Semi-classical saddles of three-dimensional gravity via holography

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We find out the complex geometries corresponding to the semi-classical saddles of three-dimensional quantum gravity by making use of the known results of dual conformal field theory (CFT), which is effectively given by Liouville field theory. We examine both the cases with positive and negative cosmological constants. We determine the set of semi-classical saddles to choose from the homotopy argument in the Chern-Simons formulation combined with CFT results and provide strong supports from the mini-superspace approach to the quantum gravity. For the case of positive cosmological constant, partial results were already obtained in our previous works, and they are consistent with the current ones. For the case of negative cosmological constant, we identify the geometry corresponding a semi-classical saddle with three-dimensional Euclidean anti-de Sitter space dressed with imaginary radius three-dimensional spheres. The geometry is generically unphysical, but we argue that the fact itself does not lead to any problems.

I. INTRODUCTION

One of the important issues in the path integral formulation of quantum gravity is to determine which geometries should be integrated over. The path integral may be written as a sum over non-perturbative saddles dressed with perturbative corrections. This implies that the problem can be rephrased as identifying the geometries corresponding to the non-perturbative saddles chosen. We attack this problem in a simple setup, i.e., three-dimensional pure gravity with positive or negative cosmological constant by applying the holographic duality [1]. We consider the path integral over geometries analytically continued to take complex values for the following reasons. Firstly, the path integral of quantum gravity over real geometries would give divergent results and the analytic continuation provides a way to regularize the divergence. Secondly, the holography for the case with positive cosmological constant may be derived via an analytic continuation from the case with negative cosmological constant [2], see also [3, 4].

Gravity theory in three-dimensional anti-de Sitter (AdS_3) space is supposed to be dual to a two-dimensional conformal field theory (CFT_2). We are interested in the semi-classical saddles with small Newton constant, whose universal features are supposed to be captured by Liouville field theory with large central charge. We can examine gravity theory on three-dimensional de Sitter (dS_3) space from dual CFT_2 as well. In previous works [5–8], it was also proposed that classical gravity on dS_3 can be examined by Liouville field theory with large but imaginary central charge as the result of analytic continuation, see also [4, 9]. In fact, we have classified semi-classical saddles of gravity theory on dS_3 black holes and determined possible saddles by comparing the exact expressions of dual CFT at the imaginary large central charge [10, 11]. As an evidence, we have computed Gibbons-Hawking entropy [12–15] from the both sides and find agreements.

In this letter, we continue to investigate these semi-classical saddles in more systematic ways. The possible semi-classical saddles in the Chern-Simons formulation of three-dimensional gravity can be classified by the homotopy group as discussed in [16]. We construct the geometry corresponding to the semi-classical saddles in the quantum gravity. We then provide its strong supports by carefully examining the mini-superspace model of quantum gravity, see, e.g., [17–20] and a review [21] for recent developments. For the case of positive cosmological constant, we reproduce the previous result in [10, 11]. For the case of negative cosmological constant, we claim that the semi-classical saddles are given by Euclidean AdS_3 attached with imaginary radius three-spheres ($\mathbb{S}^3$'s). Such geometries are seemingly unphysical but we will argue in the following that these are indeed relevant ones.

II. NON-PERTURBATIVE SADDLES IN LIOUVILLE THEORY

We examine the semi-classical saddles in three-dimensional gravity by making use of the exact results of Liouville theory obtained in [22, 23]. The CFT consists of a bosonic field ϕ with background charge $Q = b + b^{-1}$. Its central charge is related to the parameter b as $c = 1 + 6Q^2$ and the large central charge may be realized by the limit $b \to 0$. We are interested in the correlation functions of vertex operators of the form $\exp(2\alpha\phi)$, where α behaves as $\alpha = \eta/b$ with fixed η. These operators are usually referred as heavy. At this limit, the insertions of vertex operators can be regarded as a part of action, where the equation of motion is reduced to

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\phi_c = 2\lambda e^{\phi_c} - 2\pi \sum_i \eta_i \delta^2(z - z_i)\,. \qquad (1)$$

Here $z, \bar{z}$ are complex coordinates and λe^{ϕ_c} with $\phi_c = 2b\phi$ comes from the Liouville interaction. We can easily see that

$$\phi_{c(n)} = \phi_{c(0)} + 2\pi i n \tag{2}$$

with integer n is a solution to the equation of motion once $\phi_{c(0)}$ is one of its solutions. This means that there are saddle points labeled by an integer n.

Several exact results are available in Liouville theory [22, 23]. From them, we can obtain non-perturbative saddles of correlation functions by taking the limit mentioned above. For instance, the two point function behaves as [16]

$$\begin{aligned}&\langle V_\alpha(z_1)V_\alpha(z_2)\rangle \\ &\sim \delta(0)|z_{12}|^{-4\eta(1-\eta)/b^2}\left(e^{-\pi i(1-2\eta)/b^2} - e^{\pi i(1-2\eta)/b^2}\right)^{\pm 1} \\ &\quad\times \exp\left\{-\frac{2}{b^2}[(1-2\eta)\ln(1-2\eta)-(1-2\eta)]\right\},\end{aligned} \tag{3}$$

where $+1$ for $\operatorname{Re} b^{-2} < 0$ and -1 for $\operatorname{Re} b^{-2} > 0$.

We would like to interpret the expression in terms of dual gravity theory. We denote G as the Newton constant and Λ as the cosmological constant. For a while, we consider the simplest case with $\eta = 0$. We set $\Lambda = \ell_{\mathrm{dS}}^{-2}$ or $\Lambda = -\ell_{\mathrm{AdS}}^{-2}$, where ℓ_{dS} and ℓ_{AdS} are the radii of dS_3 and AdS_3, respectively. Their relations to the central charges are:

$$c = -ic^{(g)} = -i\frac{3\ell_{\mathrm{dS}}}{2G}\,, \quad c = \frac{3\ell_{\mathrm{AdS}}}{2G}\,. \tag{4}$$

Here we set $c^{(g)}$ as a real value [5, 6, 10]. For $\Lambda = \ell_{\mathrm{dS}}^{-2}$, we find $b^{-2} = -ic^{(g)}/6 - 13/6 + \mathcal{O}((c^{(g)})^{-1})$, which leads to $\operatorname{Re} b^{-2} < 0$. The two point function with $\eta = 0$ is related to the wave functional of universe for de Sitter holography [2], which behaves as

$$\Psi \propto e^{\frac{\pi c^{(g)}}{6}} - e^{-\frac{\pi c^{(g)}}{6}}\,, \tag{5}$$

and is given by a sum of two saddles with $n = -1, 0$ in our convention. For $\Lambda = -\ell_{\mathrm{AdS}}^{-2}$, we find $\operatorname{Re} b^{-2} \sim c/6 > 0$. AdS/CFT correspondence relate the CFT correlation function and gravity partition function as

$$\mathcal{Z} \propto e^{\frac{\pi i c}{6}} \sum_{n=0}^{\infty} e^{\frac{2n\pi i c}{6}}\,. \tag{6}$$

Here we have introduced a small cut regulator ϵ as $\ell_{\mathrm{AdS}} + i\epsilon$ with $\epsilon \to +0$. The partition function is thus given by a sum of saddles with label $n = 0, 1, 2, \ldots$.

III. SEMI-CLASSICAL SADDLES IN THREE-DIMENSIONAL GRAVITY

We examine three-dimensional gravity in the presence of cosmological constant. We use the action

$$I = I_{\mathrm{EH}} + I_{\mathrm{GH}} + I_{\mathrm{CT}}\,. \tag{7}$$

The Einstein-Hilbert action and the Gibbons-Hawking boundary terms are given by

$$\begin{aligned}I_{\mathrm{EH}} &= -\frac{1}{16\pi G}\int \sqrt{g}(R - 2\Lambda)\,, \\ I_{\mathrm{GH}} + I_{\mathrm{CT}} &= \frac{1}{8\pi G}\int \sqrt{h}(K + \sqrt{-\Lambda})\,,\end{aligned} \tag{8}$$

where K is the extrinsic curvature and I_{CT} includes a counter term. In the first order formulation, it can be rewritten in terms of Chern-Simons action as [24, 25]

$$\begin{aligned}I_{\mathrm{EH}} &= kI_{\mathrm{CS}}[A] - kI_{\mathrm{CS}}[\tilde{A}]\,, \\ I_{\mathrm{CS}}[A] &= \frac{1}{4\pi}\mathrm{tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right]\end{aligned} \tag{9}$$

up to boundary contributions. The Chern-Simons level is related to the gravitational parameters as $k = -i\kappa = -i\ell_{\mathrm{dS}}/(4G)$ or $k = \ell_{\mathrm{AdS}}/(4G)$.

We first consider the case with $\Lambda = \ell_{\mathrm{dS}}^{-2}$. As discussed in [26], we may consider the ansatz for the metric as

$$ds^2 = \ell_{\mathrm{dS}}^2\left[\left(\frac{d\theta(u)}{du}\right)^2 du^2 + \cos^2\theta(u)d\Sigma^2\right]. \tag{10}$$

We set $d\Sigma^2$ as the metric of $\mathbb{S}^2$ for a while and $\theta(u)$ as a holomorphic function of u. We are interested in the universe starting from nothing and approaching to dS_3. We thus set $\theta = (n+1/2)\pi i$ at $u = 0$ and $\theta = iu$ for $u \to \infty$. This leads to a family of geometry labeled by n. In order to make the geometry as a solution of Einstein equation, we may set $\theta = (n+1/2)\pi i(1-u)$ for $0 \le u \le 1$ and $\theta = i(u-1)$ for $1 < u$. The wave functional of universe is given by

$$\Psi[h] = \int \mathcal{D}g \exp(-I_{\mathrm{EH}} - I_{\mathrm{GH}} - I_{\mathrm{CT}})\,, \tag{11}$$

where we have integrated over the metric g such as to satisfy the boundary condition $g = h$ at the future infinity realized by $u = u_0 \gg 1$. We set h as the metric of $\mathbb{S}^2$ from the ansatz (10). The wave functional is of the form

$$\Psi[h] \sim \sum_n \exp(S_{\mathrm{GH}}^{(n)}/2 + i\mathcal{I}^{(n)})\,, \tag{12}$$

where the sum is over the family of geometries. Here $S_{\mathrm{GH}}^{(n)}$ is real and comes from the Euclidean geometry realized for $0 \le u \le 1$. On the other hand, the phase factor $\exp(i\mathcal{I}^{(n)})$ comes from the Lorentzian region realized for $1 < u$. The quantity S_{GH} is known as Gibbons-Hawking entropy [12–15] and the contribution from the geometry labeled by n is

$$S_{\mathrm{GH}} = \frac{(2n+1)\pi\ell_{\mathrm{dS}}}{2G}\,. \tag{13}$$

The CFT result can be reproduced when the geometry is summed over $n = -1, 0$.

The label n can be interpreted as a topological number in the Chern-Simons formulation. As mentioned above, the contribution to the Gibbons-Hawking entropy comes from the Euclidean geometry. Therefore, we can set $A, \tilde{A} \in \mathfrak{su}(2)$, see, e.g., [27] and appendix A of [11]. The Chern-Simons action is invariant under a small gauge transformation. However, for generic level k, a large gauge transformation is not a symmetry of the action anymore. A large gauge transformation generates gauge configuration where the value of the action has extra integer contribution as $I_{\rm CS} \to I_{\rm CS} + 2\pi i \mathbb{Z}$. In fact, the Chern-Simons action counts how many times the gauge configuration wraps $\mathbb{S}^3$ (i.e., $\pi_3(\mathbb{S}^3) = \mathbb{Z}$, see [6, 28] for examples). We can see that each geometry labeled by integer n leads to the same Chern-Simons action. Therefore, these geometries can be regarded as representatives of the gauge configurations which are not large gauge equivalent with each other.

We then move to the case with $\Lambda = -\ell_{\rm AdS}^{-2}$. Let us assume the ansatz

$$ds^2 = \ell_{\rm AdS}^2 \left[\left(\frac{d\theta(u)}{du} \right)^2 du^2 + \sinh^2 \theta(u) d\Sigma^2 \right] . \quad (14)$$

As before, we set $d\Sigma^2$ as the metric of $\mathbb{S}^2$ and $\theta(u)$ as a holomorphic function of u. We consider the geometry which approaches to Euclidean AdS$_3$ as $u \to \infty$ and truncates at $u = 0$. Thus we assign $\theta = n\pi i$ with integer n at $u = 0$ and $\theta = u$ for $u \to \infty$. In order to make the geometry as a standard solution of Einstein equation, we may set $\theta = n\pi i(1-u)$ for $0 \le u \le 1$ and $\theta = (u-1)$ for $1 < u$. For $1 < u$, the geometry is Euclidean AdS$_3$, but for $0 \le u \le 1$, the geometry becomes $\mathbb{S}^3$ with an imaginary radius or with three Lorentzian time directions. Thus, we may conclude that the geometry labeled by non-zero n is unphysical.

As mentioned above, the Chern-Simons action counts how many times gauge configuration representing the geometry wraps $\mathbb{S}^3$. We can describe each of Lorentzian dS$_3$, Euclidean dS$_3$, Euclidean AdS$_3$ and Lorentzian AdS$_3$ by a hypersurface in $\mathbb{R}^4$ with a flat metric as

$$\epsilon_0 (X^0)^2 + \epsilon_1 (X^1)^2 + \epsilon_2 (X^2)^2 + \epsilon_3 (X^3)^2 = \ell_{\rm dS}^2 . \quad (15)$$

We start from Lorentzian dS$_3$ with $-\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. We glued this geometry to Euclidean dS$_3$ (or $\mathbb{S}^3$) with replacing $\epsilon_0 = -1$ by $\epsilon_0 = 1$ at $X_0 = 0$. The Chern-Simons action equals one for the gauge configuration corresponding to $\mathbb{S}^3$. If we wrap n times $\mathbb{S}^3$, then the Chern-Simons action becomes $2\pi i n$.

Let us move to the case with AdS$_3$, where we start from Euclidean AdS$_3$ and we may analytically continue the Euclidean geometry to Lorentzian one. We may describe Euclidean AdS$_3$ by a hypersurface

$$\epsilon_0 (X^0)^2 + \epsilon_1 (X^1)^2 + \epsilon_2 (X^2)^2 + \epsilon_3 (X^3)^2 = -\ell_{\rm AdS}^2 . \quad (16)$$

with $\epsilon_0 = \epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$. As before, we may perform a Wick rotation by replacing $\epsilon_0 = 1$ by $\epsilon_0 = -1$, which leads to the Lorentzian AdS$_3$. The geometry cannot have non-trivial π_3, thus it does not lead to a non-trivial topological number. In order to have a non-trivial quantity, we may perform a different Wick rotation by replacing $\epsilon_3 = -1$ by $\epsilon_0 = 1$. The geometry is unphysical as it defines $\mathbb{S}^3$ with imaginary radius $i\ell_{\rm AdS}$. However, in terms of Chern-Simons gauge theory, there is no trouble to construct such gauge configuration and indeed leads to non-trivial Chern-Simons action with the value $2\pi i n$.

IV. MINI-SUPERSPACE APPROACH TO QUANTUM GRAVITY

As found above, the semi-classical saddles for $\Lambda = \ell_{\rm dS}^{-2}$ are sensible and indeed they are used for no-boundary proposal by Hartle and Hawking. However, the semi-classical saddles for $\Lambda = -\ell_{\rm AdS}^{-2}$ may be unphysical. Here we show that the mini-superspace approach to the quantum gravity leads to the same conclusion.

In order to confirm the validity of the approach, we first reproduce the previous result in [10, 11] for $\Lambda = \ell_{\rm dS}^{-2}$. We use the ansatz for metric as

$$ds^2 = \ell_{\rm dS}^2 \left[N(\tau)^2 d\tau^2 + a(\tau)^2 d\Sigma^2 \right] , \quad (17)$$

which slightly generalize (10). Without loss of generality, we can set $0 \le \tau \le 1$. We would like to evaluate the path integral (11). The gauge redundancy allows us to fix $N(\tau)$ to be constant N, the path integral can be reduced to [29]

$$\Psi = \int_{\mathcal{C}} dN \int \mathcal{D}a(\tau) e^{-I[a;N] - I_{\rm CT}} \quad (18)$$

with

$$I[a;N] = -\frac{\ell_{\rm dS}}{2G} \int_0^1 d\tau \, N \left[\frac{1}{N^2} \left(\frac{da}{d\tau} \right)^2 - a^2 + 1 \right] . \quad (19)$$

The contour over N is denoted by $\mathcal{C}$. Here we set the boundary conditions

$$a(0) = 0 , \quad a(1) = a_1 \quad (20)$$

and $I_{\rm CT}$ cancels the divergence proportional to a_1^2. We may assign the Neumann boundary condition at $\tau = 0$ but there will be no qualitative difference as shown in [30].

We first integrate $a(\tau)$ out. For this, we decompose $a(\tau) = \bar{a}^{(N)}(\tau) + A(\tau)$, where $\bar{a}^{(N)}(\tau)$ is a solution to the equation of motion for the action (19) and $A(\tau)$ represents small fluctuations around it. The solution to the equation of motion $d^2a/d\tau^2 + N^2 a = 0$ is given by

$$\bar{a}^{(N)}(\tau) = \frac{a_1}{\sin N} \sin(N\tau) . \quad (21)$$

Here we have imposed the boundary conditions (20). After integrating the fluctuations $A(\tau)$, the wave functional

(18) becomes (see [30] for the derivation)

$$\Psi = \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N}\sin N} \right)^{\frac{1}{2}} e^{\frac{\ell_{\rm dS}}{2G}(N + a_1^2 \cot N) - I_{\rm CT}} \tag{22}$$

up to an irrelevant overall normalization.

Now the problem is to find out a proper contour, $\mathcal{C}$, for the integration over N. For this, we first find out saddle points by solving $\partial I[\bar{a}^{(N)}; N]/\partial N = 0$. The solutions are

$$\begin{aligned} N_n^+ &= \left(n + \frac{1}{2}\right)\pi + i \ln\left(a_1 + \sqrt{a_1^2 - 1}\right) , \\ N_n^- &= \left(n + \frac{1}{2}\right)\pi - i \ln\left(a_1 + \sqrt{a_1^2 - 1}\right) , \end{aligned} \tag{23}$$

with $n \in \mathbb{Z}$, which are represented by red points in the left panel of fig. 1. We next look for the paths of steepest descent from the saddles. We find a subtlety that some of lines for the steepest descent and steepest ascent coincide with each other. In order to avoid this, we introduce a regulator as $\ell_{\rm dS} \to \ell_{\rm dS} + i\epsilon$ with $\epsilon > 0$. The necessity of regulator was discussed in [17]. In fact, the signature of the small imaginary part is highly relevant to the result. This is a specific feature in three dimensions, which cannot be observed in four dimensions [31]. As will be elaborated in [30], the direction of regularization is appropriately determined to the present one after including the one-loop correction. This can also be expected from the CFT analyses [10, 11], where the $\mathcal{O}(1)$ contribution to the central charge plays an important role to determine the relevant saddles.

The contribution of each saddle to the wave functional (22) are

$$\Psi_n^\pm \sim e^{\frac{(n+1/2)\ell_{\rm dS}\pi}{2G}} (2a_1)^{\pm i \ell_{\rm dS} \pm \epsilon} \tag{24}$$

for large a_1. The contributions from the saddles N_n^- are suppressed for large a_1, and only the contributions from the saddles N_n^+ are relevant. The paths of steepest descent are written down as solid lines in fig. 1. We denote that the path through $N_n^\pm$ by $\mathcal{J}_n^\pm$ and the orientations of paths are defined to be positive in the positive directions of the real and imaginary axes. Naively to preserve future time direction, one would choose the integration contour for N to be $i\mathbb{R}_+$. However, this contour does not reproduce the CFT partition function (5), so we need to find another contour to match with the CFT result. There are infinitely many candidates for such a contour. Among them, one can find that the proper contour that reproduces (5) is $-\mathcal{J}_0^+ + \mathcal{J}_0^- + \mathcal{J}_{-1}^+$. Finally we have identified the proper contour by using dS holography.

The geometry described by the saddle points looks complicated. However, as in [32], by Cauchy's theorem, we can introduce a time coordinate as:

$$T(\tau) = -\left(n + \frac{1}{2}\right)\pi(1-\tau)^q + i\ln(2a_1)\tau^q . \tag{25}$$

This encodes a deformation of the contour interpolating between $\tau = 0$ and $\tau = 1$, along which the geometric

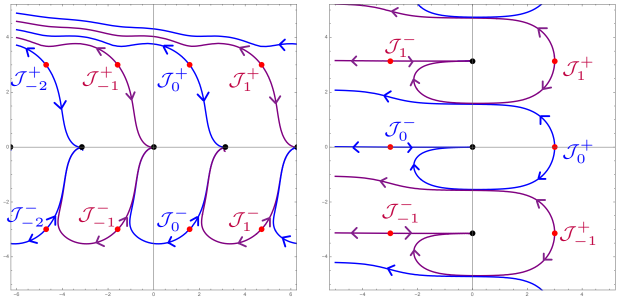

FIG. 1. Saddle points are given by red points and steepest descent paths are drawn by solid lines. Black dots denote the singularities of the integrand. Right and left figures are for $\Lambda = (\ell_{\rm dS} + i\epsilon)^{-2}$ and $\Lambda = -(\ell_{\rm AdS} + i\epsilon)^{-2}$, respectively.

criterion of [33] can be satisfied. When the index $q \to \infty$, the geometry approaches to the one discussed below (10).

We then move to the case with $\Lambda = -\ell_{\rm AdS}^{-2}$. As in the previous case, we consider the ansatz as

$$ds^2 = \ell_{\rm AdS}^2 \left[N(r)^2 dr^2 + a(r)^2 d\Sigma^2\right] \tag{26}$$

with $d\Sigma^2$ being the metric of $\mathbb{S}^2$. Here r is the radial coordinate taking a value in $0 < r < 1$. As for the previous case (18), the partition function evaluated by integrating over the metric can be reduced to

$$\mathcal{Z} = \int_{\mathcal{C}} dN \int \mathcal{D}a(r) e^{-I[a;N] - I_{\rm CT}} , \tag{27}$$

where the action is expressed as

$$I[a; N] = -\frac{\ell_{\rm AdS}}{2G} \int_0^1 dr\, N \left(\frac{1}{N^2}\frac{d^2 a}{dr^2} + a^2 + 1 \right) . \tag{28}$$

We assign the Dirichlet boundary conditions $a(0) = 0$ and $a(1) = a_1$, see [30] for the case where the Neumann boundary condition is assigned at $r = 1$.

We first integrate over $a(r) = \bar{a}^{(N)}(r) + A(r)$, where $a^{(N)}(r)$ is the saddle point and $A(r)$ are fluctuations around it. The equation of motion for $a(r)$ is given by $d^2a/d\tau^2 - N^2 a = 0$ and the solution subject to the Dirichlet boundary conditions is

$$\bar{a}^{(N)}(r) = \frac{a_1}{\sinh N} \sinh(Nr) . \tag{29}$$

We further integrate out the fluctuations, $A(r)$. The partition function is now written as (see [30] for the derivation)

$$\mathcal{Z} = \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N}\sinh N} \right)^{\frac{1}{2}} e^{\frac{\ell_{\rm dS}}{2G}(N + a_1^2 \coth N) - I_{\rm CT}} \tag{30}$$

up to an overall factor.

We then determine the contour $\mathcal{C}$ for the integration over N. Following the standard recipe, we first find out the stationary points satisfying $\partial I[\bar{a}^{(N)}; N]/\partial N = 0$. They are listed as

$$N_n^+ = n\pi i + \ln\left(a_1 + \sqrt{a_1^2+1}\right) , \\ N_n^- = n\pi i - \ln\left(a_1 + \sqrt{a_1^2+1}\right) , \tag{31}$$

with $n \in \mathbb{Z}$. The steepest descent lines originating these saddle points can be obtained. However, as in the previous case, some of lines for the steepest descent and ascent coincides with each other, which may require a regularization as, say, $\ell_{\rm AdS} \to \ell_{\rm AdS} + i\epsilon$. The steepest descent paths after the shifts are depicted in the right panel of fig. 1. We denote that the steepest descent path through $N_n^\pm$ by $\mathcal{J}_n^\pm$ and its orientations are defined to be positive in the positive directions of the real and imaginary axes.

In this case, it is natural to integrate over N along the positive real line. This integration contour can be deformed into $\sum_{n=0}^\infty \mathcal{J}_n^+ - \sum_{n=1}^\infty \mathcal{J}_n^-$. We thus take the saddle points N_n^+ with $n = 0, 1, 2 \ldots$ and N_n^- with $n = 1, 2, \ldots$. Each contribution from the saddle point to the partition function is

$$\mathcal{Z}_n^\pm \sim e^{\frac{n\pi i(\ell_{\rm AdS}+i\epsilon)}{2G}} (2a_1)^{\pm\frac{\ell_{\rm AdS}}{2G}} \tag{32}$$

for large a_1. As before, we can neglect the contribution $\mathcal{Z}_n^-$ for large a_1. Thus the semi-classical limit of partition function is expressed as the convergent sum of $\mathcal{Z}_n^+$ with $n = 0, 1, 2, \ldots$, which actually reproduces the CFT partition function (6), up to an overall phase factor $e^{\frac{\pi i c}{6}}$. Therefore we have found that the natural contour we took is the proper contour that reproduces the CFT result. The infinitesimal shift $\ell_{\rm AdS} \to \ell_{\rm AdS} + i\epsilon$, in this case, corresponds to the similar shift used to perform the series expansion in (6).

As in the previous dS case, we can introduce a new radial coordinate:

$$R(t) = -n\pi i(1-r)^q + \ln(2a_1) r^q \, . \tag{33}$$

Again the geometry approaches to the one discussed below (14) for $q \to \infty$. However in contrast with the dS case (25), except for $n = 0$ case, the geometric condition of [33], [26] remains violated along this contour.

V. DISCUSSION

In this letter we proposed the complex geometries corresponding to the semi-classical saddles in three-dimensional quantum gravity with the help of dual CFT. First we constructed the geometries by using the Chern-Simons formulation of three-dimensional gravity, and then we checked that the geometries we constructed can be indeed derived from the mini-superspace approach to quantum cosmology. For the case of positive cosmological constant, the geometry is given by the one for no-boundary proposal [34], which reproduces our previous result [10, 11] and is consistent with the criteria of allowable complex geometry in [26, 33, 35]. For the case of negative cosmological constant, the geometry is claimed to be given by Euclidean AdS$_3$ attached with imaginary radius $\mathbb{S}^3$'s. The geometry should be unphysical, however this fact itself does not contradict with the unitarity of dual CFT. For instance, the partition function is real after summing over the geometry (and choosing a proper overall phase factor). Even so, it would be nice if we can relate the result to the arguments based on the criteria of [26, 33, 35]. The decomposition by semi-classical saddles could not be unique and a nice decomposition by physical semi-classical saddles may exist. See [30] for more arguments. Furthermore, it would be interesting to investigate the relation with "time-like entanglement entropy" [36, 37], which seems to capture the information of the similar attached imaginary geometry.

In this letter, we have focused on the case with two-point functions in the limit $\eta \to 0$ for simplicity. However, we can further extend the analysis by dealing with the correlation functions with heavy operator insertions satisfying $0 < \sum_i \eta_i < 1$. From the general arguments on Liouville field theory with large central charge, we can see that there are saddles labeled by integer n. The exact results are not available for generic correlation functions, so we cannot determine which saddles to choose from the CFT. Even so, the gravity analysis in this letter can be easily extended to these cases. We have assumed that $d\Sigma^2$ in the metric ansatz is given by $\mathbb{S}^2$. For generic cases, we just need to set $d\Sigma^2 = e^{\varphi(z,\bar{z})} dz d\bar{z}$, where a function $\varphi(z,\bar{z})$ satisfies $\varphi \sim -4\eta_i \ln|z - z_i|$ near the insertion point z_i and $\varphi(z,\bar{z}) \sim -4\ln|z|$ for large z. For the topological contribution, we just need to multiply $(1 - \sum_i \eta_i)$. There is another contribution to the action (7) localized at the boundary, and the boundary contribution can be shown to reduce to the Liouville action of the field $\varphi(z,\bar{z})$. This was done in [38] for AdS$_3$, and it is extended for dS$_3$ in [30].

We are planing to apply the current analysis to other cases as well. It is important to analyze the case with black holes, see [39, 40] for AdS$_3$. It is also interesting to examine higher spin extensions as in [5, 6, 10, 11]. We are interested in the cases with higher dimensions as well. At least, it is straightforward to generalize the mini-superspace approach to quantum gravity as in [17–20]. Currently, there are many techniques available for non-perturbative aspects of quantum field theory, such as, conformal bootstrap, resurgent theory, supersymmetric localization. We are currently examining the resurgent structure of Liouville field theory in [30] in order to develop a useful technique.

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