\mathcal{O}(D)\subset\mathcal{O}_X$ (by Lemma 7), so, since $\mathcal{O}(D)$ is π-torsion free and X is noetherian, (i) follows (cf. [9, I, 1.5.1]). The remaining statements are left as exercises. □

3.4. **Proof of Theorem 3.** We begin with some general considerations. Assume f is flat, of finite type, and satisfies (N). Let $\pi\in\mathcal{O}(T)$ be a uniformiser.

Consider the 1-cycle $D=\mathrm{div}_X(\pi)=\sum_i n_i[X_i]$. If N denotes a sufficiently large integer divisible by all n_i, then $\mathcal{O}(-D/N)$ is the radical of $\mathcal{O}(-D)$, i.e., the ideal sheaf of $X_0:=(X_k)_{\mathrm{red}}$. We will assume this holds.

We have a filtration

$$\begin{aligned}\mathcal{O}(-D) = \mathcal{O}(-ND/N) \subset \mathcal{O}(-(N-1)D/N) \subset \cdots \\ \cdots \subset \mathcal{O}(-2D/N) \subset \mathcal{O}(-D/N) \subset \mathcal{O}\end{aligned}$$

For an integer $0 < a < N$, define a map of sheaves of sets

$$\varphi_N : \frac{\mathcal{O}(-aD/N)}{\mathcal{O}(-(a+1)D/N)} \to \frac{\mathcal{O}(-aD)}{\mathcal{O}(-(Na+1)D/N)}$$
$$s \mapsto s^N$$

This map is well defined (exercise).

Lemma 10. *φ_N maps nonzero sections to nonzero sections.*

Proof. The claim is local. Suppose $\varphi_N(s) = 0$ for $s \in \mathcal{O}(-aD/N)$. Then $s^N \in \mathcal{O}(-(Na+1)D/N)$, so $\operatorname{div}_X(s) \geq (Na+1)D/N^2$. Hence for all $1 \leq i \leq c$ we have $v_i(s) \geq (Na+1)n_i/N^2$. Since N is divisible by n_i, we have

$$\left\lceil \frac{(Na+1)n_i}{N^2} \right\rceil = \left\lceil \frac{\lceil (Na+1)/N \rceil}{(N/n_i)} \right\rceil = \left\lceil \frac{\lceil a+1/N \rceil}{(N/n_i)} \right\rceil = \left\lceil \frac{a+1}{(N/n_i)} \right\rceil$$

(for the first equality, cf. [7, §3.2, (3.10)]). Hence $\lceil (Na+1)n_i/N^2 \rceil = \lceil (a+1)n_i/N \rceil$, and since $v_i(s) \in \mathbb{Z}_\infty$ we get $v_i(s) \geq (a+1)n_i/N$. Thus, $\operatorname{div}_X(s) \geq (a+1)D/N$ and $s \in \mathcal{O}(-(a+1)D/N)$. ☐

To simplify the notation, for $0 < a < N$ let

$$\mathcal{F}_a := \mathcal{O}(-aD/N)/\mathcal{O}(-(a+1)D/N)$$
$$\mathcal{G}_a := \mathcal{O}(-aD)/\mathcal{O}(-(Na+1)D/N)$$

We are going to analyse the global sections of $\mathcal{F}_a$ using the map φ_N.

Lemma 11. $\mathcal{G}_a \cong \mathcal{O}(-aD)|_{X_0}$

Proof. The canonical map $\mathcal{O}(-D/N) \otimes_{\mathcal{O}_X} \mathcal{O}(-aD) \to \mathcal{O}(-(Na+1)D/N)$ (Lemma 9 (iv)) gives an exact sequence

$$0 \to J \to \mathcal{O}(-aD)|_{X_0} \to \mathcal{G}_a \to 0$$

We have

$$\left\lceil \frac{(Na+1)n_i}{N} \right\rceil = \left\lceil an_i + \frac{n_i}{N} \right\rceil = an_i + \left\lceil \frac{n_i}{N} \right\rceil = an_i + 1$$

It follows that $J_{\eta_i} = 0$ for all $1 \leq i \leq c$. If $J \neq 0$, then J has an associated point which is not a maximal point. But X_0 is reduced, hence $\mathcal{O}(-aD)|_{X_0} \cong \mathcal{O}_{X_0}$ is S_1, which precludes this. Thus, $J = 0$. ☐

From now on we assume $H^0(X_0, \mathcal{O}_{X_0})$ is a field.

Corollary 1. *Assume $0 \neq s \in H^0(X, \mathcal{F}_a)$. Then*

(i) *$\varphi_N(s) \in \mathcal{G}_a(X)$ vanishes nowhere on X_t*
(ii) *$s \in \mathcal{F}_a(X)$ vanishes nowhere on X_t.*

Proof. By Lemma 11 we have $H^0(X_0, \mathcal{G}_a) \cong H^0(X_0, \mathcal{O}_{X_0})$. Since the latter is a field by assumption, (i) follows from Lemma 10.

For (ii), one shows that if $s \in I\mathcal{F}_a$ for some ideal $I \subset \mathcal{O}_X$, then $\varphi_N(s) \in I\mathcal{G}_a$ (exercise). In particular, $\varphi_N(s)$ vanishes on the support of $\mathcal{O}_X/I$. By (i), the support of $\mathcal{O}_X/I$ does not meet X_t, hence (ii). ☐

Corollary 2. *If $H^0(X, \mathcal{F}_a) \neq 0$, then $\mathcal{O}(-aD/N)$ is a line bundle on X satisfying $\mathcal{O}(-aD/N)^{\otimes N} \cong \mathcal{O}(-aD)$.*

Proof. We first work locally on X_t. Let $0 \neq s \in H^0(X, \mathcal{F}_a)$ and choose a local lift $\tilde{s} \in \mathcal{O}(-aD/N)$ of s. Then $\tilde{s}^N \in \mathcal{O}(-aD) = \pi^a \mathcal{O}_X$, so $\tilde{s}^N = \pi^a g$ for some $g \in \mathcal{O}_X$. Since $\tilde{s}^N$ vanishes nowhere on X_t (Corollary 1 (i)), up to localising we may assume $g \in \mathcal{O}_X^*$; in particular, $\tilde{s}$ is not a zerodivisor on X (as this holds for π). So, for any $f \in \mathcal{O}(-aD/N)$, we have $f/\tilde{s} = f\tilde{s}^{N-1}g^{-1}\pi^{-a} \in j_*\mathcal{O}_{X_u}$, hence $f/\tilde{s} \in \mathcal{O}_X$ by Lemma 7. Thus, $\mathcal{O}(-aD/N) \subset \mathcal{O}_X$ is the free $\mathcal{O}_X$-submodule generated by $\tilde{s}$, hence the canonical map $\mathcal{O}(-aD/N)^{\otimes N} \to \mathcal{O}(-aD)$ (Lemma 9 (iv)) is an isomorphism at the points of X_t. Since it is clearly an isomorphism over X_u, this implies the claim. □

Corollary 3. *If $H^0(X, \mathcal{F}_a) \neq 0$, then $\mathcal{O}(-aD/N)|_{X_0} \simeq \mathcal{O}_{X_0}$.*

Proof. As in the proof of Lemma 11, we have an exact sequence

$$0 \to J \to \mathcal{O}(-aD/N)|_{X_0} \to \mathcal{F}_a \to 0$$

On the other hand,

$$\left\lceil \frac{(a+1)n_i}{N} \right\rceil = \left\lceil \frac{an_i}{N} + \frac{n_i}{N} \right\rceil \leq \left\lceil \frac{an_i}{N} \right\rceil + \left\lceil \frac{n_i}{N} \right\rceil = \left\lceil \frac{an_i}{N} \right\rceil + 1 \tag{3.1}$$

If equality does not hold in (3.1), then $\lceil (a+1)n_i/N \rceil = \lceil an_i/N \rceil$, hence $(\mathcal{F}_a)_{\eta_i} = 0$. But, by Corollary 1, $\mathcal{F}_a$ has a global section vanishing nowhere on X_0, hence

$$0 \neq \mathcal{F}_a \otimes_{\mathcal{O}_X} k(\eta_i) = (\mathcal{F}_a)_{\eta_i} \otimes_{\mathcal{O}_{X,\eta_i}} k(\eta_i)$$

a contradiction. Thus, equality holds in (3.1) and it follows that $J_{\eta_i} = 0$ for all $1 \leq i \leq c$. From Corollary 2 we deduce $J = 0$, hence $\mathcal{O}(-aD/N)|_{X_0} = \mathcal{F}_a$. Now let $0 \neq s \in H^0(X, \mathcal{F}_a)$ and consider the exact sequence

$$\mathcal{O}_{X_0} \xrightarrow{s} \mathcal{F}_a \to C \to 0$$

Since s has no zeros (Corollary 1 (ii)) and $\mathcal{F}_a$ is a line bundle, the fibres of C are zero, hence $C = 0$ by Nakayama's lemma. □

Recall that we have written $D = \mathrm{div}_X(\pi) = \sum_i n_i[X_i]$; then, by Lemma 4 (iii), $\gcd(n_1, ..., n_c) = d_f$ is the (apparent) multiplicity of X_t.

Corollary 4. *Assume T is strictly henselian, f is proper, and $f_*\mathcal{O}_X = \mathcal{O}_T$. If $p \nmid d_f$ and $H^0(X, \mathcal{F}_a) \neq 0$, then $\mathcal{O}(-aD/N) \simeq \mathcal{O}_X$.*

Proof. By Corollary 2, $\mathcal{O}(-aD/N)$ is a line bundle of finite order, say e. Then

$$H^0(X, \mathcal{O}(-aD/N)^{\otimes e}) \subset H^0(X, \mathcal{O}_X) = \mathcal{O}(T)$$

so $\mathcal{O}(-aD/N)^{\otimes e} = \pi^n \mathcal{O}_X$ for some positive integer n. But $\pi\mathcal{O}_X = \mathcal{O}(-D) \subset \mathcal{O}(-aD/N)$, so $n \mid e$.

For each $1 \leq i \leq c$, let $a_i = v_i(s)$, where $s \in \mathcal{O}(-aD/N)_{\eta_i}$ is a generator. Then $E := \sum_i a_i[X_i]$ is a 1-cycle satisfying $\frac{e}{n}E = D$. Clearly, $\mathcal{O}(-E) = \mathcal{O}(-aD/N)$, hence $\mathcal{O}(-aD/N)$ has order dividing e/n. Thus, $n = 1$ and it follows that $e | d_f$. In particular, e is prime to p. Since T is strictly henselian and f proper, we have $H^1(X, \mu_e) = \mathrm{Pic}(X)[e]$ and similarly for X_0. By the proper base change theorem, it follows that $\mathrm{Pic}(X)[e] = \mathrm{Pic}(X_0)[e]$ (cf. [3, Exp. XII, 5.5]). Now the claim follows from Corollary 3. □

Corollary 5. *Assume T is strictly henselian, f is proper, and $f_*\mathcal{O}_X = \mathcal{O}_T$. If $p \nmid d_f$, then $H^0(X, \mathcal{F}_a) = 0$.*

Proof. If not, then $\mathcal{O}(-aD/N)$ is a trivial line bundle and $H^0(X, \mathcal{O}(-aD/N)) \subset H^0(X, \mathcal{O}_X) = \mathcal{O}(T)$, so $\mathcal{O}(-aD/N) = \pi\mathcal{O}_X$ (because $\mathcal{O}(-D) \subset \mathcal{O}(-aD/N)$). In particular, the inclusions $\mathcal{O}(-D) \subset \mathcal{O}(-(a+1)D/N) \subset \mathcal{O}(-aD/N)$ are equalities and $\mathcal{F}_a = 0$. □

Proof of Theorem 3. Note that X_t is connected since $H^0(X_0, \mathcal{O}_{X_0})$ is. So, by Lemma 6, $H^0(X, \mathcal{O}_X)$ is a finite normal $\mathcal{O}(T)$-algebra with connected special fibre. Hence it must be local, whence a discrete valuation ring. Then, by Lemma 4 (iv) and Lemma 6, we may assume $f_*\mathcal{O}_X = \mathcal{O}_T$. By Proposition 1 (i), Lemma 4 (vi), and Lemma 5 we may also assume T is strictly henselian.

In this case, by Corollary 5, we have

$$H^1(X, \mathcal{O}(-(a+1)D/N)) \subset H^1(X, \mathcal{O}(-aD/N))$$

for all $0 < a < N$. Since $\mathcal{O}(-D/N)$ is the ideal sheaf of X_0 and $H^0(X, \mathcal{O}_{X_0}) = k(t)$, we also have $H^1(X, \mathcal{O}(-D/N)) \subset H^1(X, \mathcal{O}_X)$. We deduce that the map $H^1(X, \mathcal{O}(-D)) \to H^1(X, \mathcal{O}_X)$ is injective, hence $H^0(X, \mathcal{O}_X) \otimes_{\mathcal{O}(T)} k(t) \cong H^0(X_t, \mathcal{O}_{X_t})$. Since X is flat over T, it follows from Proposition 1 (iv) that f is cohomologically flat. □

Now we drop the assumption on $H^0(X_0, \mathcal{O}_{X_0})$. From Theorem 3 we deduce a higher dimensional generalisation of Raynaud's theorem.

Corollary 6. *Let $f : X \to T$ be a proper and flat morphism satisfying (N). If the special fibre of f is connected and $p \nmid \delta_f$, then f is cohomologically flat.*

Proof. Since X_t is connected by assumption, arguing as in the proof of Theorem 3 we reduce to the case $f_*\mathcal{O}_X = \mathcal{O}_T$ and T strictly henselian.

Now, since $p \nmid \delta_f$, X_t has a component which is geometrically reduced by Lemma 4 (ii). In particular, X_0 has a nonempty open subscheme which is smooth over t, hence has a point in the separably closed field $k(t)$. Since X_t is connected, it follows that $H^0(X_0, \mathcal{O}_{X_0}) = k(t)$ and we can apply Theorem 3. □

Remark 1. For f projective and Cohen–Macaulay, one can also prove Corollary 6 by induction on the dimension, the base case being Raynaud's Theorem 2, with a suitably chosen hyperplane section for the induction step.

Remark 2. The first named author can generalise Theorem 3 to formal schemes over (not necessarily discrete) valuation rings.

4. Logarithmic criterion

Throughout this section, $f_{\log} : (X, M_X) \to (T, M_T)$ is a morphism of fs log schemes whose underlying morphism of scheme $f : X \to T$ is flat and of finite type, and (T, M_T) is log regular.

We begin with some preliminaries that will be of use later on in the section.

Definition 1. *Let S be a scheme. A* normal crossings scheme over S *is a morphism of locally finite presentation $D \to S$ such that, locally for the étale topology, D is the scheme-theoretic union of closed subschemes $D_1, ..., D_r$ and there is an integer d such that, for all $J \subset \{1, ..., r\}$, $D_J := \cap_{j \in J} D_j$ is a smooth S-scheme of relative dimension $d + 1 - |J|$.*

Lemma 12. *Let $S \to T$ be an extension of spectra of discrete valuation rings such that S_t is the spectrum of a field. If X is a regular, proper and flat T-scheme and $X_0 := (X_t)_{\mathrm{red}}$ is a normal crossings scheme over t, then $X \times_T S$ is regular and $X_0 \times_T S \subset X \times_T S$ is a normal crossings divisor.*

Proof. We claim that $X \times_T S$ is regular at the points of its special fibre. This is local for the étale topology on X and so we may assume $X_0 = \cup_{i=1}^r X_i$ with $X_i \to t$ smooth. Then $X_i \times_T S$ is smooth over S_t, hence regular, and since X is regular and $S \to T$ is flat, the closed immersions $X_i \times_T S \to X \times_T S$ are regular ([9, IV$_4$, 19.1.5]). So $X \times_T S$ is regular along $X_i \times_T S$ for $1 \le i \le r$ ([9, IV$_4$, 19.1.1]), hence the claim. Now, since the regular locus of a S-scheme of finite type is open ([9, IV$_2$, 6.12.6]) and $X \times_T S \to S$ is proper, it follows that $X \times_T S$ is regular. We leave the other claim to the reader. □

Lemma 13. *Let $k(t) \subset k(s)$ be a field extension. There is strict morphism of log regular schemes $(S, M_S) \to (T, M_T)$ with S the spectrum of a discrete valuation ring and $k(s) \cong \mathcal{O}(S) \otimes_{\mathcal{O}(T)} k(t)$, such that*

(i) *$f_{\log} \times_{(T,M_T)} (S, M_S)$ is log smooth if and only if $f_{\log}$ is*
(ii) *if f is proper, f is cohomologically flat if and only if $f \times_T S$ is.*

Moreover, $\mathcal{O}(S)$ may be taken to be henselian or complete.

Proof. Let $\pi \in \mathcal{O}(T)$ be a uniformiser. By [9, 0_{III}, 10.3.1], there exists a noetherian local ring $\mathcal{O}(S)$ containing $\mathcal{O}(T)$ such that $k(s) \cong \mathcal{O}(S)/\pi\mathcal{O}(S)$. It follows from [30, I, Prop. 2] that $\mathcal{O}(S)$ is a discrete valuation ring of uniformiser π. Endowed with the inverse image log structure of T, $S := \mathrm{Spec}(\mathcal{O}(S))$ is log regular (cf. [25, 2.6]). Since $S \to T$ is faithfully flat, (i) follows from [12] and (ii) from Proposition 1 (i). □

4.1. **Proof of Theorem 4.** In addition to the above hypotheses, assume f is proper. By Lemma 13 we may assume $k(t)$ algebraically closed.

Lemma 14. *It suffices to prove Theorem 4 for M_T the canonical log structure.*

Proof. If not, then M_T is the trivial log structure and the geometric fibres of the morphism $f_{\log}$ are log regular ([14, 8.3]), hence reduced. So, by Proposition 1 (ii), f is cohomologically flat. □

Assume $k(t)$ is algebraically closed, M_T the canonical log structure, and $f_{\log}$ log smooth. Fix a uniformiser π of $\mathcal{O}(T)$, hence a chart

$$\mathbb{N} \to M_T : 1 \mapsto \pi$$

and a morphism $T \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}])$. For a positive integer n, the homomorphism $\mathbb{N} \to \mathbb{N}$ given by multiplication by n induces a morphism of schemes $\mathbf{n} : \mathrm{Spec}(\mathbb{Z}[\mathbb{N}]) \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}])$. The base change of $T \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}])$ by $\mathbf{n}$

$$T' := T \times_{\mathrm{Spec}(\mathbb{Z}[\mathbb{N}]),\mathbf{n}} \mathrm{Spec}(\mathbb{Z}[\mathbb{N}])$$

is the spectrum of the discrete valuation ring $\mathcal{O}(T)[z]/(z^n - \pi)$. The standard log structure of the right-hand factor induces the canonical log structure $M_{T'}$ on T'. There is a natural μ_n-action on the fs log scheme $(T', M_{T'})$: see [11, §3].

Consider the fs log scheme

$$(X', M_{X'}) := (X, M_X) \times_{(T,M_T)} (T', M_{T'})$$

By base change, the action of μ_n extends to $(X', M_{X'})$. Moreover, $f'_{\log} : (X', M_{X'}) \to (T', M_{T'})$ is log smooth, hence X' is normal and Cohen–Macaulay ([32, II.4.7]), and flat over T.

Lemma 15. *$X \cong X'/\mu_n$, where the latter denotes the geometric quotient (*[23, Def. 0.6]*).*

Proof. Note that X' has a covering by μ_n-stable affine open subsets, namely, the inverse images of affine open subsets of X under the finite map $X' \to X$. Thus, the geometric quotient X'/μ_n exists by [6, Exp. V, 4.1], and $(X'/\mu_n)_u = X_u$ (since $T'_u \to u$ is a μ_n-torsor). So the canonical map $X'/\mu_n \to X$ is a finite and birational, hence is an isomorphism since X is normal. □

By [13, 4.4 (ii)], $f'_{\log}$ is integral since $\overline{M}_{T'} = \mathbb{N}$. From [32, II.3.4], we deduce that $f'_{\log} : (X', M_{X'}) \to (T', M_{T'})$ is a saturated morphism for n sufficiently divisible. Fix n such that this holds. Then, by [32, II.4.2], f' has reduced, hence geometrically reduced special fibre. By [9, IV_3, 12.2.1], $f' : X' \to T'$ has geometrically reduced fibres. So, by Proposition 1 (ii), f' is cohomologically flat.

Since f' is cohomologically flat, so is the composition $h : X' \xrightarrow{f'} T' \to T$ (Prop. 1 (v)), hence the natural map

$$(h_* \mathcal{O}_{X'}) \otimes_{\mathcal{O}_T} k(t) \to h_* \mathcal{O}_{X' \times_T t} \tag{4.1}$$

is an isomorphism.

Lemma 16. *The canonical maps*

$$(f_* \mathcal{O}_X) \otimes_{\mathcal{O}_T} k(t) \to (h_* \mathcal{O}_{X'})^{\mu_n} \otimes_{\mathcal{O}_T} k(t) \to (h_* \mathcal{O}_{X' \times_T t})^{\mu_n} \leftarrow f_* \mathcal{O}_{X \times_T t}$$

are isomorphisms.

Proof. This follows from Lemma 15, (4.1), and the exactness of the μ_n-action, cf. [6, Exp. I, §§4–6]. □

From Lemma 16 we get that the canonical map

$$(f_* \mathcal{O}_X) \otimes_{\mathcal{O}_T} k(t) \to f_* \mathcal{O}_{X \times_T t}$$

is an isomorphism. By Proposition 1 (iv), this implies that f is cohomologically flat, which completes the proof of Theorem 4.

Remark 3. It is also possible to deduce Theorem 4 from Theorem 3 (or, more precisely, Corollary 6). One first reduces to the case $f_* \mathcal{O}_X \cong \mathcal{O}_T$, $k(t)$ is algebraically closed, and X is regular. Then, by considering the torsion in $M_X / f^* M_T$, one shows that, for suitable n, $X' \to T'$ satisfies the hypothesis of Corollary 6 and the quotient map $q : X' \to X$ is a μ_n-torsor. So X'/T is cohomologically flat and, since $\mathcal{O}_X$ is a direct summand of $q_* \mathcal{O}_{X'}$, one deduces the cohomological flatness of f.

4.2. **An example.** Let M_T be the canonical log structure. To illustrate the proof of Theorem 4, we give an example of a log smooth morphism to (T, M_T) with nonreduced fibres, which acquires smooth fibres after base change $(T', M_{T'}) \to (T, M_T)$, where $(T', M_{T'})$ is as in §4.1. Let n be the degree of $T' \to T$ and let $A \to T$ be an abelian scheme containing $(\mu_n)_T$ as a subgroup. For example, this holds if $\mathcal{O}(T)$ is complete, $k(t)$ is algebraically closed, and A_t is ordinary (cf. [15, p. 150]). Then one can show that the fppf quotient of $A \times_T T'$ by the diagonal action of μ_n is representable by a regular projective T-scheme X which is naturally log smooth over (T, M_T). Moreover, the multiplicity of X_t is n, hence X_t is not reduced.

4.3. **Application: torsors under abelian varieties with good reduction.** Let $A \to T$ be an fppf commutative group scheme and Y a u-scheme which is a torsor under A_u.

Definition 2. *An* A-model *of* Y *is a pair* (X, μ)*, where* X *is an fppf* T*-scheme with an* A*-action* $\mu : A \times_T X \to X$*, such that*

(a) *the fibre of* (X, μ) *over* u *is isomorphic to* Y *with its* A_u*-action*
(b) *the morphism* $\mu \times \mathrm{pr}_X : A \times_T X \to X \times_T X$ *is surjective.*

Lemma 17. *Let* (X, μ) *be an* A*-model of an* A_u*-torsor* Y*.*

(i) *The surjective morphism* $\mu \times \mathrm{pr}_X : A \times_T X \to X \times_T X$ *is quasi-finite.*
(ii) *If* $\bar{t} \to T$ *is a geometric point with algebraically closed residue field, then* $X_{\bar{t}}$ *is homeomorphic to a quotient of* $A_{\bar{t}}$ *by a finite subgroup.*

Write $X_0 := (X_t)_{\mathrm{red}}$*.*

(iii) *If* $A \to T$ *is smooth, then the fppf quotient of* X_0 *by* A_t *is representable by a finite field extension of* t *and the quotient map* $X_0 \to X_0/A_t$ *is smooth.*

Let $A^0 \subset A_t$ *be the connected component of the identity.*

(iv) *If* $A \to T$ *is proper and smooth, then so is* $X_0 \to \mathrm{Spec}(H^0(X_0, \mathcal{O}_{X_0}))$*; in fact, its geometric fibers are abelian varieties isogenous to* A^0*.*

Proof. (Cf. [19, 8.1] for the case A is an abelian scheme) Since $\mu \times \mathrm{pr}_X$ is surjective, the fibres of $X \to T$ are separated by [6, Exp. $\mathrm{VI_B}$, 5.3] and [6, Exp. $\mathrm{VI_A}$, 2.6.1 (i)]. We claim that for any point $x \in X$ the induced morphism $A \times_T x \to X \times_T x$ has finite fibres. Let $G \subset A \times_T x$ be the stabiliser of the point $x \in X \times_T x$; it is a closed subgroup scheme of $A \times_T x$ ([6, Exp. $\mathrm{VI_B}$, 6.2.4 (a)]). By [6, Exp. V, 10.1.2], we have an induced monomorphism $(A \times_T x)/G \hookrightarrow X \times_T x$, which must be a homeomorphism (cf. [6, Exp. $\mathrm{VI_A}$, proof of 2.5.4]). Since $X \to T$ is fppf and all of the irreducible components of its fibres have the same dimension ([6, Exp. $\mathrm{VI_A}$, 2.5.4 (i)]), it follows that $\dim X \times_T x = \dim Y$. Similarly, $\dim A \times_T x = \dim A_u$, hence $\dim X \times_T x = \dim A \times_T x$. Thus, $\dim G = 0$ by [6, Exp. $\mathrm{VI_A}$, 2.5.4], proving (i) and (ii).

Now assume $A \to T$ is smooth; then A_t acts on X_0. Consider the stabiliser F of the diagonal $\Delta \subset X_0 \times_t X_0$, i.e., $F := (A \times_T X_0) \times_{\mu \times \mathrm{pr}_{X_0}} \Delta$. It is a closed group scheme of A_{X_0}. Since X_0 is reduced, F is flat over a dense open subscheme of X_0, which must be all of X_0 because the action of A_t on X_0 is transitive (consider the diagonal action of A_t on F and Δ). Thus, by [6, Exp. V, 10.4.1] the fppf quotient A_{X_0}/F is representable by a group scheme, which is smooth over X_0 (cf. [6, Exp. $\mathrm{VI_A}$, 3.3.2]). Moreover, $A_{X_0}/F \hookrightarrow X_0 \times_t X_0$ is a monomorphism, so the transitivity of the A_t-action on X_0 implies that the fppf quotient X_0/A_t is representable by the quotient of X_0 by the groupoid A_{X_0}/F (cf. [6, Exp. V, 10.4.2]). In particular, the quotient map $X_0 \to X_0/A_t$ is smooth. By transitivity of the A_t-action, X_0/A_t is a point, hence it is a finite field extension of t since it is reduced and fppf, proving (iii). Similarly, the fppf quotient X_0/A^0 is representable by a finite reduced t-scheme and the geometric fibres of the quotient map $X_0 \to X_0/A^0$ are finite quotients of A^0. So, if A^0 is an abelian variety, then $H^0(X_0/A^0, \mathcal{O}) \cong H^0(X_0, \mathcal{O})$, which implies (iv). $\square$

From now on we assume A is an abelian scheme over T. In this case, by a result of Raynaud, there exists a regular, projective A-model X of Y, which, up to isomorphism, is the unique proper regular A-model (cf. [27, (c) p. 82], [18]). We endow X with the canonical fs log structure M_X given by $X_0 \subset X$, where $X_0 = (X_t)_{\mathrm{red}}$. By Lemma 17 (iv), (X, M_X) is log regular. Denote $f_{\log} : (X, M_X) \to (T, M_T)$ the resulting morphism.

Theorem 5. *The following are equivalent:*

(i) $f_{\log} : (X, M_X) \to (T, M_T)$ *is log smooth*

(ii) $f : X \to T$ *is cohomologically flat and* M_T *is the canonical log structure.*

Moreover, X_0 *is smooth over* t *in this case and its conormal sheaf in* X *is a line bundle of order equal to the multiplicity of* X_t.

Proof. By Theorem 4 we may assume f is cohomologically flat. Note that $\mathcal{O}(T) = H^0(X, \mathcal{O}_X)$ (since X is normal).

Let $I = \mathcal{O}_X(-X_0)$ be ideal sheaf of X_0 in X; it is a line bundle of order dividing the multiplicity $d := d_f$ of X_t. For $n \in \mathbb{N}$, let $X_n \subset X$ be the closed subscheme of ideal sheaf I^{n+1}. Then $X_t = X_{d-1}$ and we have exact sequences

$$0 \to I^n/I^{n+1} \to \mathcal{O}_{X_n} \to \mathcal{O}_{X_{n-1}} \to 0$$

where $I^n/I^{n+1} \cong (I/I^2)^{\otimes n}$ is a line bundle on X_0.

Since f is cohomologically flat, we have $k(t) = \Gamma(\mathcal{O}_{X_{d-1}}) \hookrightarrow \Gamma(\mathcal{O}_{X_0})$, hence $\Gamma(I^{d-1}/I^d) = 0$. It follows from [22, II, §8, (vii) p. 76] that $H^1(X_0, I^{d-1}/I^d) = 0$ (cf. Lemma 17 (iv)), so we obtain an isomorphism $k(t) = \Gamma(\mathcal{O}_{X_{d-1}}) \cong \Gamma(\mathcal{O}_{X_{d-2}})$. Continuing this way we get $\Gamma(I^n/I^{n+1}) = 0$ for $n < d$ and $\Gamma(\mathcal{O}_{X_0}) = k(t)$. Thus, $(I/I^2)^{\otimes n}$ is not a trivial line bundle for $n < d$, so I/I^2 has order at least d, hence equal to d. Moreover, since $\Gamma(\mathcal{O}_{X_0}) = k(t)$, X_0 is smooth and geometrically connected over t by Lemma 17. In particular, X_0 is a normal crossings scheme (Def. 1) over t. It follows from Lemma 12 and Lemma 13 that we may assume $k(t)$ algebraically closed and T complete.

Now, if (i) holds and M_T is the trivial log structure, then by log smoothness we have $X(T) \neq \emptyset$, hence $X \simeq A$ by uniqueness of the Raynaud model. In particular, f is smooth and we leave it to the reader to check that M_T cannot be trivial by definition of M_X. This proves (i) $\Rightarrow$ (ii).

For the converse, if $p \nmid d$, then $f_{\log}$ is log smooth by [20, Prop. 1 (d)] (cf. Prop. 4 (v) below). So assume $p \mid d$. Let $\nu^1 = \mathbb{G}_m/p$. Given a uniformiser $\pi \in \mathcal{O}(T)$, locally on X we may write $\pi = vx^d$ with $v \in \mathcal{O}_X^*$ and $x \in I$ a local generator. One checks without difficulty that v extends to a global section of $H^0(X, \nu^1)$, hence defines a global section $\alpha \in H^0(X_0, \nu^1)$.

The exact sequence of (Zariski) sheaves on X_0

$$0 \to \mathbb{G}_{m,X_0} \xrightarrow{p} \mathbb{G}_{m,X_0} \to \nu^1_{X_0} \to 0$$

provides an isomorphism $H^0(X_0, \nu^1) \cong \mathrm{Pic}(X_0)[p]$. It is straightforward to check that this maps α to the class of $\frac{d}{p}(I/I^2)$, thus $\alpha \neq 0$.

On the other hand, since X_0 is smooth, the map $\mathrm{dlog} : \nu^1_{X_0} \to \Omega^1_{X_0/t}$ is injective, hence we obtain a nonzero global 1-form $\mathrm{dlog}(\alpha)$ on X_0. Now, a global 1-form on an abelian variety vanishes nowhere (cf. [22, (iii) p. 42]), so the same is true for $\mathrm{dlog}(\alpha)$ (cf. Lemma 17 (iv)). Finally, by [20, Prop. 1 (c)], $\Omega^1_{X_0/t}$ is a locally direct summand in $\omega^1_X|_{X_0}$ (where ω^1 denotes the sheaf of logarithmic 1-forms), and hence the image of $\mathrm{dlog}(\alpha)$ in the latter is a nowhere vanishing section. But this image is none other than the image of $\mathrm{dlog}(\pi)$, and applying [20, Prop. 1 (d)] we deduce the log smoothness of $f_{\log}$. □

We deduce the following higher-dimensional generalisation of a result for genus 1 curves of [21] and [20] (with no restriction on $k(t)$).

Corollary 7. *Assume M_T is the canonical log structure. Let $A \to T$ be an abelian scheme and Y an A_u-torsor. The following are equivalent:*

- (i) *Y can be extended to a proper, log smooth morphism over (T, M_T)*
- (ii) *Y can be extended to a surjective, log smooth morphism over (T, M_T)*
- (iii) *the proper regular A-model of Y is cohomologically flat over T.*

Proof. (i) ⇒ (ii) trivially and (iii) ⇒ (i) by Theorem 5, so it remains to show (ii) ⇒ (iii). Then it suffices to show that $f_{\log} : (X, M_X) \to (T, M_T)$ is log smooth, where X is the proper regular A-model of Y and M_X its canonical log structure.

Let $g_{\log} : (Z, M_Z) \to (T, M_T)$ be a surjective log smooth morphism extending $Y \to u$. Let $\zeta \in Z_t$ be a maximal point; since (Z, M_Z) is log regular, $\mathcal{O}_{Z,\zeta}$ is a discrete valuation ring. By [25, 2.6], we have $(\overline{M}_Z)_\zeta = \mathbb{N}$ and, since M_T is canonical, $g_{\log}$ is Kummer in an open neighbourhood V of ζ. Since X/T is proper, up to further localizing at ζ, we may assume there is a morphism $V \to X$ which is an open immersion on generic fibres over T. By [25, 2.6], this morphism naturally extends to a morphism of log schemes.

The fibre product $(X, M_X) \times_{(T,M_T)} (V, M_Z|_V)$ is log smooth over the log regular scheme (X, M_X), hence is itself log regular. In particular, its underlying scheme is normal. On the other hand, base-changing the morphism $\mu \times \mathrm{pr}_X$ by $(V, M_Z|_V) \to (X, M_X)$, we get a morphism

$$A \times_T (V, M_Z|_V) \to (X, M_X) \times_{(T,M_T)} (V, M_Z|_V)$$

On underlying schemes, this is a finite morphism of normal flat T-schemes which is an isomorphism on generic fibres, hence it must be an isomorphism. Now it follows easily from [25, 2.6] that it is an isomorphism of log schemes. In particular, $f_{\log} \times_{(T,M_T)} (V, M_Z|_V)$ is log smooth, hence, by log flat descent [12], so is $f_{\log}$. □

4.4. **Application: curves.** We can improve the main result of [20] by removing the perfectness assumption on $k(t)$ and allowing the log structure to contain some horizontal components (cf. [29, 31]).

Theorem 6. *Assume X is regular of dimension 2, the generic fibre of $f_{\log}$ is log smooth, and M_T is the canonical log structure. Let $g : U \hookrightarrow X$ be the largest open subset on which M_X is trivial. If f is proper, then $f_{\log}$ is log smooth if and only if*

- (i) $M_X = g_* \mathcal{O}_U^* \cap \mathcal{O}_X$
- (ii) *$(X \setminus U) \subset X$ is a normal crossings divisor*
- (iii) *$H^i_{\text{ét}}(U_{\bar{u}}, \mathbb{Q}_l)$ is tamely ramified for $i \leq 1$ (where $\bar{u} \to u$ is a geometric point)*
- (iv) *f is cohomologically flat*
- (v) *$X_0 := (X_t)_{\mathrm{red}}$ is normal crossings scheme over t (Def. 1), such that any component of X_t of multiplicity divisible by p has no self-intersection and no two such components intersect.*

The next result is similar to [20, Prop. 1], but without restriction on $k(t)$.[3]

Proposition 4. *Assume X is regular, the generic fibre of $f_{\log}$ is log smooth, and M_T is the canonical log structure. Let $j : X_u \to X$ be the inclusion of the generic fibre of f and define a log structure $N_X := j_*\mathcal{O}^*_{X_u} \cap \mathcal{O}_X$.*

(i) *(X, M_X) is log regular if and only if*
  (1) *$M_X = g_*\mathcal{O}^*_U \cap \mathcal{O}_X$, where $g : U \hookrightarrow X$ is the largest open subset on which M_X is trivial*
  (2) *$X \setminus U \subset X$ is a normal crossings divisor.*

Moreover, if (X, M_X) is log regular at a geometric point $x \to X_t$, then

(ii) *$\overline{M}_{X,x} \simeq \mathbb{N}^r$, and preimages $x_1, ..., x_r \in \mathcal{O}_{X,x}$ of a basis of $\overline{M}_{X,x}$ form part of a regular system of parameters*
(iii) *there is a canonical map $N_{X,x} \to M_{X,x}$; in particular, a uniformiser $\pi \in \mathcal{O}(T)$ can be written $\pi = v \prod_{i=1}^r x_i^{a_i}$ with $v \in \mathcal{O}^*_{X,x}$ and $a_i \in \mathbb{N}$ (possibly zero).*

Furthermore, with the notation of (ii)–(iii), *we have*

(iv) *if $N_{X,x} \neq M_{X,x}$, then $a_i = 0$ for some $1 \leq i \leq r$*
(v) *if $p \nmid a_i$ for some $1 \leq i \leq r$ and $k(t) \subset k(x)$ is separable, then $f_{\log}$ is log smooth at x*
(vi) *if $f_{\log}$ is log smooth and $p \mid a_i$ for all $1 \leq i \leq r$, then $r < \dim \mathcal{O}_{X,x}$*
(vii) *if $f_{\log}$ is log smooth, then locally at x there is a smooth morphism*

$$X \to \operatorname{Spec}(\mathcal{O}_T[y, x_1, ..., x_r]/(\pi - y \prod_{i=1}^r x_i^{a_i}))$$

(viii) *if $f_{\log}$ is log smooth, then X_0 is a normal crossings scheme over t.*

Proof. (i) is local at a geometric point $x \to X$. First assume (X, M_X) is log regular. Then $M_X = g_*\mathcal{O}^*_U \cap \mathcal{O}_X$ by [25, 2.6]. By [14, Prop. 3.2] there is a sharp fs monoid P of rank $r \leq d := \dim \mathcal{O}_{X,x}$ inducing the log structure M_X, a regular local ring R of dimension at most 1, and elements $x_{r+1}, ..., x_d$ forming part of a regular system of parameters such that $\widehat{\mathcal{O}}_{X,x} = R[[P]][[x_{r+1}, ..., x_d]]/(\theta)$, where θ has constant term equal to $\operatorname{char}(k(x))$. Using the regularity of X, one easily shows there are $x_1, ..., x_r \in P$ such that $P = \oplus_{i=1}^r \mathbb{N}x_i$ and $x_1, ..., x_d$ form a regular sequence of parameters (cf. [25, 5.2] or [20, proof of Prop. 1]). Then $\prod_{i=1}^r x_i$ defines a normal crossings divisor whose support is equal to $X \setminus U$. This implies the necessity of the conditions.

For the sufficiency, let $x_1, ..., x_d \in \mathcal{O}_{X,x}$ be a regular system of parameters such that $\prod_{i=1}^r x_i$ defines the normal crossings divisor $X \setminus U$. The elements $x_1, ..., x_r \in g_*\mathcal{O}^*_U \cap \mathcal{O}_X = M_X$ correspond to the irreducible components of $X \setminus U$ and we have $g_*\mathcal{O}^*_U/\mathcal{O}^*_X = \oplus_{i=1}^r \mathbb{Z}x_i$. Thus, the monoid $\oplus_{i=1}^r \mathbb{N}x_i$ induces M_X at x, and (X, M_X) is log regular at x by definition, proving (i) as well as (ii).

Since M_T is the canonical log structure, we have $U \subset X_u$, hence $N_X \subset M_X$ (by (i)), proving the first assertion of (iii); the second follows immediately. Note that the prime divisors of π are the x_i for which $a_i > 0$, so if $N_{X,x} \neq M_{X,x}$ then $a_i = 0$ for some $1 \leq i \leq r$, hence (iv).

[3] We point out an inaccuracy in [20]: the first statement of part (e) of [20, Prop. 1] is (rather trivially) false but holds if one assumes f to be log smooth (which is what was needed in that paper)—see part (vi) of Proposition 4.

For (v), taking an a_ith root of v we have $\pi = (v^{1/a_i}x_i)^{a_i}\prod_{1\leq j\leq r, j\neq i} x_j^{a_j}$ from which we can easily construct a log smooth chart as in [13, 3.5]. The details are left to the reader.

For the proof of (vi)–(viii), assume $f_{\log}$ is log smooth and X connected of dimension $d = \dim \mathcal{O}_{X,x}$. Consider the two cases

(a) $p \mid a_i$ for all $1 \leq i \leq r$
(b) $p \nmid a_i$ for some $1 \leq i \leq r$.

In case (b), we may assume $v^{1/a_i} \in \mathcal{O}_X$, hence, up to relabeling we may assume $v = 1$ in this case.

For a T-scheme S with log structure, write ω^1_S for the logarithmic differentials over T with the trivial log structure. If S is a log scheme over (T, M_T), then write $\omega^1_{S/T}$ for the relative log differentials over (T, M_T), and similarly for morphisms of log schemes. Since $f_{\log}$ is smooth, by [13, 3.12] we have an exact sequence

$$0 \to f^*\omega^1_T \to \omega^1_X \to \omega^1_{X/T} \to 0$$

with $\omega^1_{X/T}$ a vector bundle of rank $d-1$. Pulling back to X_t, we deduce an exact sequence

$$0 \to f^*\omega^1_t \to \omega^1_{X_t} \to \omega^1_{X_t/t} \to 0$$

Since $\omega^1_T = k(t)\,\mathrm{dlog}(\pi)$, it follows that $\omega^1_{X_t}$ is locally free of rank d with $\mathrm{dlog}(\pi)$ forming part of a basis.

Consider the closed subscheme $Y \subset X$ of ideal $I = (x_1, ..., x_r)$. There is an $\mathcal{O}_Y$-linear residue map $\rho : \omega^1_X|_Y \to \mathcal{O}^r_Y$ mapping $\mathrm{dlog}(x_1), ... \mathrm{dlog}(x_r)$ to a basis of the target, whose kernel contains the usual differentials $\Omega^1_{X/T}|_Y$ ([20, Prop. 1], cf. [26, IV, 2.3.5]). So $\mathrm{dlog}(x_1), ... \mathrm{dlog}(x_r)$ form part of a basis of $\omega^1_X|_Y$. In case (a) we have $\mathrm{dlog}(\pi) \equiv \mathrm{dlog}(v) \mod p$, hence $\rho(\mathrm{dlog}(\pi)) = 0$. It follows that $\mathrm{d}v, \mathrm{dlog}(x_1), ..., \mathrm{dlog}(x_r)$ form part of a basis of $\omega^1_X|_Y$ in this case.

Endowing Y with the inverse image log structure of X, by [26, IV, 2.3.2] we have an exact sequence of log differentials

$$I/I^2 \to \omega^1_X|_Y \to \omega^1_Y \to 0$$

and clearly the left hand map is zero. Hence $\omega^1_Y \cong \omega^1_X|_Y$.

Let y be an indeterminate in case (a) (resp. $y = 1$ in case (b)). Define a log scheme (U, M_U) by

$$U = \mathrm{Spec}(\mathcal{O}(T)[y^{\pm 1}, x_1, ..., x_r]/(\pi - y\prod_{i=1}^r x_i^{a_i}))$$

and M_U the natural log structure induced by $\oplus_{i=1}^r \mathbb{N}x_i$. Let $V \subset U$ be the closed subscheme of ideal $(x_1, ..., x_r)$. The natural morphism $X \to U$ mapping y to v induces strict morphisms of log schemes $(X, M_X) \to (U, M_U)$ and $(Y, M_X|_Y) \to (V, M_U|_V)$. Denoting the underlying morphism $h : Y \to V$, we have a right-exact sequence

$$0 \to h^*\Omega^1_{V/t} \to \Omega^1_{Y/t} \to \Omega^1_{Y/V} \to 0 \tag{4.2}$$

Since $\omega^1_V \cong \omega^1_U|_V$, ω^1_V is a free $\mathcal{O}_V$-module generated by $\mathrm{d}y, \mathrm{dlog}(x_1), ..., \mathrm{dlog}(x_r)$. It follows that $\omega^1_{Y/V} \cong \Omega^1_{Y/V}$ is locally free of rank $d-(r+1)$ in case (a) (resp. $d-r$ in case (b)). Since $\mathrm{d}v$ vanishes nowhere in case (a) (resp. $V = t$ in case (b)) and Y is regular, hence reduced, the sequence (4.2) must also be left exact. Hence, $\Omega^1_{Y/t}$ is locally free of rank $d-r$. Since $\dim Y = d - r$, by [9, IV$_4$, 17.15.5] Y is smooth over t. It now follows from (4.2) that h is smooth. Applying [9, 0_{IV}, 15.1.21], we see that $X \to U$ is flat along Y, hence smooth, proving (vii) as well as (vi). Finally, (viii) follows from (vii). □

For the proof of Theorem 6 we will need a couple of lemmata.

Lemma 18. *Assume the generic fibre of $f_{\log}$ is log smooth and f is proper and satisfies (N). Let $X \to T' \to T$ be the Stein factorisation of f. If $\bar{u} \to T$ is a geometric point lying over u and $H^0_{\text{ét}}(X_{\bar{u}}, \mathbb{Z}/n)$ is tamely ramified for some integer $n > 1$ prime to p, then $T' \to T$ is a tamely ramified covering. In particular, this holds if $f_{\log}$ is log smooth.*

Proof. By assumption, the geometric generic fibre of $f_{\log}$ is log regular, hence reduced. This implies that T'_u is the spectrum of a product of separable extensions of $k(u)$ ([9, IV_2, 4.6.1]).

We have $H^0_{\text{ét}}(T'_{\bar{u}}, \mathbb{Z}/n) = H^0_{\text{ét}}(X_{\bar{u}}, \mathbb{Z}/n)$, and T' is normal (Lemma 6). Now apply the next lemma to deduce the first statement. For the second statement, note that X is normal in this case and, by Nakayama's theorem [24], the tame ramification condition holds. □

Lemma 19. *Let $k(u) \subset K$ be a finite separable extension and $S = \operatorname{Spec}(K)$. If $H^0_{\text{ét}}(S \times_u \bar{u}, \mathbb{Z}/n)$ is tamely ramified for some $n > 1$ prime to p, then $k(u) \subset K$ is a tamely ramified extension. The same holds for $\mathbb{Z}_l$ or $\mathbb{Q}_l$ coefficients, where $l \neq p$ is prime.*

Proof. Let $I \subset \operatorname{Gal}(k(\bar{u})/k(u))$ be the inertia group and $P \subset I$ the wild inertia group. Then $H^0_{\text{ét}}(S_{\bar{u}}, \mathbb{Z}/n)^P = (\mathbb{Z}/n)^e$, where e is the number of connected components of $S_{\bar{u}}/P = \operatorname{Spec}(K \otimes_{k(u)} k(\bar{u})^P)$. If P acts trivially on $H^0_{\text{ét}}(S_{\bar{u}}, \mathbb{Z}/n)$, then any connected component of $S_{\bar{u}}/P$ is geometrically connected over $k(\bar{u})^P$ so must be isomorphic to $\operatorname{Spec}(k(\bar{u})^P)$, hence $K \subset k(\bar{u})^P$. This proves the first statement and the second can be reduced to this one. □

Proof of Theorem 6. We first note that the conditions are necessary: (i) follows from [25, 2.6], (ii) from Proposition 4, (iii) from Nakayama's theorem [24], (iv) from Theorem 4, and (v) follows from (ii) and Proposition 4.

To show the sufficiency, first note that since X_u is log smooth over u it is geometrically normal, hence smooth. In fact, locally for the étale topology its log structure can be given by the monoid $\mathbb{N}$ and an étale map $k(u)[\mathbb{N}] \to \mathcal{O}_{X_u}$. This implies that the log structure of X_u is given by a finite set of closed points whose residue fields are separable extensions of $k(u)$. Moreover, (iii) implies that these extensions are tamely ramified: indeed, we have an exact sequence

$$H^1_{\text{ét}}(U_{\bar{u}}, \mathbb{Q}_l) \to H^0_{\text{ét}}((X \setminus U)_{\bar{u}}, \mathbb{Q}_l)(-1) \to H^2_{\text{ét}}(X_{\bar{u}}, \mathbb{Q}_l)$$

and $H^2_{\text{ét}}(X_{\bar{u}}, \mathbb{Q}_l) = \oplus \mathbb{Q}_l(-1)$ is unramified, so, since the wild inertia group is pro-p (cf. [30, IV]), $H^0_{\text{ét}}((X \setminus U)_{\bar{u}}, \mathbb{Q}_l)$ is tamely ramified and the claim follows from Lemma 19.

Let N_X be as in the statement of Proposition 4. If $x \to X_t$ is a geometric point, then $\overline{N}_{X,x} \neq 0$. If $\overline{M}^{\mathrm{gp}}_{X,x} \neq \overline{N}^{\mathrm{gp}}_{X,x}$, then we must have $\overline{M}_{X,x} \cong \mathbb{N}^2$ and, by Proposition 4 (iv), there is are $x_1, x_2 \in M_{X,x}$ mapping to a basis of $\overline{M}_{X,x}$ such that $\pi = v x_2^a$ for some $v \in \mathcal{O}^*_{X,x}$. The closed subscheme of $T' \subset X_{(x)}$ of equation $x_1 = 0$ is regular, flat over T, and T'_u is a point of $X_u \setminus U$, hence $T' \to T$ is tamely ramified by the above. In particular, this implies that $k(x)$ is a separable extension of $k(t)$. Since x_1, x_2 generate the maximal ideal of $\mathcal{O}_{X,x}$, the image of x_2 in $\mathcal{O}_{T'}$ is a uniformiser, hence $p \nmid a$. By Proposition 4 (v), $f_{\log}$ is log smooth at this point.

Now suppose $x \to X_t$ is a geometric point such that $\overline{M}^{\mathrm{gp}}_{X,x} = \overline{N}^{\mathrm{gp}}_{X,x}$. In this case we have $(g_*\mathcal{O}^*_U)_x = (j_*\mathcal{O}^*_{X_u})_x$, hence $M_{X,x} = N_{X,x}$ by (i). We reduce this case to the case $k(t)$ is algebraically closed as follows. First of all, we may assume T to be strictly henselian. Moreover, by Lemma 18 and Lemma 6 we may assume $f_*\mathcal{O}_X = \mathcal{O}_T$, hence the fibres of f are geometrically connected. Let S be the spectrum of a henselian discrete valuation ring with algebraically closed residue field as in by Lemma 13 and let $\bar{v} \to S_u$ be a geometric point lying above $\bar{u}$. Then, $X_0 \times_T S \subset X \times_T S$ is a normal crossings divisor and X_S is regular by Lemma 12. Moreover, the action of the wild inertia subgroup of $\pi_1(S_u, \bar{v})$ on $H^i_{\text{ét}}(U_{\bar{v}}, \mathbb{Q}_l) \cong H^i_{\text{ét}}(U_{\bar{u}}, \mathbb{Q}_l)$ factors through that of the wild inertia subgroup of $\pi_1(u, \bar{u})$, hence (iii) holds for X_S. Applying [20, Thm. 1], we deduce that the induced morphism $(X_S, N_{X_S}) \to (S, M_S)$ is log smooth. Hence $f_{\log} \times_T S$ is log smooth above x and so is $f_{\log}$ by Lemma 13. $\square$

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