Some Probabilistic aspects of $(q, 2)-$Fock space

Yungang Lu

Department of Mathematics, University of Bari "Aldo Moro"

Via E. Orabuna, 4, 70125, Bari, Italy

ABSTRACT

This paper primarily focuses on the investigation of the distribution of certain crucial operators with respect to significant states on the $(q, 2)-$Fock space, for instance, the vacuum distribution of the field operator.

Keywords: Interacting Fock space; Pair partition; Vacuum distribution of field operator.

1. Introduction

In this paper, we study mainly probabilistic aspects of a specific interacting Fock space, namely, the $(q, 2)-$Fock space, which is a particular $(q, m)-$Fock space originally introduced in [11].

The $(q, m)-$Fock space over a given Hilbert space $\mathcal{H}$ servers to concretize the Quan–algebra, a natural generalization of the traditional $q-$algebra. Here, the creation and annihilation operators adhere to an analogue of the usual $q-$commutation relation with q being an operator, rather than a scalar. In a previous work [11], we established the foundation of the Quan–algebra and examined some of its fundamental properties, particular,focusing on Wick's theorem.

In [12] and [13], we explored combinatorial aspects of the $(q, 2)-$Fock space. Our primary objective was to characterize the set of pair partitions essential for determining the vacuum expectation of any product of the creation-annihilation operators defined on $(q, 2)-$Fock space. We revealed as well a strong connection between the cardinality of this set of pair partitions and the Catalan's convolution formula introduced in [8] (refer to [16] and its cited references for various proofs).

Now we shift our focus to the probabilistic aspects of $(q, 2)-$Fock space. Our central result is presented in Theorem (3.8), which provides the explicit formulation of the probability measure $L_{q,f}$:=the vacuum distribution of the filed operator $Q(f)$ (i.e., the sum of the creation and annihilation operators) defined on the $(q, 2)-$Fock space with a test function f.

The symmetric nature of the distribution $L_{q,f}$, as demonstrated in Proposition 2.4 (i.e., the all odd moments of $L_{q,f}$ are zero), guarantees that $L_{q,f}$ is entirely determined by the vacuum distribution of the operator $Q^2(f)$. Moreover, due to the boundedness of the creation and annihilation operators as confirmed in Proposition 2.4, this distribution can be fully characterized by its moments, or equivalently, by its moment–generating function. Section 2 is primarily dedicated to the calculation the moment–generating function of $Q^2(f)$ with respect to the vacuum.

After obtaining the moment-generating function, our focus shifts to the investigation of the corresponding distribution. This investigation is carried out in Section 3.

2. $(q, 2)-$Fock space and the generating function of field operator with respect to the vacuum state

2.1. $(q, 2)-$**Fock space.** Now we turn out attention to the $(q, 2)-$Fock space, a specific instance of the $(q, m)-$Fock space introduced in [11]. Let $\mathcal{H}$ be a Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle$, the dimensional of $\mathcal{H}$ is assumed to be greater than or equal to 2 and we'll maintain this convention throughout our discussion.

For any $n \geq 2$, consider $n-$fold tensor product of $\mathcal{H}$, denoted as $\mathcal{H}^{\otimes n}$. Now let's define the operators for any $q \in [-1, 1]$:

- λ_1 :=the identity operator on $\mathcal{H}$, represented by $\mathbf{1}_{\mathcal{H}}$;
- for any $n \in \mathbb{N}$, λ_{n+2} is a **linear** operator on $\mathcal{H}^{\otimes(n+2)}$ and is characterized by the following:

$$\lambda_{n+2} := \mathbf{1}_{\mathcal{H}}^{\otimes n} \otimes \lambda_2 \text{ and } \lambda_2(f \otimes g) := f \otimes g + qg \otimes f\,, \quad \forall f, g \in \mathcal{H}$$

As discussed in [11], it is easy to check the positivity of λ_n's and so

$$\mathcal{H}_n := \text{ the completion of the } (\mathcal{H}^{\otimes n}, \langle \cdot, \lambda_n \cdot \rangle_{\otimes n})/Ker\langle \cdot, \lambda_n \cdot \rangle_{\otimes n}$$

is a Hilbert space, where $\langle \cdot, \cdot \rangle_{\otimes n}$ is the usual tensor scalar product. The scalar product of $\mathcal{H}_n$ will be denoted by $\langle \cdot, \cdot \rangle_n$, then $\langle \cdot, \cdot \rangle_1 := \langle \cdot, \cdot \rangle$ and for any $n \geq 2$,

$$\langle F, G \rangle_n := \langle F, \lambda_n G \rangle_{\otimes n}\,, \quad \forall F, G \in \mathcal{H}^{\otimes n}$$

or equivalently for any $n \in \mathbb{N}$,

$$\begin{aligned}\langle F, G \otimes f \otimes g \rangle_n :=& \langle F, G \otimes f \otimes g \rangle_{\otimes n} + q\langle F, G \otimes g \otimes f \rangle_{\otimes n} \\ & \forall F \in \mathcal{H}^{\otimes(n+2)},\, G \in \mathcal{H}^{\otimes n} \text{ and } f, g \in \mathcal{H}\end{aligned}$$

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space, let, for any $n \in \mathbb{N}^*$, $\mathcal{H}_n$ be the Hilbert space defined earlier and let $\mathcal{H}_0 := \mathbb{C}$, as usual,*

- *the Hilbert space $\Gamma_{q,2}(\mathcal{H}) := \bigoplus_{n=0} \mathcal{H}_n$ is named as the $(q, 2)-$**Fock space** over $\mathcal{H}$;*
- *$\Phi := 1 \oplus 0 \oplus 0 \oplus \ldots$ is termed as the **vacuum vector** of $\Gamma_{q,2}(\mathcal{H})$;*
- *for any $n \in \mathbb{N}^*$, $\mathcal{H}_n$ is called as the $n-$**particle space**.*

Throughout, we'll use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to represent denote the scalar product and the induced norm, both in $\Gamma_{q,2}(\mathcal{H})$ and in $\mathcal{H}_n$'s if there is no confusion.

It is straightforward to observe the following **consistency** of $\langle \cdot, \cdot \rangle_n$'s: for any $0 \neq f \in \mathcal{H}$ and for any $n \in \mathbb{N}^*$,

$$\|f \otimes F\| = 0 \text{ whenever } F \in \mathcal{H}_n \text{ verifying } \|F\| = 0$$

This consistency guarantees that for any $f \in \mathcal{H}$, the operator that maps $F \in \mathcal{H}_n$ to $f \otimes F \in \mathcal{H}_{n+1}$ is a well–defined linear operator from $\mathcal{H}_n$ to $\mathcal{H}_{n+1}$.

Definition 2.2. *For any $f \in \mathcal{H}$, the $(q, 2)-$**creation operator** (with the test function f), denoted as $A^+(f)$, is defined as a **linear** operator on $\Gamma_{q,2}(\mathcal{H})$ with the following properties:*

$$A^+(f)\Phi := f\,, \quad A^+(f)F := f \otimes F, \quad \forall n \in \mathbb{N}^* \text{ and } F \in \mathcal{H}_n \tag{2.1}$$

Throughout this paper, we denote as usual

$$\{-1, 1\}^m := \text{the set of all } \{-1, 1\}-\text{valued function defined on } \{1, \ldots, m\}, \quad \forall m \in \mathbb{N}^*$$

$$\{-1, 1\}_+^{2n} := \{\varepsilon \in \{-1, 1\}^{2n} : \sum_{h=1}^{2n} \varepsilon(h) = 0,\, \sum_{h=p}^{2n} \varepsilon(h) \geq 0,\, \forall p \in \{1, \ldots, 2n\}\}, \quad \forall n \in \mathbb{N}^*$$

$$NCPP(2n) := \text{the set of all non–crossing pair partitions of } \{1, \ldots, 2n\}, \quad \forall n \in \mathbb{N}^*$$

It is well known (see, e.g., [3]) that for any $\varepsilon \in \{-1,1\}_+^{2n}$, there exists unique non–crossing pair partition $\{(l_h^\varepsilon, r_h^\varepsilon)\}_{h=1}^n \in NCPP(2n)$ such that $\varepsilon^{-1}(\{-1\}) = \{l_h^\varepsilon : h \in \{1,\dots,n\}\}$ (or equivalently, $\varepsilon^{-1}(\{1\}) = \{r_h^\varepsilon : h \in \{1,\dots,n\}\}$). Thus, we can set a bijection τ from $\{-1,1\}_+^{2n}$ to $NCPP(2n)$ defined as $\tau(\varepsilon) := \{(l_h^\varepsilon, r_h^\varepsilon)\}_{h=1}^n$. We'll refer to

- $\tau(\varepsilon)$ as the **counterpart** of $\varepsilon \in \{-1,1\}_+$;
- $\tau^{-1}(\theta)$ as the **counterpart** of $\theta := \{(l_h, r_h)\}_{h=1}^n \in NCPP(2n)$.

Moreover, by denoting in further

$$\{-1,1\}_{+,*}^{2n} := \{\varepsilon \in \{-1,1\}_+^{2n},\ \sum_{h=p}^{2n} \varepsilon(h) = 0 \text{ only for } p = 1\}$$

$$NCPP_*(2n) := \{\{(l_h, r_h)\}_{h=1}^n \in NCPP(2n) :\ r_1 = 2n\}$$

the above τ induces a bijection between $\{-1,1\}_{+,*}^{2n}$ and $NCPP_*(2n)$.

Remark 2.3. *Let $n \in \mathbb{N}^*$, the following assets are easily checked.*

- *For any $\varepsilon \in \{-1,1\}_{+,*}^{2n}$, if we define $\varepsilon'(k) := \varepsilon(k+1)$ for all $k \in \{1,\dots,2n-2\}$, then it follows that $\varepsilon' \in \{-1,1\}_+^{2(n-1)}$. Moreover, ε' runs over $\{-1,1\}_+^{2(n-1)}$ as ε running over $\{-1,1\}_{+,*}^{2n}$.*
- *For any $\{(l_h, r_h)\}_{h=1}^n \in NCPP_*(2n)$, id we defined $l'_h := l_{h+1} - 1$ and $r'_h := r_{h+1} - 1$ for all $h \in \{1,\dots,n-1\}$, then it follows that $\{(l'_h, r'_h)\}_{h=1}^{n-1} \in NCPP(2(n-1))$. Moreover, $\{(l'_h, r'_h)\}_{h=1}^{n-1}$ runs over $NCPP(2(n-1))$ as $\{(l_h, r_h)\}_{h=1}^n$ running over $NCPP_*(2n)$.*

As a consequence, we obtain:

$$\left|\{-1,1\}_{+,*}^{2n}\right| = \left|\{-1,1\}_+^{2(n-1)}\right| = \left|NCPP_*(2n)\right| = \left|NCPP(2(n-1))\right| = C_{n-1} \tag{2.2}$$

The following results, namely Proposition 2.4 and Corollary 2.5, provide elementary–fundamental properties of the $(q,2)-$Fock space and the creation–annihilation on it. The proof will be omitted and they can be found in [12] (as well as [11]).

Proposition 2.4. *Let $\mathcal{H}$ be a Hilbert space and let $q \in [-1,1]$. For any $f \in \mathcal{H}$, $A^+(f)$ is bounded:*

$$\|A^+(f)\| = \|f\| \cdot \begin{cases} \sqrt{1+q}, & \text{if } q \in [0,1]; \\ 1, & \text{if } q \in [-1,0) \end{cases}$$

Moreover,

*1) the $(q,2)-$**annihilation operator** (with the test function f) $A(f) := \left(A^+(f)\right)^*$ is well–defined and for any $n \in \mathbb{N}^*$, $\{g_1,\dots,g_n\} \subset \mathcal{H}$, the following properties hold:*

$$A(f)\Phi = 0, \quad A(f)(g_1 \otimes \ldots \otimes g_n) = \begin{cases} \langle f, g_1\rangle \Phi, & \text{if } n = 1; \\ \langle f, g_1\rangle g_2 + q\langle f, g_2\rangle g_1, & \text{if } n = 2; \\ \langle f, g_1\rangle g_2 \otimes \ldots \otimes g_n, & \text{if } n > 2 \end{cases} \tag{2.3}$$

2) for any $n \in \mathbb{N}^*$ and $f \in \mathcal{H}$,

$$\|A(f)|_{\mathcal{H}_1}\| = \|A^+(f)|_{\mathcal{H}_0}\| = \|f\|, \quad \|A(f)|_{\mathcal{H}_{n+1}}\| = \|A^+(f)|_{\mathcal{H}_n}\|$$

$$\|A(f)A^+(f)|_{\mathcal{H}_n}\| = \|A^+(f)A(f)|_{\mathcal{H}_n}\| = \|A(f)|_{\mathcal{H}_n}\|^2$$

and additionally,

$$\|A(f)\| = \|A^+(f)\|; \quad \|A(f)A^+(f)\| = \|A^+(f)A(f)\| = \|A(f)\|^2$$

3) by denoting, as usual,

$$A^\epsilon(f) := \begin{cases} A^+(f), & \text{if } \epsilon = 1 \\ A(f), & \text{if } \epsilon = -1 \end{cases}; \quad \forall \epsilon \in \{-1,1\} \text{ and } f \in \mathcal{H}$$

the vacuum expectation

$$\langle \Phi, A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(m)}(f_m)\Phi\rangle$$

differs from zero only if $m = 2n$ (i.e. m is even) and $\varepsilon \in \{-1,1\}^{2n}_+$;

4) for any $n \in \mathbb{N}^*$ and $\varepsilon \in \{-1,1\}^{2n}_+$, for any $c \in \mathbb{C}$,

$$A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(2n)}(f_{2n})\Phi = c\Phi \iff \langle \Phi, A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(2n)}(f_{2n})\Phi\rangle = c \tag{2.4}$$

Corollary 2.5. *Let $m \in \mathbb{N}$, $\{f_h\}_{h=1}^m \subset \mathcal{H}$ and $\varepsilon \in \{-1,1\}^m$.*

1) If $\sum_{h=p}^m \varepsilon(h) \geq 0$ for any $p \in \{1,\dots,n\}$, then the restriction of the operator $A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(m)}(f_m)$ to $\oplus_{r=2}^\infty \mathcal{H}_r$ is equal to $b^{\varepsilon(1)}(f_1)\dots b^{\varepsilon(m)}(f_m)$ (recall that b^+ and b are the usual **free** *creator and annihilator respectively).*

2) If $\sum_{h=1}^m \varepsilon(h) = 0$ (it requires necessarily that m is even), for any $r \in \mathbb{N}$, $0 \oplus \mathcal{H}_r \oplus 0$ (in particular, $\mathbb{C} \oplus 0$) is invariant under the action of $A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(m)}(f_m)$.

3) In the case of $m = 2n$

$$A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(2n)}(f_{2n})\Phi = \langle A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(2n)}(f_{2n})\rangle\Phi$$

and the restriction of the operator $A^{\varepsilon(1)}(f_1)\dots A^{\varepsilon(2n)}(f_{2n})$ to the subspace $\oplus_{r=2}^\infty \mathcal{H}_r$ is $\prod_{h=1}^n \langle f_{l_h^\varepsilon}, f_{r_h^\varepsilon}\rangle \cdot$ **1**.

2.2. The vacuum expectation in (2.4). To determine the distribution of the field operator $Q(f) := A(f) + A^+(f)$ with respect to the vacuum state, the boundedness of $Q(f)$ (which also holds for both $A(f)$ and $A^+(f)$) simplifies the problem to finding its moments.

When $f = 0$, it is trivial that $A(0) = A^+(0) = 0$ and so the distribution of $Q(0)$ is clearly the one point distribution centred at the original. Therefore we shall take $f \neq 0$. As previously mentioned, our initial step is to compute all moments of $Q(f)$ with respect to the vacuum state. Furthermore, in light of Proposition 2.4, we can observe that all odd moments of field operator $Q(f)$ are equal to zero. Consequently, the distribution of $Q(f)$ is fully determined by even moments:

$$u_n := \langle \Phi, Q(f)^{2n}\Phi\rangle = \sum_{\varepsilon\in\{-1,1\}^{2n}_+} \langle \Phi, A^{\varepsilon(1)}(f)\dots A^{\varepsilon(2n)}(f)\Phi\rangle, \quad n \in \mathbb{N} \tag{2.5}$$

Let's denote furthermore,

$$v_0 := 1, \quad v_n := \sum_{\varepsilon\in\{-1,1\}^{2n}_{+,*}} \left\langle \Phi, A^{\varepsilon(1)}(f)\dots A^{\varepsilon(2n)}(f)\Phi\right\rangle, \quad n \in \mathbb{N} \tag{2.6}$$

Proposition 2.6. *u_n's verify the system*

$$u_{n+1} = \sum_{k=1}^{n+1} v_k u_{n+1-k} = \sum_{h=0}^{n} v_{h+1} u_{n-h}, \quad \forall n \in \mathbb{N} \tag{2.7}$$

with the initial condition $u_0 = 1$ and $u_1 = \|f\|^2$. While, v_n's can be rewritten to

$$v_n = \sum_{\varepsilon\in\{-1,1\}^{2(n-1)}_+} \left\langle \Phi, A(f)A^{\varepsilon(1)}(f)\dots A^{\varepsilon(2n-2)}(f)A^+(f)\Phi\right\rangle, \quad \forall n \in \mathbb{N}^* \tag{2.8}$$

and in particular, $v_1 = \|f\|^2$.

Proof: The statements $u_0 = 1$ and $u_1 = v_1 = \|f\|^2$ directly follow from their respective definitions given in (2.5) and (2.6).

By defining $\varepsilon'(j) := \varepsilon(j+1)$ for all $j \in \{1,\dots,2n-2\}$, we can observe that ε' runs over $\{-1,1\}^{2(n-1)}_+$ as ε varying over $\{-1,1\}^{2n}_{+,*}$. consequently, (2.8) is obtained.

The second equality in (2.7) is trivial and we'll now prove the first.

The non–crossing principle guarantees that for any $\{(l_h, r_h)\}_{h=1}^n \in NCPP(2n)$, r_1 must be an even number. Therefore, we can partition the set $NCPP(2n)$ into $\bigcup_{k=1}^{n} NCPP_k(2n)$, where for any $k \in \{1, \ldots, n\}$, $NCPP_k(2n)$ is a generalization of $NCPP_*(2n)$ defined as:

$$NCPP_k(2n) := \{\{(l_h, r_h)\}_{h=1}^n \in NCPP(2n) : r_1 = 2k\}$$

The first equality in (2.7) is obtained by observing the following easily checked facts:

- the sets $NCPP_k(2n)$'s are pairwise disjoint;
- for any $k = 1, \ldots, n$, as $\{(l_h, r_h)\}_{h=1}^n$ ranges over $NCPP_k(2n)$, $\{(l_h, r_h)\}_{h=1}^k$ represents the set $NCPP_*(2k)$ and $\{(l_h, r_h)\}_{h=k+1}^n$ represents the set of all non–crossing pair partitions of the set $\{2k + 1, \ldots, 2n\}$. $\square$

Proposition 2.6 establishes a strong dependence of the moments u_n's on the v_n's. The specific characteristics of the v_n's are a key focus of [13], which presents the following assertion:

Theorem 2.7. *For any $n \in \mathbb{N}$ and $f \in \mathcal{H}$,*

$$\begin{aligned} v_{n+1} &= \|f\|^{2(n+1)} \sum_{r=0}^{n} (1+q)^r \sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_1+\ldots+i_r = n-r}} C_{i_1} \ldots C_{i_r} \\ &= \|f\|^{2(n+1)} \sum_{r=0}^{n} (1+q)^r \frac{r}{2n-r} \binom{2n-r}{n} \end{aligned} \tag{2.9}$$

Throughout this text and in the following discussions, C_m is the $m-$th Catalan number and

$$\sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_1+\ldots+i_r = n-r}} C_{i_1} \ldots C_{i_r} \, (1+q)^r \Big|_{r=0} := \delta_{n,0} \tag{2.10}$$

2.3. The moment–generating function of $Q(f)^2$. Due to the boundedness of the field operator, for any $f \in \mathcal{H}$, there exists a $\delta_f > 0$, such that the series

$$\sum_{n=0}^{\infty} x^n \langle \Phi, Q(f)^{2n} \Phi \rangle$$

converges when $x \in (-\delta_f, \delta_f)$. The function $S : (-\delta_f, \delta_f) \longmapsto \mathbb{R}$ defined by the above series is usually named as the *moment–generating function* of the random variable $Q(f)^2$. Moreover, the boundedness of the field operator ensures that the distribution of the random variable $Q(f)^2$ is fully determined by its moment–generating function S.

Theorem 2.8. *For any $f \in \mathcal{H}$, by denoting $\delta := \min\{\frac{1}{4\|f\|^2}, \delta_f\}$, we have:*

$$S(x) = \frac{1 - q + (1+q)\sqrt{1 - 4\|f\|^2 x}}{1 - q + (1+q)\sqrt{1 - 4\|f\|^2 x} - 2\|f\|^2 x}, \qquad \forall x \in (-\delta, \delta) \tag{2.11}$$

Consequently, the function $x \longmapsto S(x^2)$ is the moment–generating function of $Q(f)$.

Proof: The second statement is self-evident, so let's focus on proving the first. First of all,

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} x^n u_n = 1 + \sum_{n=0}^{\infty} x^{n+1} u_{n+1} \overset{(2.7)}{=} 1 + \sum_{n=0}^{\infty} x^{n+1} \sum_{h=0}^{n} v_{h+1} u_{n-h} \\ &= 1 + \sum_{h=0}^{\infty} x^{h+1} v_{h+1} \sum_{n=h}^{\infty} x^{n-h} u_{n-h} = 1 + S(x) \sum_{h=0}^{\infty} x^{h+1} v_{h+1} \\ &\overset{(2.9)}{=} 1 + S(x) \sum_{n=0}^{\infty} x^{n+1} \, \|f\|^{2(n+1)} \sum_{r=0}^{n} (1+q)^r \sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_1+\ldots+i_r = n-r}} C_{i_1} \ldots C_{i_r} \end{aligned}$$

By using the convention (2.10) to the above formula, we find that

$$
\begin{aligned}
S(x) &= 1 + x\|f\|^2 S(x)\Big(1 + \sum_{n=1}^{\infty} \big(x\|f\|^2\big)^n \sum_{r=1}^{n} (1+q)^r \sum_{\substack{i_1,\ldots,i_r\geq 0 \\ i_1+\ldots+i_r=n-r}} C_{i_1}\ldots C_{i_r}\Big) \\
&= 1 + x\|f\|^2 S(x)\Big(1 + \sum_{r=1}^{\infty} \big(x\|f\|^2(1+q)\big)^r \\
&\qquad \cdot \sum_{n=r}^{\infty} \big(x\|f\|^2\big)^{n-r} \sum_{\substack{i_1,\ldots,i_r\geq 0 \\ i_1+\ldots+i_r=n-r}} C_{i_1}\ldots C_{i_r}\Big) \qquad (2.12)
\end{aligned}
$$

Thanks to the following well–known facts:

- the generating function of the Catalan's sequence $\{C_n\}_{n=0}^{\infty}$ is

$$
C(x) := \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1-4x}}, \quad \forall x \in \Big(-\frac{1}{4}, \frac{1}{4}\Big) \qquad (2.13)
$$

- for any $m \in \mathbb{N}$ and $\{\alpha_k\}_{k=0}^{\infty} \subset \mathbb{C}$,

$$
\Big(\sum_{k=0}^{\infty} \alpha_k x^k\Big)^m = \sum_{k=0}^{\infty} x^k \sum_{\substack{i_1,\ldots,i_m\geq 0 \\ i_1+\ldots+i_m=k}} \alpha_{i_1}\ldots\alpha_{i_m} \qquad (2.14)
$$

we know that for any $x \in \big(-\frac{1}{4\|f\|^2}, \frac{1}{4\|f\|^2}\big)$,

$$
\begin{aligned}
&\sum_{n=r}^{\infty} \big(x\|f\|^2\big)^{n-r} \sum_{\substack{i_1,\ldots,i_r\geq 0 \\ i_1+\ldots+i_r=n-r}} C_{i_1}\ldots C_{i_r} \overset{k:=n-r}{=} \sum_{k=0}^{\infty} \big(x\|f\|^2\big)^k \sum_{\substack{i_1,\ldots,i_r\geq 0 \\ i_1+\ldots+i_r=k}} C_{i_1}\ldots C_{i_r} \\
\overset{(2.14)}{=} &\Big(\sum_{k=0}^{\infty} \big(x\|f\|^2\big)^k C_k\Big)^r \overset{(2.13)}{=} \frac{\big(1-\sqrt{1-4\|f\|^2 x}\big)^r}{\big(2x\|f\|^2\big)^r}
\end{aligned}
$$

Thank to this equality, (2.12) simplifies to the following: for any $x \in (-\delta, \delta)$ with $\delta := \min\big\{\frac{1}{4\|f\|^2}, \delta_f\big\}$,

$$
\begin{aligned}
S(x) =& 1 + x\|f\|^2 S(x)\Big(1 + \sum_{r=1}^{\infty} \big(x\|f\|^2(1+q)\big)^r \frac{\big(1-\sqrt{1-4\|f\|^2 x}\big)^r}{\big(2x\|f\|^2\big)^r}\Big) \\
=& 1 + \frac{2x\|f\|^2 S(x)}{2-(1+q)\big(1-\sqrt{1-4\|f\|^2 x}\big)}
\end{aligned}
$$

Therefore, (2.11) is obtained by resolving this equation. $\square$

3. Distribution of field operator

Now we are ready to find the distribution of field operator. We begin by presenting the following result, which provides the distribution of the field operator for $q \in \{-1, 0\}$.

Proposition 3.1. *For any $0 \neq f \in \mathcal{H}$, the vacuum distribution of $Q(f)$ is*

1) $\frac{1}{2}\big(\delta_{-\|f\|} + \delta_{\|f\|}\big)$ if $q=-1$; here and throughout, for any $c \in \mathbb{R}$, δ_c is the Dirac measure centred at c;

2) the Wigner distribution over the interval $(-2\|f\|, 2\|f\|)$ if $q=0$.

Proof: The moment–generating function of $Q(f)^2$ equals, thanks to Theorem 2.8, to

$$x \longmapsto \frac{1}{1 - \|f\|^2 x}$$

when $q = -1$ and

$$x \longmapsto \frac{1 + \sqrt{1 - 4\|f\|^2 x}}{1 + \sqrt{1 - 4\|f\|^2 x} - 2\|f\|^2 x} \tag{3.1}$$

when $q = 0$. So

• when $q = -1$, the vacuum distribution of $Q(f)^2$ is the one–point distribution centered at $\|f\|^2$. So, the vacuum distribution of $Q(f)$ is the two points distribution on $\{-\|f\|, \|f\|\}$ with the equi–probability $\frac{1}{2}$.

• when $q = 0$, It can be readily verified that the function in (3.1) is equivalent to:

$$x \longmapsto \frac{1 - \sqrt{1 - 4\|f\|^2 x}}{2\|f\|^2 x}$$

As a result, the vacuum distribution of the field operator $Q(f)$ follows the Wigner law on the interval $(-2\|f\|, 2\|f\|)$. $\square$

Moving forward, we'll focus on determining the distribution of $Q(f)$ for $q \in (-1, 1] \setminus \{0\}$. For technical reason, we prefer to introduce a new parameter, denoted as $a := 1 + q$. With this new parameter a, the moment–generating function of $Q(f)^2$ takes the following form:

$$x \longmapsto \frac{2 - a + a\sqrt{1 - 4\|f\|^2 x}}{2 - a + a\sqrt{1 - 4\|f\|^2 x} - 2\|f\|^2 x}$$

Specifically, for $f \in \mathcal{H}$ such that $\|f\| = \frac{1}{2}$, the above moment–generating function is

$$\begin{aligned} S_a(x) :=& \left. \frac{2 - a + a\sqrt{1 - 4\|f\|^2 x}}{2 - a + a\sqrt{1 - 4\|f\|^2 x} - 2\|f\|^2 x} \right|_{\|f\| = \frac{1}{2}} \\ =& \frac{16(a-1) - 2(2a^2 + a - 2)x + 2ax\sqrt{1-x}}{16(a-1) - 4(a-1)(a+2)x - x^2} \end{aligned} \tag{3.2}$$

This function will be the starting point of our investigation of the distribution of field operator.

Additionally, an important measurable function for computing the vacuum distribution of $Q(f)^2$ is given by:

$$g_a(x) := \frac{\sqrt{1-x}}{\sqrt{x}(x^2 - \frac{a+2}{4}x - \frac{1}{16(a-1)})} \chi_{(0,1)}(x), \quad \forall x \in \mathbb{R} \tag{3.3}$$

where, $a \in (0, 1) \cup (1, 2]$.

Lemma 3.2. *Let, for any $a \in (0, 1) \cup (1, 2]$,*

$$h_a(x) := x^2 - \frac{a+2}{4}x - \frac{1}{16(a-1)}, \quad \forall x \in \mathbb{R}$$

then

i) $h_a > 0$ (so $g_a \geq 0$ and $g_a\big|_{(0,1)} > 0$) if $a \in (0, 1)$;

ii) if $a \in (1, 2]$, we have $h_a < 0$ on the interval $(0, 1)$ (so $g_a \leq 0$ and $g_a\big|_{(0,1)} < 0$).

Proof: It is trivial to have

$$h_a'(x) = 2x - \frac{a+2}{4} \in \begin{cases} (0, +\infty), & \text{if } x > \frac{a+2}{8} \\ (-\infty, 0), & \text{if } x < \frac{a+2}{8} \end{cases} ; \qquad h_a'(x)\Big|_{x = \frac{a+2}{8}} = 0$$

so h_a is decreasing on the interval $\left(-\infty, \frac{a+2}{8}\right)$ and increasing on the interval $\left(\frac{a+2}{8}, +\infty\right)$. Consequently, h_a reaches its (global) minimum at $\frac{a+2}{8}$:

$$h_a(x) \geq h_a(x)\Big|_{x=\frac{a+2}{8}} = \frac{(a+2)^2}{64} - \frac{(a+2)^2}{32} - \frac{1}{16(a-1)} = \frac{a^3+3a^2}{64(1-a)}, \quad \forall x \in \mathbb{R} \tag{3.4}$$

Moreover, since $0 < a \leq 2$, the global minimum point of h_a (i.e. $\frac{a+2}{8}$) falls within the interval $(0,1)$. Therefore,

$$h_a(x) < \min\{h_a(0), h_a(1)\}, \qquad \forall x \in (0,1) \tag{3.5}$$

In the case of $a \in (0,1)$, (3.4) implies:

$$h_a(x) \geq \frac{a^3+3a^2}{64(1-a)} > 0, \qquad \forall x \in \mathbb{R}$$

For $a \in (1,2]$, it is clear that $a-1 > 0$ and so $h_a(0) = -\frac{1}{16(a-1)} < 0$. Combining this fact with (3.5), the thesis will be proved if we can show that $h_a(1) \leq 0$.

This is indeed the case since

$$h_a(1) = 1 - \frac{a+2}{4} - \frac{1}{16(a-1)} \leq 0 \iff \frac{1}{16(a-1)} \geq \frac{2-a}{4} \iff 4a^2 - 12a + 9 \geq 0$$

and since function $f(a) := 4a^2 - 12a + 9$ has the global minimum $f(\frac{3}{2}) = 0$. □

3.1. The distribution of $Q(f)^2$ for $a \in (1,2]$ (i.e., $q \in (0,1]$) and $\|f\| = \frac{1}{2}$. For any $a > 1$, we introduce

$$a_1 := \frac{\sqrt{(a+2)^2 + \frac{4}{a-1}} + a + 2}{8}; \qquad a_2 := \frac{\sqrt{(a+2)^2 + \frac{4}{a-1}} - a - 2}{8} \tag{3.6}$$

$$A_1 := 16(a-1) - \frac{2a}{a_1+a_2}\left(\sqrt{\frac{a_2+1}{a_2}} - \sqrt{\frac{a_1-1}{a_1}}\right) \tag{3.7}$$

and

$$A_2 := \frac{2a}{a_1+a_2}\left(a_2\sqrt{\frac{a_1-1}{a_1}} + a_1\sqrt{\frac{a_2+1}{a_2}}\right) - 2(2a^2+a-2) \tag{3.8}$$

Lemma 3.3. *For any $a > 1$,*

i) $0 < a_2 < a_1$;

ii) $a_1 \geq 1$ and the equality hold if and only if $a = \frac{3}{2}$;

iii) the following equalities hold

$$a_1 a_2 = \frac{1}{16(a-1)}; \quad a_1 - a_2 = \frac{a+2}{4}; \quad a_1 + a_2 = \frac{a\sqrt{a+3}}{4\sqrt{a-1}} \tag{3.9}$$

$$(a_1a_2 + a_1)(a_1a_2 - a_2) = \frac{\left(a - \frac{3}{2}\right)^2}{64(a-1)^2}\ ; \quad a_1^2 + a_2^2 = \frac{a^2(a+3) - 2}{16(a-1)} \tag{3.10}$$

$$\sqrt{\frac{a_2+1}{a_2}} - \sqrt{\frac{a_1-1}{a_1}} = \frac{2a\sqrt{(a-1)(a+3)}}{\sqrt{a^2 + a - \frac{3}{2} + |a - \frac{3}{2}|}} \tag{3.11}$$

and

$$\left(a_2\sqrt{\frac{a_1-1}{a_1}}+a_1\sqrt{\frac{a_2+1}{a_2}}\right)^2$$
$$=\frac{1}{4(a-1)}\begin{cases}a^2(a-1)\left(a+\frac{5}{2}\right)^2-2a+3, & \text{if } a\in(1,\frac{3}{2}]\\ a^2(a-1)\left(a+\frac{5}{2}\right)^2, & \text{if } a\in(\frac{3}{2},2]\end{cases} \tag{3.12}$$

iv) one has

$$0\le A_1=\begin{cases}0, & \text{if } \ a\in\left(1,\frac{3}{2}\right]\\ 16(a-1)\left(1-\frac{a}{\sqrt{a^2+2a-3}}\right) & \text{if } \ a\in\left(\frac{3}{2},2\right]\end{cases} \tag{3.13}$$

and

$$A_2=a_2A_1 \tag{3.14}$$

Proof: The affirmation i) is evident due to the positivity of $a+2$ (in fact, $a+2>3$ in case $a>1$) and the fact that $\sqrt{(a+2)^2+\frac{4}{a-1}}>a+2$ holds whenever $a>1$.

It is straightforward to observe, using the definition of a_1 and noticing the fact $a+2>a-1>0$, that:

$$a_1\ge 1 \iff \sqrt{(a+2)^2+\frac{4}{a-1}}>8-(a+2) \iff \frac{1}{a-1}\ge 8-4a$$
$$\iff -4a^2+12a-8\le 1 \iff \left(a-\frac{3}{2}\right)^2\ge 0$$

Moreover one inequality becomes an equality if and only if the other does, if and only if $a=\frac{3}{2}$. Thus the affirmation ii) is proved. Now we turn to prove affirmation iii).

The three equalities in (3.9) are obtained directly from the definitions of a_1 and a_2. Therefore, we have the two equalities in (3.10) as follows:

$$(a_1a_2+a_1)(a_1a_2-a_2)=a_1a_2(a_1a_2+a_1-a_2-1)\overset{(3.9)}{=}\frac{(a-\frac{3}{2})^2}{64(a-1)^2}$$
$$a_1^2+a_2^2=(a_1-a_2)^2+2a_1a_2\overset{(3.9)}{=}\left(\frac{a+2}{4}\right)^2+\frac{2}{16(a-1)}=\frac{a^2(a+3)-2}{16(a-1)}$$

The equality (3.11) is obtained through the following calculation (notice that $\frac{a_1+a_2}{\sqrt{a_1a_2}}\overset{(3.9)}{=}a\sqrt{a+3}$):

$$\sqrt{\frac{a_2+1}{a_2}}-\sqrt{\frac{a_1-1}{a_1}}=\frac{\sqrt{a_1(a_2+1)}-\sqrt{a_2(a_1-1)}}{\sqrt{a_1a_2}}$$
$$=\frac{a_1+a_2}{\sqrt{a_1a_2}(\sqrt{a_1a_2+a_1}+\sqrt{a_1a_2-a_2})}=\frac{a\sqrt{a+3}}{\sqrt{(\sqrt{a_1a_2+a_1}+\sqrt{a_1a_2-a_2})^2}} \tag{3.15}$$

(3.9) and (3.10) ensure that

$$(\sqrt{a_1a_2+a_1}+\sqrt{a_1a_2-a_2})^2$$
$$=a_1a_2+a_1+a_1a_2-a_2+2\sqrt{(a_1a_2+a_1)(a_1a_2-a_2)}$$
$$=\frac{1}{8(a-1)}+\frac{a+2}{4}+\frac{|a-\frac{3}{2}|}{4(a-1)}=\frac{1}{4(a-1)}(a^2+a-\frac{3}{2}+|a-\frac{3}{2}|)$$

Applying this formula to (3.15), we obtain (3.11).

To prove (3.12), we can observe, since the first equality in (3.10), that:

$$\sqrt{a_1a_2(a_1-1)(a_2+1)} = \sqrt{(a_1a_2+a_1)(a_1a_2-a_2)} = \frac{\left|a-\frac{3}{2}\right|}{8(a-1)} \tag{3.16}$$

Additionally, the expression on the left hand side of (3.12) is equal to:

$$a_1^2 + a_2^2 + \frac{a_1^2}{a_2} - \frac{a_2^2}{a_1} + 2\sqrt{a_1a_2(a_1-1)(a_2+1)}$$

where,

- (3.16) and the second equality in (3.10) guarantee that:

$$a_1^2 + a_2^2 + 2\sqrt{a_1a_2(a_1-1)(a_2+1)} = \frac{a^2(a+3) - 2 + 4\left|a-\frac{3}{2}\right|}{16(a-1)}$$

- by rewriting $a_1^3 - a_2^3$ as $(a_1-a_2)(a_1^2+a_1a_2+a_2^2)$, the first two equality in (3.9) and the second equality in (3.10) ensure that:

$$\frac{a_1^3-a_2^3}{a_1a_2} = \frac{(a_1-a_2)(a_1^2+a_1a_2+a_2^2)}{a_1a_2} = \frac{(a+2)(a^2(a+3)-1)}{4}$$

Therefore,

$$\left(a_2\sqrt{\frac{a_1-1}{a_1}} + a_1\sqrt{\frac{a_2+1}{a_2}}\right)^2 = \frac{a^2(a-1)\left(a+\frac{5}{2}\right)^2 - a + \frac{3}{2} + \left|a-\frac{3}{2}\right|}{4(a-1)}$$

which is nothing else than the expression on the right hand side of (3.12).

Finally we consider the statement iv) which will be proven in three steps.

Step 1: to show the formula (3.13) Once we establish the equality in (3.13), the subsequent fact implies the inequality in (3.13), i.e. $A_1 \geq 0$:

$$\sqrt{a^2+2a-3} \geq a, \quad \forall a \in \left(\frac{3}{2}, 2\right]$$

So we only need to show the equality in (3.13). Just by the definition of A_1, along with the formulae (3.9) and (3.11), we can derive:

$$\begin{aligned} A_1 &\overset{(3.7)}{=} 16(a-1) - \frac{2a}{a_1+a_2}\left(\sqrt{\frac{a_2+1}{a_2}} - \sqrt{\frac{a_1-1}{a_1}}\right) \\ &= 16(a-1) - \frac{16a(a-1)}{\sqrt{a^2+a-\frac{3}{2}+\left|a-\frac{3}{2}\right|}} \end{aligned}$$

So, the equality in (3.13) is proven by observing that

$$\sqrt{a^2+a-\frac{3}{2}+|a-\frac{3}{2}|} = \begin{cases} a, & \text{if } a \in \left(1, \frac{3}{2}\right] \\ \sqrt{a^2+2a-3}, & \text{if } a \in \left(\frac{3}{2}, 2\right] \end{cases}$$

Step 2: to prove the formula (3.14) for $a \in \left(1, \frac{3}{2}\right]$ Thanks to (3.8) and (3.13), it can be observed that for $a \in \left(1, \frac{3}{2}\right]$, the following equivalences hold:

$$(3.14) \text{ holds} \iff A_2 = 0 \iff a\left(a_2\sqrt{\frac{a_1-1}{a_1}} + a_1\sqrt{\frac{a_2+1}{a_2}}\right) = (2a^2+a-2)(a_1+a_2)$$

i.e., thanks to (3.9), $A_2 = 0$ if and only if

$$\left(a_2\sqrt{\frac{a_1-1}{a_1}} + a_1\sqrt{\frac{a_2+1}{a_2}}\right)^2 = \frac{(2a^2+a-2)^2(a+3)}{16(a-1)} \tag{3.17}$$

Noticing that in case of $a \in \left(1, \frac{3}{2}\right]$, (3.12) says that the left hand side of (3.17) is equal to

$$\frac{4a^2(a-1)\left(a+\frac{5}{2}\right)^2 - 8a + 12}{16(a-1)}$$

and which is nothing else than the right hand side of (3.17).

Step 3: to prove the formula (3.14) for $a \in \left(\frac{3}{2}, 2\right]$ In the case of $a \in \left(\frac{3}{2}, 2\right]$, (3.12) gives that:

$$\left(a_2\sqrt{\frac{a_1-1}{a_1}} + a_1\sqrt{\frac{a_2+1}{a_2}}\right)^2 = \frac{a^2\left(a+\frac{5}{2}\right)^2}{4} = \frac{a^2(2a+5)^2}{16}$$

i.e., thanks to the positivity of the above terms,

$$a_2\sqrt{\frac{a_1-1}{a_1}} + a_1\sqrt{\frac{a_2+1}{a_2}} = \frac{a(2a+5)}{4}$$

Therefore,

$$A_2 = \frac{2a(2a+5)\sqrt{a-1}}{\sqrt{a+3}} - 2(2a^2+a-2) \tag{3.18}$$

On the other hand, (3.6) and (3.13) yield

$$\begin{aligned} a_2 A_1 =& \frac{\sqrt{(a+2)^2+\frac{4}{a-1}}-(a+2)}{8} \cdot 16(a-1) \cdot \left(1 - \frac{a}{\sqrt{a^2+2a-3}}\right) \\ =& \frac{2}{\sqrt{a+3}} \cdot \left(a(2a+5)\sqrt{a-1} - \sqrt{a+3}(2a^2+a-2)\right) \end{aligned}$$

and which is exactly equal to A_2 as shown in (3.18). $\square$

Remark 3.4. *As an easy corollary of the first two equalities in* (3.9)*, we can deduce, using the function h_a introduced in Lemma 3.2, the following results:*

$$\begin{aligned} & \frac{1}{a_1+a_2}\left(\frac{1}{a_1-x} + \frac{1}{a_2+x}\right)\sqrt{\frac{1-x}{x}}\chi_{(0,1)}(x) \\ =& \frac{1}{(a_1-x)(a_2+x)}\sqrt{\frac{1-x}{x}}\chi_{(0,1)}(x) = \frac{1}{a_1a_2+(a_1-a_2)x-x^2}\sqrt{\frac{1-x}{x}}\chi_{(0,1)}(x) \\ \overset{(3.9)}{=}& \frac{1}{\frac{1}{16(a-1)}+\frac{a+2}{4}x-x^2}\sqrt{\frac{1-x}{x}}\chi_{(0,1)}(x) = -\frac{1}{h_a(x)}\sqrt{\frac{1-x}{x}}\chi_{(0,1)}(x) = -g_a(x) \end{aligned} \tag{3.19}$$

here, g_a is introduced in (3.3)*. Thanks to the affirmation ii) of Lemma 3.2 and the first two affirmations of Lemma 3.3 (which essentially state that $0 < a_2 < a_1 \geq 1$), we can be certain of the positivity of the function $-g_a$.*

Now, let us introduce

$$\mu_a(B) := \frac{a}{8\pi(a-1)}\int_B(-g_a)(x)dx + \frac{A_1}{16(a-1)}\delta_{a_1}(B), \qquad \forall B \in \mathcal{B} \tag{3.20}$$

here and throughout, $\mathcal{B}$ is the Borel $\sigma-$algebra on $\mathbb{R}$. Then the positivity of the function $-g_a$, the scalars A_1 and $a-1$ (remembering that now we are considering the case of $a > 1$) ensure that μ_a is a measure. Moreover, it's worth noting that μ_a is **absolutely continuous** if and only if $A_1 = 0$.

In the following discussion, the sphere in $\mathcal{H}$ consisting of elements with the norm $\frac{1}{2}$ will play a significant role and we shall denote it as $\mathcal{H}_{1/2}$.

Theorem 3.5. *For any $a \in (1, 2]$ and $f \in \mathcal{H}_{1/2}$,*

$$\int \frac{\mu_a(dx)}{1 - tx} = S_a(t), \qquad \forall\ t \in (-1, 1) \tag{3.21}$$

i.e. μ_a is the vacuum distribution of $Q(f)^2$ with $f \in \mathcal{H}_{1/2}$.

Proof: By utilizing the definitions of A_1 and A_2, and referring to the following well–known formulae (as found in, for example, Section 17.3.12 of [1]):

$$\begin{aligned}
&\int_0^1 \frac{\sqrt{1-x^2}}{1-\alpha x^2} dx = \frac{\pi}{2} \cdot \frac{1-\sqrt{1-\alpha}}{\alpha}, \quad \forall \alpha \in (-1, 1] \\
&\int_0^1 \frac{\sqrt{1-x^2}}{\alpha + x^2} dx = \frac{\pi}{2} \cdot \Big(\sqrt{\frac{1+\alpha}{\alpha}} - 1\Big), \quad \forall \alpha > 0
\end{aligned} \tag{3.22}$$

We can derive, for sufficiently small $|t|$, the following result:

$$\begin{aligned}
&\int_0^1 \frac{1}{1-tx} \sqrt{\frac{1-x}{x}} dx \\
=&2 \int_0^1 \frac{1}{1-ty^2} \sqrt{1-y^2} dy \overset{(3.22)}{=} \frac{\pi}{t}(1 - \sqrt{1-t}) = \frac{\pi}{1+\sqrt{1-t}}
\end{aligned} \tag{3.23}$$

Moreover,

- by noticing that $0 < \frac{1}{a_1} \le 1$, we have:

$$\int_0^1 \frac{1}{a_1 - x} \sqrt{\frac{1-x}{x}} dx \overset{y:=\sqrt{x}}{=} \frac{2}{a_1} \int_0^1 \frac{1}{1 - \frac{y^2}{a_1}} \sqrt{1-y^2} dy \overset{(3.22)}{=} \pi \Big(1 - \sqrt{1 - \frac{1}{a_1}}\Big) \tag{3.24}$$

and consequently,

$$\begin{aligned}
&\int_0^1 \frac{1}{1-tx} \frac{1}{a_1 - x} \sqrt{\frac{1-x}{x}} dx = \frac{1}{1-a_1 t} \int_0^1 \Big(\frac{1}{a_1 - x} - \frac{t}{1-tx}\Big) \sqrt{\frac{1-x}{x}} dx \\
\overset{(3.23),(3.24)}{=}& \frac{\pi}{1-a_1 t} \Big(\sqrt{1-t} - \sqrt{1 - \frac{1}{a_1}}\Big)
\end{aligned}$$

- by noticing that $a_2 > 0$, we obtain:

$$\int_0^1 \frac{1}{a_2 + x} \sqrt{\frac{1-x}{x}} dx \overset{y:=\sqrt{x}}{=} 2 \int_0^1 \frac{1}{a_2 + y^2} \sqrt{1-y^2} dy \overset{(3.22)}{=} \pi \Big(\sqrt{1 + \frac{1}{a_2}} - 1\Big)$$

and consequently,

$$\begin{aligned}
&\int_0^1 \frac{1}{1-tx} \frac{1}{a_2 + x} \sqrt{\frac{1-x}{x}} dx = \frac{1}{1+a_2 t} \int_0^1 \Big(\frac{1}{a_2 + x} + \frac{t}{1-tx}\Big) \sqrt{\frac{1-x}{x}} dx \\
\overset{(3.23),(3.24)}{=}& \frac{\pi}{1+a_2 t} \Big(\sqrt{1 + \frac{1}{a_2}} - \sqrt{1-t}\Big)
\end{aligned}$$

Summing up, we find that:

$$\int_{\mathbb{R}} \frac{-g_a(x)}{1-tx}dx = \int_0^1 \frac{1}{a_1+a_2}\left(\frac{1}{a_1-x}+\frac{1}{a_2+x}\right)\frac{1}{1-tx}\sqrt{\frac{1-x}{x}}dx$$
$$= \frac{\pi}{(1+a_2t)(1-a_1t)(a_1+a_2)}\cdot$$
$$\cdot\Big((1-a_1t)\Big(\sqrt{1+a_2^{-1}}-\sqrt{1-t}\Big)+(1+a_2t)\Big(\sqrt{1-t}-\sqrt{1-a_1^{-1}}\Big)\Big) \tag{3.25}$$

In the expression

$$\frac{1}{a_1+a_2}\Big((1-a_1t)\Big(\sqrt{1+a_2^{-1}}-\sqrt{1-t}\Big)+(1+a_2t)\Big(\sqrt{1-t}-\sqrt{1-a_1^{-1}}\Big)\Big)$$

- the coefficient of $t\sqrt{1-t}$ is obviously 1;
- the coefficient of $\sqrt{1-t}$ is trivially zero;
- the coefficient of t is $\frac{-1}{a_1+a_2}\Big(a_1\sqrt{1+a_2^{-1}}+a_2\sqrt{1-a_1^{-1}}\Big)$, which equals, thanks to (3.8) (i.e., the definition of A_2), to $\frac{-1}{2a}(A_2+2\,(2a^2+a-2)\,)$;
- the constant term is $\frac{1}{a_1+a_2}\left(\sqrt{1+a_2^{-1}}-\sqrt{1-a_1^{-1}}\right)$, which equals, thanks to (3.7) (i.e., the definition of A_1), to $\frac{1}{2a}(16(a-1)-A_1)$.

By applying these facts to (3.25), we conclude that:

$$\int_{\mathbb{R}} \frac{-g_a(x)}{1-tx}dx$$
$$= \frac{\pi}{2a}\cdot\frac{16(a-1)-2(2a^2+a-2)t+2at\sqrt{1-t}}{(1+a_2t)(1-a_1t)}-\frac{\pi}{2a}\cdot\frac{A_1+A_2t}{(1+a_2t)(1-a_1t)}$$
$$\overset{(3.14)}{=}\frac{\pi}{2a}\cdot\frac{16(a-1)-2(2a^2+a-2)t+2at\sqrt{1-t}}{(1+a_2t)(1-a_1t)}-\frac{\pi}{2a}\cdot\frac{A_1}{(1-a_1t)} \tag{3.26}$$

Moreover, since

$$(1+a_2t)(1-a_1t) = 1-(a_1-a_2)t-a_1a_2t^2$$
$$\overset{(3.9)}{=}1-\frac{a+2}{4}t-\frac{1}{16(a-1)}t^2 = \frac{1}{16(a-1)}\Big(16(a-1)-4(a-1)(a+2)t-t^2\Big)$$

we know that the first term in the right hand side of (3.26) is equal to, thanks to (3.2),

$$\frac{8\pi(a-1)}{a}\cdot S_a(t)$$

And so (3.26) simplifies to

$$\int_{\mathbb{R}} \frac{-g_a(x)}{1-tx}dx = \frac{8\pi(a-1)}{a}\cdot S_a(t)-\frac{\pi}{2a}\cdot\frac{A_1}{1-a_1t}$$
$$= \frac{8\pi(a-1)}{a}\cdot S_a(t)-\frac{\pi A_1}{2a}\int_{\mathbb{R}}\frac{d\delta_{a_1}}{1-tx}$$

i.e.,

$$\int_{\mathbb{R}}\frac{1}{1-tx}\mu_a(dx) = S_a(t) = \int_{\mathbb{R}}\Big(-g_a(x)\frac{a}{8\pi(a-1)}dx+\frac{A_1}{16(a-1)}d\delta_{a_1}\Big)$$

This clearly proves the thesis. □

Now let's examine two specific cases: $a=2$ and $a=\frac{3}{2}$ (which corresponds to $q=1$ and $q=\frac{1}{2}$, respectively).

In the case of $a = 2$, according to (3.6) and (3.13), we have the following results:

$$a_1 = \frac{\sqrt{5}+1}{2}, \qquad a_2 = \frac{\sqrt{5}-1}{2}, \qquad A_1 = \frac{16(\sqrt{5}-2)}{\sqrt{5}}$$

Hence,

$$-g_2(x) \overset{(3.19)}{=} \frac{\sqrt{1-x}}{\sqrt{x}\left(\frac{\sqrt{5}+1}{2} - x\right)\left(\frac{\sqrt{5}-1}{2} + x\right)}\chi_{(0,1)}(x)$$

Theorem 3.5 tells us that the distribution of $Q(f)^2$ for any $f \in \mathcal{H}_{1/2}$ is given by the probability measure μ_2, defined as follows:

$$\mu_2(B) := \int_B \frac{-g_2(x)}{4\pi}dx + \frac{\sqrt{5}-2}{\sqrt{5}}\delta_{\frac{\sqrt{5}+1}{2}}(B)\ , \quad \forall B \in \mathcal{B}$$

In the case of $a = \frac{3}{2}$, we find that

$$a_1 = 1, \quad a_2 = \frac{1}{8}\ ; \quad A_1 = 0$$

This means that the distribution of $Q(f)^2$ for any $f \in \mathcal{H}_{1/2}$ is *absolutely continuous* with the probability density function $-\frac{3}{8\pi}g_{\frac{3}{2}}$, where $-g_{\frac{3}{2}}$ is defined as:

$$-g_{\frac{3}{2}}(x) = \frac{\sqrt{1-x}}{\sqrt{x}(1-x)\left(x+\frac{1}{8}\right)}\chi_{(0,1)}(x)$$

3.2. The distribution of $Q(f)^2$ with $q \in (-1,0)$ and $\|f\| = \frac{1}{2}$. Now, let's turn to treat the case of $0 < a < 1$ (or equivalently $q \in (-1,0)$). In this case, our main result is given as follows:

Theorem 3.6. *For any $a \in (0,1)$ and $f \in \mathcal{H}_{1/2}$, the distribution of $Q(f)^2$ is absolutely continuous, with the probability density function being the previously introduced g_a in (3.3).*

Proof: Recall from (3.3) that

$$g_a(x) = \frac{\sqrt{1-x}}{\sqrt{x}\left(x^2 - \frac{a+2}{4}x - \frac{1}{16(a-1)}\right)}\chi_{(0,1)}(x), \quad \forall x \in \mathbb{R}$$

With the assistance of formula (3.22), an elementary calculation shows that for any $a \in (0,1)$,

$$\int_0^1 \frac{\sqrt{x(1-x)}}{x^2 - \frac{a+2}{4}x - \frac{1}{16(a-1)}}dx = 2\pi\left(\frac{1}{a} - 1\right)$$

and

$$\int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}\left(x^2 - \frac{a+2}{4}x - \frac{1}{16(a-1)}\right)}dx = 8\pi\left(\frac{1}{a} - 1\right)$$

By utilizing these equalities and (3.22), we can deduce that, for any $a \in (0,1)$ and any $t \in (-1,1)$,

$$\int_0^1 \frac{g_a(x)}{1-tx}dx = \frac{8\pi(1-a)}{a} \cdot S_a(t)$$

$\square$

Corollary 3.7. *For any $a \in (0,2]$, we define*

$$p_a(x) := \frac{2a}{\pi} \cdot \frac{\sqrt{1-x}}{\sqrt{x}\,(1 + 4(a-1)(a+2)x - 16(a-1)x^2)}\chi_{(0,1)}(x), \quad \forall x \in \mathbb{R} \tag{3.27}$$

Then p_a is positive and measurable. Moreover, *1) for $a \in (0, \frac{3}{2}]$, p_a is the probability density function of the vacuum distribution of $Q(f)^2$ with $f \in \mathcal{H}_{\frac{1}{2}}$;*

2) for any $a \in \left(\frac{3}{2}, 2\right]$, p_a is **not** *a probability density function and the vacuum distribution of $Q(f)^2$ with $f \in \mathcal{H}_{\frac{1}{2}}$ is:*

$$B \longmapsto \int_B p_a(x)dx + \frac{A_1}{16(a-1)}\delta_{a_1}(B), \quad \forall B \in \mathcal{B}$$

Proof: Since $p_1(x) = \frac{2}{\pi} \cdot \frac{\sqrt{1-x}}{\sqrt{x}}$ is nothing else than the probability density function of ξ^2 with ξ having the Wigner distribution on the interval $(-1, 1)$, Proposition 3.1 confirms our result for $a = 1$. Now, let's consider the case of $1 \neq a \in (0, 2]$. In this case, we have, for any $x \in \mathbb{R}$, that

$$p_a(x) = \frac{a}{8\pi(1-a)} \frac{\sqrt{1-x}}{\sqrt{x}\left(x^2 - \frac{a+2}{4}x - \frac{1}{16(a-1)}\right)} \chi_{(0,1)}(x) = \frac{a}{8\pi(1-a)} g_a(x) \tag{3.28}$$

and moreover, Lemma 3.2 guarantees the positivity of the function p_a for any $a \in (0, 1) \cup (1, 2]$. Finally, by utilizing (3.28), Theorem 3.5 and Theorem 3.6, we can derive the results in affirmations 1) and 2) $\square$

3.3. The distribution of $Q(f)$. Now we are ready to provide the distribution of the field operator $Q(f)$ with a general test function $0 \neq f \in \mathcal{H}$. Our main result in this section is as follows:

Theorem 3.8. *For any $q \in (-1, 1]$ (equivalently, $a := 1 + q \in (0, 2]$), and for any $0 \neq f \in \mathcal{H}$, we denote*

- *$L_{q,f}$ as the vacuum distribution of the field operator $Q(f)$;*
- *function $l_{q,f}$ as*

$$l_{q,f}(x) := \frac{(1+q)\|f\|^2}{2\pi} \cdot \frac{\sqrt{4\|f\|^2 - x^2}}{(\|f\|^4 + q(q+3)\|f\|^2x^2 - qx^4)} \chi_{(-2\|f\|, 2\|f\|)}(x), \quad \forall x \in \mathbb{R} \tag{3.29}$$

Then the following statements are true:

1) for any $q \in \left(-1, \frac{1}{2}\right]$, $L_{q,f}$ is absolutely continuous, and $l_{q,f}$ is its probability density function;

2) for any $q \in \left(\frac{1}{2}, 1\right]$, $l_{q,f}$ is **not** *probability density function, and in this case,*

$$L_{q,f}(B) = \int_B l_{q,f}(x)dx + \frac{1}{2}\Big(1 - \frac{1+q}{\sqrt{q(q+4)}}\Big)\Big(\delta_{-\frac{\sqrt{a_1}}{2\|f\|^2}} + \delta_{\frac{\sqrt{a_1}}{2\|f\|^2}}\Big)(B)$$

for all Borel sets $B \in \mathcal{B}$, where a_1 is defined as in (3.6).

Remark 3.9. *It is evident that*

- *$\frac{1+q}{\sqrt{q(q+4)}} = \frac{q}{\sqrt{a^2+2a-3}}$ because $a = 1 + q$;*
- *for any $q \in \left(\frac{1}{2}, 1\right]$, we have $q(q+4) > q^2 + 2q + 1$, which implies $0 < \frac{1+q}{\sqrt{q(q+4)}} < 1$.*

Proof: [Proof of Theorem 3.8] (3.29) reveals that when $f \in \mathcal{H}_{1/2}$, $l_{q,f}$ is given by the function:

$$x \longmapsto \frac{2a}{\pi} \cdot \frac{\sqrt{1-x^2}}{1 + 4(a-1)(a+2)x^2 - 16(a-1)x^4} \chi_{(-1,1)}(x)$$

This is nothing else than the function $x \longmapsto |x|p_a(x^2)$, where p_a is introduced in (3.27). Therefore, for $f \in \mathcal{H}_{1/2}$, the two conclusions of Theorem 3.8 are guaranteed by Theorem 3.5, Theorem 3.6 and the following well–known facts:

If ξ is a 1–dimensional random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and it is **symmetric** *(meaning that ξ and $-\xi$ have the same distributed), then its distribution can be determined by the distribution of ξ^2. In particular, if the distribution of ξ^2 can be expressed as a sum $\nu_c + \nu_d$, where*

- *ν_c is absolutely continuous with a (sub–probability) density function f,*
- *ν_d is a discrete measure of the form $\nu_d = \sum_k p_k \delta_{b_k}$ with $\{p_k, b_k\} \subset (0, +\infty)$ for all k,*

then, the distribution of ξ can be expressed as $\nu_c' + \nu_d'$, where

- *ν_c' is absolutely continuous with the (sub–probability) density function $x \longmapsto |x| f(x^2) \chi_{(-\sqrt{c}, \sqrt{c})}(x)$;*
- *ν_d' is a discrete measure of the form $\nu_d' = \sum_k \frac{p_k}{2} \big(\delta_{-\sqrt{b_k}} + \delta_{\sqrt{b_k}}\big)$.*

In general, for any $f \in \mathcal{H} \setminus \{0\}$ and $a \in (0, 2]$, we can express $Q(f)$ as $2\|f\| Q\Big(\frac{f}{2\|f\|}\Big)$. Thus, our main result is obtained, as $\frac{f}{2\|f\|}$ belongs to $\mathcal{H}_{\frac{1}{2}}$ and in consideration of the following well known fact:

Let η be a 1–dimensional random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and let $c > 0$. If the distribution of η has the form $\nu_c + \nu_d$, where

- *ν_c is absolutely continuous with a (sub–probability) density function denoted as f,*
- *ν_d is a discrete measure of the form $\nu_d = \sum_k p_k \delta_{b_k}$ with $\{p_k, b_k\} \subset (0, +\infty)$ for all k,*

then the distribution of $c\eta$ follows the form of $\nu_c' + \nu_d'$ and where

- *ν_c' is absolutely continuous and characterized by the (sub–probability) density function $x \longmapsto \frac{1}{c} f(\frac{x}{c})$;*
- *ν_d' is a discrete measure of the form $\nu_d' = \sum_k p_k \delta_{cb_k}$.* $\square$

References

[1] M. Abramowitz, I.A. Stegun: Handbook of Mathematical Functions, Dover Publications (1965).

[2] L. Accardi, M. Bożejko: Interacting Fock space and Gaussianization of probability measures, Infin. Dimens. Anal. Quantum Probab. Relat. Top., v.1, n.1, pp.663–670 (1998).

[3] L. Accardi, Y.G. Lu: The Wigner semi-circle law in quantum electrodynamics, Commun. Math. Phys. 180, pp.605–632 (1996).

[4] L. Accardi, Y.G. Lu: The quantum moment problem for a classical random variable and a classification of interacting Fock spaces, Infin. Dimens. Anal. Quantum. Probab. Relat. Top., v.25, n.1, 2250003 (2022). DOI:10.1142/S0219025722500035

[5] M. Bożejko, R. Speicher: An example of a generalized Brownian motion. Commun. Math. Phys. 137, pp.519–531 (1991).

[6] D.F. Bailey: Counting Arrangements of 1's and -1's". Mathematics Magazine, 69 (2), pp.128–131 (1996). DOI:10.1080/0025570X.1996.11996408

[7] D. Bowman, A. Regev: Counting symmetry: classes of dissections of a convex regular polygon, Adv. Appl. Math. v.56, pp.35–55 (2014).

[8] E. Catalan: Sur les nombres de Segner, Rendiconti del Circolo Matematico di Palermo, 1, pp.190–201 (1887).

[9] J. Katriel, M. Kibler: Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers, J. Phys. A: Math. Gen., v.25, n.9, pp.2683–2691 (1992).

[10] P.J. Larcombe, D.R. French: The Catalan number k-fold self-convolution identity: the original formulation, J. Combin. Math. Combin. Comput., v.46, pp.191–204 (2003).

[11] Y.G. Lu: A New Interacting Fock Space, The Quon Algebra with Operator Parameter and Its Wick's Theorem, submitted to Reviews in Mathematical Physics, arXiv:5439654[math-ph]

[12] Y.G. Lu: Some combinatorial aspects of $(q, 2)-$Fock space, accepted for publication in Journal of Combinatorics.

[13] Y.G. Lu: Weighted Catalan convolution and $(q, 2)-$Fock space, submitted to Electronic Journal of Combinatorics, arXiv:5441085[math.CO]

[14] T. Mansour, M. Schork: On the normal ordering of multi-mode Boson operators Wick's theorem for q-deformed Boson operators. Russian J. Math. Phys., v.15, n.1, pp.77–88 (2008).

[15] T. Mansour, M. Schork, S. Severini: Wick's theorem for q-deformed Boson operators. J. Phys. A: Math. Gen., v.40, n.29, pp.8393–8401 (2007). DOI:10.1088/1751-8113/40/29/014

[16] A. Regev: A proof of Catalan's Convolution formula, Integers, v.12, n.5, pp.929–934 (2012). DOI.org/10.1515/integers-2012-0014, arxiv:1109.0571v1 [math.CO] 2 Sep 2011

[17] S. Reuveni: Catalan's Trapezoids. Probability in the Engineering and Informational Sciences, v.28 (03), pp.353–361 (2014). DOI:doi.org/10.1017/S0269964814000047
[18] L.W. Shapiro: A Catalan triangle, Discrete Math. 14, n.1, pp.83–90 (1976).
[19] S.J. Tedford: Combinatorial interpretations of convolutions of the Catalan numbers, Integers 11 (2011).