

A Type Theory with a Tiny Object

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Abstract

We present an extension of Martin-Löf Type Theory that contains a tiny object; a type for which there is a *right* adjoint to the formation of function types as well as the expected left adjoint. We demonstrate the practicality of this type theory by proving various properties related to tininess internally and suggest a few potential applications.

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1 Introduction

Tiny objects are central to Lawvere’s account of differential forms in synthetic differential geometry [Law80]; a tiny object $\mathbb{T}$ in a category $\mathcal{C}$ is one with the amazing property that the internal hom functor $(\mathbb{T} \rightarrow -) : \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint $\sqrt{-} : \mathcal{C} \rightarrow \mathcal{C}$. In synthetic differential geometry, the *infinitesimal interval* defined by $D := \{x : \mathbb{R} \mid x^2 = 0\}$ is one such object. This object corepresents tangent spaces so that $D \rightarrow X$ is the tangent space of X , and elements of $(D \rightarrow X) \rightarrow \mathbb{R}$ are therefore (not-necessarily linear) 1-forms on X . By tininess, such a form corresponds to an element of $X \rightarrow \sqrt{\mathbb{R}}$, that is, 1-forms are simply functions on X valued in a highly non-standard space. Additional properties like linearity, closedness, etc. may be imposed by carving out an appropriate subobject of $\sqrt{\mathbb{R}}$. The notion of tininess was anticipated by [KR79; Law79]

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but its utility was first made explicit in [Law80, §3], and it appears in much of the ensuing work on synthetic differential geometry [Law97; Law98; Law02; Law04; Law11; KR98; KR99; Koc06].

Tinness is simultaneously unusual and abundant. Unusual because, in the category of sets, only the singletons are tiny. This is easy to check: $(\mathbb{T} \rightarrow -)$ can only preserve coproducts when $\mathbb{T} \cong 1$. And abundant because, for any category $\mathcal{C}$ with finite products, the representable presheaves on $\mathcal{C}$ are all tiny. An explicit formula for the right adjoint to $(\mathbb{T} \rightarrow -)$ with $\mathbb{T}$ representable is $\sqrt{Y}(c) \equiv \text{PSh}(\mathcal{C})(\mathbb{T} \rightarrow yc, Y)$, as may be verified by a little (co)end calculus. In favourable cases representable sheaves are also tiny [MR91, Appendix 4], but this is less common.

In this paper we tackle a challenge set by Lawvere, to produce a formal system for working with tiny objects: “This possibility does not seem to have been contemplated by combinatory logic; the formalism should be extended to enable treatment of so basic a situation” [Law04, Section 3]. We describe an extension of Martin-Löf Type Theory that makes a fixed type $\mathbb{T}$ tiny by introducing a type former $\sqrt{}$ for the amazing right adjoint to $(\mathbb{T} \rightarrow -)$. The situation is a little subtle. Freyd proves that for any tiny object $\mathbb{T}$ in a topos, the pullback of $\mathbb{T}$ to any slice is also tiny [Yet87, Theorem 1.4]. But the right adjoint witnessing this tininess is *not* stable under base-change. We therefore have to add to the context structure of type theory in order to support this operation.

We add $\sqrt{}$ as a ‘Fitch style’ modality [Clo18; BCMEPS20], where the type former $\sqrt{}$ is made right adjoint to an operation on contexts. Such modalities are particularly nice when they are a *double* right adjoint, and FitchTT [GCKGB22] is designed for adding modalities of this kind to MLTT. We are in such a situation, of course, because

$$(- \times \mathbb{T}) \dashv (\mathbb{T} \rightarrow -) \dashv \sqrt{-},$$

and so we could use [GCKGB22] directly to produce a type theory.

But there is a special feature of $\sqrt{}$ which impels us to create a specialised theory for it. Specifically, the leftmost adjoint $(- \times \mathbb{T})$ already exists as an operation on contexts: it is simply context extension with $i : \mathbb{T}$. We allow $\mathbb{T}$ to be an *ordinary type*, rather than a pre-type or special piece of syntax. This flexibility is essential for applications in synthetic differential geometry where many important tiny objects may be constructed internally as ordinary types. The only new context former needed is something corresponding to $(\mathbb{T} \rightarrow -)$, which we write as context extension with a $\blacksquare$.

To make $(-, i : \mathbb{T}) \dashv (-, \blacksquare)$ on contexts, we need unit $\Gamma \rightarrow (\Gamma, i : \mathbb{T}, \blacksquare)$ and counit $(\Gamma, \blacksquare, i : \mathbb{T}) \rightarrow \Gamma$ substitutions. In [GCKGB22] these are added axiomatically, and the type theory is presented in a variable-free CwF style where these explicit substitutions are pushed around manually. The downside of the approach is that figuring out how to use a variable could be potentially challenging: one may have to devise by hand an explicit substitution that extracts the variable from the context. One of our aims is to give a fully explicit variable rule which builds in the normal forms of these ‘stuck substitutions’.

The reason this gets interesting is that, because $\mathbb{T}$ is an ordinary type, we can substitute *any* term $t : \mathbb{T}$ for i in the counit substitution. And so, the admissible counit rule is parameterised by a genuine term $t : \mathbb{T}$. The counit rule commutes with all type and term formers and becomes stuck on the ordinary variable rule, so every use of a variable in this theory will have (possibly many) attached terms of $\mathbb{T}$ corresponding to these stuck counits.

Related Work. In [LOPS18], a tiny interval $\mathbb{I}$ is used to construct, internal to a 1-topos, a universe that classifies fibrations. This is performed in ‘crisp type theory’, a fragment of Shulman’s spatial type theory [Shu18]. The $\sqrt{}$ type former is described by a collection of axioms, and the fact that the adjunction is external is enforced by requiring the inputs to these axioms to be ‘crisp’, roughly, protected by a use of the global sections/discrete inclusion modality $\flat$. Internally, this manifests as an equivalence $\flat(A \rightarrow \sqrt{B}) \simeq \flat((\mathbb{T} \rightarrow A) \rightarrow B)$. An alternative axiomatisation that is coherent for higher types is given in [Mye22, Appendix A], by instead asserting for each crisp type B a counit map $(\mathbb{T} \rightarrow \sqrt{B}) \rightarrow B$ such that the induced map $\flat(A \rightarrow \sqrt{B}) \rightarrow \flat((\mathbb{T} \rightarrow A) \rightarrow B)$ is an equivalence.

These equivalences guarded by $\flat$ are more restrictive than necessary: we will prove an equivalence $(A \rightarrow \sqrt{B}) \simeq \sqrt{((\mathbb{T} \rightarrow A) \rightarrow B)}$, where A and B do not have to be ‘global types’ (but the dependency of B is still somewhat restricted).

The most comparable type theory to ours is presented in [ND21]. The authors give rules for a right adjoint to the *dependent product* $\Pi_{\mathbb{T}} : \mathcal{E}/\mathbb{T} \rightarrow \mathcal{E}$ rather than the non-dependent function type, an operation which they call *transpension*. It is a theorem of Freyd [Yet87, Proposition 1.2] that this is equivalent to the apparently weaker notion of tininess. Because we target the non-dependent function type, our new judgemental structures and rules are much simpler than those for transpension, and let us maintain admissibility of substitution and (conjecturally) normalisation; for transpension it is unclear whether these are achievable.

The context extension with the tiny type is also treated specially, similar to accounts of internal parametricity [CH20; Cav21] [GCKGB22, Section 5] where the $\Gamma, i : \mathbb{T}$ context extension is a special piece of syntax, and so only special ‘terms’ may be substituted for it.

A benefit of these previous systems is that the precise notion of tininess is easier to tweak. For example, whenever $\mathcal{C}$ is monoidal, representable presheaves on $\mathcal{C}$ are *monoidally tiny* in that $(\mathbb{T} \multimap -)$ has a right adjoint, where $\multimap$ denotes the Day internal hom. This non-cartesian tininess is central to forthcoming work on Higher Observational Type Theory [SAK22], whose intended semantics are in a non-cartesian cube category.

Contributions.

- In Section 2, we introduce the new context structure required to support the type former and in Section 3 the type-former itself.
- In Section 4, we give a couple of useful constructions that are definable internally. Section 4.1 shows that we can derive induction principles for functions out of $\mathbb{T}$, and Section 4.2 recovers the transpension operation from our non-dependent right adjoint, providing a new construction that applies in the univalent setting.
- In Section 5, we discuss some potential applications of the type theory and a prototype type-checker utilising an extension of the usual normalisation by evaluation algorithm.

Terminology. There is a constellation of terminology around tininess and similar notions, with “amazingly tiny”, “atomic”, “infinitesimal”, “satisfying the ATOM property”, “internally/externally projective” and “small-projective” all used to refer to one of $\mathcal{C}(\mathbb{T}, -) : \mathcal{C} \rightarrow \mathbf{Set}$ or $(\mathbb{T} \rightarrow -) : \mathcal{C} \rightarrow \mathcal{C}$ preserving epimorphisms, finite colimits, colimits or having a right adjoint. With additional assumptions many of these notions coincide, but we follow [Yet87] in settling on simply “tiny” as the name for objects whose internal hom has a right adjoint. The other properties play no role in this work. There are also various notations for the right adjoint, including $(-)_\mathbb{T}$ [Yet87] and $(-)^{1/\mathbb{T}}$ [Law97], and even $\nabla_\mathbb{T}$ [Yet87] and $\mathbb{Q}-$ for transpension [ND21]. We follow [LOPS18] in using $\sqrt{-}$.

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2 Contexts and Variables

We take as our starting point the bare judgements of Martin-Löf Type Theory. When using our theory we will also assume dependent sums, products, intensional identity types and an infinite hierarchy of universes, but the extension of type theory with a tiny object does not require the presence of any other type-formers.

The $\sqrt{}$ type-former will be added by making it right adjoint to a judgemental version of “ $\mathbb{T} \rightarrow \Gamma$ ” which we write as $\Gamma, \mathbf{\Delta}_{\mathcal{L}}$. Here, to refer to it later, the lock is annotated with a ‘lock name’ $\mathcal{L}$. This first section will be spent adding the necessary rules for the context $\Gamma, \mathbf{\Delta}_{\mathcal{L}}$ to behave like functions into Γ .

A simple way to achieve this would be to assert axiomatic unit and counit substitutions for the $(- \times \mathbb{T}) \dashv (\mathbb{T} \rightarrow -)$ adjunction as in the following:

$$\begin{array}{c} \text{UNIT-SUB} \quad \frac{\Gamma \text{ ctx}}{\Gamma \vdash \text{unit}_{\Gamma, i, \mathcal{L}} : \Gamma, i : \mathbb{T}, \mathbf{\Delta}_{\mathcal{L}}} \qquad \text{COUNIT-SUB} \quad \frac{\Gamma \text{ ctx}}{\Gamma, \mathbf{\Delta}_{\mathcal{L}}, i : \mathbb{T} \vdash \text{counit}_{\Gamma, \mathcal{L}, i} : \Gamma} \end{array}$$

together with equations that explain how these explicit substitutions are pushed around and annihilated.

We can do better, however, and give normal forms for the placement of these explicit substitutions. The examples in later sections show that it is quite feasible to work in the resulting type theory by hand.

Besides the context lock, we only need one additional base rule: a modified version of the variable rule that builds in stuck instances of the counit substitution. The rule is completely structural and independent of the rules for types (besides the existence of $\mathbb{T}$). There are two new admissible rules, corresponding to precomposition with the counit or unit substitutions.

We now describe the additions to MLTT one rule at a time.

The Tiny Type. There is a closed type $\mathbb{T}$.

$$\text{TINY-FORM} \frac{}{\Gamma \vdash \mathbb{T} \text{ type}}$$

In applications, this will typically be an already-existing closed type rather than a new one asserted axiomatically.

Context Locks. There is a special context extension,

$$\text{CTX-LOCK} \frac{\Gamma \text{ ctx}}{\Gamma, \mathbf{a}_{\mathcal{L}} \text{ ctx}}$$

to be thought of as $\mathbb{T} \rightarrow \Gamma$. We call $\mathcal{L}$ a ‘lock name’, all lock names in a context are unique.

The Counit. Because a locked context represents functions into what comes before the lock, we can use variables to the left of a lock if we can provide an argument to the function: this corresponds to precomposition with the counit substitution, with a substitution for $\mathbb{T}$ and some contractions built-in.

The simplest situation we can encounter is

$$\text{COUNIT?} \frac{\Gamma \vdash a : A \quad \Gamma, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash t : \mathbb{T} \quad \mathbf{a}_{\mathcal{L}} \notin \Gamma'}{\Gamma, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash a[t/\mathbf{a}_{\mathcal{L}}] : A[t/\mathbf{a}_{\mathcal{L}}]}$$

corresponding (non-dependently) to the composite

$$(\mathbb{T} \rightarrow \Gamma) \times \Gamma' \xrightarrow{[\text{id}, t]} (\mathbb{T} \rightarrow \Gamma) \times \Gamma' \times \mathbb{T} \xrightarrow{\text{pr}} (\mathbb{T} \rightarrow \Gamma) \times \mathbb{T} \xrightarrow{\varepsilon} \Gamma \xrightarrow{a} A$$

The new piece of term syntax is a stuck instance of this $[t/\mathbf{a}_{\mathcal{L}}]$ rule which we build into the variable rule below. To distinguish the admissible rule from the stuck rule, we will write the admissible rule as $[t/\mathbf{a}_{\mathcal{L}}]$ and the actual stuck syntax as $\llbracket t/\mathbf{a}_{\mathcal{L}} \rrbracket$. Roughly, the admissible $[t/\mathbf{a}_{\mathcal{L}}]$ will add a $\llbracket t/\mathbf{a}_{\mathcal{L}} \rrbracket$ to every *free* variable usage in a , much like the underlining operation in the type theory for $\mathfrak{h}$ [RFL21].

We need to generalise this rule in a couple of ways. First, we need to allow an additional telescope to be carried along, so that we may go under binders in the term $a : A$.

Additionally, we may need to apply the counit to multiple locks simultaneously. To facilitate this, for a context $\Gamma, \Gamma' \text{ ctx}$ let $\text{locks}(\Gamma')$ denote the (ordered) list of context locks $\mathbf{a}_{\mathcal{L}_1}, \dots, \mathbf{a}_{\mathcal{L}_n}$ that appear in the telescope Γ' .

The generalised rule is then:

$$\begin{array}{c} \text{COUNIT-TELE} \frac{\Gamma \vdash \Gamma'' \text{ tele} \quad \Gamma, \Gamma' \vdash t_i : \mathbb{T} \text{ for } \mathcal{L}_i \in \text{locks}(\Gamma')}{\Gamma, \Gamma' \vdash \Gamma''[t_1/\mathbf{a}_{\mathcal{L}_1}, \dots, t_n/\mathbf{a}_{\mathcal{L}_n}] \text{ tele}} \\ \\ \text{COUNIT} \frac{\Gamma, \Gamma'' \vdash a : A \quad \Gamma, \Gamma' \vdash t_i : \mathbb{T} \text{ for } \mathcal{L}_i \in \text{locks}(\Gamma')}{\Gamma, \Gamma', \Gamma''[t_1/\mathbf{a}_{\mathcal{L}_1}, \dots, t_n/\mathbf{a}_{\mathcal{L}_n}] \vdash a[t_1/\mathbf{a}_{\mathcal{L}_1}, \dots, t_n/\mathbf{a}_{\mathcal{L}_n}] : A[t_1/\mathbf{a}_{\mathcal{L}_1}, \dots, t_n/\mathbf{a}_{\mathcal{L}_n}]} \end{array}$$

Because we will so often be working with a sequence of terms and locks, we will abbreviate $[t_1/\mathbf{q}_{\mathcal{L}_1}, \dots, t_n/\mathbf{q}_{\mathcal{L}_n}]$ to simply $[\vec{t}/\mathbf{q}_{\mathcal{L}}]$.

Formally, the telescope Γ'' should be annotated in the syntax of the operation, as it is not inferable from the raw syntax.

Variable Usage. To use a variable $x : A$ that is behind some context locks, we must provide a term $t_i : \mathbb{T}$ for each lock between that variable and the front of the context.

$$\text{VAR} \frac{\Gamma, x : A, \Gamma' \vdash \vec{t} : \mathbb{T} \text{ for } \mathcal{L} \in \text{locks}(\Gamma')}{\Gamma, x : A, \Gamma' \vdash x[\vec{t}/\mathbf{q}_{\mathcal{L}}] : A[\vec{t}/\mathbf{q}_{\mathcal{L}}]}$$

The type of the variable usage has the admissible counit rule applied to it: typically these counits will not be stuck on the type unless A is itself a variable.

Substitution. Ordinary substitution into a variable continues into the keys associated with the variable. When we find the variable we are substituting for, the ‘stuck’ counits are turned back into admissible ones, which then proceed into the substituted term:

$$\begin{aligned} \Gamma, \Gamma'[a/x] \vdash (y[\vec{t}/\mathbf{q}_{\mathcal{L}}])[a/x] &\equiv y[\vec{t}[a/x]/\mathbf{q}_{\mathcal{L}}] \\ \Gamma, \Gamma'[a/x] \vdash (x[\vec{t}/\mathbf{q}_{\mathcal{L}}])[a/x] &\equiv a[\vec{t}[a/x]/\mathbf{q}_{\mathcal{L}}] \end{aligned}$$

Computing the Cunit. We will now explain how the cunit operation is actually computed on syntax. We begin with the simplest possible instance: the addition a single additional key with using a closed term of $\mathbb{T}$:

$$\text{CUNIT} \frac{\Gamma, \Gamma'' \vdash b : B \quad \cdot \vdash t : \mathbb{T}}{\Gamma, \blacksquare_{\mathcal{L}}, \Gamma''[t/\mathbf{q}_{\mathcal{L}}] \vdash b[t/\mathbf{q}_{\mathcal{L}}] : B[t/\mathbf{q}_{\mathcal{L}}]}$$

This is calculated by induction on b until we reach an instance of the variable rule, say for a variable $x : A$.

As in the definition of single-variable substitution, there are cases depending on where in the context x lies. The typical case is when $x : A$ is in Γ , so the variable usage looks like

$$\Gamma_1, x : A, \Gamma_2, \Gamma'' \vdash x[\vec{j}/\mathbf{q}_{\mathcal{J}}, \vec{k}/\mathbf{q}_{\mathcal{K}}] : A[\vec{j}/\mathbf{q}_{\mathcal{J}}, \vec{k}/\mathbf{q}_{\mathcal{K}}]$$

where $\mathcal{J}$ are the locks in Γ_2 and $\mathcal{K}$ are the locks in Γ'' . The new $[t/\mathbf{q}_{\mathcal{L}}]$ is slot into place, and left stuck:

$$\Gamma_1, x : A, \Gamma_2, \blacksquare_{\mathcal{L}}, \Gamma''[t/\mathbf{q}_{\mathcal{L}}] \vdash x[\vec{j}/\mathbf{q}_{\mathcal{J}}, \vec{k}/\mathbf{q}_{\mathcal{K}}][t/\mathbf{q}_{\mathcal{L}}] \equiv x[\vec{j}/\mathbf{q}_{\mathcal{J}}, t/\mathbf{q}_{\mathcal{L}}, \vec{k}/\mathbf{q}_{\mathcal{K}}]$$

When working informally, this means counting how many $\blacksquare_{\mathcal{K}}$ are created between the application of $[t/\mathbf{q}_{\mathcal{L}}]$ and the variable usage (by the use of some of the typing rules to be introduced later), and placing $t/\mathbf{q}_{\mathcal{L}}$ to the left of all these freshly created $\blacksquare_{\mathcal{K}_i}$.

If instead $x : A$ is in Γ'' , the variable usage looks like

$$\Gamma, \Gamma''_1, x : A, \Gamma''_2 \vdash x[\vec{k}/\mathbf{q}_{\mathcal{K}}] : A[\vec{k}/\mathbf{q}_{\mathcal{K}}]$$

where $\mathcal{K}_1, \dots, \mathcal{K}_n$ are the locks in Γ''_2 . Access to x is not affected by the addition of the lock $\blacksquare_{\mathcal{L}}$ (which is placed to the left of it), so the variable usage is left unchanged:

$$\Gamma, \blacksquare_{\mathcal{L}}, \Gamma''_1[t/\mathbf{q}_{\mathcal{L}}], x : A[t/\mathbf{q}_{\mathcal{L}}], \Gamma''_2[t/\mathbf{q}_{\mathcal{L}}] \vdash x[\vec{k}/\mathbf{q}_{\mathcal{K}}][t/\mathbf{q}_{\mathcal{L}}] \equiv x[\vec{k}/\mathbf{q}_{\mathcal{K}}]$$

The cunit is more complex to evaluate when arbitrary open terms of $\mathbb{T}$ are involved. Two complications can arise:

- First, $t : \mathbb{T}$ may not be closed, so that in $\Gamma_1, x : A, \Gamma_2, \Gamma' \vdash t : \mathbb{T}$, the variables used in the term t are now locked behind the fresh locks in Γ'' by the time we reach the variable x . We therefore apply $[\vec{k}/\mathbf{a}_K]$ to the term t , so:

$$x[\vec{j}/\mathbf{a}_J, \vec{k}/\mathbf{a}_K][t/\mathbf{a}_L] \equiv x[\vec{j}/\mathbf{a}_J, t[\vec{k}/\mathbf{a}_K]/\mathbf{a}_L, \vec{k}/\mathbf{a}_K]$$

- Instead, the terms $\vec{j}$ and $\vec{k}$ may not be closed, and so some of the the variables used in them might now lie behind the new lock $\mathbf{a}_L$. The term t is used to unlock them:

$$x[\vec{j}/\mathbf{a}_J, \vec{k}/\mathbf{a}_K][t/\mathbf{a}_L] \equiv x[\vec{j}[t/\mathbf{a}_L]/\mathbf{a}_J, t/\mathbf{a}_L, \vec{k}[t/\mathbf{a}_L]/\mathbf{a}_K]$$

- Finally, and slightly horrifyingly, the two complications can happen at the same time, yielding the final, general definition:

$$x[\vec{j}/\mathbf{a}_J, \vec{k}/\mathbf{a}_K][t/\mathbf{a}_L] \equiv x[\vec{j}[t/\mathbf{a}_L]/\mathbf{a}_J, t[\vec{k}[t/\mathbf{a}_L]/\mathbf{a}_K]/\mathbf{a}_L, \vec{k}[t/\mathbf{a}_L]/\mathbf{a}_K]$$

Examples of the Countit. The fully general rule is the worst-case scenario; the vast majority of cases encountered in practice are simpler. First, the countit commutes with ordinary term-formers such as pairing:

$$\begin{aligned} x : \mathbb{N}, y : \mathbb{N} & \vdash (x, y) : \mathbb{N} \times \mathbb{N} \\ x : \mathbb{N}, y : \mathbb{N}, \mathbf{a}_L, i : \mathbb{T} & \vdash (x, y)[i/\mathbf{a}_L] \\ & \equiv (x[i/\mathbf{a}_L], y[i/\mathbf{a}_L]) : \mathbb{N} \times \mathbb{N} \end{aligned}$$

Any variables which are bound in a term do not get annotated with the new countit:

$$\begin{aligned} x : \mathbb{N} & \vdash (\lambda y. x + y) : \mathbb{N} \rightarrow \mathbb{N} \\ x : \mathbb{N}, \mathbf{a}_L, i : \mathbb{T} & \vdash (\lambda y. x + y)[i/\mathbf{a}_L] \\ & \equiv (\lambda y. x[i/\mathbf{a}_L] + y) : \mathbb{N} \rightarrow \mathbb{N} \end{aligned}$$

When the variable $y : \mathbb{N}$ is bound, it is added to the context *after* the lock $\mathbf{a}_L$, the variable usage therefore needs no annotation to be well-formed. In particular, the rule has no effect on closed terms:

$$\begin{aligned} \cdot & \vdash 2 : \mathbb{N} \\ \mathbf{a}_L, i : \mathbb{T} & \vdash 2[i/\mathbf{a}_L] \\ & \equiv 2 : \mathbb{N} \end{aligned}$$

The countit is also applied to the type of a term, and is computed in a similar way.

$$\begin{aligned} A : \mathcal{U}, B : A \rightarrow \mathcal{U}, f : \prod_{(x:A)} B(x) & \vdash f : \prod_{(x:A)} B(x) \\ A : \mathcal{U}, B : A \rightarrow \mathcal{U}, f : \prod_{(x:A)} B(x), \mathbf{a}_L, i : \mathbb{T} & \vdash f[i/\mathbf{a}_L] : \left(\prod_{(x:A)} B(x) \right) [i/\mathbf{a}_L] \\ & \equiv f[i/\mathbf{a}_L] : \prod_{(x:A[i/\mathbf{a}_L])} (B(x))[i/\mathbf{a}_L] \\ & \equiv f[i/\mathbf{a}_L] : \prod_{(x:A[i/\mathbf{a}_L])} B[i/\mathbf{a}_L](x) \end{aligned}$$

Because x is bound in the Π type-former, its usage is not annotated in the codomain $B[i/\mathbf{a}_L](x)$. The type-family $B[i/\mathbf{a}_L]$ has type $(A \rightarrow \mathcal{U})[i/\mathbf{a}_L] \equiv (A[i/\mathbf{a}_L] \rightarrow \mathcal{U})$, so applying it to $x[i/\mathbf{a}_L]$ is well-formed.

When a variable already has a stuck countit attached, the result of applying an additional countit will depend on the relative position of the two locks. For the simplest possible case, the variable usage

$$x : A, \mathbf{a}_K, k : \mathbb{T} \vdash x[k/\mathbf{a}_K] : A,$$

there are already a few possibilities.

- If the new lock is placed at the end of the context, the terms used on the existing lock will also end up with the stuck count on their variables.

$$\begin{aligned}
x : A, \mathbf{\bullet}_{\mathcal{K}}, k : \mathbb{T} & \quad \vdash x[k/\mathbf{q}_{\mathcal{K}}] & : A \\
x : A, \mathbf{\bullet}_{\mathcal{K}}, k : \mathbb{T}, \mathbf{\bullet}_{\mathcal{L}}, i : \mathbb{T} & \vdash x[k/\mathbf{q}_{\mathcal{K}}][i/\mathbf{q}_{\mathcal{L}}] \\
& \equiv x[k[i/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}, i/\mathbf{q}_{\mathcal{L}}] \\
& \equiv x[k[i/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}, i/\mathbf{q}_{\mathcal{L}}] : A
\end{aligned}$$

- Moving one step to the left, we may place the new lock in between $\mathbf{\bullet}_{\mathcal{K}}$ and $k : \mathbb{T}$. In this case neither of the variables i or k are locked, and so do not need stuck count of their own to be used.

$$\begin{aligned}
x : A, \mathbf{\bullet}_{\mathcal{K}}, k : \mathbb{T} & \quad \vdash x[k/\mathbf{q}_{\mathcal{K}}] & : A \\
x : A, \mathbf{\bullet}_{\mathcal{K}}, \mathbf{\bullet}_{\mathcal{L}}, i : \mathbb{T}, k : \mathbb{T} & \vdash x[k/\mathbf{q}_{\mathcal{K}}][i/\mathbf{q}_{\mathcal{L}}] \\
& \equiv x[k[i/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}, i/\mathbf{q}_{\mathcal{L}}] \\
& \equiv x[k/\mathbf{q}_{\mathcal{K}}, i/\mathbf{q}_{\mathcal{L}}] & : A
\end{aligned}$$

- One step further, we may place the new lock *before* an existing one, in which case the term i is the one which gains a stuck count in order to be used:

$$\begin{aligned}
x : A, \mathbf{\bullet}_{\mathcal{K}}, k : \mathbb{T} & \quad \vdash x[k/\mathbf{q}_{\mathcal{K}}] & : A \\
x : A, \mathbf{\bullet}_{\mathcal{L}}, i : \mathbb{T}, \mathbf{\bullet}_{\mathcal{K}}, k : \mathbb{T} & \vdash x[k/\mathbf{q}_{\mathcal{K}}][i/\mathbf{q}_{\mathcal{L}}] \\
& \equiv x[i[k[i/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}]/\mathbf{q}_{\mathcal{L}}, k/\mathbf{q}_{\mathcal{K}}] \\
& \equiv x[i[k/\mathbf{q}_{\mathcal{K}}]/\mathbf{q}_{\mathcal{L}}, k/\mathbf{q}_{\mathcal{K}}] \\
& \equiv x[i[k/\mathbf{q}_{\mathcal{K}}]/\mathbf{q}_{\mathcal{L}}, k/\mathbf{q}_{\mathcal{K}}] & : A
\end{aligned}$$

- Finally, we may place the new lock at the start of the context, in which case the variable usage is not changed at all:

$$\begin{aligned}
x : A, \mathbf{\bullet}_{\mathcal{K}}, k : \mathbb{T} & \quad \vdash x[k/\mathbf{q}_{\mathcal{K}}] & : A \\
\mathbf{\bullet}_{\mathcal{L}}, i : \mathbb{T}, x : A, \mathbf{\bullet}_{\mathcal{K}}, k : \mathbb{T} & \vdash x[k/\mathbf{q}_{\mathcal{K}}][i/\mathbf{q}_{\mathcal{L}}] \\
& \equiv x[k/\mathbf{q}_{\mathcal{K}}] & : A
\end{aligned}$$

Unit. To complete the implementation of the adjunction, we have an admissible rule representing the unit map. The following rule is precomposition with the substitution $\Gamma \rightarrow (\Gamma, i : \mathbb{T}, \mathbf{\bullet}_{\mathcal{L}})$, suitably generalised with a telescope Γ' :

$$\begin{array}{c}
\text{UNIT-TELE} \quad \frac{\Gamma, i : \mathbb{T}, \mathbf{\bullet}_{\mathcal{L}} \vdash \Gamma' \text{ tele}}{\Gamma \vdash \Gamma'[\gamma i. / \mathbf{\bullet}_{\mathcal{L}}] \text{ tele}} \qquad \text{UNIT} \quad \frac{\Gamma, i : \mathbb{T}, \mathbf{\bullet}_{\mathcal{L}}, \Gamma' \vdash a : A}{\Gamma, \Gamma'[\gamma i. / \mathbf{\bullet}_{\mathcal{L}}] \vdash a[\gamma i. / \mathbf{\bullet}_{\mathcal{L}}] : A[\gamma i. / \mathbf{\bullet}_{\mathcal{L}}]}
\end{array}$$

As with the $[t/\mathbf{q}_{\mathcal{L}}]$ operation, we use substitution-like syntax for the unit map because, like substitution, it commutes past everything until it reaches a variable. The $\gamma i.$ notation is chosen to match the syntax that is used for the type-former later.

The lock and key then ‘click together’, yielding an ordinary substitution. In the simplest case, when $t : \mathbb{T}$ is closed and x has a single matching stuck COUNTER.

$$x[t/\mathbf{q}_{\mathcal{L}}][\gamma i. / \mathbf{\bullet}_{\mathcal{L}}] \equiv x[t/i]$$

If t is an open term, the unit rule must continue into t :

$$x[t/\mathbf{q}_{\mathcal{L}}][\gamma i. / \mathbf{\bullet}_{\mathcal{L}}] \equiv x[t[\gamma i. / \mathbf{\bullet}_{\mathcal{L}}]/i]$$

Of course, because x is already a variable there are only two possibilities: either it is equal to the variable i bound in the unit rule or it is not.

$$\begin{aligned} i\llbracket t/\mathbf{q}_{\mathcal{L}} \rrbracket[\gamma i./\mathbf{a}_{\mathcal{L}}] &::= t[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ x\llbracket t/\mathbf{q}_{\mathcal{L}} \rrbracket[\gamma i./\mathbf{a}_{\mathcal{L}}] &::= x \quad \text{otherwise} \end{aligned}$$

If x has several stuck counits, the operation needs to continue into the other stuck counits.

$$\begin{aligned} i\llbracket \vec{j}/\mathbf{q}_{\mathcal{J}}, t/\mathbf{q}_{\mathcal{L}}, \vec{k}/\mathbf{q}_{\mathcal{K}} \rrbracket[\gamma i./\mathbf{a}_{\mathcal{L}}] &::= t[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ x\llbracket \vec{j}/\mathbf{q}_{\mathcal{J}}, t/\mathbf{q}_{\mathcal{L}}, \vec{k}/\mathbf{q}_{\mathcal{K}} \rrbracket[\gamma i./\mathbf{a}_{\mathcal{L}}] &::= x\llbracket \vec{j}[\gamma i./\mathbf{a}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{J}}, \vec{k}[\gamma i./\mathbf{a}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}} \rrbracket \end{aligned}$$

In contrast to the counit, this rule is never stuck. After any application of $[\gamma i./\mathbf{a}_{\mathcal{L}}]$, the lock $\mathcal{L}$ and the variable i no longer appear in the resulting term $a[\gamma i./\mathbf{a}_{\mathcal{L}}]$.

Examples of the Unit. Now, applying the unit rule. Similar to the counit, the unit commutes with ordinary type formers (in these cases having no effect, because the variables used come after the lock).

$$\begin{aligned} i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, x : \mathbb{N}, y : \mathbb{N} &\vdash (x, y) && : \mathbb{N} \times \mathbb{N} \\ x : \mathbb{N}, y : \mathbb{N} &\vdash (x, y)[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv (x[\gamma i./\mathbf{a}_{\mathcal{L}}], y[\gamma i./\mathbf{a}_{\mathcal{L}}]) \\ &\equiv (x, y) && : \mathbb{N} \times \mathbb{N} \\ i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, x : \mathbb{N} &\vdash (\lambda y. x + y) && : \mathbb{N} \rightarrow \mathbb{N} \\ x : \mathbb{N} &\vdash (\lambda y. x + y)[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv (\lambda y. x[\gamma i./\mathbf{a}_{\mathcal{L}}] + y) \\ &\equiv (\lambda y. x + y) && : \mathbb{N} \rightarrow \mathbb{N} \end{aligned}$$

If the variable used is not the one that the unit is being applied to then it remains unchanged:

$$\begin{aligned} x : A, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, k : \mathbb{T} &\vdash x\llbracket k/\mathbf{q}_{\mathcal{L}} \rrbracket : A \\ x : A, k : \mathbb{T} &\vdash (x\llbracket k/\mathbf{q}_{\mathcal{L}} \rrbracket)[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv x\llbracket k/i \rrbracket \\ &\equiv x : A \end{aligned}$$

The simplest interesting case is a variable usage that matches the variable that the unit is being applied to.

$$\begin{aligned} i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, k : \mathbb{T} &\vdash i\llbracket k/\mathbf{q}_{\mathcal{L}} \rrbracket : \mathbb{T} \\ k : \mathbb{T} &\vdash (i\llbracket k/\mathbf{q}_{\mathcal{L}} \rrbracket)[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv i\llbracket k[\gamma i./\mathbf{a}_{\mathcal{L}}] \rrbracket/i \\ &\equiv k[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv k : \mathbb{T} \end{aligned}$$

That is not the only way we could have gained access to i . We can use the term of $\mathbb{T}$ just constructed to gain access to i a second time:

$$\begin{aligned} i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, k : \mathbb{T} &\vdash i\llbracket i\llbracket k/\mathbf{q}_{\mathcal{L}} \rrbracket/\mathbf{q}_{\mathcal{L}} \rrbracket : \mathbb{T} \\ k : \mathbb{T} &\vdash (i\llbracket i\llbracket k/\mathbf{q}_{\mathcal{L}} \rrbracket/\mathbf{q}_{\mathcal{L}} \rrbracket)[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv i\llbracket k/\mathbf{q}_{\mathcal{L}} \rrbracket[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv k[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv k : \mathbb{T} \end{aligned}$$

Nothing stops us from iterating this forever:

$$i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, k : \mathbb{T} \vdash i[\![i[\![k/\mathbf{a}_{\mathcal{L}}]\!]/\mathbf{a}_{\mathcal{L}}]\!] : \mathbb{T}$$

Semantically, the context corresponds to $i : \mathbb{T} \rightarrow \mathbb{T}$ and $k : \mathbb{T}$, and the term is an incarnation of the iterated application $i(i(\dots i(k)))$. The unit $[\gamma i./\mathbf{a}_{\mathcal{L}}]$ replaces i with the identity, and so the entire term reduces to k regardless of how many iterations we do.

The result of applying the counit may be different for each variable, depending on what terms are used in the attached counit:

$$\begin{aligned} i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, j : \mathbb{T}, k : \mathbb{T} &\vdash (i[\![j/\mathbf{a}_{\mathcal{L}}]\!], i[\![k/\mathbf{a}_{\mathcal{L}}]\!]) : \mathbb{T} \times \mathbb{T} \\ j : \mathbb{T}, k : \mathbb{T} &\vdash (i[\![j/\mathbf{a}_{\mathcal{L}}]\!], i[\![k/\mathbf{a}_{\mathcal{L}}]\!])[\gamma i./\mathbf{a}_{\mathcal{L}}] \\ &\equiv (i[\![j/\mathbf{a}_{\mathcal{L}}]\!][\gamma i./\mathbf{a}_{\mathcal{L}}], i[\![k/\mathbf{a}_{\mathcal{L}}]\!][\gamma i./\mathbf{a}_{\mathcal{L}}]) \\ &\equiv (i[j/i], i[k/i]) \\ &\equiv (j, k) : \mathbb{T} \times \mathbb{T} \end{aligned}$$

At this point it may appear as though the counits are just delayed substitutions that are ‘activated’ by the unit, but this is not the case. Suppose we have global elements $1, 2, 3, 4 : \mathbb{T}$, and consider a term

$$x : \mathbb{T}, y : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, \mathbf{a}_{\mathcal{K}} \vdash (x[\![1/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], y[\![4/\mathbf{a}_{\mathcal{K}}]\!]) : \mathbb{T} \times \mathbb{T}$$

Applying two different substitutions allows us to select which of the stuck keys to apply:

$$\begin{aligned} &(x[\![1/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], y[\![4/\mathbf{a}_{\mathcal{K}}]\!])[\gamma x./\mathbf{a}_{\mathcal{L}}][\gamma y./\mathbf{a}_{\mathcal{K}}] \\ &\equiv (i[\![1/\mathbf{a}_{\mathcal{L}}]\!], j[\![3/\mathbf{a}_{\mathcal{L}}]\!], j[\![3/\mathbf{a}_{\mathcal{L}}]\!], j[\![4/\mathbf{a}_{\mathcal{K}}]\!])[\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma j./\mathbf{a}_{\mathcal{K}}] \\ &\equiv (i[\![1/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], j[\![4/\mathbf{a}_{\mathcal{K}}]\!])[\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma j./\mathbf{a}_{\mathcal{K}}] \\ &\equiv (1, 4) \end{aligned}$$

and

$$\begin{aligned} &(x[\![1/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], y[\![4/\mathbf{a}_{\mathcal{K}}]\!])[\gamma x./\mathbf{a}_{\mathcal{L}}][\gamma y./\mathbf{a}_{\mathcal{K}}] \\ &\equiv (j[\![1/\mathbf{a}_{\mathcal{L}}]\!], i[\![3/\mathbf{a}_{\mathcal{L}}]\!], i[\![3/\mathbf{a}_{\mathcal{L}}]\!], j[\![4/\mathbf{a}_{\mathcal{K}}]\!])[\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma j./\mathbf{a}_{\mathcal{K}}] \\ &\equiv (j[\![1/\mathbf{a}_{\mathcal{L}}]\!], y[\![3/\mathbf{a}_{\mathcal{L}}]\!], i[\![3/\mathbf{a}_{\mathcal{L}}]\!], j[\![4/\mathbf{a}_{\mathcal{K}}]\!])[\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma j./\mathbf{a}_{\mathcal{K}}] \\ &\equiv (2, 3) \end{aligned}$$

We will be able to cause these different substitutions to occur once we have the type-former. So the *user* of a term containing stuck keys gets to choose which of the terms is eventually used, but doesn’t get to choose the actual term that gets plugged in. And the stored terms for the other locks are completely lost!

These rules are peculiar, but if we insist on pushing the admissible rules to the leaves it seems something like this is forced on us by the setting we are trying to capture.

3 The Amazing Right Adjoint

The type-former is made right adjoint to $\mathbf{a}_{\mathcal{L}}$ in the ‘obvious’ way, following the pattern of [Clo18; BCMEPS20; GCKGB22].

$$\begin{aligned} \sqrt{\text{-FORM}} \quad &\frac{\Gamma, \mathbf{a}_{\mathcal{L}} \vdash A : \mathcal{U}}{\Gamma \vdash \sqrt{\mathcal{L}} A : \mathcal{U}} & \sqrt{\text{-INTRO}} \quad &\frac{\Gamma, \mathbf{a}_{\mathcal{L}} \vdash a : A}{\Gamma \vdash \mathbf{a}_{\mathcal{L}}.a : \sqrt{\mathcal{L}} A} & \sqrt{\text{-ELIM}} \quad &\frac{\Gamma, i : \mathbb{T} \vdash r : \sqrt{\mathcal{L}} A}{\Gamma \vdash r(\gamma i.) : A[\gamma i./\mathbf{a}_{\mathcal{L}}]} \\ &(\mathbf{a}_{\mathcal{L}}.a)(\gamma i.) \equiv a[\gamma i./\mathbf{a}_{\mathcal{L}}] & & r \equiv \mathbf{a}_{\mathcal{L}}.(r[i/\mathbf{a}_{\mathcal{L}}](\gamma i.)) \end{aligned}$$

In words:

- $\sqrt{\text{-FORM}}$ and $\sqrt{\text{-INTRO}}$: For any term $a : A$, that uses all ambient variables locked behind $\mathbf{a}_{\mathcal{L}}$, we can bind the lock $\mathbf{a}_{\mathcal{L}}$ and form a term $\mathbf{a}_{\mathcal{L}}.a : \sqrt{\mathcal{L}}A$. The term syntax is intended to be reminiscent of an ordinary λ -abstraction.

Reading upwards, going under $\sqrt{\mathcal{L}}$ or $\mathbf{a}_{\mathcal{L}}.-$ means that all extant variables become locked by $\mathbf{a}_{\mathcal{L}}$, so every future use of those variables in a must have an attached $-[i/\mathbf{a}_{\mathcal{L}}]$ to be well formed.

These are precisely the formation and introduction rule for a dependent right adjoint type [BCMEPS20].

If A is a closed type, we will just write $\sqrt{A}$ rather than binding an unused lock name.

- $\sqrt{\text{-ELIM}}$: If using an additional assumption $i : \mathbb{T}$ we can produce a term $r : \sqrt{\mathcal{L}}A$, then we can amazingly apply r to the fresh i to form $r(\gamma i.) : A[\gamma i./\mathbf{a}_{\mathcal{L}}]$. We are free to ignore the new assumption i in the term r if we wish, just as a constant function may ignore its argument.

The syntax is, put mildly, unorthodox: to keep the analogy with ordinary function application, the variable i is bound in the parentheses on the right, but then is *in scope in the body of the ‘function’ to the left*¹.

- $\sqrt{\text{-BETA}}$: Binding a lock $\mathbf{a}_{\mathcal{L}}.a$ and amazingly applying the result to $\gamma i.$ reduces to $a[\gamma i./\mathbf{a}_{\mathcal{L}}]$, with $(\mathbf{a}_{\mathcal{L}}.a)(\gamma i.) \equiv a[\gamma i./\mathbf{a}_{\mathcal{L}}]$ in perfect analogy with $(\lambda x.b)(a) \equiv b[a/x]$.
- $\sqrt{\text{-ETA}}$: Any term $r : \sqrt{\mathcal{L}}A$ is equal to the term that binds a lock and immediately amazingly applies r to it, in analogy with $f \equiv (\lambda x.f(x))$.

3.1 Basic Examples

To begin, some maps that are easily definable from the rules. The functor $\mathbb{T} \rightarrow -$ is a monad, and its right adjoint $\sqrt{}$ is therefore a comonad.

Definition 3.1. If A is (for now) a closed type, define an extract map $e_A : \sqrt{A} \rightarrow A$ by

$$\begin{aligned} e_A &: \sqrt{A} \rightarrow A \\ e_A(r) &:= r(\gamma i.) \end{aligned}$$

In words, to get a term of A we need a term of $\sqrt{A}$ that we can amazingly evaluate on an additional assumption $i : \mathbb{T}$. But we already have the variable $r : \sqrt{A}$, so we have no need to use the additional assumption.

We have no way of quantifying over *closed* types internally, so any definition mentioning closed types should be considered provisional: a corresponding fully internal definition will be given shortly after.

Definition 3.2. If $f : A \rightarrow B$ is a (for now) closed function, define $\sqrt{f}$ by

$$\begin{aligned} \sqrt{f} &: \sqrt{A} \rightarrow \sqrt{B} \\ (\sqrt{f})(r) &:= \mathbf{a}_{\mathcal{L}}.f(r[i/\mathbf{a}_{\mathcal{L}}])(\gamma i.) \end{aligned}$$

This should be compared with the definition of function post-composition, as the shape of the definition is the same:

$$\begin{aligned} f \circ - &: (C \rightarrow A) \rightarrow (C \rightarrow B) \\ (f \circ -)(r) &:= \lambda c.f(r(c)) \end{aligned}$$

In words, we start with $r : \sqrt{A}$. To produce a term of $\sqrt{B}$ we have to produce term of B , but with all our present assumptions locked behind $\mathbf{a}_{\mathcal{L}}$. There is a function $f : A \rightarrow B$ available, so we just need access to a term of A . We cannot amazingly evaluate $r : \sqrt{A}$ to get this term of A , because r has become trapped behind $\mathbf{a}_{\mathcal{L}}$. But we *can* amazingly evaluate $r[i/\mathbf{a}_{\mathcal{L}}] : \sqrt{A}$ on an additional assumption $i : \mathbb{T}$. This gets us an

¹If we took Lawvere seriously [Law06] and wrote function application as xf , then we would not have this problem!

argument of type A that we can apply f to. This time, the presence of the additional assumption $i : \mathbb{T}$ is crucial: it is exactly what we needed to gain access to r .

If A is not a closed type, then the type of $e_A : \sqrt{A} \rightarrow A$ is not well-formed: once we go under $\sqrt{}$, we lose access to the variable A . Similarly, if f is not a closed function, then we cannot produce $\sqrt{f} : \sqrt{A} \rightarrow \sqrt{B}$, even if the types A and B are themselves closed, because the function f (as a variable) also becomes locked behind $\blacksquare_{\mathcal{L}}$ when the argument r does.

However: if *all* the inputs to the above constructions are provided under $\sqrt{}$, then we can have access to them where we need it. Suppose instead that $A : \sqrt{\mathcal{U}}$. Then, we may use A as an input to $\sqrt{}$ by forming $\sqrt{\mathcal{L}}A[i/\mathbf{q}_{\mathcal{L}}](\gamma i.) : \mathcal{U}$. Our map e_A can now be correctly typed:

Definition 3.3. For $A : \sqrt{\mathcal{U}}$, define a map e_A

$$\begin{aligned} e_A : \sqrt{\mathcal{L}}A[i/\mathbf{q}_{\mathcal{L}}](\gamma i.) &\rightarrow A(\gamma i.) \\ e_A(r) &\equiv r(\gamma i.) \end{aligned}$$

And similarly for functoriality:

Definition 3.4. For $A, B : \sqrt{\mathcal{U}}$ and $f : \sqrt{\mathcal{L}}((\gamma i.A[i/\mathbf{q}_{\mathcal{L}}]) \rightarrow (\gamma i.B[i/\mathbf{q}_{\mathcal{L}}]))$, define $\sqrt{f}$ by

$$\begin{aligned} \sqrt{f} : \sqrt{\mathcal{L}}(\gamma i.A[i/\mathbf{q}_{\mathcal{L}}]) &\rightarrow \sqrt{\mathcal{L}}(\gamma i.B[i/\mathbf{q}_{\mathcal{L}}]) \\ \sqrt{f}(r) &\equiv \blacksquare_{\mathcal{L}}.(\gamma j.f[j/\mathbf{q}_{\mathcal{L}}])(\gamma i.r[i/\mathbf{q}_{\mathcal{L}}]) \end{aligned}$$

This simple example is already very noisy, so to cut down on symbols we introduce two pieces of syntactic sugar.

First: Like ordinary $\rightarrow$ -INTRO, and as with any negative type-former, the $\sqrt{}$ -INTRO rule is right-invertible and so we will define terms of it via copattern matching. For example, we will write

$$r(\blacksquare_{\mathcal{L}}, \blacksquare_{\mathcal{K}}) \equiv \mathcal{J} \quad \text{to mean} \quad r \equiv \blacksquare_{\mathcal{L}}.\blacksquare_{\mathcal{K}}.\mathcal{J}$$

This integrates nicely with ordinary copattern-style definitions of functions: we write

$$f(x, y, \blacksquare_{\mathcal{L}}, z) \equiv \mathcal{J} \quad \text{to mean} \quad f \equiv \lambda x.\lambda y.\blacksquare_{\mathcal{L}}.\lambda z.\mathcal{J}$$

This syntax makes it easy to see at a glance which locks affect each variable in the term $\mathcal{J}$: the x and y here are to the left of $\blacksquare_{\mathcal{L}}$ in the list of arguments and so are locked by $\blacksquare_{\mathcal{L}}$, whereas z is not locked at all.

Similarly, we might evaluate some combination of $\rightarrow$ and $\sqrt{}$ by applying it to a tuple of arguments, as in $f(a, b, \gamma i., c)$. The ‘amazing’ argument to a lock is a new bound variable i as in $\sqrt{\text{-ELIM}}$, but remember that this variable is now in scope to the left of that argument in the tuple *and* in the function itself. This form of evaluation computes in the ‘obvious’ way, with a β -reduction for each entry in the list of arguments. If f is defined via the pattern match above, we find

$$\begin{aligned} f(a(i), b(i), \gamma i., c) &\equiv f(a(i))(b(i))(\gamma i.)(c) \\ &\equiv (\lambda x.\lambda y.\blacksquare_{\mathcal{L}}.\lambda z.\mathcal{J})(a(i))(b(i))(\gamma i.)(c) \\ &\equiv (\blacksquare_{\mathcal{L}}.\lambda z.\mathcal{J}[a(i)/x][b(i)/y])(\gamma i.)(c) \\ &\equiv (\lambda z.\mathcal{J}[a(i)/x][b(i)/y][\gamma i./\blacksquare_{\mathcal{L}}])(c) \\ &\equiv \mathcal{J}[a(i)/x][b(i)/y][\gamma i./\blacksquare_{\mathcal{L}}][c/z] \end{aligned}$$

We have arranged the syntax of our admissible rules so that function definitions and uses click together, as in ordinary function evaluation: comparing $f(a(i), b(i), \gamma i., c)$ and $f(x, y, \blacksquare_{\mathcal{L}}, z)$, the first arguments become substitutions $[a(i)/x][b(i)/y]$ and the amazing $\gamma i.$ ‘argument’ becomes a ‘substitution’ $[\gamma i./\blacksquare_{\mathcal{L}}]$.

Any ordinary function $f : A \rightarrow B$ is tautologically equal to the definition

$$f(x) \equiv f(x)$$

by η -expansion. For $\sqrt{\cdot}$, this η -expansion introduces an instance of the counit. That is, any term $r : \sqrt{\mathcal{L}}A$ is tautologically equal to the definition

$$r(\blacksquare_{\mathcal{L}}) := r[i/\mathbf{q}_{\mathcal{L}}](\gamma i.)$$

Remark 3.5. Using copattern syntax could also apply to other right adjoint modalities, like $\sharp$ [Shu18]. In the original syntax, functoriality of $\sharp$ on a function $f : A \rightarrow B$ is defined by $\sharp f(u) := f(u_{\sharp})^{\sharp}$. In a copattern-style syntax, this might be written $\sharp f(u, \sharp) := f(u(\sharp))$, or $\sharp f(u, \sharp_i) := f(u(\sharp_e))$ if we wish to distinguish the symbols for introduction and elimination. Here, the interpretation is that a variable placed to the left of $\sharp_i$ becomes crisp in the body of the function, and an ‘application’ $s(\sharp_e)$ is only well-typed when s is crisp. In this case, u becomes crisp and so it is valid to apply it to $\sharp_e$.

Second: We will often need to extract terms of A from assumptions $r : \sqrt{A}$ using the extract map $\mathbf{e}_A$ defined above. When such assumptions are behind many context locks, the $\sqrt{\cdot}$ -ELIM rule provides a term $i : \mathbb{T}$ that can be used to unlock all these locks at once, giving us access to A regardless of where r lies in the context.

Definition 3.6. If a variable $x : \sqrt{\mathcal{L}}A$ is behind several locks $\blacksquare_{\mathcal{K}_1} \dots \blacksquare_{\mathcal{K}_n}$, then let

$$x_{\downarrow} := (x[\![i/\mathbf{q}_{\mathcal{K}_1}, \dots, i/\mathbf{q}_{\mathcal{K}_n}]\!])(\gamma i.) : A[i/\mathbf{q}_{\mathcal{K}_1}, \dots, i/\mathbf{q}_{\mathcal{K}_n}][\gamma i./\blacksquare_{\mathcal{L}}]$$

in particular, if $x : \sqrt{\mathcal{L}}A$ is behind no locks in the context then $x_{\downarrow} := \mathbf{e}_A(x)$.

This notation $x_{\downarrow}$ is intended to evoke the elimination rule $x_{\sharp}$ used in cohesive HoTT (and similar). It cannot be internalised as a function, it is just an admissible piece of syntax.

Using these shorthands, the definition of functoriality is much less noisy: For $A, B : \sqrt{\mathcal{U}}$ and $f : \sqrt{\mathcal{L}}(A_{\downarrow} \rightarrow B_{\downarrow})$ we have

$$\begin{aligned} \sqrt{f} : \sqrt{\mathcal{L}}A_{\downarrow} &\rightarrow \sqrt{\mathcal{L}}B_{\downarrow} \\ \sqrt{f}(r, \blacksquare_{\mathcal{L}}) &\equiv f_{\downarrow}(r_{\downarrow}) \end{aligned}$$

The comultiplication of $\sqrt{\cdot}$ can be also written nicely using this notation. For a type $A : \sqrt{\mathcal{U}}$, define

$$\begin{aligned} \delta_A : \sqrt{A}_{\downarrow} &\rightarrow \sqrt{\sqrt{A}_{\downarrow}} \\ \delta_A(r, \blacksquare_{\mathcal{L}}, \blacksquare_{\mathcal{K}}) &\equiv r_{\downarrow} \end{aligned}$$

We can show that, as a right adjoint, $\sqrt{\cdot}$ preserves 1 and pullbacks.

Proposition 3.7. $\sqrt{1} \simeq 1$

Proof. Any term $r : \sqrt{1}$ will η -expand as $r \equiv \blacksquare_{\mathcal{L}}.r[\![i/\mathbf{q}_{\mathcal{L}}]\!](\gamma i.) \equiv \blacksquare_{\mathcal{L}}.\star$ and so $\sqrt{1}$ is contractible. $\square$

Proposition 3.8. If $A : \sqrt{\mathcal{U}}$ and $B : \sqrt{(A_{\downarrow} \rightarrow \mathcal{U})}$ there is an equivalence

$$\sqrt{\left(\sum_{(x:A_{\downarrow})} B_{\downarrow}(x)\right)} \simeq \sum_{(x:\sqrt{A_{\downarrow}})} \sqrt{(B_{\downarrow}(x_{\downarrow}))}$$

Proof. Define maps either way by

$$\begin{aligned} r &\mapsto (\blacksquare_{\mathcal{L}}.\mathbf{pr}_1 r_{\downarrow}, \blacksquare_{\mathcal{K}}.\mathbf{pr}_2 r_{\downarrow}) \\ (s, t) &\mapsto \blacksquare_{\mathcal{L}}.(s_{\downarrow}, t_{\downarrow}) \end{aligned}$$

$\square$

Proposition 3.9. For any $A : \sqrt{\mathcal{U}}$ and $r, s : \sqrt{A}_{\downarrow}$, there is an equivalence

$$(r = s) \simeq \sqrt{(r_{\downarrow} = s_{\downarrow})}$$

Proof. By the fundamental theorem of **Id**-types, we just have to check that for fixed $r : \sqrt{A}_{\downarrow}$, the type

$$\sum_{(s : \sqrt{A}_{\downarrow})} \sqrt{(r_{\downarrow} = s_{\downarrow})}$$

is contractible. By Proposition 3.8, this type is equivalent to

$$\sqrt{\left(\sum_{(x : A_{\downarrow})} r_{\downarrow} = x\right)}$$

and the interior is a contractible pair. □

This is more recognisable as a statement of left-exactness with the left side η -expanded:

Corollary 3.10. $(\blacksquare_{\mathcal{L}}.r_{\downarrow} = \blacksquare_{\mathcal{L}}.s_{\downarrow}) \simeq \sqrt{\mathcal{L}}(r_{\downarrow} = s_{\downarrow})$

3.2 Adjointness

We now investigate how $\sqrt{}$ relates to $(\mathbb{T} \rightarrow -)$. It is well-known that, although $(\mathbb{T} \rightarrow -)$ externally has a right adjoint, it is not typically a *fibred* right adjoint unless $\mathbb{T} \simeq 1$ [Yet87; Koc05].

There is, however, a useful internal statement. To begin, unit and counit maps for the adjunction.

Definition 3.11. If $A, B : \mathcal{U}$ are (for now) closed types, there are maps

$$\begin{aligned} \eta_A &: A \rightarrow \sqrt{(\mathbb{T} \rightarrow A)} \\ \varepsilon_B &: (\mathbb{T} \rightarrow \sqrt{B}) \rightarrow B \end{aligned}$$

given by

$$\begin{aligned} \eta_A(a, \blacksquare_{\mathcal{L}}, t) &:= a \llbracket t / \mathfrak{A}_{\mathcal{L}} \rrbracket \\ \varepsilon_B(f) &:= f(i, \gamma i.) \end{aligned}$$

Allowing the maximum amount of dependency, for *any* $A : \mathcal{U}$, there is a map

$$\eta_A : A \rightarrow \sqrt{\mathcal{L}} \prod_{(t : \mathbb{T})} A \llbracket t / \mathfrak{A}_{\mathcal{L}} \rrbracket$$

and for *any* $B : \mathbb{T} \rightarrow \sqrt{\mathcal{U}}$ a map

$$\varepsilon_B : \left(\prod_{(t : \mathbb{T})} \sqrt{\mathcal{K}} B(t)_{\downarrow} \right) \rightarrow B(j, \gamma j.)$$

with the same definitions as in the closed case.

Besides ε_B not being definable for an arbitrary type B , we have also seen that $\sqrt{-}$ is not functorial on arbitrary maps $A \rightarrow B$. So to what extent is $(\mathbb{T} \rightarrow -) \dashv \sqrt{-}$? We can prove an internal adjointness statement analogous to one in [FHM03, §3] for adjunctions in which the left adjoint is strong monoidal.

Proposition 3.12. *If A and B are closed types then there is an equivalence*

$$\Phi : \sqrt{((\mathbb{T} \rightarrow A) \rightarrow B)} \rightarrow (A \rightarrow \sqrt{B})$$

Happily, we can allow A and B to be dependent types:

Proposition 3.13. *For any $A : \mathcal{U}$ and $B : \sqrt{\mathcal{U}}$, there is an equivalence*

$$\sqrt{\mathcal{L}}((\prod_{(t : \mathbb{T})} A \llbracket t / \mathfrak{A}_{\mathcal{L}} \rrbracket) \rightarrow B_{\downarrow}) \rightarrow (A \rightarrow \sqrt{\mathcal{L}} B_{\downarrow})$$

And in fact, we can also let B depend on A . In full generality:

Proposition 3.14. *For any $A : \mathcal{U}$ and $B : A \rightarrow \sqrt{\mathcal{U}}$, there is an equivalence*

$$\Phi : \sqrt{\mathcal{K}} \left(\prod_{(h : \prod_{(t : \mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}])} B[i/\mathfrak{A}_{\mathcal{L}}](h(i), \gamma i.) \right) \rightarrow \prod_{(a : A)} \sqrt{\mathcal{L}} B(a)$$

given by

$$\begin{aligned} \Phi(r, a, \mathfrak{A}_{\mathcal{L}}) &:= r[i/\mathfrak{A}_{\mathcal{L}}](\gamma i., \lambda t. a[t/\mathfrak{A}_{\mathcal{L}}]) \\ \Phi^{-1}(f, \mathfrak{A}_{\mathcal{K}}, h) &:= f[k/\mathfrak{A}_{\mathcal{K}}](h(k), \gamma k.) \end{aligned}$$

Proof. All types involved have definitional η -principles, so this is a matter of crunching through the composites in both directions.

First, we η -expand $\Phi^{-1}(\Phi(r))$ to $\mathfrak{A}_{\mathcal{J}}. \lambda h. \Phi^{-1}(\Phi(r[j/\mathfrak{A}_{\mathcal{J}}]), \gamma j., h)$ and simplify the body:

$$\begin{aligned} &\Phi^{-1}(\Phi(r[j/\mathfrak{A}_{\mathcal{J}}]), \gamma j., h) \\ &\equiv \Phi(r[j/\mathfrak{A}_{\mathcal{J}}])[k/\mathfrak{A}_{\mathcal{K}}][\gamma j./\mathfrak{A}_{\mathcal{K}}](h(k), \gamma k.) && \text{(Definition of } \Phi^{-1}) \\ &\equiv \Phi(r[k/\mathfrak{A}_{\mathcal{J}}])(h(k), \gamma k.) && \text{(Computing } [\gamma j./\mathfrak{A}_{\mathcal{K}}]) \\ &\equiv r[k/\mathfrak{A}_{\mathcal{K}}, i/\mathfrak{A}_{\mathcal{L}}](\gamma i., \lambda t. h(k)[t/\mathfrak{A}_{\mathcal{L}}])[\gamma k./\mathfrak{A}_{\mathcal{L}}] && \text{(Definition of } \Phi) \\ &\equiv r[i/\mathfrak{A}_{\mathcal{K}}](\gamma i., \lambda t. h(t)) && \text{(Computing } [\gamma k./\mathfrak{A}_{\mathcal{L}}]) \\ &\equiv r[i/\mathfrak{A}_{\mathcal{K}}](\gamma i., h) && (\eta \text{ for functions}) \end{aligned}$$

which is precisely the η -expansion of r .

The other way, $\Phi(\Phi^{-1}(f))$ expands to $\lambda a. \mathfrak{A}_{\mathcal{J}}. \Phi(\Phi^{-1}(f[j/\mathfrak{A}_{\mathcal{J}}]), a, \gamma j.)$ and:

$$\begin{aligned} &\Phi(\Phi^{-1}(f[j/\mathfrak{A}_{\mathcal{J}}]), a, \gamma j.) \\ &\equiv \Phi^{-1}(f[j/\mathfrak{A}_{\mathcal{J}}])[i/\mathfrak{A}_{\mathcal{L}}](\gamma i., \lambda t. a[j/\mathfrak{A}_{\mathcal{J}}][t/\mathfrak{A}_{\mathcal{L}}])[\gamma j./\mathfrak{A}_{\mathcal{L}}] && \text{(Definition of } \Phi) \\ &\equiv \Phi^{-1}(f[i/\mathfrak{A}_{\mathcal{J}}])(\gamma i., \lambda t. a[t/\mathfrak{A}_{\mathcal{J}}]) && \text{(Computing } [\gamma j./\mathfrak{A}_{\mathcal{L}}]) \\ &\equiv f[i/\mathfrak{A}_{\mathcal{J}}, k/\mathfrak{A}_{\mathcal{K}}][\gamma i./\mathfrak{A}_{\mathcal{K}}](\lambda t. a[t/\mathfrak{A}_{\mathcal{J}}])(k), \gamma k.) && \text{(Definition of } \Phi^{-1}) \\ &\equiv f[k/\mathfrak{A}_{\mathcal{J}}](\lambda t. a[t/\mathfrak{A}_{\mathcal{J}}])(k), \gamma k.) && \text{(Computing } [\gamma i./\mathfrak{A}_{\mathcal{K}}]) \\ &\equiv f[k/\mathfrak{A}_{\mathcal{J}}](a[k/\mathfrak{A}_{\mathcal{J}}], \gamma k.) && (\beta \text{ for functions}) \end{aligned}$$

is the η -expansion of f . □

Remark 3.15. Internal equivalences of this kind for an arbitrary adjoint modality have been defined in MTT, where the modality is instead a positive type-former, see [ND21, Proposition 3.4] and [GKNB20, Section 10.4]. The fact that this internal formulation circumvents the ‘no-go’ theorem of [LOPS18] was also noted in [ND21, Section 10.1].

4 Constructions

4.1 Higher-Dimensional Induction

Since we have given $\mathbb{T} \rightarrow -$ a right adjoint, it now preserves colimits. In particular, we support the same kind of “higher-dimensional pattern matching” on functions out of $\mathbb{T}$, as suggested in [ND21, §2.4]. In this section we derive an induction principle for functions of $\mathbb{T}$ into coproducts, though we expect easy generalisations to other inductive types.

Proposition 4.1. *Suppose types $A, B : \mathcal{U}$ and a type family*

$$P : \sqrt{\mathcal{L}} \left(\left(\prod_{(t : \mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}] + B[t/\mathfrak{A}_{\mathcal{L}}] \right) \rightarrow \mathcal{U} \right).$$

Then there is a map

$$\begin{aligned} \text{hind} &: \sqrt{\mathcal{L}} \left(\prod_{(l:\prod_{(t:\mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}])} P_{\downarrow}(\text{inl} \circ l) \right) \\ &\quad \times \sqrt{\mathcal{L}} \left(\prod_{(r:\prod_{(t:\mathbb{T})} B[t/\mathfrak{A}_{\mathcal{L}}])} P_{\downarrow}(\text{inr} \circ r) \right) \\ &\rightarrow \sqrt{\mathcal{L}} \left(\prod_{(f:\prod_{(t:\mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}] + B[t/\mathfrak{A}_{\mathcal{L}}])} P_{\downarrow}(f) \right) \end{aligned}$$

such that $\text{hind}(g, h)$ computes via

$$\begin{aligned} \text{hind}(g, h)(\mathfrak{A}_{\mathcal{L}}, \lambda t. \text{inl}(l_{\downarrow}(t))) &\equiv g_{\downarrow}(l_{\downarrow}) \\ \text{hind}(g, h)(\mathfrak{A}_{\mathcal{L}}, \lambda t. \text{inr}(r_{\downarrow}(t))) &\equiv h_{\downarrow}(r_{\downarrow}) \end{aligned}$$

for any $l : \sqrt{\mathcal{L}}(\prod_{(t:\mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}])$ and $r : \sqrt{\mathcal{L}}(\prod_{(t:\mathbb{T})} B[t/\mathfrak{A}_{\mathcal{L}}])$.

If we like, we can always extract the underlying function $\text{hind}(g, h)(\gamma i.) : \prod_{(f:\mathbb{T} \rightarrow A+B)} P_{\downarrow}(f)$ of the result.

Proof. There is a chain of equivalences

$$\begin{aligned} &\sqrt{\mathcal{L}} \left(\prod_{(l:\prod_{(t:\mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}])} P_{\downarrow}(\text{inl} \circ l) \right) \times \sqrt{\mathcal{L}} \left(\prod_{(r:\prod_{(t:\mathbb{T})} B[t/\mathfrak{A}_{\mathcal{L}}])} P_{\downarrow}(\text{inr} \circ r) \right) \\ &\quad (\text{Dependent adjointness}) \\ &\simeq \left(\prod_{(a:A)} \sqrt{\mathcal{L}} P_{\downarrow}(\lambda i. \text{inl}(a[i/\mathfrak{A}_{\mathcal{L}}])) \right) \times \left(\prod_{(b:B)} \sqrt{\mathcal{L}} P_{\downarrow}(\lambda i. \text{inr}(b[i/\mathfrak{A}_{\mathcal{L}}])) \right) \\ &\quad (\text{Universal property of } +) \\ &\simeq \prod_{(s:A+B)} \sqrt{\mathcal{L}} P_{\downarrow}(\lambda i. s[i/\mathfrak{A}_{\mathcal{L}}]) \\ &\quad (\text{Dependent adjointness}) \\ &\simeq \sqrt{\mathcal{L}} \left(\prod_{(f:\prod_{(t:\mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}] + B[t/\mathfrak{A}_{\mathcal{L}}])} P_{\downarrow}(f) \right) \end{aligned}$$

As a term, the forward map is

$$\begin{aligned} \text{hind}(g, h)(\mathfrak{A}_{\mathcal{L}}, f) &:= \text{case}_+(f(i), a. \mathfrak{A}_{\mathcal{K}}. g_{\downarrow}(\lambda t. a[t/\mathfrak{A}_{\mathcal{K}}]), \\ &\quad b. \mathfrak{A}_{\mathcal{K}}. h_{\downarrow}(\lambda t. b[t/\mathfrak{A}_{\mathcal{K}}]))(\gamma i.) \end{aligned}$$

We can show directly that this map has the desired computational properties:

$$\begin{aligned} &\text{hind}(g, h)(\mathfrak{A}_{\mathcal{L}}, \lambda t. \text{inl}(l_{\downarrow}(t))) \\ &\equiv \text{case}_+(\text{inl}(l_{\downarrow}(i)), a. \mathfrak{A}_{\mathcal{K}}. g_{\downarrow}(\lambda t. a[t/\mathfrak{A}_{\mathcal{K}}]), b. \mathfrak{A}_{\mathcal{K}}. h_{\downarrow}(\lambda t. b[t/\mathfrak{A}_{\mathcal{K}}]))(\gamma i.) \\ &\equiv (\mathfrak{A}_{\mathcal{K}}. g_{\downarrow}(\lambda t. l_{\downarrow}[t/\mathfrak{A}_{\mathcal{K}}])(i[t/\mathfrak{A}_{\mathcal{K}}]))(\gamma i.) \\ &\equiv g_{\downarrow}(\lambda t. l_{\downarrow}[t/\mathfrak{A}_{\mathcal{K}}])(i[t/\mathfrak{A}_{\mathcal{K}}])[\gamma i./\mathfrak{A}_{\mathcal{K}}] \\ &\equiv g_{\downarrow}(\lambda t. l_{\downarrow}(t)) \\ &\equiv g_{\downarrow}(l_{\downarrow}) \end{aligned}$$

and similarly when evaluated at $\lambda t. \text{inr}(r_{\downarrow}(t))$. □

The upshot of the above is that we support higher-dimension pattern matching of the following form: to define a function $p : \sqrt{\mathcal{L}} \left(\prod_{(f:\prod_{(t:\mathbb{T})} A[t/\mathfrak{A}_{\mathcal{L}}] + B[t/\mathfrak{A}_{\mathcal{L}}])} P_{\downarrow}(f) \right)$, it suffices to give cases

$$\begin{aligned} p(\mathfrak{A}_{\mathcal{L}}, \lambda t. \text{inl}(l_{\downarrow}(t))) &:= \dots \\ p(\mathfrak{A}_{\mathcal{L}}, \lambda t. \text{inr}(r_{\downarrow}(t))) &:= \dots \end{aligned}$$

An immediate application is:

Proposition 4.2. *For any types $A, B : \mathcal{U}$, the map*

$$\text{split}_{A+B} : ((\mathbb{T} \rightarrow A) + (\mathbb{T} \rightarrow B)) \rightarrow (\mathbb{T} \rightarrow A + B)$$

defined by case-splitting is an equivalence.

Proof. The inverse map is definable using this new pattern matching, with motive the constant type family

$$P(\bullet_{\mathcal{L}}, i) := \left(\prod_{(t:\mathbb{T})} A[t/\mathbf{q}_{\mathcal{L}}] \right) + \left(\prod_{(t:\mathbb{T})} B[t/\mathbf{q}_{\mathcal{L}}] \right)$$

The cases we have to provide are:

$$\begin{aligned} \text{unsplit}_{A+B}(\bullet_{\mathcal{L}}, \lambda t. \text{inl}(l_{\downarrow}(t))) &:= \text{inl}(l_{\downarrow}) \\ \text{unsplit}_{A+B}(\bullet_{\mathcal{L}}, \lambda t. \text{inr}(r_{\downarrow}(t))) &:= \text{inr}(r_{\downarrow}) \end{aligned}$$

In one direction, the composite computes definitionally:

$$\text{unsplit}_{A+B}(\text{split}_{A+B}(\text{inl}(f))) \equiv \text{unsplit}_{A+B}(\lambda i. \text{inl}(f(i))) \equiv \text{inl}(f)$$

In the other direction, we use higher-dimensional pattern matching again, this time with motive

$$P'(\bullet_{\mathcal{L}}, f) := \prod_{(t:\mathbb{T})} \text{split}_{A[t/\mathbf{q}_{\mathcal{L}}] + B[t/\mathbf{q}_{\mathcal{L}}]} \left(\text{unsplit}_{A[t/\mathbf{q}_{\mathcal{L}}] + B[t/\mathbf{q}_{\mathcal{L}}]}(f) \right) (t) = f(t)$$

In the branches, the left-hand side of the path computes away:

$$\text{split}(\text{unsplit}(\lambda t. \text{inl}(l_{\downarrow}(t)))) \equiv \text{split}(\text{inl}(l_{\downarrow})) \equiv \lambda t. \text{inl}(l_{\downarrow}(t))$$

and similarly for inr , so we may inhabit the type family by

$$\begin{aligned} \text{inv}_{A+B}(\bullet_{\mathcal{L}}, \lambda t. \text{inl}(l_{\downarrow}(t)), t) &:= \text{refl}_{\text{inl}(l_{\downarrow}(t))} \\ \text{inv}_{A+B}(\bullet_{\mathcal{L}}, \lambda t. \text{inr}(r_{\downarrow}(t)), t) &:= \text{refl}_{\text{inr}(r_{\downarrow}(t))} \end{aligned}$$

proving (via function extensionality) that unsplit_{A+B} is indeed an inverse. $\square$

Notice that we did not need the input types A and B to be guarded by $\sqrt{}$ in the above Proposition. With a little more care, we could obtain a more refined map

$$\sqrt{}_{\mathcal{L}} \left(\left(\left(\prod_{(t:\mathbb{T})} A[t/\mathbf{q}_{\mathcal{L}}] \right) + \left(\prod_{(t:\mathbb{T})} B[t/\mathbf{q}_{\mathcal{L}}] \right) \right) \rightarrow \left(\prod_{(t:\mathbb{T})} A[t/\mathbf{q}_{\mathcal{L}}] + B[t/\mathbf{q}_{\mathcal{L}}] \right) \right)$$

whence the original equivalence may be extracted.

4.2 Transpension

An equivalent characterisation of a tiny object in a 1-topos, due to Freyd [Yet87, Proposition 1.2], is an object $\mathbb{T}$ such that the dependent product functor $\Pi_{\mathbb{T}} : \mathcal{E}/\mathbb{T} \rightarrow \mathcal{E}$ has a right adjoint $\nabla_{\mathbb{T}} : \mathcal{E} \rightarrow \mathcal{E}/\mathbb{T}$. This operation was given the name *transpension* in [ND20; ND21].

If $\mathbb{T}$ is tiny in the ordinary sense, then in a 1-topos this adjoint $\nabla_{\mathbb{T}}$ can be defined by sending each object B to the left map in the pullback

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad \lrcorner \quad} & \sqrt{} \left(\sum_{(P:\Omega)} P \rightarrow B \right) \\ \nabla_{\mathbb{T}} B \downarrow & & \downarrow \sqrt{\text{pr}_1} \\ \mathbb{T} & \xrightarrow{(\chi_{\text{id}_{\mathbb{T}}})^{\vee}} & \sqrt{\Omega} \end{array}$$

In the univalent setting this construction is no longer correct, but simply replacing Ω with the universe works: for $B : \sqrt{\mathcal{U}}$, define the type family $B^{1/-} : \mathbb{T} \rightarrow \mathcal{U}$ by the pullback

$$\begin{array}{ccc} \sum_{(t:\mathbb{T})} B^{1/t} & \longrightarrow & \sqrt{\left(\sum_{(X:\mathcal{U})} X \rightarrow B_{\downarrow}\right)} \\ \downarrow & \lrcorner & \downarrow \sqrt{\text{pr}_1} \\ \mathbb{T} & \xrightarrow{(-=\text{id}_{\mathbb{T}})^{\vee}} & \sqrt{\mathcal{U}} \end{array}$$

where the map along the bottom is the transpose of $(\lambda f. f = \text{id}_{\mathbb{T}}) : (\mathbb{T} \rightarrow \mathbb{T}) \rightarrow \mathcal{U}$, namely

$$t \mapsto \blacksquare_{\mathcal{K}} \cdot \prod_{(s:\mathbb{T})} t \llbracket s / \mathfrak{A}_{\mathcal{K}} \rrbracket = s$$

We can simplify this description a fair amount. Because $\sqrt{}$ commutes with Σ -types, the map on the right is equivalent to the projection

$$\text{pr}_1 : \sum_{(X:\sqrt{\mathcal{U}})} \sqrt{\mathcal{K}}(X_{\downarrow} \rightarrow B_{\downarrow}) \rightarrow \sqrt{\mathcal{U}}$$

The fibre of the pullback over a fixed $t : \mathbb{T}$ is then easy to calculate: it is equivalent to the type of the second component of the sum:

$$\begin{aligned} & \sqrt{\mathcal{L}}(X_{\downarrow} \rightarrow B_{\downarrow})[(\blacksquare_{\mathcal{K}} \cdot \prod_{(s:\mathbb{T})} t \llbracket s / \mathfrak{A}_{\mathcal{K}} \rrbracket = s) / X] \\ & \equiv \sqrt{\mathcal{L}}\left(\left(\prod_{(s:\mathbb{T})} t \llbracket s / \mathfrak{A}_{\mathcal{L}} \rrbracket = s\right) \rightarrow B_{\downarrow}\right) \end{aligned}$$

and so we take this as our definition.

Definition 4.3. For $B : \sqrt{\mathcal{U}}$, define the *transpension* $B^{1/-} : \mathbb{T} \rightarrow \mathcal{U}$ to be

$$B^{1/t} := \sqrt{\mathcal{L}}\left(\left(\prod_{(s:\mathbb{T})} t \llbracket s / \mathfrak{A}_{\mathcal{L}} \rrbracket = s\right) \rightarrow B_{\downarrow}\right)$$

The whole point of this construction was to be a right adjoint to the dependent product with $\mathbb{T}$, and indeed it is:

Proposition 4.4. For $A : \mathbb{T} \rightarrow \mathcal{U}$ and $B : \sqrt{\mathcal{U}}$

$$\sqrt{\mathcal{L}}\left(\left(\prod_{(s:\mathbb{T})} A \llbracket s / \mathfrak{A}_{\mathcal{L}} \rrbracket(s)\right) \rightarrow B_{\downarrow}\right) \simeq \left(\prod_{(t:\mathbb{T})} A(t) \rightarrow B^{1/t}\right)$$

Proof. To apply dependent adjointness (Proposition 3.14) we need to massage the interior of the $\sqrt{\mathcal{L}}$ a little. Let P denote the type family

$$\begin{aligned} P & : \left(\sum_{(t:\mathbb{T})} A(t)\right) \rightarrow \sqrt{\mathcal{U}} \\ P((t, a), \blacksquare_{\mathcal{K}}) & := \left(\prod_{(s:\mathbb{T})} t \llbracket s / \mathfrak{A}_{\mathcal{K}} \rrbracket t = s\right) \rightarrow B_{\downarrow} \end{aligned}$$

Then

$$\begin{aligned} & \sqrt{\mathcal{L}}\left(\left(\prod_{(s:\mathbb{T})} (A \llbracket s / \mathfrak{A}_{\mathcal{L}} \rrbracket)(s)\right) \rightarrow B_{\downarrow}\right) \\ & \simeq \sqrt{\mathcal{L}}\left(\left(\prod_{(s:\mathbb{T})} \sum_{(t:\mathbb{T})} \sum_{(a:A \llbracket s / \mathfrak{A}_{\mathcal{L}} \rrbracket(t))} (t = s)\right) \rightarrow B_{\downarrow}\right) && \text{(Contractible pair)} \\ & \simeq \sqrt{\mathcal{L}}\left(\prod_{(f:\prod_{(t':\mathbb{T})} \sum_{(t:\mathbb{T})} A \llbracket t' / \mathfrak{A}_{\mathcal{L}} \rrbracket(t))} \left(\prod_{(s:\mathbb{T})} \text{pr}_1(f(i)) = s\right) \rightarrow B_{\downarrow}\right) && \text{(Univ. property of } \Sigma) \\ & \equiv \sqrt{\mathcal{L}}\left(\prod_{(f:\prod_{(t':\mathbb{T})} \sum_{(t:\mathbb{T})} A \llbracket t' / \mathfrak{A}_{\mathcal{L}} \rrbracket(t))} P \llbracket i / \mathfrak{A}_{\mathcal{L}} \rrbracket(f(i), \gamma i.)\right) && \text{(Definition of } P) \\ & \simeq \prod_{((t,a):\sum_{(t:\mathbb{T})} A(t))} \sqrt{\mathcal{L}}(P \llbracket i / \mathfrak{A}_{\mathcal{L}} \rrbracket)(t \llbracket i / \mathfrak{A}_{\mathcal{L}} \rrbracket, a \llbracket i / \mathfrak{A}_{\mathcal{L}} \rrbracket, \gamma i.) && \text{(Proposition 3.14)} \\ & \equiv \prod_{((t,a):\sum_{(t:\mathbb{T})} A(t))} \sqrt{\mathcal{L}}\left(\left(\prod_{(s:\mathbb{T})} t \llbracket s / \mathfrak{A}_{\mathcal{L}} \rrbracket = s\right) \rightarrow B_{\downarrow}\right) && \text{(Definition of } P) \\ & \equiv \prod_{((t,a):\sum_{(t:\mathbb{T})} A(t))} B^{1/t} && \text{(Definition of } B^{1/t}) \\ & \simeq \prod_{(t:\mathbb{T})} A(t) \rightarrow B^{1/t} && \text{(Currying)} \end{aligned}$$

□

Corollary 4.5. $\sqrt{B_{\mathbb{I}}} \simeq \left(\prod_{(t:\mathbb{T})} B^{1/t} \right)$

Proof. Plug in $A(t) := 1$ in the above. □

A surprising fact is that the pullback of $\mathbb{T}$ to any slice $\mathcal{E}/X$ is itself also tiny [Yet87, Theorem 1.4], but crucially, the right adjoint witnessing the tininess of $\mathbb{T} \times X$ in $\mathcal{E}/X$ is not simply the pullback of $\sqrt{\cdot}$.

By a similar strategy as for transpension, we can describe the correct right adjoint internally. Given a family $D : X \rightarrow \mathcal{U}$ over a closed type X , this right adjoint $\sqrt{\cdot}_- D : X \rightarrow \mathcal{U}$ (to coin some notation) may be calculated by

$$\sqrt{\cdot}_x D := \sqrt{\cdot}_{\mathcal{L}} \left(\prod_{(x':X)} \left(\prod_{(k:\mathbb{T})} x' = x[k/\mathbf{q}_{\mathcal{L}}] \right) \rightarrow D(x') \right)$$

We have not yet encountered a use for this construction.

5 Applications

We end with some commentary on possible applications of our theory.

Synthetic Differential Geometry. In future work with David Jaz Myers, we intend to construct form classifiers of various kinds internal to type theory and prove the properties asserted axiomatically in [Mye21, §3] and [Mye22, Appendix A]. In particular, we plan to give a synthetic construction of the de Rham complex by constructing a long exact sequence of form classifiers $\mathbb{R} \rightarrow \Lambda^1 \rightarrow \Lambda^2 \rightarrow \dots$.

In this setting there are many important tiny objects, for example, not only is the infinitesimal interval D tiny, but also each “infinitesimal patch” around the origin in $\mathbb{R}^n$:

$$D(n) := \{x : \mathbb{R}^n \mid \forall ij \ x_i x_j = 0\}$$

Our new judgemental rules and type former have a straightforward extension to multiple tiny objects, with each a closed type. Each $\blacksquare$ and $\sqrt{\cdot}$ would be annotated with the corresponding tiny object, and the rules for the $\sqrt{\cdot}$ -type would be specialised for each tiny object.

We take this opportunity to propose some notation to be used when there are multiple tiny objects at hand. On account of the copattern notation, we suggest writing the amazing right adjoint as $\mathbb{T} \multimap A$ rather than $\sqrt{\cdot} A$ or similar. This would associate to the right like ordinary function types so that, for example, the unit of the adjunction would be written

$$\begin{aligned} \eta_A : A \rightarrow \mathbb{T} \multimap A \\ \eta_A(a, \blacksquare_{\mathcal{L}}, t) &:= a[t/\mathbf{q}_{\mathcal{L}}] \end{aligned}$$

with the argument list aligning nicely with the structure of the type.

Internal Models of Cubical Type Theory. The present type theory removes the necessity for axioms in the construction of a universe of fibrations in [LOPS18, Theorem 5.2]. Let us briefly repeat the construction to demonstrate where things become more explicit.

The ambient type theory is intensional, predicative MLTT with function extensionality and UIP: we therefore denote the ambient universe $\mathbf{Set}$ rather than $\mathcal{U}$ and aim to construct a new universe $\mathbf{U}$ that classifies fibrations. There is an interval type $\mathbb{I}$ which is assumed to be tiny.

The main input data is a *notion of composition structure*, which is a function $C : (\mathbb{I} \rightarrow \mathbf{Set}) \rightarrow \mathbf{Set}$ that parameterises the notion of fibration:

$$\begin{aligned} \text{isFib} &: \prod_{(\Gamma : \mathbf{Set})} \prod_{(A : \Gamma \rightarrow \mathbf{Set})} \mathbf{Set} \\ \text{isFib}(\Gamma)(A) &:= \prod_{(p : \mathbb{I} \rightarrow \Gamma)} C(A \circ p) \\ \mathbf{Fib} &: \mathbf{Set} \rightarrow \mathbf{Set} \\ \mathbf{Fib}(\Gamma) &:= \sum_{(A : \Gamma \rightarrow \mathbf{Set})} \text{isFib}(\Gamma)(A) \end{aligned}$$

The construction proceeds the same way as in [LOPS18, Theorem 5.2], with the universe classifying ‘crisp’ fibrations constructed as the pullback

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{\quad} & \sqrt{\sum_{(A : \mathbf{Set})} A} \\ \downarrow & \lrcorner & \downarrow \sqrt{\text{pr}_1} \\ \mathbf{Set} & \xrightarrow{\quad C^\vee \quad} & \sqrt{\mathbf{Set}} \end{array}$$

We can simplify this description by commuting $\sqrt{\quad}$ with Σ and substituting, a move not available when working with the axiomatically asserted tininess. A more convenient definition of $\mathbf{U}$ is then

$$\begin{aligned} \mathbf{U} &:= \sum_{(X : \mathbf{Set})} \sqrt{A_{\downarrow}}[C^\vee(X)/A] \\ &\equiv \sum_{(X : \mathbf{Set})} \sqrt{\mathcal{L}} C(\lambda j. X \llbracket j / \mathbf{q}_{\mathcal{L}} \rrbracket) \end{aligned}$$

But inspecting this construction, things are much simpler if we slightly change our notion of fibration. Surrounding C with $\sqrt{\quad}$ in the body of isFib , define:

$$\begin{aligned} \text{isNewFib} &: \prod_{(\Gamma : \mathbf{Set})} \prod_{(A : \Gamma \rightarrow \mathbf{Set})} \mathbf{Set} \\ \text{isNewFib}(\Gamma)(A) &:= \prod_{(g : \Gamma)} \sqrt{\mathcal{L}} C(\lambda i. A \llbracket i / \mathbf{q}_{\mathcal{L}} \rrbracket (g \llbracket i / \mathbf{q}_{\mathcal{L}} \rrbracket)) \\ \mathbf{NewFib} &: \mathbf{Set} \rightarrow \mathbf{Set} \\ \mathbf{NewFib}(\Gamma) &:= \sum_{(A : \Gamma \rightarrow \mathbf{Set})} \text{isNewFib}(\Gamma)(A) \end{aligned}$$

Unchanged, the type $\mathbf{U}$ defined above classifies *non-crisp* fibrations of this tweaked kind, in that there is a completely unrestricted equivalence between the type of functions $\Gamma \rightarrow \mathbf{U}$ and $\mathbf{NewFib}(\Gamma)$. We can construct an element of $\mathbf{NewFib} \mathbf{U}$ tautologically, running the identity $\mathbf{U} \rightarrow \mathbf{U}$ backwards through that equivalence. The proofs of fibrancy for each type constructor would need to be re-done to account for the presence of $\sqrt{\quad}$ in isNewFib , something we also leave for future work. There is the intriguing prospect of these proofs amounting to a constructive *implementation* of cubical type theory internal to MLTT with a tiny object, if enough of the non-computational axioms used in [LOPS18] can be avoided.

Higher-Order Abstract Syntax. I am indebted to Jon Sterling for pointing out the following application. In [FPT99; FT01], the authors work in the internal language of the category of functors $\mathbf{FinSet} \rightarrow \mathbf{Set}$ to give semantics for higher-order abstract syntax. The inclusion $\mathbb{V} : \mathbf{FinSet} \rightarrow \mathbf{Set}$ is thought of as an object of *abstract variables*. It is tiny, because it is represented by $1^{\text{op}} \in \mathbf{FinSet}^{\text{op}}$.

Suppose we have an inductive type $\mathbf{Tm}$ representing the abstract syntax of the untyped λ -calculus, and want to implement single-variable substitution $\text{subst} : (\mathbb{V} \rightarrow \mathbf{Tm}) \rightarrow \mathbf{Tm} \rightarrow \mathbf{Tm}$ so that $\text{subst}(\lambda i. v, u)$ substitutes u for i wherever it occurs in v . We need to induct on v “under the binder” for i , and the tininess of $\mathbb{V}$ allows us to do this by similar techniques to Section 4.1. Applying adjointness, it will be enough to define the conjugate operation with type $\mathbf{Tm} \rightarrow \sqrt{(\mathbf{Tm} \rightarrow \mathbf{Tm})}$, which can be done by ordinary induction on $\mathbf{Tm}$.

Implementation. Finally, we have a prototype implementation of a type-checker available at <https://github.com/mvr/tiny>, using a novel extension of normalisation-by-evaluation to handle the new context structure and type former. The key insight is that the semantic environment corresponding to a context $\Gamma, \mathbf{a}_c$ can be a *meta-level function* from semantic values of type $\mathbb{T}$ to semantic environments for the context Γ . When looking up a variable in the environment, the values attached to the variable as keys are used as the arguments to these functions. Our type-checker verifies that the two round-trips in Proposition 3.14 are indeed proven by `refl` in both directions (<https://github.com/mvr/tiny/blob/master/adjunction.tiny>). The normalisation algorithm works in practice, but it is left for future work to verify that it works in theory.

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A Proofs

In this appendix, we provide proofs that the admissible rules described in Section 2 preserve typing.

Establishing metatheoretic properties of our type theory is somewhat easier than modal theories like, for example, [GSB19] and [RFL21], because the new rules only ever extend the context on the right rather than performing some manipulation potentially at any point in the middle. In particular, it is not necessary to prove versions of ‘lock strengthening/weakening/exchange’ as in [GSB19], or ‘marking/unmarking’ operations as in [RFL21].

Throughout, we continue to let $\vec{t}/\mathbf{a}_{\mathcal{L}}$ stand in for a sequence such as $t_1/\mathbf{a}_{\mathcal{L}_1}, \dots, t_n/\mathbf{a}_{\mathcal{L}_n}$.

Like substitution, our new admissible rules are definable purely on well-scoped syntax.

Definition A.1. A *scope* is an ordered list of variable names and lock names, where the variable names have no associated types. Unlike ordinary type theory, the relative position of variables and locks is important. Every ordinary context and telescope has an underlying scope.

The judgement $G \vdash r$ **rawterm** denotes a raw term in scope G . These are defined as usual, but importantly, in a well-scoped term, all variable uses have the correct number of associated keys for their position in the scope.

Definition A.2. The unit and counit operations

$$\begin{array}{c} \text{COUNIT} \frac{G, G'' \vdash r \text{ rawterm} \quad G, G' \vdash t \text{ rawterm for } \mathcal{L} \in \text{locks}(G')}{G, G', G'' \vdash r[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ rawterm}} \quad \text{UNIT} \frac{G, i, \mathbf{a}_{\mathcal{L}}, G' \vdash r \text{ rawterm}}{G, G' \vdash r[\gamma i./\mathbf{a}_{\mathcal{L}}] \text{ rawterm}} \end{array}$$

are defined on raw syntax as described in Section 2.

These operations on raw syntax satisfy a number of associativity properties whose mind-numbing proofs we delay until later.

Proposition A.3. *The counit operation is admissible on context telescopes, types and terms.*

$$\begin{array}{c} \text{COUNIT-TELE} \frac{\Gamma \vdash \Gamma'' \text{ tele} \quad \Gamma, \Gamma' \vdash t : \mathbb{T} \text{ for } \mathcal{L} \in \text{locks}(\Gamma')}{\Gamma, \Gamma' \vdash \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ tele}} \\ \text{COUNIT} \frac{\Gamma, \Gamma'' \vdash a : A \quad \Gamma, \Gamma' \vdash t : \mathbb{T} \text{ for } \mathcal{L} \in \text{locks}(\Gamma')}{\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash a[\vec{t}/\mathbf{a}_{\mathcal{L}}] : A[\vec{t}/\mathbf{a}_{\mathcal{L}}]} \end{array}$$

Proof. On telescopes:

Case:

$$\frac{\Gamma \vdash \Gamma'' \text{ tele} \quad \Gamma, \Gamma'' \vdash A \text{ type}}{\Gamma \vdash \Gamma'', x : A \text{ tele}}$$

By induction $\Gamma, \Gamma' \vdash \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ tele}$ and $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash A[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ type}$, so $\Gamma, \Gamma' \vdash \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}], A[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ tele}$

Case:

$$\frac{\Gamma \vdash \Gamma'' \text{ tele}}{\Gamma \vdash \Gamma'', \mathbf{a}_{\mathcal{L}} \text{ tele}}$$

By induction $\Gamma, \Gamma' \vdash \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ tele}$ so $\Gamma, \Gamma' \vdash \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}], \mathbf{a}_{\mathcal{L}} \text{ tele}$.

On terms:

Case:

$$\frac{\begin{array}{l} \Gamma_1, x : A, \Gamma_2, \Gamma'' \vdash s : \mathbb{T} \text{ for } \mathcal{J} \in \text{locks}(\Gamma_2) \\ \Gamma_1, x : A, \Gamma_2, \Gamma'' \vdash s'' : \mathbb{T} \text{ for } \mathcal{K} \in \text{locks}(\Gamma'') \end{array}}{\Gamma_1, x : A, \Gamma_2, \Gamma'' \vdash x[\vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}'/\mathbf{a}_{\mathcal{K}}] : A[\vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}'/\mathbf{a}_{\mathcal{K}}]}$$

Inductively, we have

$$\begin{array}{l} \Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma'' \vdash \vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}] : \mathbb{T} \\ \Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma'' \vdash \vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}] : \mathbb{T} \end{array}$$

Also inductively, we may use the latter to form

$$\Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma'' \vdash \vec{t}[\vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] : \mathbb{T}$$

These are all the keys required to form

$$\begin{array}{l} \Gamma_1, x : A, \Gamma_2, \Gamma', \Gamma'' \vdash x[\vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \vec{t}[\vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}], \vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\ : A[\vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \vec{t}[\vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}], \vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \end{array}$$

And finally $A[\vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \vec{t}[\vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \equiv A[\vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}'/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]$ by Lemma A.5.

Case:

$$\frac{\Gamma, \Gamma_1'', x : A, \Gamma_2'' \vdash s'' : \mathbb{T} \text{ for } \mathcal{K} \in \text{locks}(\Gamma_2'')}{\Gamma, \Gamma_1'', x : A, \Gamma_2'' \vdash x[\vec{s}'/\mathbf{a}_{\mathcal{K}}] : A[\vec{s}'/\mathbf{a}_{\mathcal{K}}]}$$

Inductively $\Gamma, \Gamma', \Gamma_1''[\vec{t}/\mathbf{a}_{\mathcal{L}}], x : A[\vec{t}/\mathbf{a}_{\mathcal{L}}], \Gamma_2''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash s''[\vec{t}/\mathbf{a}_{\mathcal{L}}] : \mathbb{T}$, and we may form $\Gamma, \Gamma', \Gamma_1''[\vec{t}/\mathbf{a}_{\mathcal{L}}], x : A[\vec{t}/\mathbf{a}_{\mathcal{L}}], \Gamma_2''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash x[\vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] : A[\vec{t}/\mathbf{a}_{\mathcal{L}}][\vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]$. Finally, $A[\vec{t}/\mathbf{a}_{\mathcal{L}}][\vec{s}'[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \equiv A[\vec{s}'/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]$ by Lemma A.6.

Case:

$$\frac{\Gamma, \Gamma'' \vdash A \text{ type} \quad \Gamma, \Gamma'', x : A \vdash B \text{ type}}{\Gamma, \Gamma'' \vdash \prod_{(x:A)} B \text{ type}}$$

Inductively $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash A[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ type}$ and also, with the telescope Γ'' extended, $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}], x : A[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash B[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ type}$, and we may form $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash \prod_{(x:A[\vec{t}/\mathbf{a}_{\mathcal{L}}])} B[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ type}$.

Case:

$$\frac{\Gamma, \Gamma'', \mathbf{a}_{\mathcal{K}} \vdash A \text{ type}}{\Gamma, \Gamma'' \vdash \sqrt{\mathcal{K}} A \text{ type}}$$

Inductively $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}], \mathbf{a}_{\mathcal{K}} \vdash A[\vec{t}/\mathbf{a}_{\mathcal{L}}] \text{ type}$ and so we may form $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash \sqrt{\mathcal{K}}(A[\vec{t}/\mathbf{a}_{\mathcal{L}}]) \text{ type}$.

Case:

$$\frac{\Gamma, \Gamma'', \mathbf{a}_{\mathcal{K}} \vdash a : A}{\Gamma, \Gamma'' \vdash \mathbf{a}_{\mathcal{K}}.a : \sqrt{\mathcal{K}} A}$$

Similar: Inductively $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}], \mathbf{a}_{\mathcal{K}} \vdash a[\vec{t}/\mathbf{a}_{\mathcal{L}}] : A[\vec{t}/\mathbf{a}_{\mathcal{L}}]$ and so we may form $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash \mathbf{a}_{\mathcal{K}}.a[\vec{t}/\mathbf{a}_{\mathcal{L}}] : \sqrt{\mathcal{K}}(A[\vec{t}/\mathbf{a}_{\mathcal{L}}])$.

Case:

$$\frac{\Gamma, \Gamma'', i : \mathbb{T} \vdash r : \sqrt{\mathcal{K}} A}{\Gamma, \Gamma'' \vdash r(\gamma i.) : A[\gamma i./\mathbf{a}_{\mathcal{K}}] \text{ type}}$$

Inductively $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}], i : \mathbb{T} \vdash r[\vec{t}/\mathbf{a}_{\mathcal{L}}] : \sqrt{\mathcal{K}}(A)[\vec{t}/\mathbf{a}_{\mathcal{L}}]$. By definition $(\sqrt{\mathcal{K}} A)[\vec{t}/\mathbf{a}_{\mathcal{L}}] \equiv \sqrt{\mathcal{K}}(A[\vec{t}/\mathbf{a}_{\mathcal{L}}])$, and so reapplying the rule gives $\Gamma, \Gamma', \Gamma''[\vec{t}/\mathbf{a}_{\mathcal{L}}] \vdash (r[\vec{t}/\mathbf{a}_{\mathcal{L}}])(\gamma i.) : A[\vec{t}/\mathbf{a}_{\mathcal{L}}][\gamma i./\mathbf{a}_{\mathcal{K}}]$. Finally, $A[\vec{t}/\mathbf{a}_{\mathcal{L}}][\gamma i./\mathbf{a}_{\mathcal{K}}] \equiv A[\gamma i./\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]$ by Lemma A.7.

□

Proposition A.4. *The unit operation is admissible on context telescopes, types and terms.*

$$\text{UNIT-TELE} \frac{\Gamma, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}} \vdash \Gamma' \text{ tele}}{\Gamma \vdash \Gamma'[\gamma i./\mathbf{a}_{\mathcal{L}}] \text{ tele}} \quad \text{UNIT} \frac{\Gamma, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash a : A}{\Gamma, \Gamma'[\gamma i./\mathbf{a}_{\mathcal{L}}] \vdash a[\gamma i./\mathbf{a}_{\mathcal{L}}] : A[\gamma i./\mathbf{a}_{\mathcal{L}}]}$$

Proof. On telescopes:

Case:

$$\frac{\Gamma, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}} \vdash \Gamma' \text{ tele} \quad \Gamma, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash A \text{ type}}{\Gamma, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}} \vdash \Gamma', x : A \text{ tele}}$$

Inductively $\Gamma \vdash \Gamma'[\gamma i./\mathbf{a}_{\mathcal{L}}] \text{ tele}$ and $\Gamma, \Gamma'[\gamma i./\mathbf{a}_{\mathcal{L}}] \vdash A[\gamma i./\mathbf{a}_{\mathcal{L}}] \text{ type}$, and we may form $\Gamma \vdash \Gamma'[\gamma i./\mathbf{a}_{\mathcal{L}}], x : A[\gamma i./\mathbf{a}_{\mathcal{L}}] \text{ tele}$.

Case:

$$\frac{\Gamma, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}} \vdash \Gamma' \text{ tele}}{\Gamma, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}} \vdash \Gamma', \mathbf{a}_{\mathcal{K}} \text{ tele}}$$

Inductively $\Gamma \vdash \Gamma'[\gamma i./\mathbf{a}_{\mathcal{L}}] \text{ tele}$ so also $\Gamma \vdash \Gamma'[\gamma i./\mathbf{a}_{\mathcal{L}}], \mathbf{a}_{\mathcal{K}} \text{ tele}$.

On terms:

Case: When $x \neq i$ and x comes before i in the context:

$$\frac{\begin{array}{c} \Gamma_1, x : A, \Gamma_2, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash s : \mathbb{T} \text{ for } \mathcal{J} \in \text{locks}(\Gamma_2) \\ \Gamma_1, x : A, \Gamma_2, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash l : \mathbb{T} \\ \Gamma_1, x : A, \Gamma_2, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash s' : \mathbb{T} \text{ for } \mathcal{K} \in \text{locks}(\Gamma') \end{array}}{\Gamma_1, x : A, \Gamma_2, i : \mathbb{T}, \mathbf{a}_{\mathcal{L}}, \Gamma' \vdash x[\vec{s}/\mathbf{a}_{\mathcal{J}}, l/\mathbf{a}_{\mathcal{L}}, \vec{s}'/\mathbf{a}_{\mathcal{K}}] : A[\vec{s}/\mathbf{a}_{\mathcal{J}}, l/\mathbf{a}_{\mathcal{L}}, \vec{s}'/\mathbf{a}_{\mathcal{K}}]}$$

Inductively, $\Gamma_1, x : A, \Gamma_2, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash s[\gamma i. / \blacksquare_{\mathcal{L}}] : \mathbb{T}$ and $\Gamma_1, x : A, \Gamma_2, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash s'[\gamma i. / \blacksquare_{\mathcal{L}}] : \mathbb{T}$, so

$$\begin{aligned} \Gamma_1, x : A, \Gamma_2, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash x[\vec{s}[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{J}}, \vec{s}'[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{K}}] \\ : A[\vec{s}[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{J}}, \vec{s}'[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{K}}] \end{aligned}$$

Finally, $A[\vec{s}[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{J}}, \vec{s}'[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{K}}] \equiv A[\vec{s} / \mathfrak{Q}_{\mathcal{J}}, l / \mathfrak{Q}_{\mathcal{L}}, \vec{s}' / \mathfrak{Q}_{\mathcal{K}}][\gamma i. / \blacksquare_{\mathcal{L}}]$, by Lemma A.8.

Case: When $x \neq i$ and x comes after i in the context:

$$\frac{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma'_1, x : A, \Gamma'_2 \vdash s' : \mathbb{T} \text{ for } \mathcal{K} \in \text{locks}(\Gamma'_2)}{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma'_1, x : A, \Gamma'_2 \vdash x[\vec{s}' / \mathfrak{Q}_{\mathcal{K}}] : A[\vec{s}' / \mathfrak{Q}_{\mathcal{K}}]}$$

Inductively, $\Gamma, \Gamma'_1[\gamma i. / \blacksquare_{\mathcal{L}}], x : A[\gamma i. / \blacksquare_{\mathcal{L}}], \Gamma'_2[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash s'[\gamma i. / \blacksquare_{\mathcal{L}}] : \mathbb{T}$ and so $\Gamma, \Gamma'_1[\gamma i. / \blacksquare_{\mathcal{L}}], x : A[\gamma i. / \blacksquare_{\mathcal{L}}], \Gamma'_2[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash x[\vec{s}'[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{K}}] : A[\vec{s}'[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{K}}]$. Finally, by Lemma A.7, $A[\gamma i. / \blacksquare_{\mathcal{L}}][\vec{s}'[\gamma i. / \blacksquare_{\mathcal{L}}] / \mathfrak{Q}_{\mathcal{K}}] \equiv A[\vec{s}' / \mathfrak{Q}_{\mathcal{K}}][\gamma i. / \blacksquare_{\mathcal{L}}]$.

Case:

$$\frac{\begin{array}{c} \Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash l : \mathbb{T} \\ \Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash s' : \mathbb{T} \text{ for } \mathcal{K} \in \text{locks}(\Gamma') \end{array}}{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash i[l / \mathfrak{Q}_{\mathcal{L}}, \vec{s}' / \mathfrak{Q}_{\mathcal{K}}] : \mathbb{T}}$$

Inductively, $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash l[\gamma i. / \blacksquare_{\mathcal{L}}] : \mathbb{T}$ and we are done.

Case:

$$\frac{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash A \text{ type} \quad \Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma', x : A \vdash B \text{ type}}{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash \prod_{(x:A)} B \text{ type}}$$

Inductively $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash A[\gamma i. / \blacksquare_{\mathcal{L}}] \text{ type}$ and $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}], x : A[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash B[\gamma i. / \blacksquare_{\mathcal{L}}] \text{ type}$, so we may form $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash \prod_{(x:A[\gamma i. / \blacksquare_{\mathcal{L}}])} B[\gamma i. / \blacksquare_{\mathcal{L}}] \text{ type}$.

Case:

$$\frac{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma', \blacksquare_{\mathcal{K}} \vdash A \text{ type}}{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash \sqrt{\mathcal{K}} A \text{ type}}$$

Inductively $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}], \blacksquare_{\mathcal{K}} \vdash A[\gamma i. / \blacksquare_{\mathcal{L}}] \text{ type}$ and so we may form $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash \sqrt{\mathcal{K}}(A[\gamma i. / \blacksquare_{\mathcal{L}}]) \text{ type}$.

Case:

$$\frac{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma', \blacksquare_{\mathcal{K}} \vdash a : A}{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash \blacksquare_{\mathcal{K}}.a : \sqrt{\mathcal{K}} A}$$

Similar: Inductively $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}], \blacksquare_{\mathcal{K}} \vdash a[\gamma i. / \blacksquare_{\mathcal{L}}] : A[\gamma i. / \blacksquare_{\mathcal{L}}]$ and so we may form $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash \blacksquare_{\mathcal{K}}.a[\gamma i. / \blacksquare_{\mathcal{L}}] : \sqrt{\mathcal{K}}(A[\gamma i. / \blacksquare_{\mathcal{L}}]) \text{ type}$.

Case:

$$\frac{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma', k : \mathbb{T} \vdash r : \sqrt{\mathcal{K}} A}{\Gamma, i : \mathbb{T}, \blacksquare_{\mathcal{L}}, \Gamma' \vdash r(\gamma k.) : A[\gamma k. / \blacksquare_{\mathcal{K}}] \text{ type}}$$

Inductively $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}], k : \mathbb{T} \vdash r[\gamma i. / \blacksquare_{\mathcal{L}}] : \sqrt{\mathcal{K}}(A)[\gamma i. / \blacksquare_{\mathcal{L}}]$. By definition $\sqrt{\mathcal{K}}(A)[\gamma i. / \blacksquare_{\mathcal{L}}] \equiv \sqrt{\mathcal{K}}(A[\gamma i. / \blacksquare_{\mathcal{L}}])$, and so reapplying the rule gives $\Gamma, \Gamma'[\gamma i. / \blacksquare_{\mathcal{L}}] \vdash (r[\gamma i. / \blacksquare_{\mathcal{L}}])(\gamma k.) : A[\gamma i. / \blacksquare_{\mathcal{L}}][\gamma k. / \blacksquare_{\mathcal{K}}]$. Finally, $A[\gamma i. / \blacksquare_{\mathcal{L}}][\gamma k. / \blacksquare_{\mathcal{K}}] \equiv A[\gamma k. / \blacksquare_{\mathcal{K}}][\gamma i. / \blacksquare_{\mathcal{L}}]$ by Lemma A.9.

□

Now, the proofs of the equations used in the previous two propositions, all of which hold on the level of well-scoped raw terms. For each equation, the variable rule is the only case where something interesting happens.

Lemma A.5.

$$\begin{array}{c}
G_1 \vdash \mathcal{J} \text{ rawterm} \\
G_1, G_2, G'' \vdash s \text{ rawterm for } \mathcal{J} \in \text{locks}(G_2) \\
G_1, G_2, G' \vdash t \text{ rawterm for } \mathcal{L} \in \text{locks}(G') \\
G_1, G_2, G'' \vdash s'' \text{ rawterm for } \mathcal{K} \in \text{locks}(G'') \\
\hline
G_1, G_2, G', G'' \vdash \mathcal{J}[\vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \equiv \mathcal{J}[\vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \vec{t}[\vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]
\end{array}$$

Proof. Suppose $G_1 \equiv G_{11}, x, G_{12}$ and the variable usage is

$$\frac{G_{11}, x, G_{12} \vdash m \text{ rawterm for } \mathcal{M} \in \text{locks}(G_{12})}{G_{11}, x, G_{12} \vdash x[\vec{m}/\mathbf{a}_{\mathcal{M}}] \text{ rawterm}}$$

Then

$$\begin{aligned}
& x[\vec{m}/\mathbf{a}_{\mathcal{M}}][\vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \\
& \equiv x[\vec{m}[\vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}''/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{M}}, \vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \\
& \equiv x[\vec{m}[\vec{s}/\mathbf{a}_{\mathcal{J}}, \vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{M}}, \vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \vec{t}[\vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
& \equiv x[\vec{m}[\vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \vec{t}[\vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{M}}, \\
& \quad \vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \\
& \quad \vec{t}[\vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \\
& \quad \vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
& \equiv x[\vec{m}/\mathbf{a}_{\mathcal{M}}][\vec{s}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{J}}, \vec{t}[\vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]
\end{aligned}$$

by inductive hypothesis on m . □

Lemma A.6.

$$\begin{array}{c}
G, G''_1 \vdash \mathcal{J} \text{ rawterm} \\
G, G' \vdash t \text{ rawterm for } \mathcal{L} \in \text{locks}(G') \\
G, G''_1, G''_2 \vdash s'' \text{ rawterm for } \mathcal{K} \in \text{locks}(G''_2) \\
\hline
G, G', G''_1, G''_2 \vdash \mathcal{J}[\vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \equiv \mathcal{J}[\vec{t}/\mathbf{a}_{\mathcal{L}}][\vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]
\end{array}$$

Proof. There are two places where the variable may lie.

Case: If x lies in G , so $G \equiv G_1, x, G_2$:

$$\frac{G_1, x, G_2, G''_1 \vdash m \text{ rawterm for } \mathcal{M} \in \text{locks}(G_2) \quad G_1, x, G_2, G''_1 \vdash n \text{ rawterm for } \mathcal{N} \in \text{locks}(G''_1)}{G_1, x, G_2, G''_1 \vdash x[\vec{m}/\mathbf{a}_{\mathcal{M}}, \vec{n}/\mathbf{a}_{\mathcal{N}}] \text{ rawterm}}$$

And then

$$\begin{aligned}
& x[\vec{m}/\mathbf{a}_{\mathcal{M}}, \vec{n}/\mathbf{a}_{\mathcal{N}}][\vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \\
& \equiv x[\vec{m}[\vec{s}''/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{M}}, \vec{n}[\vec{s}''/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}, \vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \\
& \equiv x[\vec{m}[\vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{M}}, \\
& \quad \vec{t}[\vec{n}[\vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}, \vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \\
& \quad \vec{n}[\vec{s}''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}, \\
& \quad \vec{s}''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]
\end{aligned}$$

Each of these keys attached to x can be rewritten by induction:

$$\begin{aligned}
\vec{m}[s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] &\equiv \vec{m}[\vec{t}/\mathbf{a}_{\mathcal{K}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
\vec{t}[\vec{n}[s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}, s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] &\equiv \vec{t}[\vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}, s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
\vec{n}[s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] &\equiv \vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]
\end{aligned}$$

And so, continuing the chain of equations:

$$\begin{aligned}
&x[\vec{m}/\mathbf{a}_{\mathcal{M}}, \vec{n}/\mathbf{a}_{\mathcal{N}}][s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \\
&\equiv \dots \\
&\equiv x[\vec{m}[\vec{t}/\mathbf{a}_{\mathcal{L}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{M}}, \\
&\quad \vec{t}[\vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}, s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \\
&\quad \vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}, \\
&\quad s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
&\equiv x[\vec{m}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{M}}, \vec{t}[\vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}]/\mathbf{a}_{\mathcal{L}}, \vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
&\equiv x[\vec{m}/\mathbf{a}_{\mathcal{M}}, \vec{n}/\mathbf{a}_{\mathcal{N}}][\vec{t}/\mathbf{a}_{\mathcal{L}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]
\end{aligned}$$

Case: If x lies in G''_1 , so $G''_1 \equiv G''_{11}, x, G''_{12}$:

$$\frac{G, G''_{11}, x, G''_{12} \vdash n \text{ rawterm for } \mathcal{N} \in \text{locks}(G''_{12})}{G, G''_{11}, x, G''_{12} \vdash x[\vec{n}/\mathbf{a}_{\mathcal{N}}] \text{ rawterm}}$$

The reasoning is essentially the same:

$$\begin{aligned}
&x[\vec{n}/\mathbf{a}_{\mathcal{N}}][s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \\
&\equiv x[\vec{n}[s''/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}, s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}] \\
&\equiv x[\vec{t}[\vec{n}[s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}, s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \vec{n}[s''/\mathbf{a}_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}, s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
&\equiv x[\vec{t}[\vec{n}[s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}, s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \\
&\quad \vec{n}[s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}, \\
&\quad s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
&\equiv x[\vec{t}[\vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}]/\mathbf{a}_{\mathcal{L}}, \vec{n}[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}] \\
&\equiv x[\vec{n}/\mathbf{a}_{\mathcal{N}}][\vec{t}/\mathbf{a}_{\mathcal{L}}][s''[\vec{t}/\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]
\end{aligned}$$

again by induction in the middle step.

□

Lemma A.7. *The unit and counit commute, depending on the relative position in the context the two operations are being applied to.*

$$\frac{G, G''_1, i, \blacksquare_{\mathcal{K}}, G''_2 \vdash \mathcal{J} \text{ rawterm} \quad G, G' \vdash t \text{ rawterm for } \mathcal{L} \in \text{locks}(G')}{G, G', G''_1, G''_2 \vdash \mathcal{J}[\vec{t}/\mathbf{a}_{\mathcal{L}}][\gamma i./\blacksquare_{\mathcal{K}}] \equiv \mathcal{J}[\gamma i./\blacksquare_{\mathcal{K}}][\vec{t}/\mathbf{a}_{\mathcal{L}}]}$$

$$\frac{G_1, i, \blacksquare_{\mathcal{L}}, G_2, G'' \vdash \mathcal{J} \text{ rawterm} \quad G_1, i, \blacksquare_{\mathcal{L}}, G_2, G' \vdash s' \text{ rawterm for } \mathcal{K} \in \text{locks}(G')}{G_1, i, \blacksquare_{\mathcal{L}}, G_2, G', G'' \vdash \mathcal{J}[s'/\mathbf{a}_{\mathcal{K}}][\gamma i./\blacksquare_{\mathcal{L}}] \equiv \mathcal{J}[\gamma i./\blacksquare_{\mathcal{L}}][s'[\gamma i./\blacksquare_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}]}$$

Proof. For the first, there are several places a variable could be, but the case of interest is the variable rule for i :

$$\frac{G, G_1'', i, \blacksquare_{\mathcal{K}}, G_2'' \vdash k \text{ rawterm for } \mathcal{K} \quad G, G_1'', i, \blacksquare_{\mathcal{K}}, G_2'' \vdash o \text{ rawterm for } \mathcal{O} \in \text{locks}(G_2'')}{G, G_1'', i, \blacksquare_{\mathcal{K}}, G_2'' \vdash i[k/\mathbf{q}_{\mathcal{K}}, \vec{o}/\mathbf{q}_{\mathcal{O}}] \text{ rawterm}}$$

in which case

$$\begin{aligned} & i[k/\mathbf{q}_{\mathcal{K}}, \vec{o}/\mathbf{q}_{\mathcal{O}}][\vec{t}/\mathbf{q}_{\mathcal{L}}][\gamma i./\blacksquare_{\mathcal{K}}] \\ & \equiv i[t[k[\vec{t}/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}, \vec{o}[\vec{t}/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{O}}]/\mathbf{q}_{\mathcal{L}}, k[\vec{t}/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}, \vec{o}[\vec{t}/\mathbf{q}_{\mathcal{L}}]/\mathbf{q}_{\mathcal{O}}][\gamma i./\blacksquare_{\mathcal{K}}] \\ & \equiv k[\vec{t}/\mathbf{q}_{\mathcal{L}}][\gamma i./\blacksquare_{\mathcal{K}}] \\ & \equiv k[\gamma i./\blacksquare_{\mathcal{K}}][\vec{t}/\mathbf{q}_{\mathcal{L}}] \\ & \equiv i[k/\mathbf{q}_{\mathcal{K}}, \vec{o}/\mathbf{q}_{\mathcal{O}}][\gamma i./\blacksquare_{\mathcal{K}}][\vec{t}/\mathbf{q}_{\mathcal{L}}] \end{aligned}$$

For the second, again the variable rule for i is the interesting case:

$$\frac{G_1, i, \blacksquare_{\mathcal{L}}, G_2, G'' \vdash l \text{ rawterm for } \mathcal{L} \quad G_1, i, \blacksquare_{\mathcal{L}}, G_2, G'' \vdash m \text{ rawterm for } \mathcal{M} \in \text{locks}(G_2) \quad G_1, i, \blacksquare_{\mathcal{L}}, G_2, G'' \vdash n \text{ rawterm for } \mathcal{N} \in \text{locks}(G'')}{G_1, i, \blacksquare_{\mathcal{L}}, G_2, G'' \vdash i[l/\mathbf{q}_{\mathcal{L}}, \vec{m}/\mathbf{q}_{\mathcal{M}}, \vec{n}/\mathbf{q}_{\mathcal{N}}] \text{ rawterm}}$$

in which case

$$\begin{aligned} & i[l/\mathbf{q}_{\mathcal{L}}, \vec{m}/\mathbf{q}_{\mathcal{M}}, \vec{n}/\mathbf{q}_{\mathcal{N}}][\vec{s'}/\mathbf{q}_{\mathcal{K}}][\gamma i./\blacksquare_{\mathcal{L}}] \\ & \equiv i[l[\vec{s'}/\mathbf{q}_{\mathcal{K}}]/\mathbf{q}_{\mathcal{L}}, \vec{m}[\vec{s'}/\mathbf{q}_{\mathcal{K}}]/\mathbf{q}_{\mathcal{M}}, \vec{s'}[\vec{n}[\vec{s'}/\mathbf{q}_{\mathcal{K}}]/\mathbf{q}_{\mathcal{N}}]/\mathbf{q}_{\mathcal{K}}, \vec{n}[\vec{s'}/\mathbf{q}_{\mathcal{K}}]/\mathbf{q}_{\mathcal{N}}][\gamma i./\blacksquare_{\mathcal{L}}] \\ & \equiv l[\vec{s'}/\mathbf{q}_{\mathcal{K}}][\gamma i./\blacksquare_{\mathcal{L}}] \\ & \equiv l[\gamma i./\blacksquare_{\mathcal{L}}][\vec{s'}[\gamma i./\blacksquare_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}] \\ & \equiv i[l/\mathbf{q}_{\mathcal{L}}, \vec{m}/\mathbf{q}_{\mathcal{M}}, \vec{n}/\mathbf{q}_{\mathcal{N}}][\gamma i./\blacksquare_{\mathcal{L}}][\vec{s'}[\gamma i./\blacksquare_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}] \end{aligned}$$

□

Lemma A.8.

$$\frac{G_1 \vdash \mathcal{J} \text{ rawterm} \quad G_1, G_2, i, \blacksquare_{\mathcal{L}}, G' \vdash s \text{ rawterm for } \mathcal{M} \in \text{locks}(G_2) \quad G_1, G_2, i, \blacksquare_{\mathcal{L}}, G' \vdash l \text{ rawterm for } \mathcal{L} \quad G_1, G_2, i, \blacksquare_{\mathcal{L}}, G' \vdash s' \text{ rawterm for } \mathcal{N} \in \text{locks}(G')}{G_1, G_2, G' \vdash \mathcal{J}[\vec{s'}/\mathbf{q}_{\mathcal{J}}, l/\mathbf{q}_{\mathcal{L}}, \vec{s'}/\mathbf{q}_{\mathcal{K}}][\gamma i./\blacksquare_{\mathcal{L}}] \equiv \mathcal{J}[\vec{s'}[\gamma i./\blacksquare_{\mathcal{L}}]/\mathbf{q}_{\mathcal{J}}, \vec{s'}[\gamma i./\blacksquare_{\mathcal{L}}]/\mathbf{q}_{\mathcal{K}}]}$$

Proof. Straightforward induction, key is that i does not occur in $\mathcal{J}$. □

Lemma A.9.

$$\frac{G, i, \blacksquare_{\mathcal{L}}, G', k, \blacksquare_{\mathcal{K}}, G'' \vdash \mathcal{J} \text{ rawterm}}{G, G', G'' \vdash \mathcal{J}[\gamma i./\blacksquare_{\mathcal{L}}][\gamma k./\blacksquare_{\mathcal{K}}] \equiv \mathcal{J}[\gamma k./\blacksquare_{\mathcal{K}}][\gamma i./\blacksquare_{\mathcal{L}}]}$$

Proof. The interesting cases are, of course, the variable rules for i and k :

Case:

$$\begin{array}{c}
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash t \text{ rawterm for } \mathcal{L} \\
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash m \text{ rawterm for } \mathcal{M} \in \text{locks}(G') \\
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash s \text{ rawterm for } \mathcal{K} \\
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash n \text{ rawterm for } \mathcal{N} \in \text{locks}(G'') \\
\hline
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash i[t/\mathbf{a}_{\mathcal{L}}, \vec{m}/\mathbf{a}_{\mathcal{M}}, s/\mathbf{a}_{\mathcal{K}}, \vec{n}/\mathbf{a}_{\mathcal{N}}] \text{ rawterm}
\end{array}$$

Then:

$$\begin{aligned}
& i[t/\mathbf{a}_{\mathcal{L}}, \vec{m}/\mathbf{a}_{\mathcal{M}}, s/\mathbf{a}_{\mathcal{K}}, \vec{n}/\mathbf{a}_{\mathcal{N}}][\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma k./\mathbf{a}_{\mathcal{K}}] \\
& \equiv t[\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma k./\mathbf{a}_{\mathcal{K}}] \\
& \equiv t[\gamma k./\mathbf{a}_{\mathcal{K}}][\gamma i./\mathbf{a}_{\mathcal{L}}] \\
& \equiv i[t[\gamma k./\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{L}}, \vec{m}[\gamma k./\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{M}}, \vec{n}[\gamma k./\mathbf{a}_{\mathcal{K}}]/\mathbf{a}_{\mathcal{N}}][\gamma i./\mathbf{a}_{\mathcal{L}}] \\
& \equiv i[t/\mathbf{a}_{\mathcal{L}}, \vec{m}/\mathbf{a}_{\mathcal{M}}, s/\mathbf{a}_{\mathcal{K}}, \vec{n}/\mathbf{a}_{\mathcal{N}}][\gamma k./\mathbf{a}_{\mathcal{K}}][\gamma i./\mathbf{a}_{\mathcal{L}}]
\end{aligned}$$

Case:

$$\begin{array}{c}
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash s \text{ rawterm for } \mathcal{K} \\
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash n \text{ rawterm for } \mathcal{N} \in \text{locks}(G'') \\
\hline
G, i, \mathbf{a}_{\mathcal{L}}, G', k, \mathbf{a}_{\mathcal{K}}, G'' \vdash k[s/\mathbf{a}_{\mathcal{K}}, \vec{n}/\mathbf{a}_{\mathcal{N}}] \text{ rawterm}
\end{array}$$

Then:

$$\begin{aligned}
& k[s/\mathbf{a}_{\mathcal{K}}, \vec{n}/\mathbf{a}_{\mathcal{N}}][\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma k./\mathbf{a}_{\mathcal{K}}] \\
& \equiv k[s[\gamma i./\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{K}}, \vec{n}[\gamma i./\mathbf{a}_{\mathcal{L}}]/\mathbf{a}_{\mathcal{N}}][\gamma k./\mathbf{a}_{\mathcal{K}}] \\
& \equiv s[\gamma i./\mathbf{a}_{\mathcal{L}}][\gamma k./\mathbf{a}_{\mathcal{K}}] \\
& \equiv s[\gamma k./\mathbf{a}_{\mathcal{K}}][\gamma i./\mathbf{a}_{\mathcal{L}}] \\
& \equiv k[s/\mathbf{a}_{\mathcal{K}}, \vec{n}/\mathbf{a}_{\mathcal{N}}][\gamma k./\mathbf{a}_{\mathcal{K}}][\gamma i./\mathbf{a}_{\mathcal{L}}]
\end{aligned}$$

□