Particle systems with sources and sinks

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Dated: March 5, 2024

Abstract

Local perturbations in conservative particle systems can have a non-local influence on the stationary measure. To capture this phenomenon, we analyze in this paper two toy models. We study the symmetric exclusion process on a countable set of sites V with a source at a given point (called the origin), starting from a Bernoulli product measure with density ρ. We prove that when the underlying random walk on V is recurrent, then the system evolves towards full occupation, whereas in the transient case we obtain a limiting distribution which is not product and has long-range correlations. For independent random walkers on V, we analyze the same problem, starting from a Poissonian measure. Via intertwining with a system of ODE's, we prove that the distribution is Poissonian at all later times $t > 0$, and that the system "explodes" in the limit $t \to \infty$ if and only if the underlying random walk is recurrent. In the transient case, the limiting density is a simple function of the Green's function of the random walk.

1 Introduction

Introducing local perturbations in conservative particle systems can have a drastic, i.e., non-local influence on the stationary measure, which can change from a product form to a measure with long-range correlations. This

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has already been studied in [4, 7] where anisotropic perturbations of the symmetric exclusion process were shown to exhibit long-range correlations, using a formal series expansion method. See also the more recent work [8], where an exclusion process with a driven bond is studied, and [5] where the growth of the total number of particles in a symmetric exclusion process with source is studied. The abelian sandpile model [1] is another well-known important example showing that a conservative diffusive dynamics combined with sources leads to a self-organized critical state, i.e., a stationary measure which has power law decay of correlations.

To capture this phenomenon, we analyze in this paper two toy models. We first consider the symmetric exclusion process with a source at the origin, in the setting of an infinite graph, and start it from a Bernoulli product measure of constant density, which is the stationary probability measure for the dynamics without source. In case the underlying random walk of the exclusion process is recurrent, we show that the system becomes fully occupied, whereas in the transient case, a limiting measure is obtained. This measure is the microscopic analogue of the solution of the Poisson equation, and it is shown to have long-range correlations. We then study a similar setting where the source at the origin is replaced by the combination of a source and a sink, which can be thought of as a coupling to a reservoir. For the density corresponding to the reservoir density the stationary measure is a product measure, but for other initial densities, in the transient case another limiting measure (with identical limiting density) with long-range correlations is obtained. Finally, we consider the same setting in the case of independent random walkers, where we show that starting from a homogeneous product of Poissonian distributions, the distribution at any later time is still a Poisson product measure, with a density which diverges in the recurrent case, and has a limit in the transient case. The main tool for the proof is an intertwining between the independent walkers with a source and a deterministic system of coupled linear differential equations which can be solved explicitly.

The rest of our paper is organized as follows. In section 2 we introduce the exclusion process with source (and possibly sink) at the origin, and explain the main questions. In section 3 we state and prove the main result, that is, about invariant measures for the exclusion processes with source or with source and sink. In section 4 we prove more on these measures for the process with a source, namely negative correlations and computation of covariances. In section 5 we consider the case of independent random walkers.

2 Setting, notations and definitions

Let V denote a countable set of vertices and $(p(x,y), x, y \in V)$ irreducible and symmetric random walk transition rates on V, i.e., for every $x, y \in V$, $p(x,y) = p(y,x) \geq 0$, and there exists $n \in \mathbb{N}$ such that $p^{(n)}(x,y) > 0$, where $p^{(n)}(x,y)$ denotes the corresponding n-step transition rate, i.e.,

$$p^{(n)}(x,y) = \sum_{z_1,\ldots,z_{n-1} \in V} p(x,z_1) \ldots p(z_{n-1},y)$$

Note that we do not assume a priori that $(p(x,y), x, y \in V)$ is a probability transition. In order to avoid existence problems, we will assume that

$$\sup_{x \in V} \sum_{y \in V} p(x,y) < \infty$$

2.1 Single particle dynamics

We denote by $\{X_t : t \geq 0\}$ a continuous-time random walk moving with rate $(p(x,y) : x, y \in V)$ over the (oriented) edge xy, i.e., the random walk on V with generator

$$L_{\mathrm{RW}} f(x) = \sum_{y \in V} p(x,y)(f(y) - f(x)) \tag{1}$$

for a function $f : V \to \mathbb{R}$. We denote by $\mathbb{E}_x^{\mathrm{RW}}$ expectation for this random walk starting at $X_0 = x$.

2.2 Exclusion process

Next, we consider the symmetric exclusion process on V based on $(p(x,y) : x, y \in V)$, which intuitively speaking consists of independent walkers moving each one according to the generator L_{RW} and subject to the restriction that at any instant of time multiple occupancies are forbidden, i.e., all jumps leading to more than one particle per site are forbidden.

We thus consider configurations of particles with at most one particle per site and denote the corresponding configuration space by $\Omega = \{0,1\}^V$. Elements of Ω are denoted by η, ξ, ζ, and for $\eta \in \Omega$ we denote by η_x the occupation at vertex $x \in V$, i.e, $\eta_x = 1$ (resp. $\eta_x = 0$) means x is occupied (resp. empty).

The exclusion process based on $(p(x,y) : x, y \in V)$ is then defined as the unique Markov process $\{\eta(t) : t \geq 0\}$ on $\{0,1\}^V$ with generator given by

$$L_{\text{SEP}} f(\eta) = \sum_{x,y \in V} p(x,y)\eta_x(1-\eta_y)(f(\eta^{x,y}) - f(\eta)) \tag{2}$$

for a local function $f : \Omega \to \mathbb{R}$ (i.e., depending on a finite number of coordinates $\eta_i, i \in V$), where $\eta^{x,y}$ is the configuration obtained from η by removing a particle at x and putting it at y, that is,

$$(\eta^{x,y})_z = \begin{cases} \eta_x - 1 & \text{if } z = x \\ \eta_y + 1 & \text{if } z = y \\ \eta_z & \text{otherwise} \end{cases} \tag{3}$$

The existence of this process is proved in [6, Chapter VIII], that also contains many properties of the process, including self-duality (see next subsection). The interpretation of the generator is that particles move in continuous time according to the hopping rates $(p(x,y) : x, y \in V)$ but jumps to already occupied sites are suppressed.

If in the exclusion process we start from a finite number of particles, initially located at sites in a set $A = \{x_1, \ldots, x_n\} \subset V$ then at any later time, the set of occupied vertices is a finite set $A(t) = \{x_1(t), \ldots, x_n(t)\}$. For $n = 2$ we write $A = \{x, y\}$, $A(t) = \{x(t), y(t)\}$. Thus we equivalently consider the exclusion process as taking values in the set of finite subsets of V and (with an abuse of notation) we may write its generator also as

$$L_{\text{SEP}} f(A) = \sum_{x,y \in V} p(x,y) I(x \in A, y \notin A)(f(A^{x,y}) - f(A)) \tag{4}$$

for a local function f, where I denotes the indicator function and $A^{x,y}$ is the set obtained by removing x from A and adding y to A. We denote by $\mathbb{E}^{\text{SEP}}_A$ the expectation in the process $\{A(t), t \geq 0\}$, by $\mathbb{E}^{\text{SEP}}_{x_1,\ldots,x_n}$ expectation, by $\mathscr{V}_n$ the generator and by $V_n(t)$ the semi-group in the corresponding labeled process $(x_1(t), \ldots, x_n(t))$, where we choose the initial labels, which are preserved in the course of time. We thus write, for a local function f,

$$V_n(t) f(x_1, \ldots, x_n) = \mathbb{E}^{\text{SEP}}_{x_1,\ldots,x_n} f(x_1(t), \ldots, x_n(t))$$

A function of the configuration $\eta(t)$ can then alternatively be viewed as a symmetric function of $x_1(t), \ldots, x_n(t)$.

For n independent random walkers (each one with generator (1)), we denote respectively by $\mathscr{U}_n$, $U_n(t)$ and $\mathbb{E}^{\text{IRW}}_{x_1,\ldots,x_n}$ the corresponding generator, semi-group and expectation. More generally, the independent random walkers process is then a process on $\mathbb{N}^V$ with formal generator

$$L_{\text{IRW}} f(\eta) = \sum_{x,y\in V} p(x,y)\,(\eta_x(f(\eta^{x,y}) - f(\eta)) + \eta_y(f(\eta^{y,x}) - f(\eta))) \tag{5}$$

working on local functions $f : \mathbb{N}^V \to \mathbb{R}$. We denote by $\mathbb{E}^{\text{IRW}}_\eta$ the corresponding expectation, when starting from the configuration η.

2.3 Self-duality for exclusion and random walkers

An important property of the (symmetric) exclusion process is self-duality (see [6, Chapter VIII]), which is formulated as follows. Let $\xi \in \Omega$ denote a finite configuration, i.e., $\sum_x \xi_x < \infty$, and, for $\eta \in \Omega$, define

$$D(\xi, \eta) = I(\xi \leq \eta) \tag{6}$$

where $\xi \leq \eta$ refers to coordinatewise order, i.e., if there is a particle in ξ at a site x, then there must also be a particle in η at x. Then we have, for any finite configuration ξ, for any $\eta \in \Omega$ and $t > 0$

$$\mathbb{E}^{\text{SEP}}_\eta D(\xi, \eta(t)) = \mathbb{E}^{\text{SEP}}_\xi D((\xi(t), \eta) \tag{7}$$

Let us denote by $\xi = e_x$ the configuration with a single particle at x and no particles elsewhere, then, under the exclusion process $\{\xi(t), t \geq 0\} = \{e_{X_t} : t \geq 0\}$, where X_t is the random walk with generator L_{RW} of (1), the duality relation (7) reads

$$\mathbb{E}^{\text{SEP}}_\eta \eta_x(t) = \mathbb{E}^{\text{RW}}_x(\eta_{X_t})$$

The symmetric exclusion process has as reversible probability measures homogeneous Bernoulli product measures. We denote these reversible product measures by

$$\nu_\rho, \quad \rho \in [0,1], \qquad \text{with } \nu_\rho(\eta_x = 1) = \rho, \;\text{ for all } x \in V \tag{8}$$

Similarly, there is a self-duality relation for independent random walkers, which reads as follows. Define the polynomials

$$D_{\text{IRW}}(\xi, \eta) = I(\xi \leq \eta) \prod_{i\in V} \frac{\eta_i!}{(\eta_i - \xi_i)!}$$

where $\xi \in \mathbb{N}^V$ is a finite configuration of the independent random walkers process, and $\eta \in \mathbb{N}^V$. Then we have

$$\mathbb{E}_\eta^{\mathrm{IRW}}(D_{\mathrm{IRW}}(\xi, \eta(t)) = \mathbb{E}_\xi^{\mathrm{IRW}}(D_{\mathrm{IRW}}(\xi(t), \eta)) \tag{9}$$

For the proof of this self-duality relation, we refer to [2].

2.4 The model with a source (or with source and sink)

To define the process with a source (or a source and a sink), we fix a vertex $0 \in V$, that we call the origin, and call the process *symmetric exclusion process with source at* 0, *of intensity* $\lambda \geq 0$ the process with generator

$$L_{\mathrm{SEP},0} f(\eta) = \sum_{x,y} p(x,y)\eta_x(1-\eta_y)(f(\eta^{x,y}) - f(\eta)) + \lambda(1-\eta_0)(f(\eta^0) - f(\eta)) \tag{10}$$

for a local function f, where η^0 is the configuration obtained from η by flipping the occupation variable at the origin, that is,

$$(\eta^0)_z = \begin{cases} 1-\eta_0 & \text{if } z = 0 \\ \eta_z & \text{otherwise} \end{cases} \tag{11}$$

In other words: particles move according to the symmetric exclusion process, and whenever the origin is empty at rate λ a particle is added. We denote by $S_0(t)$ the semi-group of this process. We start the process with source at 0 from ν_ρ (defined in (8)), we denote by $\nu_{\rho,0}(t) := \nu_\rho S_0(t)$ the measure at time $t > 0$.

Note that we use the sub-index 0 to refer that the process has a source at the origin; later on, we will use the sub-index 1 for the process with a source and a sink.

We then study the following two questions.

1. When is $\lim_{t\to\infty} \nu_{\rho,0}(t)$ equal to $\delta_{\underline{1}}$, the Dirac measure concentrated on the fully occupied configuration (that is, such that $\underline{1}(x) = 1$ for any site $x \in V$)?

2. If $\lim_{t\to\infty} \nu_{\rho,0}(t) \neq \delta_{\underline{1}}$, what is the limiting measure? What is its density, and are there non-trivial correlations?

The same questions will also be asked for a model with a source and a sink at 0, i.e., the process with generator

$$\begin{aligned} L_{\mathrm{SEP},1} f(\eta) &= \sum_{x,y} p(x,y)\eta_x(1-\eta_y)(f(\eta^{x,y}) - f(\eta)) \\ &+ \lambda(1-\eta_0)(f(\eta^0) - f(\eta)) + \mu\eta_0(f(\eta^0) - f(\eta)) \end{aligned} \tag{12}$$

for an intensity $\mu > 0$, for a local function f. We denote by $S_1(t)$ the semi-group of this process. We start the process with source and sink at 0 from ν_ρ, we denote by $\nu_{\rho,1}(t) := \nu_\rho S_1(t)$ the measure at time $t > 0$. Notice that for this process the Bernoulli product measure with density

$$\rho_R := \frac{\lambda}{\lambda + \mu} \tag{13}$$

is reversible, but this fact does not imply that this measure is the only invariant probability measure, i.e., the two questions asked for the models with source can be asked for the model with source and sink as well. The idea is that the source and sink site corresponds to a "reservoir" and that in the transient case the system can "miss the reservoir", and therefore converge to a limiting density different from the density ρ_R imposed by the reservoir.

3 Invariant measures

Consider the process with a source, that is, with generator (10), and denote by $\mathbb{E}_\eta$ expectation in this process starting from configuration $\eta \in \Omega$. We denote by $\mathbb{E}_{\nu_\rho} = \int \mathbb{E}_\eta \nu_\rho(d\eta)$ expectation starting from an initial configuration η which is ν_ρ distributed. First we show the existence of $\lim_{t\to\infty} \nu_\rho S_0(t) =: \mu_{\rho,0}$. This is proved with the help of a dual process, where particles are moving according to an exclusion process with a sink at the origin.

3.1 Two duals of the model with source and convergence to an invariant measure

In this subsection we introduce two alternative duality relations for the process with a source, analogous to the ones introduced in [6, Chapter III] for spin systems. For the first one, that is a "killed random walkers dual", denote, for $A \subset V$ a finite set,

$$H(A, \eta) = \prod_{x \in A} \eta_x \tag{14}$$

with the convention $H(\emptyset, \eta) = 1$. Then, we compute

$$\lambda(1-\eta_0)\left(H(A,\eta^0) - H(A,\eta)\right) = \lambda I(0 \in A)(H(A \setminus \{0\}, \eta) - H(A, \eta)) \tag{15}$$

where we used

$$H(A, \eta^0) = \begin{cases} H(A, \eta) & \text{if } 0 \notin A \\ \left(\prod_{x \neq 0, x \in A} \eta_x\right)(1-\eta_0) & \text{if } 0 \in A \end{cases} \tag{16}$$

By combining (14)–(16) with the self-duality relation (7) of the symmetric exclusion process we find

$$L_{\mathrm{SEP},0}H(A,\eta) = \overline{\mathscr{L}}H(A,\eta) \tag{17}$$

where the generator $\overline{\mathscr{L}}$ works on the A-variable (i.e., on finite subsets of V) and is given by

$$\begin{aligned} \overline{\mathscr{L}}f(A) &= \sum_{x,y\in V} p(x,y)I(x\in A, y\notin A)(f(A^{x,y})-f(A)) \\ &\quad +\lambda I(0\in A)(f(A\setminus\{0\})-f(A)) \end{aligned} \tag{18}$$

for a local function f. The dual process $\{\overline{A}(t), t\geq 0\}$ with generator $\overline{\mathscr{L}}$ is a process taking values in the set of finite subsets of V, and can be described as follows: particles initially located at the sites of A perform the symmetric exclusion process starting from A (i.e., there are particles at the sites of A and no particles elsewhere), and are killed with rate λ when they are at the origin. Let us denote $\mathbb{E}_A^{dual}$ expectation in this process starting from A, and further denote $p_t^\lambda(A,B)$ the transition probability in this dual process to go from A to B in time t; hence note that here, as each time we introduce a transition $p_t(.,.)$, either for exclusion or for random walker, we assume that $(p(x,y), x,y\in V)$ is a probability transition.

As a consequence, we have the following.

Theorem 3.1. *Let $A\subset V$ be a finite set. Then we have, for every $t>0$*

$$\int H(A,\eta)\nu_{\rho,0}(t)(d\eta) = \sum_{\substack{B\subset V\\ |B|\leq|A|, B\neq\emptyset}} p_t^\lambda(A,B)\rho^{|B|} + p_t^\lambda(A,\emptyset) = \mathbb{E}_A^{dual}(\rho^{|\overline{A}(t)|}) \tag{19}$$

and as $t\to\infty$ the limiting measure $\mu_{\rho,0}=\lim_{t\to\infty}\nu_{\rho,0}(t)$ exists, is invariant and satisfies

$$\int H(A,\eta)\mu_{\rho,0}(d\eta) = \mathbb{E}_A^{dual}(\rho^{|\overline{A}_\infty|}) \tag{20}$$

where $|\overline{A}_\infty| = \lim_{t\to\infty}|\overline{A}(t)|$.

Proof. By the duality relation (17) we obtain

$$\mathbb{E}_\eta H(A,\eta(t)) = \mathbb{E}_A^{dual} H(\overline{A}(t),\eta) \tag{21}$$

Integrating this relation over ν_ρ (in the η-variable) yields (19). The limit $t\to\infty$ is well-defined because the cardinality of $\overline{A}(t)$ is non-increasing in t.

The invariance of $\mu_{\rho,0}$ follows because it is equal to the limit $\lim_{t\to\infty}\nu_{\rho,0}(t)$ (by [6, Chapter I, Proposition 1.8]). $\square$

There is also an alternative duality relation with a Feynman-Kac term, that can be obtained as follows. If we define, for $A \subset V$ a finite set,

$$\widetilde{H}(A,\eta) = \prod_{x\in A}(1-\eta_x) \tag{22}$$

then we find

$$L_{\mathrm{SEP},0}\widetilde{H}(A,\cdot)(\eta) = \mathscr{R}\widetilde{H}(\cdot,\eta)(A)$$

where the operator $\mathscr{R}$ is in Schrödinger operator form and given by

$$\mathscr{R}f(A) = \sum_{x,y\in V} I(x\in A, y\notin A)p(x,y)(f(A^{x,y})-f(A)) - \lambda I(0\in A)f(A)$$

for a local function f. As a consequence of the Feynman-Kac formula we then have

$$\mathbb{E}_\eta \widetilde{H}(A,\eta(t)) = \mathbb{E}_A^{\mathrm{SEP}}\left(e^{-\lambda\int_0^t I(0\in A(s))ds}\widetilde{H}(A(t),\eta)\right) \tag{23}$$

Upon integrating (23) over the Bernoulli measure ν_ρ gives

$$\int \widetilde{H}(A,\eta)\nu_{\rho,0}(t)(d\eta) = (1-\rho)^{|A|}\mathbb{E}_A^{\mathrm{SEP}}\left(e^{-\lambda\int_0^t I(0\in A(s))ds}\right)$$

Then we have the following result on the limiting measure $\mu_{\rho,0}$ for the dynamics with a source, as well as on $\mu_{\rho,1}$ for the dynamics with source and sink.

Theorem 3.2. *I)* ***The model with source.*** *Let $\{\eta(t), t\geq 0\}$ denote the process with generator* (10).

1. *If $(p(x,y): x,y\in V)$ is recurrent then $\mu_{\rho,0} = \delta_{\underline{1}}$. Moreover for any configuration $\eta\in\Omega$, we have $\lim_{t\to\infty}\delta_\eta S_0(t) = \delta_{\underline{1}}$, where δ_η denotes the Dirac measure on configuration η. As a consequence in that case $\delta_{\underline{1}}$ is the unique invariant probability measure.*
2. *If $(p(x,y): x,y\in V)$ is transient then $\lim_{t\to\infty}\nu_{\rho,0}(t) = \mu_{\rho,0}$ with the following properties*
   a) Limiting density:

$$\lim_{t\to\infty}\mathbb{E}_{\nu_\rho}(1-\eta_x(t)) = (1-\rho)\mathbb{E}_x^{\mathrm{RW}}e^{-\lambda\int_0^\infty \delta_{X_s,0}\; ds} \tag{24}$$

where $\delta_{\cdot,\cdot}$ denotes the Kronecker symbol.

b) Covariances:

$$\begin{aligned} cov_{\mu_{\rho,0}}(\eta_x,\eta_y) &= (1-\rho)^2 \Big(\mathbb{E}^{\mathrm{SEP}}_{x,y} e^{-\lambda \int_0^\infty (\delta_{x(s),0}+\delta_{y(s),0})\ ds} \\ &\qquad -\mathbb{E}^{\mathrm{IRW}}_{x,y} e^{-\lambda \int_0^\infty (\delta_{X_s,0}+\delta_{Y_s,0})\ ds}\Big) \end{aligned} \tag{25}$$

II) ***The model with source and sink.*** *Let $\{\eta(t), t \geq 0\}$ denote the process with generator* (12)*.*

1. *If $(p(x,y) : x,y \in V)$ is recurrent then $\lim_{t\to\infty} \nu_{\rho,1}(t) = \nu_{\rho_R}$ (recall* (13)*). The same holds for any initial configuration, i.e., $\lim_{t\to\infty} \delta_\eta(t) = \nu_{\rho_R}$ for every $\eta \in \Omega$.*
2. *If $(p(x,y) : x,y \in V)$ is transient then $\nu_{\rho,1}(t) \to \mu_{\rho,1}$ with the following properties*
   a) Limiting density:

$$\lim_{t\to\infty} \mathbb{E}_{\nu_\rho}(\eta_x(t)) = \rho_R + (\rho-\rho_R)\, \mathbb{E}^{\mathrm{RW}}_x e^{-(\lambda+\mu)\int_0^\infty \delta_{X_s,0}\ ds} \tag{26}$$

   b) We have the following formula for the covariances

$$\begin{aligned} cov_{\mu_{\rho,1}}(\eta_x,\eta_y) &= (\rho-\rho_R)^2 \Big(\mathbb{E}^{\mathrm{SEP}}_{x,y} e^{-\lambda \int_0^\infty (\delta_{x(s),0}+\delta_{y(s),0})\ ds} \\ &\qquad -\mathbb{E}^{\mathrm{IRW}}_{x,y} e^{-\lambda \int_0^\infty (\delta_{X_s,0}+\delta_{Y_s,0})\ ds}\Big) \end{aligned} \tag{27}$$

Proof. **Part I, case 1.** Let us start by the computation starting from the generator (10) for $x \in V$,

$$L_{\mathrm{SEP},0}(1-\eta_x) = \sum_{y\in V} p(x,y)((1-\eta_y)-(1-\eta_x)) - \lambda\delta_{x,0}(1-\eta_x) \tag{28}$$

Denoting

$$\psi(x,\eta) = (1-\eta_x),\ \psi_x = \psi(x,\cdot) \tag{29}$$

we can write (28) in the form

$$L_{\mathrm{SEP},0}\psi_x = \sum_{y\in V} p(x,y)(\psi_y-\psi_x) - \lambda\delta_{x,0}\psi_x \tag{30}$$

Let us denote $\mathscr{A}$ the operator working on the x-variable as

$$\mathscr{A}\varphi(x) = \sum_{y\in V} p(x,y)(\varphi(y)-\varphi(x)) - \lambda\delta_{x,0}\varphi(x)$$

which is the sum of the random walk generator L_{RW} and a multiplication operator with the "potential" $-\lambda\delta_{x,0}$. As a consequence, by the Feynman-Kac formula we obtain

$$e^{t\mathscr{A}}\varphi(x) = \mathbb{E}_x^{\mathrm{RW}}\left(e^{-\lambda\int_0^t \delta_{X_s,0}\ ds}\varphi(X_t)\right)$$

This identity, combined with (30) gives

$$\mathbb{E}_\eta(1-\eta_x(t)) = e^{t\mathscr{A}}\psi(\cdot,\eta)(x) = \mathbb{E}_x^{\mathrm{RW}}\left(e^{-\lambda\int_0^t \delta_{X_s,0}\ ds}(1-\eta_{X_t})\right) \tag{31}$$

Integrating over ν_ρ gives

$$\int \nu_\rho(d\eta)\mathbb{E}_\eta(1-\eta_x(t)) = (1-\rho)\mathbb{E}_x^{\mathrm{RW}}\left(e^{-\lambda\int_0^t \delta_{X_s,0}\ ds}\right) \tag{32}$$

Taking the limit $t\to\infty$ gives statements 1 (because the integral on the right hand side of (32) goes to $+\infty$), and 2a) of Part I of Theorem 3.2.

Part II, case 1. To prove the case with source and sink, start again from the computation of $L_{\mathrm{SEP},1}\eta_x$ for the generator (12) and $x\in V$; we get, with the notation $\chi_x = \eta_x$

$$L_{\mathrm{SEP},1}\chi_x = \sum_{y\in V} p(x,y)(\chi_y-\chi_x) - (\lambda+\mu)\delta_{x,0}\chi_x + \lambda\delta_{x,0} \tag{33}$$

Let us consider the equation

$$\frac{d}{dt}\varphi(x,t) = \mathscr{B}\varphi(\cdot,t)(x) + \lambda\delta_{x,0} \tag{34}$$

where

$$\mathscr{B}f(x) = \sum_{y\in V} p(x,y)(f(y)-f(x)) - (\lambda+\mu)\delta_{x,0}f(x).$$

Then by the variation of constants method, we find the solution

$$\varphi(x,t) = e^{t\mathscr{B}}\varphi(x,0) + \int_0^t e^{(t-s)\mathscr{B}}\delta_{x,0}\ ds \tag{35}$$

where $\mathscr{B}$ as well as $e^{t\mathscr{B}}$ work on the x-variable. Because the semigroup $e^{t\mathscr{B}}$ can be computed using the Feynman-Kac formula we obtain, combining (33) with (35)

$$
\begin{aligned}
& \mathbb{E}_\eta(\eta_x(t)) \\
= \ & \mathbb{E}_x^{\mathrm{RW}}\left(e^{-(\lambda+\mu)\int_0^t \delta_{X_s,0}\ ds}\eta_{X_t}\right) + \lambda\int_0^t \mathbb{E}_x^{\mathrm{RW}}\left(e^{-(\lambda+\mu)\int_0^s \delta_{X_r,0}\ dr}\delta_{X_s,0}\right)\ ds \\
= \ & \mathbb{E}_x^{\mathrm{RW}}\left(e^{-(\lambda+\mu)\int_0^t \delta_{X_s,0}\ ds}\eta_{X_t}\right)\ ds \\
& + \frac{\lambda}{\lambda+\mu}\int_0^t \mathbb{E}_x^{\mathrm{RW}}\left(e^{-(\lambda+\mu)\int_0^s \delta_{X_r,0}\ dr}(\lambda+\mu)\delta_{X_s,0}\right)\ ds \\
= \ & \mathbb{E}_x^{\mathrm{RW}}\left(e^{-(\lambda+\mu)\int_0^t \delta_{X_s,0}\ ds}\eta_{X_t}\right)\ ds + \rho_R\int_0^t (-1)\frac{d}{ds}(e^{s\mathscr{B}}1)(x) \\
= \ & \mathbb{E}_x^{\mathrm{RW}}\left(e^{-(\lambda+\mu)\int_0^t \delta_{X_s,0}\ ds}\eta_{X_t}\right)\ ds \\
& + \rho_R\left(1 - \mathbb{E}_x^{\mathrm{RW}}\left(e^{-(\lambda+\mu)\int_0^t \delta_{X_s,0}\ ds}\right)\right)
\end{aligned} \tag{36}
$$

which yields items 1 and 2a) of Part II of Theorem 3.2.

Part I, case 2. We now focus on the transient case and prove statement 2b) of part I of Theorem 3.2. We compute the expectation

$$\mathbb{E}_\eta(\psi(x,\eta(t))\psi(y,\eta(t)))$$

for $x \in V$, where we remind the reader the notation (29). First, a generator computation yields, using self-duality of the symmetric exclusion process (see (7)), for $x, y \in V$,

$$L\psi_x\psi_y = \mathscr{V}_2\psi_x\psi_y - \lambda(\delta_{x,0} + \delta_{y,0})\psi_x\psi_y \tag{37}$$

for $x \in V$, where $\mathscr{V}_2$ denotes the generator of two exclusion particles initially starting from x, y (cf. subsection 2.2). Using once more the Feynman-Kac formula, this leads to

$$\mathbb{E}_\eta(\psi(x,\eta(t))\psi(y,\eta(t))) = \mathbb{E}_{x,y}^{\mathrm{SEP}}\left(e^{-\lambda\int_0^t(\delta_{x(s),0}+\delta_{y(s),0})\ ds}\psi(x(t),\eta)\psi(y(t),\eta)\right) \tag{38}$$

Integrating this equality over ν_ρ (in the η-variable) gives

$$\int \mathbb{E}_\eta(\psi(x,\eta(t))\psi(y,\eta(t)))\ \nu_\rho(d\eta) = (1-\rho)^2\mathbb{E}_{x,y}^{\mathrm{SEP}}\left(e^{-\lambda\int_0^t(\delta_{x(s),0}+\delta_{y(s),0})\ ds}\right) \tag{39}$$

Taking the limit $t \to \infty$ gives

$$\lim_{t\to\infty}\int \mathbb{E}_\eta(\psi(x,\eta(t))\psi(y,\eta(t)))\ \nu_\rho(d\eta) = (1-\rho)^2\mathbb{E}_{x,y}^{\mathrm{SEP}}\left(e^{-\lambda\int_0^\infty(\delta_{x(s),0}+\delta_{y(s),0})\ ds}\right)$$

which proves 2b) of part I of the theorem. Item 2b) of part II is proved along the same lines. $\square$

Using the alternative duality relation (23), we can prove the following more general statement

COROLLARY 3.1. *Let $(p(x,y) : x, y \in V)$ be transient, then for $\mu_{\rho,0} = \lim_{t\to\infty} \nu_\rho S_0(t)$ we have the following. For all $x_1, \ldots, x_n$, n distinct points in V, we have*

$$\begin{aligned}
&\int \prod_{i=1}^{n}(1-\eta_{x_i})\mu_{\rho,0}(d\eta) - \prod_{i=1}^{n}\int(1-\eta_{x_i})\mu_{\rho,0}(d\eta) \\
= \quad & (1-\rho)^n \left(\mathbb{E}^{\mathrm{SEP}}_{x_1,\ldots,x_n} e^{-\lambda\sum_{i=1}^n \int_0^\infty \delta_{x_i(s),0}ds} - \mathbb{E}^{\mathrm{IRW}}_{x_1,\ldots,x_n} e^{-\lambda\sum_{i=1}^n \int_0^\infty \delta_{x_i(s),0}ds}\right)
\end{aligned} \tag{40}$$

4 Further properties of the invariant measures

4.1 Negative correlations

To derive more properties of the invariant measure $\mu_{\rho,0}$, we compare the evolutions of exclusion process and of independent random walkers. We first prove the following lemma, which will imply negative correlations in the measure $\mu_{\rho,0}$. Recall the notation introduced in Subsection 2.2.

LEMMA 4.1. *For all $t \geq 0$, and for all $x_1, \ldots, x_n$, n distinct points in V*

$$\left(\mathbb{E}^{\mathrm{SEP}}_{x_1,\ldots,x_n} e^{-\lambda\sum_{i=1}^n \int_0^t \delta_{x_i(s),0}ds} - \mathbb{E}^{\mathrm{IRW}}_{x_1,\ldots,x_n} e^{-\lambda\sum_{i=1}^n \int_0^t \delta_{x_i(s),0}ds}\right) \leq 0 \tag{41}$$

PROOF. Let us denote $\Psi(x_1, \ldots, x_n) = \sum_{i=1}^n \delta_{x_i,0}$. Then we have, using the Feynman-Kac formula

$$\begin{aligned}
&\left(\mathbb{E}^{\mathrm{SEP}}_{x_1,\ldots,x_n} e^{-\lambda\sum_{i=1}^n \int_0^t \delta_{x_i(s),0}ds} - \mathbb{E}^{\mathrm{IRW}}_{x_1,\ldots,x_n} e^{-\lambda\sum_{i=1}^n \int_0^t \delta_{x_i(s),0}ds}\right) \\
= \quad & (e^{t(\mathscr{V}_n+\Psi)}1 - e^{t(\mathscr{U}_n+\Psi)}1)(x_1, \ldots, x_n) \\
= \quad & \int_0^t (e^{(t-s)(\mathscr{V}_n+\Psi)}(\mathscr{V}_n - \mathscr{U}_n)e^{s(\mathscr{U}_n+\Psi)}1)(x_1, \ldots, x_n)ds
\end{aligned} \tag{42}$$

Because the function 1 is positive definite, and the semigroup $(e^{s(\mathscr{U}_n+\Psi)}1)(x_1, \ldots, x_n)$ factorizes, i.e.,

$$(e^{s(\mathscr{U}_n+\Psi)}1)(x_1, \ldots, x_n) = \sum_{z_1,\ldots,z_n}\prod_{i=1}^n k_s(x_i, z_i)$$

where

$$k_s(u,v) = \mathbb{E}_u^{\mathrm{IRW}}\left(e^{-\lambda\int_0^t \delta_{x(s),0}} I(x(t)=v)\right)$$

we have that $(e^{s(\mathscr{U}_n+\Psi)}1)(x_1,\ldots,x_n)$ is a positive definite symmetric function of $x_1,\ldots,x_n$. Therefore, by Liggett's inequality (see [6, Chapter VIII, Proposition 1.7]), we have $((\mathscr{V}_n - \mathscr{U}_n)e^{s(\mathscr{U}_n+\Psi)}1)(x_1,\ldots,x_n) \le 0$. The result then follows from (42) using that $e^{(t-s)(\mathscr{V}_n+\Psi)}$ is a positive semigroup (i.e., maps non-negative functions to non-negative functions). $\square$

Then we have the following corollary for the process with a source.

Proposition 4.1 (Negative correlations)**.** *Let $(p(x,y): x,y \in V)$ be transient, then for $\mu_{\rho,0} = \lim_{t\to\infty} \nu_\rho S_0(t)$ we have the following. For all $x_1,\ldots,x_n$, n distinct points in V, we have*

$$\int \prod_{i=1}^n (1-\eta_{x_i})\mu_{\rho,0}(d\eta) - \prod_{i=1}^n \int (1-\eta_{x_i})\mu_{\rho,0}(d\eta) \le 0 \tag{43}$$

Proof. This follows by combining Corollary 3.1 with Lemma 4.1. $\square$

To complement this result, we show that in general $\mu_{\rho,0}$ is not a product measure.

Proposition 4.2. *Let $(p(x,y): x,y \in V)$ be transient, then for $0<\rho<1$, $\mu_{\rho,0} = \lim_{t\to\infty} \nu_\rho S_0(t)$ is not a product measure.*

Proof. We have

$$\mathbb{E}_{\nu_\rho}(1-\eta_x(t)) = (1-\rho)\mathbb{E}_x^{\mathrm{RW}}\left(e^{-\lambda\int_0^t \delta_{X_s,0}\; ds}\right) =: h(x,t)$$

Then $h(x,t)$ satisfies

$$\frac{dh(x,t)}{dt} = L_{\mathrm{RW}} h(x,t) - \lambda\delta_{x,0} h(x,t)$$

with $h(x,0) = 1-\rho$. As a consequence, $u(x) = \lim_{t\to\infty} h(x,t)$ exists because $(p(x,y): x,y \in V)$ is transient, and it satisfies

$$L_{\mathrm{RW}} u(x) = \lambda\delta_{x,0} u(x) \tag{44}$$

Let us denote by Λ the product measure with $\int (1-\eta_x)\Lambda(d\eta) = u(x)$. Now fix $x \neq 0$ such that $p(0,x) > 0$ and compute, using the generator (10) and

(37).

$$\begin{aligned}
& \int (L_{\mathrm{SEP},0}(1-\eta_x)(1-\eta_0))\Lambda(d\eta) \\
= \; & (L_{\mathrm{RW}}u(0))u(x) + (L_{\mathrm{RW}}u(x))u(0) - p(0,x)(u(0)-u(x))^2 - \lambda u(0)u(x) \\
= \; & -p(0,x)(u(0)-u(x))^2 \neq 0
\end{aligned}$$

Here in the last step we used (44). So we conclude that Λ is not invariant. Because we proved earlier that $\mu_{\rho,0}$ is invariant, it cannot be equal to Λ. $\square$

4.2 The covariance

To understand better the covariance in the limiting measure $\mu_{\rho,0}$, we approximate $\left(\mathbb{E}^{\mathrm{SEP}}_{x,y} e^{-\lambda\int_0^\infty(\delta_{x(s),0}+\delta_{y(s),0})\, ds} - \mathbb{E}^{\mathrm{IRW}}_{x,y} e^{-\lambda\int_0^\infty(\delta_{X_s,0}+\delta_{Y_s,0})\, ds}\right)$ around $\lambda = 0$ up to second order in λ (first and zeroth order being zero). In order to prepare this computation, let us denote by $e_x + e_y$ the configuration with one particle at x and one at y. Then $\delta_{x(s),0} + \delta_{y(s),0}$ is the occupation at zero at time s starting from $e_x + e_y$ initially.

Let us denote by $p_t^{\mathrm{SEP}}(e_x + e_y, e_u + e_v)$ the transition probability for two exclusion particles to arrive at time t at the configuration $e_u + e_v$ when initially started from $e_x + e_y$. and $p_t^{\mathrm{IRW}}(e_x + e_y, e_u + e_v)$ for the corresponding independent particles. Let us restrict to the transient case, where the associated Green's functions are well defined by

$$\begin{aligned}
G^{\mathrm{SEP}}(x,y;u,v) &= \int_0^\infty p_t^{\mathrm{SEP}}(e_x + e_y, \delta_u + \delta_v)\; dt \\
G^{\mathrm{IRW}}(x,y;u,v) &= \int_0^\infty p_t^{\mathrm{IRW}}(e_x + e_y, e_u + e_v)\; dt \\
G(x,y) &= \int_0^\infty p_t(x,y)\; dt \qquad (45)
\end{aligned}$$

We then have the following.

Proposition 4.3. *Assume that the single particle random walk is transient. In the notation of* (25)*, for the symmetric exclusion process with source at the origin, we have, as* $\lambda \to 0$,

$$cov_{\mu_{\rho,0}}(\eta_x, \eta_y) = \lambda^2(1-\rho)^2\psi(x,y) + o(\lambda^2) \qquad (46)$$

where

$$\begin{aligned}\psi(x,y) &= 2\sum_{z\neq 0} G(0,z)\left(G^{\mathrm{SEP}}(x,y;0,z)-G^{\mathrm{IRW}}(x,y;0,z)\right)\\ &- 2G(0,0)G^{\mathrm{IRW}}(0,0;x,y) \qquad (47)\end{aligned}$$

Proof. First notice that when initially with two particles at locations x,y, the expression $\int_0^\infty(\delta_{x(s),0}+\delta_{y(s),0})ds$ can be rewritten as $\int_0^\infty \eta_0(s)ds$ where $\eta_0(s)$ denotes the number of particles at 0 at time s. Therefore, by expanding the exponential $e^{-\lambda\int_0^\infty(\delta_{x(s),0}+\delta_{y(s),0})ds}$ up to second order in λ, we observe that we have to compute

$$\begin{aligned}&\int_0^\infty\int_0^\infty \mathbb{E}^{\mathrm{SEP}}_{e_x+e_y}\eta_0(s)\eta_0(r)\ dsdr - \int_0^\infty\int_0^\infty \mathbb{E}^{\mathrm{IRW}}_{e_x+e_y}\eta_0(s)\eta_0(r)\ dsdr\\ =\ & 2\int_0^\infty\int_0^\infty \mathbb{E}^{\mathrm{SEP}}_{e_x+e_y}\eta_0(s)\eta_0(s+r)\ dsdr - 2\int_0^\infty\int_0^\infty \mathbb{E}^{\mathrm{IRW}}_{e_x+e_y}\eta_0(s)\eta_0(s+r)\ dsdr \qquad (48)\end{aligned}$$

For the symmetric exclusion process, by self-duality (see (7)), and using also that $\eta_0(s)^2=\eta_0(s)$ because $\eta_0(s)\in\{0,1\}$, we obtain the following.

$$\begin{aligned}&\mathbb{E}^{\mathrm{SEP}}_{e_x+e_y}\eta_0(s)\eta_0(s+r)\\ =\ & \sum_{z\in V, z\neq 0} p_r(0,z)\mathbb{E}^{\mathrm{SEP}}_{e_x+e_y}(\eta_0(s)\eta_z(s)) + p_r(0,0)\mathbb{E}^{\mathrm{SEP}}_{e_x+e_y}(\eta_0(s))\\ =\ & \sum_{z\in V, z\neq 0} p_r(0,z)p_s^{\mathrm{SEP}}(e_x+e_y;e_0+e_z)\\ +\ & p_r(0,0)(p_s(0,x)+p_s(0,y)) \qquad (49)\end{aligned}$$

By self-duality of independent random walkers (see Subsection 2.3) we have

$$\begin{aligned}&\mathbb{E}^{\mathrm{IRW}}_{e_x+e_y}(\eta_0(s)\eta_0(s+r))\\ =\ & \sum_{z\in V, z\neq 0} p_r(0,z)\mathbb{E}^{\mathrm{IRW}}_{e_x+e_y}(\eta_0(s)\eta_z(s)) + p_r(0,0)\mathbb{E}^{\mathrm{IRW}}_{e_x+e_y}(\eta_0(s)\eta_0(s))\\ =\ & \sum_{z\in V, z\neq 0} p_r(0,z)p_s^{\mathrm{IRW}}(e_x+e_y;0,z) + p_r(0,0)\mathbb{E}^{\mathrm{IRW}}_{e_x+e_y}(\eta_0(s)(\eta_0(s)-1))\\ +\ & p_r(0,0)(p_s(0,x)+p_s(0,y))\\ =\ & \sum_{z\in V, z\neq 0} p_r(0,z)p_s^{\mathrm{IRW}}(x,y;0,z) + p_r(0,0)(p_s^{\mathrm{IRW}}(0,0;x,y))\\ +\ & p_r(0,0)(p_s(0,x)+p_s(0,y)) \qquad (50)\end{aligned}$$

Subtracting (50) from (49) gives

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \mathbb{E}^{\mathrm{SEP}}_{e_x+e_y}\left(\eta_0(s)\eta_0(r)\right)\, dsdr - \int_0^\infty \int_0^\infty \mathbb{E}^{\mathrm{IRW}}_{e_x+e_y}\left(\eta_0(s)\eta_0(r)\right)\, dsdr \\
= &\sum_{z\in V, z\neq 0} G(0,z)\left(G^{\mathrm{SEP}}(x,y;0,z) - G^{\mathrm{IRW}}(x,y;0,z)\right) \\
&- G(0,0)G^{\mathrm{IRW}}(0,0;x,y) \qquad (51)
\end{aligned}$$

□

Remark 4.1. *Proposition 4.3 shows that the correlations in the limiting measure have long-range character, because $G(0,0)G^{\mathrm{IRW}}(0,0;x,y)$ decays as a power law when $|x|,|y|\to\infty$, which suggests that the stationary measure $\mu_{\rho,0}$ has properties of a self-organized critical state.*

5 Independent random walkers

In this section, we consider the same problem of adding a source in the context of independent random walkers. When initially started from a homogeneous Poisson measure, we show that at any later time $t>0$, we still have a Poisson measure, and depending on the transience/recurrence of the underlying random walkers, we obtain a limiting non-homogeneous density profile, or the density (expected number of particles at each site) blows up in the limit $t\to\infty$. This result is proved via an intertwining relation with a deterministic process, which we believe is of independent interest.

5.1 Independent random walkers: generator

As in the previous section, let V denote a countable set of vertices and $(p(x,y): x,y\in V)$ an irreducible and symmetric random walk transition probability on V. Recall that the independent random walkers process is a process on $\mathbb{N}^V$ with formal generator (5). The independent random walkers process with source at $0\in V$ is then defined via the formal generator

$$\begin{aligned}
L_{\mathrm{IRW},0} f(\eta) &= \sum_{x,y\in V} p(x,y)\left(\eta_x(f(\eta^{x,y}) - f(\eta)) + \eta_y(f(\eta^{y,x}) - f(\eta))\right) \\
&+ \lambda(f(\eta+e_0) - f(\eta)) \qquad (52)
\end{aligned}$$

The generators (5) and (52) can be rewritten in terms of creation and annihilation operators defined as follows. For a function $f:\mathbb{N}\to\mathbb{R}$ we define

$$af(n) = nf(n-1)I(n\geq 1), \qquad a^\dagger f(n) = f(n+1) \qquad (53)$$

For a local function $f : \mathbb{N}^V \to \mathbb{R}$ we then define the operators $a_i, a_i^\dagger$ for $i \in V$ as

$$\begin{aligned} a_i f(\eta) &= \eta_i f(\eta - e_i) \\ a_i^\dagger f(\eta) &= f(\eta + e_i) \end{aligned} \tag{54}$$

where $\eta_i f(\eta - e_i)$ is a shorthand for $a_i f(\eta) = 0$ if $\eta_i = 0$ and $a_i f(\eta) = \eta_i f(\eta - e_i)$ otherwise. The generator (52) can then be rewritten as follows

$$L_{\mathrm{IRW},0} = -\sum_{x,y\in V} p(x,y)(a_x - a_y)(a_x^\dagger - a_y^\dagger) + \lambda(a_0^\dagger - Id) \tag{55}$$

where Id denotes the identity operator.

5.2 Intertwining operators

In this subsection we prove that the process with generator (52) starting from ν_ρ is intertwined with a deterministic process. This amounts to find a differential operator representation for the creation and annihilation operators (54), in the spirit of [3]. In order to define the intertwiner, we first introduce for a function $f : \mathbb{N} \to \mathbb{R}$ the associated generating function

$$Gf(z) = \sum_{n=0}^{\infty} f(n) \frac{z^n}{n!} e^{-z} \tag{56}$$

where we implicitly assume that the function f is such that the series defining $Gf(z)$ is absolutely convergent in an open interval around the origin. Notice that for $z \geq 0$, $Gf(z)$ is precisely the expectation of f w.r.t. Poisson distribution with parameter z. The following proposition collects intertwining relations between G and the creation and annihilation operators defined above.

Proposition 5.1. *Let $a, a^\dagger$ be defined as above in* (53)*, and G the generating function* (56)*. Then we have*

$$\begin{aligned} G(af)(z) &= zGf(z) \\ G(a^\dagger f)(z) &= \frac{\partial Gf}{\partial z}(z) + Gf(z) \end{aligned} \tag{57}$$

Proof. This follows from a simple computation. □

We now extend the generating function G to the multi-variate setting by tensorization. For $f : \mathbb{N}^V \to \mathbb{R}$ a local function depending on $\eta_i, i \in \Lambda$ with $\Lambda \subset V$ finite, we define

$$\mathscr{G} f(z) = \sum_{\eta_i, i\in\Lambda} \prod_{i\in\Lambda} \frac{z_i^{\eta_i}}{\eta_i!} e^{-z_i} f(\eta_i, i \in \Lambda) \tag{58}$$

Notice that if $z > 0$, then $\mathscr{G} f(z) = \nu_z(f)$ with ν_z the product of Poisson measures with parameter z.

We have the following intertwining relation. Let $\widehat{\Omega}$ denote the configuration space $\mathbb{R}^V$. With small abuse of notation we denote by z configurations in $\widehat{\Omega}$.

Denote, for f local and smooth, and $z \in \widehat{\Omega}$

$$\mathscr{L} f(z) = \sum_{x,y\in V} -p(x,y)(z_x - z_y)(\partial_x - \partial_y) f(z) + \lambda(\partial_0 f)(z) \tag{59}$$

with $\partial_x = \frac{\partial}{\partial z_x}$. Then $\mathscr{L}$ is the generator of a deterministic system of differential equations

$$\frac{dz_x(t)}{dt} = \sum_{y\in V} p(x,y)(z_y(t) - z_x(t)) + \lambda\delta_{x,0} \tag{60}$$

using the generator (1) and the associated semi-group, we can rewrite (60) as follows

$$\frac{dz_x(t)}{dt} = (L_{\mathrm{RW}} z)_x(t) + \lambda\delta_{x,0} \tag{61}$$

This is an equation which can be solved by the classical variation of constants method, which gives

$$z_x(t) = \mathbb{E}_x(z_{X_t}(0)) + \lambda \int_0^t \mathbb{E}_x(\delta_{X_s,0}) ds \tag{62}$$

Using proposition 5.1 we then obtain the following intertwining relation and as a consequence evolution of Poisson product measures.

Proposition 5.2. *Let $L_{\mathrm{IRW},0}$ denote the generator of the independent random walkers with source at the origin given in* (52) *and $\mathscr{L}$ the generator of the deterministic system given in* (59). *Then we have, for every local function $f : \mathbb{N}^V \to \mathbb{R}$, such that $\mathscr{G} f$ exists and is smoothly depending on z:*

$$\mathscr{G} L_{\mathrm{IRW},0} f = \mathscr{L}\mathscr{G} f \tag{63}$$

As a consequence for any $\rho : V \to [0, \infty)$, if we denote ν_ρ the product of Poisson measures with parameter $\rho(x)$ at $x \in V$, and $\nu_\rho(t)$ the measure obtained at time t when starting the Markov process with generator (52)*, then we have*

$$\nu_\rho(t) = \nu_{\rho_t} \tag{64}$$

with

$$\rho_t(x) = \mathbb{E}_x(\rho(X_t)) + \int_0^t \mathbb{E}_x(\delta_{X_s,0})ds \tag{65}$$

the solution of (62) *with initial condition $z_x(0) = \rho(x)$. The density $\rho_t(x)$ diverges as $t \to \infty$ if and only if the random walk with generator $L_{\rm RW}$ is recurrent. If this random walk is transient, starting from a homogeneous Poisson product measure with density ρ, $\nu_\rho(t)$ converges, as $t \to \infty$ to the Poisson product measure with density*

$$\rho_\infty(x) = \rho + \lambda \int_0^\infty \mathbb{E}_x(\delta_{X_s,0})ds.$$

which is a solution of the equation

$$L_{\rm RW} f(x) = \lambda \delta_{x,0}$$

Acknowledgements We thank Mario Ayala, Jochem Hogendijk and Mitchell Maassen Van den Brink for useful discussions. F.R. thanks MAP5 lab. for financial support and hospitality.

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