A particle model that conserves the measure in the phase space, but does not conserve the kinetic energy

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Abstract

We consider a particular model of hard spheres that collide inelastically, losing a fixed amount of kinetic energy at each collision. We show that the transport associated to this hard sphere dynamics preserves locally the measure in the phase space. We prove the analog of Alexander's theorem for our model, providing the global well-posedness of the trajectories, for almost every initial datum.

Keywords. Inelastic Hard Spheres; Hard Ball Systems; Billiard Systems; Particle Systems.

1 Introduction

In this note we consider a two-dimensional model of inelastic particles exhibiting a surprising property: on the one hand during each collision which is sufficiently energetic the system will lose a positive amount of kinetic energy, on the other hand the flow induced by such a dynamics will preserve the measure in the phase space. Intuitively, these two properties look contradictory, we will show that they are actually independent from each other. We provide also examples of such models in dimension $d \geq 2$, with d arbitrary.
As a consequence of the conservation of the measure in the phase space by the flow of the particle dynamics, we deduce an Alexander's theorem ([1], [7]) for such a model: the dynamics of the particle system is globally well-posed for almost every initial datum in the phase space. To the best of our knowledge, this is the first example of an Alexander's theorem for a model of particles different from the classical elastic hard spheres. In the case of the inelastic particles with a fixed restitution coefficient, the global well-posedness of the dynamics for almost every initial datum is still an open problem.

2 Presentation of the model

We consider a system of N spherical particles, of diameter 1, evolving in $\mathbb{R}^d$, where $d \geq 2$ is a positive integer. The position, respectively the velocity, of each of the particles will be denoted by $x_i \in \mathbb{R}^d$, respectively $v_i \in \mathbb{R}^d$. The *configuration* of the system is the vector

$$Z_N = (x_1, v_1, \dots, x_N, v_N) \in \mathbb{R}^{2dN}, \tag{1}$$

collecting the positions and velocities of all of the N particles of the system. We assume that when the particles are at a positive distance one from another ($|x_i - x_j| > 1$), they move in straight line with constant velocity (that is, the particles evolve according to the free flow). When two particles collide, that is, when there exists a pair (i,j) with $1 \leq i \neq j \leq N$ such that $|x_i - x_j| = 1$, the velocities v_i, v_j of such colliding particles are immediately changed into v_i', v_j' according to the reflection law:

$$\left\{ \begin{array}{rcl} v_i' & = & \frac{v_i+v_j}{2} - \sigma\sqrt{\frac{|v_j-v_i|^2}{4} - \varepsilon_0}, \\ v_j' & = & \frac{v_i+v_j}{2} + \sigma\sqrt{\frac{|v_j-v_i|^2}{4} - \varepsilon_0}, \end{array} \right. \tag{2}$$

where $\varepsilon_0 > 0$ is a *fixed, positive* real number, and σ is the symmetry of the normalized relative velocity $(v_j - v_i)/|v_j - v_i|$ with respect to $\omega^\perp$, where ω is the line of contact between the two colliding particles, that is:

$$\sigma = \frac{v_j - v_i}{|v_j - v_i|} - 2\left(\frac{(v_j - v_i)}{|v_j - v_i|}\cdot\omega\right)\omega, \tag{3}$$

and

$$\omega = \frac{x_j - x_i}{|x_j - x_i|}\cdot \tag{4}$$

A fixed quantity, equal to $\varepsilon_0 > 0$, of kinetic energy is lost during each collision of colliding pairs of particles that are energetic enough:

$$\frac{|v_i'|^2}{2} + \frac{|v_j'|^2}{2} = \frac{|v_i|^2}{2} + \frac{|v_j|^2}{2} - \varepsilon_0. \tag{5}$$

Such a model is a simple generalization of the elastic hard spheres, that one recovers by prescribing $\varepsilon_0 = 0$ (and in this case (2) is the σ-representation of the elastic collision between two hard spheres). One may interpret the model as a system of particles that emit one photon at each collision, such a photon being always emitted with the same frequency.
Of course, (2) defines a well-posed law of interaction only when $|v_j - v_i|^2 \geq 4\varepsilon_0$. When it is not the case, we can modify the collision law and prescribe $\varepsilon_0 = 0$ for such low energy collisions. Note that with such an extension of the definition for colliding pairs that are not energetic enough, we obtain a transport that is not injective anymore. Nevertheless, the evolution is still deterministic, everywhere it is well-defined.

We will call the particle system that we introduced above the *inelastic hard sphere with emission model.*

3 Evolution of the measure of the phase space under the action of the flow

If we consider the flow in the phase space $\left(\mathbb{R}^d\times\mathbb{R}^d\right)^N \cap \bigcap_{i\neq j}\left\{|x_i - x_j| \geq 1\right\}$ of N particles associated to the particle dynamics we introduced, we can compute the evolution of the measure in the phase space under the action of such a flow.
Between two collisions, such a flow corresponds to the free flow, which preserves the measure. Therefore, the question of the evolution of the measure lies essentially in the evolution of the measure, for the velocity variables, during the collisions. Let us denote by v and v_* the pre-collisional velocities of two colliding particles, and by v' and v_*' their post-collisional velocities. If we denote:

$$\kappa = \sqrt{\frac{|v_* - v|^2}{4} - \varepsilon_0}, \tag{6}$$

we find, for any pair of components $i, j \in \{1, \dots, d\}$:

$$\partial_{v_i}\sigma_k = -\frac{\delta_{i,k}}{|v_* - v|} + \frac{(v_{*,k} - v_k)(v_{*,i} - v_i)}{|v_* - v|^3} + 2\frac{\omega_i\omega_k}{|v_* - v|} - 2(v_* - v)\cdot\omega\frac{(v_{*,i} - v_i)\omega_k}{|v_* - v|^3} \tag{7}$$

where $\delta_{i,k} = 1$ if and only if $i = k$, 0 elsewhere, is the Kronecker's delta symbol, and:

$$\partial_{v_i}\kappa = \partial_{v_i}\sqrt{\frac{|v_* - v|^2}{4} - \varepsilon_0} = -\frac{(v_{*,i} - v_i)}{4\kappa}\cdot \tag{8}$$

Therefore, collecting (7) and (8) together we obtain:

$$\begin{aligned}\nabla_v v' &= \frac{1}{2}I_d - \frac{\kappa}{|v*-v|}I_d + \frac{\kappa}{|v_*-v|}\frac{(v_*-v)}{|v_*-v|}\otimes\frac{(v_*-v)}{|v_*-v|} + 2\frac{\kappa}{|v_*-v|}\omega\otimes\omega \\ &\quad - 2\frac{\kappa}{|v_*-v|}\frac{(v_*-v)\cdot\omega}{|v_*-v|}\frac{(v_*-v)}{|v_*-v|}\otimes\omega + \frac{(v_*-v)\otimes(v_*-v)}{4|v_*-v|\kappa} - 2\frac{(v_*-v)\cdot\omega}{|v_*-v|}\frac{(v_*-v)\otimes\omega}{4\kappa} \\ &= \frac{1}{2}I_d + \frac{\kappa}{|v_*-v|}\Big[-I_d + \left(1+\frac{|v_*-v|^2}{\kappa^2}\right)\frac{(v_*-v)}{|v_*-v|}\otimes\frac{(v_*-v)}{|v_*-v|} \\ &\qquad\qquad - 2\frac{(v_*-v)}{|v_*-v|}\cdot\omega\left(1+\frac{|v_*-v|^2}{4\kappa^2}\right)\frac{(v_*-v)}{|v_*-v|}\otimes\omega + 2\omega\otimes\omega\Big]\end{aligned} \tag{9}$$

where I_d the $d\times d$ identity matrix. Writing the formula (9) in the form:

$$\nabla_v v' = \frac{1}{2}I_d + A$$

with

$$\begin{aligned}A = \frac{\kappa}{|v_*-v|}\Big[&-I_d + \left(1+\frac{|v_*-v|^2}{\kappa^2}\right)\frac{(v_*-v)}{|v_*-v|}\otimes\frac{(v_*-v)}{|v_*-v|} \\ &- 2\frac{(v_*-v)}{|v_*-v|}\cdot\omega\left(1+\frac{|v_*-v|^2}{4\kappa^2}\right)\frac{(v_*-v)}{|v_*-v|}\otimes\omega + 2\omega\otimes\omega\Big],\end{aligned}$$

and computing the other partial derivatives $\nabla_{v_*}v'$, $\nabla_v v'_*$ and $\nabla_{v_*}v'_*$, we obtain an expression of the following form for the Jacobian matrix J of the scattering mapping $(v,v_*)\mapsto(v',v'_*)$ defined by (2):

$$J = \begin{pmatrix}\frac{1}{2}I_d + A & \frac{1}{2}I_d - A \\ \frac{1}{2}I_d - A & \frac{1}{2}I_d + A\end{pmatrix}. \tag{10}$$

The determinant of such a matrix can be computed as follows. First we obtain:

$$\det(J) = \begin{vmatrix}\frac{1}{2}I_d + A & \frac{1}{2}I_d - A \\ \frac{1}{2}I_d - A & \frac{1}{2}I_d + A\end{vmatrix} = \begin{vmatrix}\frac{1}{2}I_d + A & \frac{1}{2}I_d - A \\ I_d & I_d\end{vmatrix} = \begin{vmatrix}2A & \frac{1}{2}I_d - A \\ 0 & I_d\end{vmatrix} = \det(2A).$$

It remains to compute $\det(2A)$. Such a determinant is of the form:

$$\det(2A) = \left(-\frac{2\kappa}{|v_*-v|}\right)^d \det(I_d + \lambda u\otimes u + \mu u\otimes\omega + \nu\omega\otimes\omega), \tag{11}$$

where

$$u = \frac{(v_*-v)}{|v_*-v|}, \tag{12}$$

$$\begin{cases}\lambda &= -\left(1+\frac{|v_*-v|^2}{\kappa^2}\right), \\ \mu &= 2\frac{(v_*-v)}{|v_*-v|}\cdot\omega\left(1+\frac{|v_*-v|^2}{4\kappa^2}\right), \\ \nu &= -2.\end{cases} \tag{13}$$

We need a generalization of the well known-formula

$$\det(I_d + u\otimes\omega) = 1 + u\cdot\omega \tag{14}$$

concerning the determinant of a single tensor product. In the two-dimensional case we have the following result:

Lemma 1 (Determinant of the sum of tensors products of two vectors)**.** *We consider the two-dimensional case: $d = 2$. Let λ, μ, ν be three real numbers, and u, ω be two vectors of $\mathbb{R}^2$.*
Then:

$$\det(I_2 + \lambda u \otimes u + \mu u \otimes \omega + \nu \omega \otimes \omega) = 1 + \lambda |u|^2 + \mu u \cdot \omega + \nu |\omega|^2 + \lambda\nu \left(\det(u,\omega)\right)^2. \tag{15}$$

Applying the formula (15), (11) becomes:

$$\begin{aligned}\det(2A) = \frac{4\kappa^2}{|v_*-v|^2}\Bigg[& 1 - \left(1 + \frac{|v_*-v|}{4\kappa^2}\right) + 2\frac{(v_*-v)}{|v_*-v|}\cdot\omega\left(1+\frac{|v_*-v|^2}{4\kappa^2}\right)\left(\frac{(v_*-v)}{|v_*-v|}\cdot\omega\right) \\ & - 2 + 2\left(1 + \frac{|v_*-v|^2}{\kappa^2}\right)\det\left(\frac{(v_*-v)}{|v_*-v|},\omega\right)^2\Bigg] \\ = \frac{4\kappa^2}{|v_*-v|^2}\Bigg[& -1 - \left(1 + \frac{|v_*-v|}{4\kappa^2}\right) + 2\left(1 - \frac{|v_*-v|^2}{4\kappa^2}\right)\left(\frac{(v_*-v)}{|v_*-v|}\cdot\omega\right)^2 \\ & + 2\left(1 + \frac{|v_*-v|^2}{\kappa^2}\right)\det\left(\frac{(v_*-v)}{|v_*-v|},\omega\right)^2\Bigg],\end{aligned} \tag{16}$$

and writing:

$$\frac{(v_*-v)}{|v_*-v|}\cdot\omega = \cos\theta, \tag{17}$$

we have

$$\det\left(\frac{(v_*-v)}{|v_*-v|},\omega\right) = \sin\theta, \tag{18}$$

so that

$$\begin{aligned}\det(2A) &= \frac{4\kappa^2}{|v_*-v|^2}\left[-1 - \left(1 + \frac{|v_*-v|}{4\kappa^2}\right) + 2\left(1 - \frac{|v_*-v|^2}{4\kappa^2}\right)\cos^2\theta + 2\left(1 + \frac{|v_*-v|^2}{\kappa^2}\right)\sin^2\theta\right] \\ &= \frac{4\kappa^2}{|v_*-v|^2}\left[-1 + \left(1 + \frac{|v_*-v|}{4\kappa^2}\right)\right] = 1.\end{aligned}$$

In conclusion, we have the following result.

Theorem 1 (Measure-preserving property of the inelastic hard sphere with emission model)**.** *In dimension $d = 2$, the inelastic hard sphere with emission flow preserves locally the measure in the phase space: at t fixed, the Jacobian of the transport $Z_N \mapsto T_t(Z_N)$ of the inelastic hard spheres with emission is equal to 1.*

We introduced a model of particle system that does not conserve the kinetic energy, but that induces a flow which preserves the measure in the phase space.

Remark 1. *The flow we introduced is not injective. In particular, a pair of particles in a post-collisional configuration such that the energy of the pair is smaller or equal to ε_0 after the collision has two preimages, coming either from an elastic, or an inelastic collision. Nevertheless, in order to interpret the result of Theorem 1, for every initial datum for which the trajectory of the system is defined, we can compute the Jacobian of the transport semigroup of the inelastic hard spheres with emission.*

4 Interpretation, consequences

4.1 Global well-posedness of the flow of the inelastic hard spheres with emission

In the case of the classical model of elastic hard spheres (see for instance [11]), the question of the global well-posedness of the dynamics of the particles is addressed by Alexander's theorem [1] (see also [7] for a modern presentation). Such a well-posedness property is the first step in order to complete the proof of Lanford's theorem [8], which provides a rigorous derivation of the Boltzmann equation from the elastic hard sphere system. This well-posedness is a delicate question, for particles can experience triple collisions, which prevents to define further the dynamics, therefore such a dynamics has no chance to be globally well-posed for every initial configuration. However, Alexander's theorem establishes such a result, for *almost every* initial datum (with respect to the Lebesgue measure in the phase space of N particles).
Since one of the main ingredients in Alexander's proof is the conservation of the measure by the flow, as a direct consequence of Theorem 1, we can repeat such a proof in the case of the inelastic hard spheres with emission. The main difficulty is that, in the present case, we defined a flow of particles that is not injective in the phase space. The second key argument is an a priori uniform estimate on the number of collisions of such a system. This estimate is a consequence of the uniform bounds that are known for the systems of elastic hard spheres (see the article of Burago, Ferleger and Kononenko [3]). We can obtain the following result.

Theorem 2 (Alexander's theorem for the inelastic hard sphere with emission model)**.** *Let N be any positive integer. Then, the dynamics of the system of N inelastic hard spheres with emission is almost everywhere globally well-defined in dimension $d = 2$. In other words, for almost every initial configuration (with respect to the Lebesgue measure) $Z_N = (x_1, v_1, \dots, x_N, v_N)$ of N particles in $\left(\mathbb{R}^2\right)^{2N} \cap \bigcap_{i\neq j} \left\{|x_i - x_j| \geq 1\right\}$, the evolution of the system from such an initial datum is well-defined for all time $t \geq 0$, involving only via free flow and binary collisions, and in addition for any $T > 0$, the system of particles starting initially from Z_N experiences only a finite number of collisions in the time interval $[0, T]$.*

Proof. We start with following the classical proof of Alexander's theorem. We consider any arbitrary time $T > 0$, and two cut-off parameters $R_1, R_2 > 0$. Let us denote by $Z_N = (x_1, v_1, \dots, x_N, v_N)$ the initial configurations of the system of N particles, and we will assume that the initial positions and velocities are such that:

$$|X_N| = |(x_1, x_2, \dots, x_N)| \leq R_1, \quad |V_N| = |(v_1, v_2, \dots, v_N)| \leq R_2. \tag{19}$$

We introduce finally a third cut-off parameter δ, meant to be small, such that $T/\delta = n$ is an integer, and such that:

$$\delta \leq 1, \quad \delta \leq \frac{2}{3\sqrt{2}R_2}\cdot \tag{20}$$

Let us now define recursively the flow of inelastic hard spheres with emission, on the time interval $[0, T]$, for all initial configurations Z_N of $B_{X_N}(0, R_1) \times B_{V_N}(0, R_2)$, except for a subset $\mathcal{A}(\delta)$, the size of which we will estimate later in terms of the small cut-off parameter δ.
We introduce the set $E_0 \subset \left(\mathbb{R}^2 \times \mathbb{R}^2\right)^N \cap \bigcap_{i\neq j} \left\{|x_i - x_j| \geq 1\right\}$ defined as follows:

$$\begin{aligned} E_0 = \big\{ Z_N \in B_{X_N}(0, R_1) \times B_{V_N}(0, R_2) \ / \ \exists (i,j) \neq (k,l) \in \{1, ..., N\}^2, i < j, k < l \ / \\ |x_i - x_j| \leq 1 + \frac{3}{2}\sqrt{2}\delta R_2, |x_k - x_l| \leq 1 + \frac{3}{2}\sqrt{2}\delta R_2 \big\}. \end{aligned} \tag{21}$$

On the one hand, outside E_0 any initial configuration Z_N leads to a well-defined trajectory on the time interval $[0, \delta]$, because such an initial configuration generates a trajectory with at most one collision on the time interval $]0, 3\delta/2]$. Let us denote by $T_t : Z_N \mapsto T_t(Z_N)$ the transport of inelastic particles with emission. On the other hand we have the following estimate on the measure of E_0:

$$\begin{aligned} |E_0| &= \binom{N}{2}^2 |B_{\mathbb{R}^{2(N-2)}}(0, R_1)| \times \left(\left| B_{\mathbb{R}^2}(0, 1 + \frac{3\sqrt{2}}{2}\delta R_2) \right| - |B_{\mathbb{R}^2}(0,1)| \right)^2 \times |B_{\mathbb{R}^{2N}}(0, R_2)| \\ &\leq C(N) R_1^{2(N-2)} R_2^{2N+2} \delta^2. \end{aligned} \tag{22}$$

Let us now define the flow on $[\delta, 2\delta]$. At time δ, the trajectories are contained in $B_{X_N}(0, R_1 + \delta R_2) \times B_{V_N}(0, R_2)$. Therefore, we introduce the set E_δ, defined in the same fashion as E_0 (the only difference lies in the estimate on the positions). We have again:

$$\begin{aligned} |E_\delta| &\leq C(N)(R_1 + \delta R_2)^{2(N-2)} R_2^{2N+2} \delta^2 \\ &\leq C(N)(R_1 + T R_2)^{2(N-2)} R_2^{2N+2} \delta^2, \end{aligned} \tag{23}$$

and for any initial configuration $Z_N \notin E_0$ and such that $T_\delta(Z_N) \notin E_\delta$, Z_N generates a trajectory well-defined at least on $[0, 2\delta]$. But on the time interval $]0, \delta]$, the system experiences at most one collision, that can be either elastic, or inelastic.

Let us denote by $T^{\text{elas.}}$ the low-energy hard sphere transport, and $T^{\text{inel.}}$ the inelastic transport. Note that we will need only these flows backwards on the time intervals $[k\delta, (k+1)\delta]$. The low-energy hard sphere transport $T^{\text{elas.}}$ is defined such that when two particles collide, and the energy of the pair is smaller than ε_0, we apply the classical elastic scattering, while when the energy of the pair is strictly larger than ε_0, we apply the backwards inelastic scattering (according to the definition of the inelastic hard sphere with emission dynamics, a pair of particles with an energy strictly larger than ε_0 *right after* the collision had to collide according to the inelastic scattering (2)). The backwards inelastic scattering is defined using (2), where ε_0 is replaced by $-\varepsilon_0$ in the formula. The inverse of the inelastic transport $T^{\text{inel.}}$ is defined such that whenever two particles collide, the backwards inelastic scattering is always applied.

The flow of inelastic particles with emission is well-defined outside the set $E_0 \cup F_\delta$ on $[0, 2\delta]$, where:

$$F_\delta = T^{\text{elas.}}_{-\delta} \left(E_\delta \cap T_\delta(E_0^c) \right) \cup T^{\text{inel.}}_{-\delta} \left(E_\delta \cap T_\delta(E_0^c) \right). \tag{24}$$

We define also F_0 as E_0, and $F_{k\delta}$ is defined inductively: at time 2δ, we consider the pathological set $E_{2\delta}$, on the complement of which we can define the dynamics of the particles until 3δ, and that has his measure estimated exactly as in (23). Then, we pull back $E_{2\delta}$ at the initial time, which provides $F_{2\delta}$, with:

$$\begin{aligned} F_{2\delta} = T^{\text{elas.}}_{-\delta} &\left(T^{\text{elas.}}_{-\delta} \left(E_{2\delta} \cap T_{2\delta}(E_0^c \cap F_\delta^c) \right) \right) \cup T^{\text{elas.}}_{-\delta} \left(T^{\text{inel.}}_{-\delta} \left(E_{2\delta} \cap T_{2\delta}(E_0^c \cap F_\delta^c) \right) \right) \\ &\cup T^{\text{inel.}}_{-\delta} \left(T^{\text{elas.}}_{-\delta} \left(E_{2\delta} \cap T_{2\delta}(E_0^c \cap F_\delta^c) \right) \right) \cup T^{\text{inel.}}_{-\delta} \left(T^{\text{inel.}}_{-\delta} \left(E_{2\delta} \cap T_{2\delta}(E_0^c \cap F_\delta^c) \right) \right). \end{aligned} \tag{25}$$

In the same way, we define $F_{3\delta}$, composed of 8 terms, and so on, until $F_{(n-1)\delta}$, composed of (a priori) $2^{(n-1)}$ terms. Finally, we introduce:

$$\mathcal{A}(\delta) = \bigcup_{k=0}^{n-1} F_{k\delta}. \tag{26}$$

On the complement of $\mathcal{A}(\delta)$, the dynamics of the inelastic hard spheres with emission is well-defined on the whole time interval $[0, T]$. We now have to estimate the measure of the set $\mathcal{A}(\delta)$.

To estimate $|\mathcal{A}(\delta)|$, we need an a priori bound on the number of collisions of the trajectories we defined. For any initial configuration $Z_N \notin \mathcal{A}(\delta)$, well-defined on $[0, T]$, the maximal number of *inelastic* collisions is strictly smaller than R_2^2/ε_0. Let $K = K(\varepsilon_0, R_2)$ be the largest integer strictly smaller than R_2^2/ε_0.

In between two consecutive inelastic collisions, by definition the system can experience only elastic collisions. Now, there exists a universal bound, depending only on the number of particles N, denoted by $C_{\text{HS}}(N)$, on the maximal number of collisions that a system of elastic hard spheres can undergo (see Theorem 1.3 in [3]). Therefore, the trajectory starting from Z_N can experience at most:

$$K + (K+1) C_{\text{HS}} = C_{\text{IHS}} \tag{27}$$

collisions (elastic and inelastic). This bound is also universal, in the sense that it depends only on N and R_2, but not on the initial configuration Z_N (taken in $B_{X_N}(0, R_1) \times B_{V_N}(0, R_2)$), nor the time T, nor the cut-off parameter δ.

We can now estimate $|\mathcal{A}(\delta)|$. We have:

$$|\mathcal{A}(\delta)| \leq \sum_{k=0}^{n-1} |F_{k\delta}|. \tag{28}$$

Since both of the transports $T^{\text{elas.}}_{-\delta}$ and $T^{\text{inel.}}_{-\delta}$ preserve the measure in the phase space, the measure of $|F_{k\delta}|$ is given by $|E_{k\delta}|$, times the number of preimages of the set $E_{k\delta}$ used to define $F_{k\delta}$. Here, we do not use $|F_{k\delta}| \leq 2^k |E_{k\delta}|$, which is too rough. Instead, let us observe that the number of the possible preimages of $E_{k\delta}$ that define $F_{k\delta}$ is given by the number of the admissible compositions of the two backwards transports $T^{\text{elas.}}_{-\delta}$ and $T^{\text{inel.}}_{-\delta}$. In order to reach the time $k\delta$, we proceed to k iterations, for which at most one collision, either elastic, or inelastic, takes place. Therefore, the trajectory $t \mapsto T_t(Z_N)$ (for $t \in [0, k\delta]$) starting from a configuration Z_N is described in particular by the number p of collisions that such a trajectory undergoes on the time interval $]0, k\delta]$, where $0 \leq p \leq k$. In addition, we know that $p \leq C_{\text{IHS}}$. Besides, such a trajectory can experience at most K inelastic collisions, and after labelling the collisions $j \in \{1, \dots, p\}$, the repartition of such inelastic collisions among all the collisions is another characteristic of the trajectory. We can then partition the set of initial configuration $F^c_0 \cap \cdots \cap F^c_{(k-1)\delta}$ into cells $C_{p,q,Q}$, where

$$0 \leq p \leq \min(k, C_{\text{IHS}}), \quad 0 \leq q \leq \min(p, K), \quad Q \subset \{1, \dots, p\} \text{ with } |Q| = q,$$

characterized as follows:

$$C_{p,q,Q} = \Big\{ Z_N \in F^c_0 \cap \cdots \cap F^c_{(k-1)\delta} \ / \ t \mapsto T_t(Z_N) \text{ has } p \text{ collisions on }]0, k\delta], \ q \text{ of them are inelastic,} \\ \text{and the label of such inelastic collisions are the elements of } Q. \Big\} \tag{29}$$

We can now consider the intersection between the pathological set $E_{k\delta}$ at time $k\delta$ with the image by $T_{k\delta}$ of the cells $C_{p,q,Q}$. Restricted to each of the cells $C_{p,q,Q}$, the transport $T_{k\delta}$ is injective, and we have:

$$\big\{ Z_N \in C_{p,q,Q} \ / \ T_{k\delta}(Z_N) \in E_{k\delta} \big\} = T^{C_{p,q,Q}}_{-k\delta} \left(E_{k\delta} \cap T_{k\delta}(C_{p,q,Q}) \right), \tag{30}$$

where $T^{C_{p,q,Q}}_{-k\delta}$ is the inverse of the transport $T_{k\delta}$ restricted to the cell $C_{p,q,Q}$. Such an inverse is defined as follows: from a configuration Z_N, we apply the free transport backwards, until the first collision. If $p \in Q$, we apply the inverse inelastic scattering, which correspond to (2), where ε_0 is replaced by $-\varepsilon_0$. If $p \notin Q$, we apply the elastic scattering. We apply again the backwards free flow, until the second collision, and we choose again the scattering depending if $p-1$ belongs to Q or not. Repeating the operation, after p collisions, we will define the transport $T^{C_{p,q,Q}}_{-k\delta}$ for a time $k\delta$ on the image of the cell $C_{p,q,Q}$ by the transport $T_{k\delta}$ of inelastic particles with emission, and such that:

$$\forall\, Y_N \in C_{p,q,Q} \text{ and } \forall\, Z_N \in T_{k\delta}(C_{p,q,Q}), \qquad T_{k\delta}(Y_N) = Z_N \ \Leftrightarrow \ T^{C_{p,q,Q}}_{-k\delta}(Z_N) = Y_N. \tag{31}$$

The inverse transport $T^{C_{p,q,Q}}_{-k\delta}$ preserves the measure, and we have therefore:

$$\begin{aligned} |F_{k\delta}| &= \Big| (T_{k\delta})^{-1} \left(T_{k\delta}(F^c_{(k-1)\delta}) \cap E_{k\delta} \right) \Big| = \Big| \bigcup_{\substack{0 \leq p \leq \min(k, C_{\text{IHS}}) \\ 0 \leq q \leq \min(p,K) \\ Q \subset \{1,\dots,p\} \\ |Q| = q}} (T_{k\delta})^{-1} \left(T_{k\delta}(C_{p,q,Q}) \cap E_{k\delta} \right) \Big| \\ &\leq \sum_{p=0}^{\min(k, C_{\text{IHS}})} \sum_{q=0}^{\min(p,K)} \sum_{\substack{Q \subset \{1,\dots,p\} \\ |Q| = q}} \Big| T^{C_{p,q,Q}}_{-k\delta}(E_{k\delta}) \Big| = \sum_{p=0}^{\min(k, C_{\text{IHS}})} \sum_{q=0}^{\min(p,K)} \sum_{\substack{Q \subset \{1,\dots,p\} \\ |Q| = q}} |E_{k\delta}|, \end{aligned} \tag{32}$$

so that, using that the total number of collisions is uniformly bounded, as well as the number of inelastic collisions, we deduce in the end:

$$|\mathcal{A}(\delta)| \leq \sum_{k=0}^{n-1} \left((C_{\text{IHS}} + 1) \sum_{q=0}^{K} \begin{pmatrix} C_{\text{IHS}} \\ q \end{pmatrix} \right) |E_{k\delta}| \leq C(N, T, R_1, R_2) n \delta^2 = C(N, T, R_1, R_2) \delta. \tag{33}$$

We can now consider the intersection of the pathological sets $|\mathcal{A}(\delta)|$ when δ is sent to zero, which provides a set of zero measure in $B_{X_N}(0, R_1) \times B_{V_N}(0, R_2)$, such that the flow of inelastic hard spheres with emission is defined on its complement on $[0, T]$. Repeating the argument for three countable sequences $(R_{1,n})_n$, $(R_{2,n})_n$ and $(T_n)_n$ that all tend to infinity as n goes to infinity, we complete the proof of Theorem 2. ☐

Remark 2. *Let us note that the proof of Alexander's theorem can be directly adapted for systems of inelastic hard spheres with emission without using the strong result of [3], provided that the total energy of the system is small. Namely, if we assume that the initial energy is so small that only a single inelastic collision can take place, the proof that can be found in [7] can be adapted in the following way. The measure of the set* $\mathcal{A}(\delta)$ *goes to zero as* δ*, which is more than enough to conclude, since any rate of convergence to zero would have been enough. Then, one can consider non-uniform time steps, that depend not only on the number of sub-intervals used to decompose* $[0,T]$*, but also on the* k*-th iteration of the process that allowed to define the transport. For instance, one can consider:*

$$\delta_n(k) = \frac{T}{k\ln(n)}. \tag{34}$$

In that case, considering k *iterations of the process allows to reach the final time:*

$$\sum_{k=1}^{n} \delta_n(k) = \sum_{k=1}^{n} \frac{T}{k\ln(n)} \sim T, \tag{35}$$

while at each iteration the measure of the sets $E_{k\delta}$ *is now estimated as:*

$$|E_{k\delta}| \leq C(N,T,R_1,R_2)\left(\delta_n(k)\right)^2. \tag{36}$$

The assumption on the initial kinetic energy ensures that the sets $F_{k\delta}$ *are defined with all the compositions of* k *transports, chosen among* $T^{elas.}_{-\delta}$ *and* $T^{inel.}_{-\delta}$*, such that* at most one *inelastic collision can take place. This provides:*

$$|\mathcal{A}(\delta)| \leq \sum_{k=0}^{n-1} k\,|E_{k\delta}| \tag{37}$$

$$\leq C(N,T,R_1,R_2)\sum_{k=0}^{n-1} k\frac{1}{k^2\ln^2(n)} \sim \frac{C(N,T,R_1,R_2)}{\ln(n)}, \tag{38}$$

which is indeed a vanishing quantity as the number of the time steps n *goes to infinity.*

Remark 3. *Our proof of Theorem 2 shows that an Alexander's-like result holds for any system of particles that can undergo only a finite number of inelastic collisions, uniformly on any compact set of the phase space, and such that the scattering* S *is not shrinking too much the measure, namely, such that:*

$$|S(A)| \geq C_K |A|, \tag{39}$$

where C_K *is a local constant, that may depend on the compact set* K *of the phase space on which we consider the restriction of the scattering mapping* S*. Indeed, to prove Alexander's theorem for the system of inelastic hard spheres with emission, we did not need that the measure was preserved.*

4.2 Comparison with the model of inelastic hard spheres, with constant restitution coefficient

Concerning the "classical" model of inelastic hard spheres, the post-collisional velocities are computed according to:

$$\left\{\begin{array}{lllll} v' & = & v - \dfrac{(1+r)}{2}(v-v_*)\cdot\omega\omega & = & \dfrac{v+v_*}{2} + \dfrac{1}{2}\left[(v-v_*) - (1+r)(v-v_*)\cdot\omega\omega\right], \\ v'_* & = & v_* + \dfrac{(1+r)}{2}(v-v_*)\cdot\omega\omega & = & \dfrac{v+v_*}{2} - \dfrac{1}{2}\left[(v-v_*) - (1+r)(v-v_*)\cdot\omega\omega\right], \end{array}\right. \tag{40}$$

with $r \in [0,1[$ (the so-called restitution coefficient) is a fixed positive real number. In this case, obtaining an Alexander-like theorem is still an open question. It is an important problem, for this model is used as a simplified description of granular media at microscopic scale (see [10], [4]). The phenomenon of inelastic collapse, consisting in infinitely many collisions in a finite time, typical of the model of collision (40), is the main obstruction to obtain an Alexander-like theorem for such a model. Concerning the inelastic collapse in dimensions larger than 1, let us mention [9], [12] and [5], [6]. Even for system of three particles, the inelastic collapse can take place, and such a phenomenon is still not well understood.

4.3 Interpretation of the result of Theorem 1

The result of Theorem 1 might look surprising at first glance. However, there is a way to interpret the result, in terms of conformal mappings. Denoting indeed by m and w the respective quantities:

$$m = \frac{1}{2}(v + v_*), \qquad w = v_* - v, \tag{41}$$

the collision mapping (2) can be rewritten, in frame of the center of mass, as:

$$(m, w) \mapsto \left(m, 2\sigma(w)\sqrt{\left|\frac{w}{2}\right|^2 - \varepsilon_0} \right), \tag{42}$$

where $\sigma(w)$ is the symmetry of the unit vector $w/|w|$ with respect to the orthogonal of $\omega = (x_* - x)/|x_* - x|$. Now, if we use the polar coordinates, choosing any unit orthogonal vector to ω and defining such a vector as the first axis (i.e., of angle 0), the mapping:

$$w \mapsto 2\sigma(w)\sqrt{\left|\frac{w}{2}\right|^2 - \varepsilon_0}$$

can be rewritten as:

$$f : (\rho, \theta) \mapsto (f_\rho(\rho, \theta), f_\theta(\rho, \theta)) = \left(\sqrt{\rho^2 - 4\varepsilon_0}, -\theta \right). \tag{43}$$

The infinitesimal element of surface $[\rho_0, \rho_0 + \mathrm{d}\rho] \times [\theta_0, \theta_0 + \mathrm{d}\theta]$, written in polar coordinates, has the infinitesimal surface $|\rho_0 \,\mathrm{d}\rho \,\mathrm{d}\theta|$, and its image has the infinitesimal surface:

$$|f_\rho(\rho_0, \theta_0) \cdot \partial_\rho f_\rho(\rho_0, \theta_0) \,\mathrm{d}\rho \cdot \partial_\theta f_\theta(\rho_0, \theta_0) \,\mathrm{d}\theta| \,.$$

But since we have:

$$f_\rho(\rho_0, \theta_0)\partial_\rho f_\rho(\rho_0, \theta_0)\partial_\theta f_\theta(\rho_0, \theta_0) = \sqrt{\rho_0^2 - 4\varepsilon_0}\frac{\rho_0}{\sqrt{\rho_0^2 - 4\varepsilon_0}}(-1) = -\rho_0, \tag{44}$$

we deduce that the mapping f preserves the measure, and so does the scattering mapping (2).

Such a computation can also be carried on in larger dimension using spherical coordinates, which provides similar collision models, preserving the measure in the phase space, but losing some kinetic energy during the collisions. In dimension $d = 3$ for instance, considering the mapping f, written in spherical coordinates (ρ, θ, φ) (with $\theta \in [-\pi/2, \pi/2]$ and $\varphi \in [0, 2\pi]$) such that:

$$f : (\rho, \theta, \varphi) \mapsto \left(\left(\rho^3 - 4\varepsilon_0\right)^{1/3}, -\theta, \varphi \right), \tag{45}$$

the image by f of the infinitesimal volume $[\rho_0, \rho_0 + \mathrm{d}\rho] \times [\theta_0, \theta_0 + \mathrm{d}\theta] \times [\varphi_0, \varphi_0 + \mathrm{d}\varphi]$ has the measure:

$$\begin{aligned} \left((f_\rho)^2 \partial_\rho f_\rho \,\mathrm{d}\rho\right) \cdot (\cos\theta_0 \partial_\theta f_\theta \,\mathrm{d}\theta) \cdot (\partial_\varphi f_\varphi \,\mathrm{d}\varphi) &= \left(\rho_0^3 - 4\varepsilon_0\right)^{2/3} \frac{3\rho_0^2}{3\left(\rho_0^3 - 4\varepsilon_0\right)^{2/3}} \cdot (\cos\theta_0 \partial_\theta f_\theta \,\mathrm{d}\theta) \cdot (\partial_\varphi f_\varphi \,\mathrm{d}\varphi) \\ &= \rho_0^2 \cos\theta_0 \,\mathrm{d}\theta \,\mathrm{d}\varphi. \end{aligned} \tag{46}$$

The mapping f preserves the measure, and the same conclusion follows accordingly for the scattering mapping (2) in dimension 3. It is clear that one can construct such examples for any arbitrary dimension. Let us observe that although such particle models conserve the measure in the phase space, and loses a positive quantity of kinetic energy in any collision that is energetic enough, the way in which the particles lose kinetic energy in the particle model we consider here (defined using (45)) becomes less natural than (5), that holds only in dimension $d = 2$.

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