ON THE CHARACTER TABLES OF THE FINITE REDUCTIVE GROUPS $E_6(q)_{\mathbf{ad}}$ AND ${}^2E_6(q)_{\mathbf{ad}}$

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Abstract. We show how the character tables of the groups $E_6(q)_{\rm ad}$ and ${}^2E_6(q)_{\rm ad}$ can be constructed, where q is a power of 2. (Partial results are also obtained for any q not divisible by 3.) This is based on previous work by Hetz, Lusztig, Malle, Mizuno and Shoji, plus computations using Michel's version of CHEVIE. We also need some general results that are specific to semisimple groups which are not of simply connected type. A further crucial ingredient is the determination of the values of the unipotent characters on unipotent elements for groups of type D_4 and D_5 (in characteristic 2).

1. Introduction

Let p be a prime and $k = \overline{\mathbb{F}}_p$ be an algebraic closure of the field with p elements. Let $\mathbf{G}$ be a connected reductive algebraic group over k and assume that $\mathbf{G}$ is defined over the finite subfield $\mathbb{F}_q \subseteq k$, where q is a power of p. Let $F\colon \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius map and $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ be the finite group of fixed points. This paper is part of an ongoing project (involving various authors) to obtain as much information as possible about the character table of $\mathbf{G}^F$, where $\mathbf{G}$ is simple of exceptional type. (See the recent survey [15] for general background.) Here, we consider the case where $\mathbf{G}$ is simple, adjoint of type E_6. We shall obtain complete results for $p = 2$, and at least partial results for any $p \neq 3$. (For $p = 3$, our methods do not yield anything new.)

The general framework is provided by Lusztig's geometric approach [23], [28], starting from the generalised characters $R_{\mathbf{T},\theta}^{\mathbf{G}}$ of [3]. Our point of view will be that the values of these generalised characters are already known. Since, in general, they do not span the whole space of complex-valued class functions on $\mathbf{G}^F$, we then need to focus on determining the "missing" functions. For this purpose, one proceeds along a list $\Sigma_1, \ldots, \Sigma_N$ of the F-stable conjugacy classes of $\mathbf{G}$. For each i, the set Σ_i^F splits into conjugacy classes of $\mathbf{G}^F$ where the splitting is controlled by the finite group $A_{\mathbf{G}}(g_i) := C_{\mathbf{G}}(g_i)/C_{\mathbf{G}}^\circ(g_i)$ ($g_i \in \Sigma_i^F$). The size of $A_{\mathbf{G}}(g_i)$ is a kind of measure for the difficulty of computing the character values on elements in Σ_i^F. For example, if $A_{\mathbf{G}}(g_i) = \{1\}$, then Σ_i^F is a single $\mathbf{G}^F$-conjugacy class and the character values on g_i can be described in terms of the various $R_{\mathbf{T},\theta}^{\mathbf{G}}$. If $A_{\mathbf{G}}(g_i) \neq \{1\}$, then this is no longer true in general.

Date: May 11, 2024.

2020 *Mathematics Subject Classification.* Primary 20C33; Secondary 20G40.

Key words and phrases. Reductive groups, uniform functions, character sheaves.

In this set-up, it will be convenient to consider the vector space $\mathrm{CF}(\mathbf{G}^F \mid \Sigma_i)$ of all complex-valued class functions on $\mathbf{G}^F$ that are 0 on elements outside Σ_i^F. Then the aim is to find a basis of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma_i)$ such that

- the values of the functions in that basis are explicitly known and
- the decomposition of each function in that basis as a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$ is explicitly known.

If this problem is solved for each Σ_i, then we can construct the character table of $\mathbf{G}^F$ from the bases of the various subspaces $\mathrm{CF}(\mathbf{G}^F \mid \Sigma_i)$. Now Lusztig [28, 0.4] presents a general strategy for solving that problem (based on results in [27]), but this involves the delicate issue of fixing normalisations of characteristic functions of F-invariant character sheaves on $\mathbf{G}$. Similarly to our previous work [12] on type F_4, the special feature of groups of adjoint type E_6 in characteristic 2 is that one can largely avoid that issue, and only deal with it in some very special cases.

However, an additional complication for $\mathbf{G}$ of adjoint type E_6 arises from the fact that there are semisimple elements in $\mathbf{G}$ whose centraliser is not connected. In principle, this could be avoided by working with a simple group of simply connected type — but this would create numerous further complications of a different (and arguably more involved) nature, caused by the fact that the center is not connected. For example, the main results of Shoji [35], [36] (on which we heavily rely) are only proved for groups with a connected center. Another possibility would be to realise our group as $\mathbf{G} \cong \tilde{\mathbf{G}}/Z(\tilde{\mathbf{G}})$ where $\tilde{\mathbf{G}}$ has a simply connected derived subgroup *and* the center $Z(\tilde{\mathbf{G}})$ is connected. However, this would also create additional complications as far as characteristic functions of cuspidal character sheaves are concerned (which are needed in our argument). Thus, all in all, we have chosen to work directly with $\mathbf{G}$ simple, adjoint of type E_6.

In Section 2, we prepare some basic notation and recall (known) reduction techniques for the determination of character values. In Section 3, we consider first examples concerning the subspaces $\mathrm{CF}(\mathbf{G}^F \mid \Sigma_i)$ as above. In particular, the values of all unipotent characters on unipotent elements are determined for groups $\mathbf{G}$ of classical type D_4 and D_5 in characteristic 2. (These results, which are partly new and perhaps of independent interest, are needed later on in the discussion of groups of type E_6.) Section 4 contains the main general results of this paper. These provide some tools for dealing with the situation where $\mathbf{G}$ is semisimple but not of simply connected type.

For $\mathbf{G}$ simple, adjoint of type E_6 and $p = 2$, the values of the unipotent characters on unipotent elements have already been determined by Hetz [16], [17]. Most of the remaining work centers around a particular class Σ_i which is the supporting set for the cuspidal character sheaves on $\mathbf{G}$. In this case, we have $A_{\mathbf{G}}(g_i) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for $g_i \in \Sigma_i^F$, where the semisimple part of g_i has order 3 and a non-connected centraliser. In Section 5 we show how to find a basis of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma_i)$ which satisfies the above requirements; this relies on the results in Section 4 combined with previous results from [36], [11]. (And this part of the argument does not only work for $p = 2$ but for any $p \neq 3$; on the other hand, if $p = 3$, then there is no element g_i at all with the above properties, so our methods do not yield anything new for this case.) Once this is achieved, we can basically follow the strategy in [12] to complete the character table of $\mathbf{G}^F$; see Section 6.

Finally, we point out that we do describe a concrete procedure for constructing the character table of $\mathbf{G}^F$, but we shall not present actual tables of character values. With some more effort (mainly of a computational nature), it should certainly be possible to produce a "generic" table with the values of the unipotent characters on all elements of $\mathbf{G}^F$. I understand that Jonas Hetz (private communication) has recently produced such a "generic" table for the unipotent characters of $F_4(q)$ where q is a power of 2, following the strategy in [12]. However, there are non-unipotent characters whose values on certain elements are sums of roots of unity with $|\mathbf{W}|$ distinct terms, where $\mathbf{W}$ is the Weyl group of $\mathbf{G}$. For $\mathbf{G}$ of type E_6 we have $|\mathbf{W}| = 51840$, it is to be discussed if one really wants to write down explicit tables, or even store them on a computer.

2. Preliminaries

Let $\mathbf{G}$ be a connected reductive group and $F\colon \mathbf{G}\to\mathbf{G}$ be a Frobenius map, with respect to an $\mathbb{F}_q$-rational structure of $\mathbf{G}$. Let $\mathrm{CF}(\mathbf{G}^F)$ be the vector space of complex-valued class functions on $\mathbf{G}^F$, with standard inner product $\langle\ ,\ \rangle_{\mathbf{G}^F}$ given by

$$\langle f,f'\rangle_{\mathbf{G}^F} := |\mathbf{G}^F|^{-1}\sum_{g\in\mathbf{G}^F} f(g)\overline{f'(g)} \qquad \text{for } f,f'\in \mathrm{CF}(\mathbf{G}^F),$$

where the bar denotes complex conjugation. The set $\mathrm{Irr}(\mathbf{G}^F)$ of irreducible characters of $\mathbf{G}^F$ forms an orthonormal basis of $\mathrm{CF}(\mathbf{G}^F)$. For any subset $C\subseteq\mathbf{G}^F$ that is a union of $\mathbf{G}^F$-conjugacy classes, we denote by $\varepsilon_C\in\mathrm{CF}(\mathbf{G}^F)$ the indicator function of C, that is, for $x\in\mathbf{G}^F$ we have $\varepsilon_C(x)=1$ if $x\in C$, and $\varepsilon_C(x)=0$ otherwise. Note that, if C is a single $\mathbf{G}^F$-conjugacy class and $g\in C$, then

$$\rho(g) = |C_{\mathbf{G}}(g)^F|\langle\rho,\varepsilon_C\rangle_{\mathbf{G}^F} \qquad \text{for all } \rho\in\mathrm{Irr}(\mathbf{G}^F).$$

Let $\mathfrak{X}(\mathbf{G},F)$ be the set of all pairs $(\mathbf{T},\theta)$ where $\mathbf{T}\subseteq\mathbf{G}$ is an F-stable maximal torus and $\theta\in\mathrm{Irr}(\mathbf{T}^F)$. For such a pair $(\mathbf{T},\theta)$ let $R_{\mathbf{T},\theta}^{\mathbf{G}}\in\mathrm{CF}(\mathbf{G}^F)$ be the generalised character defined by Deligne and Lusztig [3]. It is known that every $\rho\in\mathrm{Irr}(\mathbf{G}^F)$ occurs in $R_{\mathbf{T},\theta}^{\mathbf{G}}$ for some $(\mathbf{T},\theta)\in\mathfrak{X}(\mathbf{G},F)$. Of particular interest is the case where this happens for $\theta=1$ (the trivial character): in that case, ρ is called a "unipotent" character. We refer to [6], [15] for general properties of the generalised characters $R_{\mathbf{T},\theta}^{\mathbf{G}}$.

2.1. Let $f\in\mathrm{CF}(\mathbf{G}^F)$. Then f is called a "uniform" function if f can be written as a linear combination of $R_{\mathbf{T},\theta}^{\mathbf{G}}$ for various $(\mathbf{T},\theta)\in\mathfrak{X}(\mathbf{G},F)$. In this case, we have (see [6, Prop. 10.2.4]):

$$f = |\mathbf{G}^F|^{-1}\sum_{(\mathbf{T},\theta)\in\mathfrak{X}(\mathbf{G},F)} |\mathbf{T}^F|\langle f, R_{\mathbf{T},\theta}^{\mathbf{G}}\rangle_{\mathbf{G}^F} R_{\mathbf{T},\theta}^{\mathbf{G}}.$$

For example, if Σ is an F-stable conjugacy class of $\mathbf{G}$, then Σ^F is a union of $\mathbf{G}^F$-conjugacy classes and the indicator function $\varepsilon_{\Sigma^F}\in\mathrm{CF}(\mathbf{G}^F)$ is a uniform function; see [8, Appendix] (and also [31] for a different proof).

The subspace of $\mathrm{CF}(\mathbf{G}^F)$ consisting of all uniform functions provides a first, and very useful approximation to $\mathrm{CF}(\mathbf{G}^F)$ (especially for $\mathbf{G}$ of low rank, $\leqslant 8$ say).

Example 2.2. Let C be a $\mathbf{G}^F$-conjugacy class and $g \in C$. Assume that the indicator function $\varepsilon_C \in \mathrm{CF}(\mathbf{G}^F)$ is a uniform function. Let $\rho \in \mathrm{Irr}(\mathbf{G}^F)$. Then, expressing ε_C as in 2.1, we obtain the following formula:

$$\rho(g) = |C_{\mathbf{G}}(g)^F|\langle \rho, \varepsilon_C\rangle_{\mathbf{G}^F} = |\mathbf{G}^F|^{-1} \sum_{(\mathbf{T},\theta)\in\mathfrak{X}(\mathbf{G},F)} |\mathbf{T}^F|\, \langle R_{\mathbf{T},\theta}^{\mathbf{G}}, \rho\rangle_{\mathbf{G}^F}\, R_{\mathbf{T},\theta^{-1}}^{\mathbf{G}}(g).$$

Hence, in this case, the value $\rho(g)$ is determined by the multiplicities $\langle R_{\mathbf{T},\theta}^{\mathbf{G}}, \rho\rangle_{\mathbf{G}^F}$ and the values $R_{\mathbf{T},\theta}^{\mathbf{G}}(g)$, where $(\mathbf{T},\theta)$ runs over all pairs in $\mathfrak{X}(\mathbf{G},F)$.

This situation occurs, for example, when the center $Z(\mathbf{G})$ is connected and the root system of $\mathbf{G}$ is a direct sum of root systems of type A_m, for various $m \geqslant 1$. In that case, *every* class function on $\mathbf{G}^F$ is uniform; see [15, Cor. 2.4.19] (and the references there).

Example 2.3. A class function $\gamma \in \mathrm{CF}(\mathbf{G}^F)$ is called p-constant if, for all $g \in \mathbf{G}^F$, we have $\gamma(g) = \gamma(s)$ where $s \in \mathbf{G}^F$ is the semisimple part of g. In this case, we have:

$$f \in \mathrm{CF}(\mathbf{G}^F) \text{ uniform} \qquad \Rightarrow \qquad \gamma{\cdot}f \in \mathrm{CF}(\mathbf{G}^F) \text{ uniform}.$$

(This immediately follows from [6, Prop. 10.1.6].) Let $(\mathbf{T},\theta) \in \mathfrak{X}(\mathbf{G},F)$ and $u \in \mathbf{G}^F$ be unipotent. Then the value $R_{\mathbf{T},\theta}^{\mathbf{G}}(u)$ does not depend on θ, and will be denoted by $Q_{\mathbf{T}}^{\mathbf{G}}(u)$. The function $u \mapsto Q_{\mathbf{T}}^{\mathbf{G}}(u)$ is called the Green function associated with $\mathbf{T}$. We may regard $Q_{\mathbf{T}}^{\mathbf{G}}$ as a function on all of $\mathbf{G}^F$ by letting its value be 0 on non-unipotent elements. With this convention, we have $Q_{\mathbf{T}}^{\mathbf{G}} = \gamma_{\mathrm{uni}}^{\mathbf{G}}{\cdot}R_{\mathbf{T},\theta}^{\mathbf{G}}$ where $\gamma_{\mathrm{uni}}^{\mathbf{G}} \in \mathrm{CF}(\mathbf{G}^F)$ is the class function which takes value 1 on unipotent elements, and value 0 otherwise. Then $\gamma_{\mathrm{uni}}^{\mathbf{G}}$ is p-constant and so $Q_{\mathbf{T}}^{\mathbf{G}}$ (as a function on all of $\mathbf{G}^F$) is uniform.

2.4. An F-stable closed subgroup $\mathbf{L} \subseteq \mathbf{G}$ is called a "regular subgroup" if $\mathbf{L}$ is a Levi complement in some (not necessarily F-stable) parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$. These subgroups can be used for inductive arguments, in the following situation. Let $g \in \mathbf{G}$ and $g = su = us$ be the Jordan decomposition of g, where $s \in \mathbf{G}$ is semisimple and $u \in \mathbf{G}$ is unipotent. We say that g (or its conjugacy class) is "isolated" if $C_{\mathbf{G}}^\circ(s)$ is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$. Assume now that $g \in \mathbf{G}^F$ is not isolated. Then there will be a regular subgroup $\mathbf{L} \subsetneqq \mathbf{G}$ such that $C_{\mathbf{G}}^\circ(s) \subseteq \mathbf{L}$. In this case, the values $\rho(g)$, for any $\rho \in \mathrm{Irr}(\mathbf{G}^F)$, can be computed using Schewe's formula

$$\rho(g) = \sum_{\psi\in\mathrm{Irr}(\mathbf{L}^F)} \langle R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\psi), \rho\rangle_{\mathbf{G}^F}\psi(g) \qquad \text{(see [15, Cor. 3.3.13]).}$$

Here, $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}$ is Lusztig's "twisted induction"; see [6, §9.1]. In order to evaluate the above formula, we need to assume, firstly, that the character table of the smaller group $\mathbf{L}^F \subsetneqq \mathbf{G}^F$ is known — which we can do by an inductive procedure. Secondly, we need to know the decomposition of $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\psi)$, for any $\psi \in \mathrm{Irr}(\mathbf{L}^F)$, as a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$. For the groups that we are going to consider here, the latter problem is under control as discussed in [15, §4.6, §4.7]. If ψ is a unipotent character, then the solution is implemented by the function `LusztigInductionTable` in Michel's CHEVIE [33].

2.5. Let Σ be an F-stable conjugacy class of $\mathbf{G}$. We recall how the splitting of Σ^F into conjugacy classes of $\mathbf{G}^F$ is determined. Let us fix an element $g_0 \in \Sigma^F$ and set $A_{\mathbf{G}}(g_0) := C_{\mathbf{G}}(g_0)/C_{\mathbf{G}}^\circ(g_0)$. Then F induces an automorphism of $A_{\mathbf{G}}(g_0)$ that we denote by the same symbol. Given $a \in A_{\mathbf{G}}(g_0)$, let $\dot{a} \in C_{\mathbf{G}}(g_0)$ be a representative of a. By Lang's Theorem, we can write $\dot{a} = x^{-1}F(x)$ for some $x \in \mathbf{G}$; then $g_a := xgx^{-1} \in \Sigma^F$. Let C_a be the $\mathbf{G}^F$-conjugacy class of g_a. One easily checks that C_a only depends on a (and not on the choice of x); furthermore $\Sigma^F = \bigcup_{a\in A_{\mathbf{G}}(g_0)} C_a$, where $C_a = C_{a'}$ if and only if a, a' are F-conjugate in $A_{\mathbf{G}}(g_0)$, that is, there exists some $b \in A_{\mathbf{G}}(g_0)$ such that $a' = baF(b)^{-1}$. (See [15, §2.7] for further details.) In particular, if $C_{\mathbf{G}}(g_0)$ is connected, then Σ^F is a single $\mathbf{G}^F$-conjugacy class.

Example 2.6. Let Σ be an F-stable conjugacy class of $\mathbf{G}$. Let $g_0 \in \Sigma^F$ and assume that the group $A_{\mathbf{G}}(g_0)$ is abelian. Then the map $\varphi\colon A_{\mathbf{G}}(g_0) \to A_{\mathbf{G}}(g_0)$, $a \mapsto a^{-1}F(a)$, is a group homomorphism with kernel equal to $A_{\mathbf{G}}(g_0)^F$; the F-conjugacy classes of $A_{\mathbf{G}}(g_0)$ are precisely the cosets of $A_{\mathbf{G}}(g_0)$ modulo the image of φ. Thus, we conclude:

(a) $|A_{\mathbf{G}}(g_0)^F| =$ number of F-conjugacy classes of $A_{\mathbf{G}}(g_0)$.
(b) If $A_{\mathbf{G}}(g_0)^F = \{1\}$, then $C := \Sigma^F$ is a single conjugacy class of $\mathbf{G}^F$.

In case (b), we can apply the remarks in 2.1; then the indicator function $\varepsilon_C \in \mathrm{CF}(\mathbf{G}^F)$ is a uniform function and so the values $\rho(g)$ for any $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ can be determined using the formula in Example 2.2.

2.7. Let Σ be an F-stable conjugacy class of $\mathbf{G}$. There is a natural operation on the $\mathbf{G}^F$-conjugacy classes contained in Σ^F, defined as follows. Let $g_0 \in \Sigma^F$ be fixed. If C is a $\mathbf{G}^F$-conjugacy class contained in Σ^F, then $C = C_a$ where $a \in A_{\mathbf{G}}(g_0)$. (Notation as in 2.5.) Let $g \in C$ and write $g = x^{-1}F(x)$ where $x \in \mathbf{G}$. Then one easily checks that $g' := F(x)x^{-1} = xgx^{-1} \in \mathbf{G}^F$ and that the $\mathbf{G}^F$-conjugacy class of g' does not depend on the choice of g or x; we denote the $\mathbf{G}^F$-conjugacy class of g' by $t_1(C)$; clearly, we have $t_1(C) \subseteq \Sigma^F$. The map $C \mapsto t_1(C)$ is called "twisting operator"; see, e.g., Shoji [35, 1.16] or Digne–Michel [5, Chap. IV]. One easily shows that

$$t_1(C_a) = C_{\bar{g}_0 a} \qquad \text{for all } a \in A_{\mathbf{G}}(g_0),$$

where $\bar{g}_0$ denotes the image of g_0 in $A_{\mathbf{G}}(g_0)$. We shall also need the following result.

Lemma 2.8. *In the setting of 2.7, let $a \in A_{\mathbf{G}}(g_0)$. If $g_1 \in C_a$ and $g_2 \in t_1(C_a) = C_{\bar{g}_0 a}$, then the semisimple parts of g_1, g_2 are conjugate in $\mathbf{G}^F$.*

Proof. Let $\dot{a} \in C_{\mathbf{G}}(g_0)$ be a representative of a and write $\dot{a} = y^{-1}F(y)$ where $y \in \mathbf{G}$. Then $g_a := yg_0y^{-1} \in C_a$; so we may assume that $g_1 = g_a$. Now write $g_a = s_au_a = u_as_a$ where $s_a \in \mathbf{G}^F$ is semisimple and $u_a \in \mathbf{G}^F$ is unipotent. Then $g_a \in C_{\mathbf{G}}^\circ(s_a)$; see [6, Prop. 3.5.3]. By Lang's Theorem applied to $C_{\mathbf{G}}^\circ(s_a)$, we can find some $x \in C_{\mathbf{G}}^\circ(s_a)$ such that $g_a = x^{-1}F(x)$. By 2.7, we have $g' := xg_ax^{-1} \in C_{\bar{g}_0 a}$; so we may assume that $g_2 = g'$.

Finally, note that $g' = xg_ax^{-1} = xs_au_ax^{-1} = s_a(xu_ax^{-1})$ since $x \in C^\circ_{\mathbf{G}}(s_a)$. Hence, s_a also is the semisimple part of g'. $\square$

Remark 2.9. Let Σ be an F-stable conjugacy class of $\mathbf{G}$. Let $\Sigma_{p'}$ be the set of semisimple parts of elements of Σ; then $\Sigma_{p'}$ certainly is an F-stable conjugacy class of $\mathbf{G}$. Note that, in general, not every element in $\Sigma_{p'}^F$ is the semisimple part of an element in Σ^F. But, at least, in the following special situation (which will be important for us later on; see 5.5 below) this will be the case. Let $g \in \Sigma^F$ and write $g = su = us$ where $s \in \Sigma_{p'}^F$ and $u \in \mathbf{G}^F$ is unipotent. Assume that u is regular unipotent in $C^\circ_{\mathbf{G}}(s)$. (See [6, §12.2] for general properties of regular unipotent elements.) We claim that, in this case, every element in $\Sigma_{p'}^F$ arises as the semisimple part of an element in Σ^F.

To see this, let $s' \in \Sigma_{p'}^F$ be arbitrary. It is known that regular unipotent elements always exist in connected reductive groups, and that they form a single conjugacy class. Thus, we can find a regular unipotent element $u' \in C^\circ_{\mathbf{G}}(s')$ such that $F(u') = u'$. Since $s, s' \in \Sigma_{p'}$, there exists some $x \in \mathbf{G}$ such that $s = xs'x^{-1}$; then the element $xu'x^{-1}$ is regular unipotent in $C^\circ_{\mathbf{G}}(s)$. Hence, u and $xu'x^{-1}$ are conjugate in $C^\circ_{\mathbf{G}}(s)$, and so the elements su and $s'u'$ are conjugate in $\mathbf{G}$. Thus, s' is the semisimple part of $g' := s'u' \in \Sigma^F$.

3. On the subspaces $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$

We keep the basic assumptions of the previous section. Let us now fix an F-stable conjugacy class Σ of $\mathbf{G}$. We denote by $\mathrm{CF}(\mathbf{G}^F \mid \Sigma) \subseteq \mathrm{CF}(\mathbf{G}^F)$ the subspace consisting of all class functions that take value 0 on elements outside of Σ^F. Then the general aim will be to find a basis $\{f_i\}$ of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$ such that

$(\Sigma)_1$ the values $f_i(g)$ for $g \in \Sigma^F$ can be computed explicitly and

$(\Sigma)_2$ the decomposition of each f_i as a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$ is known.

Assume that we have found such a basis $\{f_i\}$. Let $g \in \Sigma^F$ and C be the $\mathbf{G}^F$-conjugacy class of g. Let $\varepsilon_C \in \mathrm{CF}(\mathbf{G}^F)$ be the indicator function of C. Using $(\Sigma)_1$, we can express ε_C as a linear combination of the functions f_i. Now let $\rho \in \mathrm{Irr}(\mathbf{G}^F)$. By $(\Sigma)_2$, we know the inner products $\langle \rho, f_i\rangle_{\mathbf{G}^F} = \overline{\langle f_i, \rho\rangle_{\mathbf{G}^F}}$; hence, $\rho(g) = |C_{\mathbf{G}}(g)^F|\langle \rho, \varepsilon_C\rangle_{\mathbf{G}^F}$ can be computed.

As already mentioned in the introduction, Lusztig [28] presents a general strategy for finding a basis $\{f_i\}$ as above, but this involves the delicate issue of fixing normalisations of characteristic functions of F-invariant character sheaves on $\mathbf{G}$. In this section, we discuss some special cases where this issue takes a much simpler form.

First of all, a natural candidate for a basis element of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$ is the indicator function ε_{Σ^F} of the whole set Σ^F. If $s \in \Sigma_{p'}^F$ (notation as in Remark 2.9) and $C_{\mathbf{G}}(s)$ is not connected, then we obtain a slight refinement of ε_{Σ^F} by taking the indicator function on the set of those elements $g \in \Sigma^F$ such that the semisimple part of g is $\mathbf{G}^F$-conjugate to s. Again, it turns out that this refinement is a uniform function; this is the subject of the following section. Now we shall consider further candidates for functions in $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$.

3.1. We recall some facts about (cuspidal) character sheaves; see Lusztig [28] and further references there. (See also [15, 2.7.24–2.7.27] for an informal introduction.) Let A be a character sheaf on $\mathbf{G}$ such that $F^*A \cong A$. Then the choice of an isomorphism $F^*A \cong A$ gives rise to a non-zero class function $\chi_A \in \mathrm{CF}(\mathbf{G}^F)$ which is called a "characteristic function" of A; it is well-defined up to multiplication by a scalar. Assume now that

(a) $\qquad$ $\mathbf{G}$ is semisimple and A is a cuspidal character sheaf.

Then there is a unique F-stable conjugacy class Σ of $\mathbf{G}$ such that $\chi_A(g) = 0$ for all $g \in \mathbf{G}^F \setminus \Sigma^F$, that is, $\chi_A \in \mathrm{CF}(\mathbf{G}^F \mid \Sigma)$. (This follows from the fact that A is "clean" [30].) Let us fix a representative $g_0 \in \Sigma^F$. Then A corresponds to an irreducible character $\psi \in \mathrm{Irr}(A_{\mathbf{G}}(g_0))$ which is invariant under the action of F on $A_{\mathbf{G}}(g_0)$. Let us assume that ψ is a linear character. (This assumption will be satisfied in all examples that we consider; it implies that ψ is constant on the F-conjugacy classes of $A_{\mathbf{G}}(g_0)$.) Then the isomorphism $F^*A \cong A$ can be chosen such that the values of the corresponding characteristic function χ_A on Σ^F are given by

(b) $$\chi_A(g) = q^{(\dim \mathbf{G} - \dim \Sigma)/2}\psi(a) \qquad \text{if } g \in C_a \subseteq \Sigma^F \text{ where } a \in A_{\mathbf{G}}(g_0).$$

(See [11, 4.2] and the further references there.) We also set $\lambda_A := \psi(\bar{g}_0)$ where $\bar{g}_0$ denotes the image of g_0 in $A_{\mathbf{G}}(g_0)$. This number has the following alternative interpretation in terms of the twisting operator t_1 defined in 2.7:

(c) $$t_1^*(\chi_A) := \chi_A \circ t_1 = \lambda_A \chi_A; \qquad \text{see Shoji [35, 3.3, 3.8].}$$

Hence, $\chi_A \in \mathrm{CF}(\mathbf{G}^F)$ is an eigenvector for the operator $t_1^*\colon \mathrm{CF}(\mathbf{G}^F) \to \mathrm{CF}(\mathbf{G}^F)$ induced by t_1, where the corresponding eigenvalue is given by λ_A.

Thus, if Σ is the supporting set of an F-invariant cuspidal character sheaf A on $\mathbf{G}$, then χ_A is a natural candidate for a basis element of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$, where the remaining problem is to find the decomposition of χ_A as a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$. (That problem is solved in many cases, but not in complete generality; see [28], [11]).

Example 3.2. Let $p = 2$ and $\mathbf{G}$ be simple of type D_4. By [26, Prop. 19.3(d")] (and its proof), there is unique cuspidal character sheaf A_0 on $\mathbf{G}$; it is automatically F-invariant. Furthermore, the support of A_0 is given by $\Sigma = \mathscr{O}_0$ where $\mathscr{O}_0$ is the conjugacy class of regular unipotent elements. Let us fix a representative $u_0 \in \mathscr{O}_0^F$. By [20, Table 8.5a (p. 126)], we have $\dim C_{\mathbf{G}}(u_0) = 4$ and $A_{\mathbf{G}}(u_0) \cong \mathbb{Z}/2\mathbb{Z}$. Since A_0 corresponds to the non-trivial character of $A_{\mathbf{G}}(u_0)$, the function $\chi_{A_0} \in \mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F \mid \mathscr{O}_0)$ (see 3.1) is given by

$$\chi_{A_0}(u) = \begin{cases} q^2 & \text{if } u \in \mathscr{O}_0^F \text{ is } \mathbf{G}^F\text{-conjugate to } u_0, \\ -q^2 & \text{if } u \in \mathscr{O}_0^F \text{ is not } \mathbf{G}^F\text{-conjugate to } u_0. \end{cases}$$

Thus, χ_{A_0} and the indicator function $\varepsilon_{\mathscr{O}_0^F}$ form a basis of $\mathrm{CF}(\mathbf{G}^F \mid \mathscr{O}_0)$. It remains to address the following two issues (see Proposition 3.9 below):

1) find the decomposition of χ_{A_0} into irreducible characters of $\mathbf{G}^F$,
2) specify the choice of a representative $u_0 \in \mathscr{O}_0^F$.

Example 3.3. Let $p \neq 3$ and $\mathbf{G}$ be simple, adjoint of type E_6. By [26, Cor. 20.4] (and its proof) there are 6 cuspidal character sheaves on $\mathbf{G}$. They all have the same support; it is given by an F-stable conjugacy class Σ where $A_{\mathbf{G}}(g_0) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for $g_0 \in \Sigma^F$. If F acts trivially on $A_{\mathbf{G}}(g_0)$, then $\dim \mathrm{CF}(\mathbf{G}^F \mid \Sigma) = 9$. The characteristic functions of the 6 cuspidal character sheaves yield 6 basis functions in $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$. So we still need to find three further functions; these will be provided by the results in the following section.

For $\mathbf{G}$ of type E_6, we will also need some information about the subspaces $\mathrm{CF}(\mathbf{L}^F \mid \Sigma)$ where $\mathbf{L} \subseteq \mathbf{G}$ is a regular subgroup (as in 2.4) and Σ is a unipotent class of $\mathbf{L}$. Below we shall consider some examples which are relevant for the discussion in Section 6. First, some preparations.

3.4. Let $\mathbf{G}_{\mathrm{uni}}$ be the set of unipotent elements of $\mathbf{G}$ and $\mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)$ be the vector space of all functions $\mathbf{G}_{\mathrm{uni}}^F \to \mathbb{C}$ that are constant on the $\mathbf{G}^F$-conjugacy classes in $\mathbf{G}_{\mathrm{uni}}^F$. By restriction, we obtain a linear map $\pi_{\mathrm{uni}}^{\mathbf{G}} \colon \mathrm{CF}(\mathbf{G}^F) \to \mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)$. It will be convenient to regard the functions in $\mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)$ as class functions on all of $\mathbf{G}^F$, by letting their value be 0 on elements outside $\mathbf{G}_{\mathrm{uni}}^F$. With this convention, we have

$$\pi_{\mathrm{uni}}^{\mathbf{G}}(f) = \gamma_{\mathrm{uni}}^{\mathbf{G}} \cdot f \qquad \text{for any } f \in \mathrm{CF}(\mathbf{G}^F),$$

where the function $\gamma_{\mathrm{uni}}^{\mathbf{G}} \in \mathrm{CF}(\mathbf{G}^F)$ has been defined in Example 2.3.

The following two results are probably known to the experts. Since we have not found a convenient reference, we sketch the proofs.

Lemma 3.5. *Assume that $p = 2$, that the center of $\mathbf{G}$ is connected and that $\mathbf{G}/Z(\mathbf{G})$ only has simple components of type A_n, B_n, C_n or D_n. Then the restrictions of the unipotent characters of $\mathbf{G}^F$ to $\mathbf{G}_{\mathrm{uni}}^F$ span $\mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)$.*

Proof. Clearly, $\mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)$ is spanned by the functions $\pi_{\mathrm{uni}}^{\mathbf{G}}(\rho)$ for $\rho \in \mathrm{Irr}(\mathbf{G}^F)$. We must show that it is sufficient to take $\pi_{\mathrm{uni}}^{\mathbf{G}}(\rho)$ for $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ unipotent. The main point of the argument is that our assumptions on $\mathbf{G}$ imply that $C_{\mathbf{G}^*}(s)$ is a regular subgroup for every semisimple element $s \in \mathbf{G}^{*F^*}$, where $\mathbf{G}^*$ is a group dual to $\mathbf{G}$ with dual Frobenius map $F^* \colon \mathbf{G}^* \to \mathbf{G}^*$. (This follows from [13, §2.2, §2.3], and this has already been used in the proof of [22, 8.7].) Hence, we are in the "Levi case" of Lusztig's Jordan decomposition of characters; see [6, §11.4], [15, §3.3]. This means that every $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is of the form $\rho = \pm R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\lambda\cdot\psi)$ where $\mathbf{L}$ is an F-stable Levi complement of a parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, $\lambda \colon \mathbf{L}^F \to \mathbb{C}^\times$ is a p-constant linear character, and $\psi \in \mathrm{Irr}(\mathbf{L}^F)$ is a unipotent character. Now recall from 3.4 that $\pi_{\mathrm{uni}}^{\mathbf{G}}(\rho) = \gamma_{\mathrm{uni}}^{\mathbf{G}}\cdot\rho$. Since $\gamma_{\mathrm{uni}}^{\mathbf{G}}$ is p-constant, we have

$$\pi_{\mathrm{uni}}^{\mathbf{G}}(\rho) = \gamma_{\mathrm{uni}}^{\mathbf{G}}\cdot\rho = \pm\gamma_{\mathrm{uni}}^{\mathbf{G}}\cdot R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}\big(\lambda\cdot\psi\big) = \pm R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}\big(\gamma_{\mathrm{uni}}^{\mathbf{L}}\cdot(\lambda\cdot\psi)\big);$$

see [15, Prop. 3.3.16] for the last equality. Since λ is also p-constant, we certainly have $\gamma_{\mathrm{uni}}^{\mathbf{L}}\cdot(\lambda\cdot\psi) = \gamma_{\mathrm{uni}}^{\mathbf{L}}\cdot\psi$. Hence, using once more [15, Prop. 3.3.16], we conclude that

$$\pi_{\mathrm{uni}}^{\mathbf{G}}(\rho) = \pm R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}\big(\gamma_{\mathrm{uni}}^{\mathbf{L}}\cdot\psi\big) = \pm\gamma_{\mathrm{uni}}^{\mathbf{G}}\cdot R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\psi) = \pm\pi_{\mathrm{uni}}^{\mathbf{G}}\big(R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\psi)\big).$$

This completes the proof, since it is known that $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi)$ is a linear combination of unipotent characters of $\mathbf{G}^F$; see [15, Prop. 3.3.20]. $\square$

Proposition 3.6. *Let $p=2$ and $\mathbf{G}$ be simple of type A_n, B_n, C_n or D_n. Then the restrictions of the unipotent characters of $\mathbf{G}^F$ to $\mathbf{G}^F_{\rm uni}$ form a basis of $\mathrm{CF}_{\rm uni}(\mathbf{G}^F)$.*

Proof. Clearly, $\dim \mathrm{CF}_{\rm uni}(\mathbf{G}^F)$ is equal to the number of $\mathbf{G}^F$-conjugacy classes of unipotent elements of $\mathbf{G}^F$. By [22, 7.14, 8.7], the latter number is also equal to the number of unipotent characters of $\mathbf{G}^F$. So the assertion follows from Lemma 3.5. $\square$

3.7. Let $\mathscr{O}$ be an F-stable unipotent class of $\mathbf{G}$ and assume that $\mathscr{O}^F$ splits into conjugacy classes $C_1,\ldots C_r$ of $\mathbf{G}^F$. The following discussion will be helpful in distinguishing these $\mathbf{G}^F$-classes. Let $\mathbf{B}\subseteq\mathbf{G}$ be an F-stable Borel subgroup and $\mathbf{W}$ be the Weyl group of $\mathbf{G}$ with respect to an F-stable maximal torus contained in $\mathbf{B}$. Let $w\in\mathbf{W}^F$ and consider the Bruhat cell $\mathbf{B}w\mathbf{B}$. The cardinality $|\mathscr{O}^F\cap\mathbf{B}^Fw\mathbf{B}^F|$ can be explicitly determined in terms of a formula involving the Green functions of $\mathbf{G}^F$, Lusztig's non-abelian Fourier matrices and the character values of the Hecke algebra of $\mathbf{W}^F$; see [29, 1.2] for details. (In CHEVIE, this can be done as explained in [33, §6].) Assume now that $\mathscr{O}\cap\mathbf{B}w\mathbf{B}\neq\varnothing$. Then —at least in certain favorable situations (see below for examples) — the notation can be chosen such that

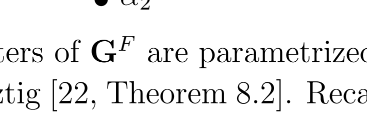
(a) $$C_1\cap\mathbf{B}^Fw\mathbf{B}^F\neq\varnothing \qquad\text{and}\qquad C_i\cap\mathbf{B}^Fw\mathbf{B}^F=\varnothing \quad\text{for } i=2,\ldots,r.$$

Consequently, we have $|C_1\cap\mathbf{B}^Fw\mathbf{B}^F|=|\mathscr{O}^F\cap\mathbf{B}^Fw\mathbf{B}^F|$, and the right hand is known; furthermore, $|C_i\cap\mathbf{B}^Fw\mathbf{B}^F|=0$ for $i=2,\ldots,r$. On the other hand, $|C_i\cap\mathbf{B}^Fw\mathbf{B}^F|$ can also be expressed in terms of a formula involving the values of certain unipotent characters of $\mathbf{G}^F$ and, again, the character values of the Hecke algebra of $\mathbf{W}^F$. If not yet all values of the unipotent characters of $\mathbf{G}^F$ on $C_1,\ldots,C_r$ are known, then that formula yields r linear relations for the unknown values. (This argument has already been used, for example, in the proof of [11, Prop. 6.5]; see also Hetz [16] and further references there.)

For groups of small rank ($\leqslant 7$, say), the Green functions $Q^{\mathbf{G}}_{\mathbf{T}}$ in Example 2.3 typically span a significant subspace of $\mathrm{CF}_{\rm uni}(\mathbf{G}^F)$, and the values of $Q^{\mathbf{G}}_{\mathbf{T}}$ can be explicitly computed using the methods in [10] (especially for the case where q is a power of a small prime p). In the examples below, we will assume that tables with the values of the functions $Q^{\mathbf{G}}_{\mathbf{T}}$ are known (but we will not print these tables).

We shall now consider groups $\mathbf{G}$ of type D_n, where we use the following labelling of simple roots in a root system of $\mathbf{G}$:

D_n
$(n\geqslant 4)$



In this case, the unipotent characters of $\mathbf{G}^F$ are parametrized by (equivalence classes of) certain symbols of rank n; see Lusztig [22, Theorem 8.2]. Recall from [22, §3] that symbols are unordered pairs $\Lambda=(S,T)$ of finite subsets of $\mathbb{Z}_{\geqslant 0}$. The rank of Λ is defined as in

[22, 2.6.4], the defect of Λ is the absolute value of $|S|-|T|$. We denote the unipotent character corresponding to $\Lambda=(S,T)$ by $\rho_{(S,T)}$.

3.8. Let $p=2$ and $\mathbf{G}$ be simple of type D_4.

(a) Assume that $\mathbf{G}^F=D_4(q)$ (split type). By [20, Table 8.5a (p. 126)], there are 12 unipotent classes in $\mathbf{G}$, which split into 14 unipotent classes in $\mathbf{G}^F=D_4(q)$. Hence, we also have $\dim \mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)=14$. The unipotent characters of $\mathbf{G}^F$ are parametrized by symbols of rank 4 and defect $d\equiv 0 \bmod 4$, where each symbol of the form (S,S) is repeated twice. We define a class function $f_0\in \mathrm{CF}(\mathbf{G}^F)$ by

$$f_0 := \tfrac{1}{2}(\rho_{(13,02)}-\rho_{(23,01)}-\rho_{(12,03)}+\rho_{(0123,-)})$$

(where $(013,2)$ stands for the symbol $\Lambda=(\{0,1,3\},\{2\})$ etc.)

(b) Assume that $\mathbf{G}^F={}^2D_4(q)$ (twisted type). Referring again to [20, Table 8.5a (p. 126)], we see that only 8 out of the 12 unipotent classes of $\mathbf{G}$ are F-stable; these split into 10 unipotent classes in $\mathbf{G}^F={}^2D_4(q)$. Hence, we also have $\dim \mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)=10$. Now the unipotent characters of $\mathbf{G}^F$ are parametrized by symbols of rank 4 and defect $d\equiv 2 \bmod 4$. We define a class function $f_0\in \mathrm{CF}(\mathbf{G}^F)$ by

$$f_0 := \tfrac{1}{2}(\rho_{(123,0)}-\rho_{(012,3)}+\rho_{(013,2)}-\rho_{(023,1)}.$$

The significance of these definitions is as follows. By Main Theorem 4.23 of [23], the decomposition of each generalised character $R_{\mathbf{T},1}^{\mathbf{G}}$ as a linear combination of unipotent characters is known. (In CHEVIE [33], this information is readily available via the functions `UnipotentCharacters` and `DeligneLusztigCharacter`; see also [15, §2.4].) For f_0 as defined above, we find that $\langle f_0, R_{\mathbf{T},1}^{\mathbf{G}}\rangle_{\mathbf{G}^F}=0$ for all $\mathbf{T}$; furthermore, every unipotent character of $\mathbf{G}^F$ is uniquely a linear combination of f_0 and the various $R_{\mathbf{T},1}^{\mathbf{G}}$. Hence, in order to determine the values of the unipotent characters on unipotent elements (and even on all elements of $\mathbf{G}^F$), it remains to determine the values of f_0.

In the split case, the following result is a very special case of Shoji [38, Theorem 6.2].

Proposition 3.9. *In the setting of 3.8, let $\mathscr{O}_0$ be the conjugacy class of regular unipotent elements of $\mathbf{G}$ and $u_0 := x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)\in\mathscr{O}_0^F$. Then, for any $g\in\mathbf{G}^F$, we have*

$$f_0(g)=\begin{cases} q^2 & \text{if } g\in\mathscr{O}_0^F \text{ and } g \text{ is } \mathbf{G}^F\text{-conjugate to } u_0,\\ -q^2 & \text{if } g\in\mathscr{O}_0^F \text{ and } g \text{ is not } \mathbf{G}^F\text{-conjugate to } u_0,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, we have $f_0=\chi_{A_0}$ where χ_{A_0} is the characteristic function in Example 3.2.

Proof. Assume first that $\mathbf{G}^F=D_4(q)$. As already discussed in [9, Example 4.4] (or in [38]), we do have $f_0=\chi_{A_0}$ in this case. Now let $\mathbf{G}^F={}^2D_4(q)$. Then we need a different argument since, in [9] and in [38], only groups of split type are considered. (The following argument would also work, almost verbatim, for the split case.) As already mentioned,

$\mathscr{O}_0^F$ splits into two classes in $\mathbf{G}^F$, with centraliser orders $2q^4, 2q^4$; let $u_0, u_1 \in \mathscr{O}_0^F$ (with u_0 as above) be representatives of these two classes. We define $f_0' \in \mathrm{CF}_{\rm uni}(\mathbf{G}^F)$ by

$$f_0'(u_0) := q^2, \qquad f_0'(u_1) := -q^2 \qquad \text{and} \qquad f_0'(u) := 0 \quad \text{for } u \in \mathbf{G}^F_{\rm uni} \setminus \mathscr{O}_0^F.$$

(Note that $f_0' = \chi_{A_0}$, but the following argument works without reference at all to the theory of character sheaves.) As usual, we regard f_0' as a function on all of $\mathbf{G}^F$ by letting its value be 0 on non-unipotent elements; then $\langle f_0', f_0'\rangle_{\mathbf{G}^F} = 1$. We must show that $f_0 = f_0'$.

Now, the distinct Green functions $Q_{\mathbf{T}}^{\mathbf{G}}$ are parametrized by the F-conjugacy classes of the Weyl group $\mathbf{W}$. (See, e.g., [15, §2.3].) One checks that there are 9 such F-conjugacy classes. Hence, since distinct Green functions are linearly independent and since $\dim \mathrm{CF}_{\rm uni}(\mathbf{G}^F) = 10$, the Green functions span a subspace of co-dimension 1 in $\mathrm{CF}_{\rm uni}(\mathbf{G}^F)$. Using the knowledge of the values of the Green functions, we find that $\langle f_0', Q_{\mathbf{T}}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = 0$ for all $\mathbf{T}$ (where, as usual, we regard each $Q_{\mathbf{T}}^{\mathbf{G}}$ as a function on all of $\mathbf{G}^F$ with value 0 on non-unipotent elements). Thus, we have:

(a) The Green functions $Q_{\mathbf{T}}^{\mathbf{G}}$ together with f_0' form a basis of $\mathrm{CF}_{\rm uni}(\mathbf{G}^F)$.

Next recall from 3.8 that $\langle f_0, R_{\mathbf{T},1}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = 0$ for all $\mathbf{T}$. Since f_0 is a linear combination of unipotent characters, it follows that f_0 is orthogonal to all uniform functions on $\mathbf{G}^F$. So, by Example 2.3, we have $\langle f_0, Q_{\mathbf{T}}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = 0$ for all $\mathbf{T}$. On the other hand, by Proposition 3.6, the restriction of f_0 to $\mathbf{G}^F_{\rm uni}$ must be non-zero. Hence, using (a) and the fact that $\langle f_0', Q_{\mathbf{T}}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = 0$ for all $\mathbf{T}$, we deduce that $\pi_{\rm uni}^{\mathbf{G}}(f_0) = cf_0'$ for some $0 \neq c \in \mathbb{C}$. In order to determine the scalar c, we use the discussion in 3.7. For $i = 0, 1$ let C_i be the $\mathbf{G}^F$-conjugacy class of u_i; thus, $\mathscr{O}_0^F = C_0 \cup C_1$. Let $w_c \in \mathbf{W}$ be a Coxeter element, such that $F(w_c) = w_c$. As discussed in [11, Example 4.9], we have

(b) $|\mathscr{O}_0^F \cap \mathbf{B}^F w_c \mathbf{B}^F| = |\mathbf{B}^F|$ and $C_i \cap \mathbf{B}^F w_c \mathbf{B}^F = \varnothing$ for $i = 0$ or $i = 1$.

By the argument of Hetz [17, Lemma 7.2] (which is based on Lusztig [29, 2.4]), one sees that a certain $\mathbf{G}^F$-conjugate of u_0 is contained in $\mathbf{B}^F w_c \mathbf{B}^F$. Hence, $C_0 \cap \mathbf{B}^F w_c \mathbf{B}^F \neq \varnothing$. Using (b), we deduce that $|C_0 \cap \mathbf{B}^F w_c \mathbf{B}^F| = |\mathbf{B}^F|$. As explained in 3.7, this yields a linear relation involving the scalar c; by explicitly working out the terms in that relation (we omit the details), we find that $c = 1$. Thus, $\pi_{\rm uni}^{\mathbf{G}}(f_0) = f_0'$. Finally, since $\langle f_0, f_0\rangle_{\mathbf{G}^F} = \langle f_0', f_0'\rangle_{\mathbf{G}^F} = 1$, we can also conclude that $f_0(g) = 0$ for all non-unipotent elements $g \in \mathbf{G}^F$; thus, $f_0 = f_0'$ as desired. □

We shall also need an analogous result for groups of type D_5. In order to avoid too much interruption of the current discussion, this will be done in an Appendix (see Section 7).

4. On non-simply-connected semisimple groups

Assume that $\mathbf{G}$ is semisimple. If $\mathbf{G}$ is not of simply connected type, then it may happen that the commutator subgroup of $\mathbf{G}^F$ is strictly smaller than $\mathbf{G}^F$; furthermore, the centraliser of a semisimple element of $\mathbf{G}$ may not be connected. In this section we provide some tools for dealing with this situation; this is mostly based on Steinberg [39].

For this purpose, it will be convenient to also fix a connected reductive algebraic group $\tilde{\mathbf{G}}$ over k and a surjective homomorphism $\pi\colon \tilde{\mathbf{G}} \to \mathbf{G}$, with the following properties:

- There is a Frobenius map $\tilde{F}\colon \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ such that $F \circ \pi = \pi \circ \tilde{F}$;
- the kernel $\ker(\pi)$ is connected and contained in the center $Z(\tilde{\mathbf{G}})$ of $\tilde{\mathbf{G}}$;
- the derived subgroup $\tilde{\mathbf{G}}_{\mathrm{der}} \subseteq \tilde{\mathbf{G}}$ is semisimple of simply connected type.

(The fact that this always exists is due to Asai; see [15, Prop. 1.7.13].) We begin by collecting some useful properties about $\pi\colon \tilde{\mathbf{G}} \to \mathbf{G}$.

Remark 4.1. The restriction of π to $\tilde{\mathbf{G}}_{\mathrm{der}}$ will be denoted by $\pi_{\mathrm{der}}\colon \tilde{\mathbf{G}}_{\mathrm{der}} \to \mathbf{G}$. Since $\mathbf{G}$ is semisimple, π_{der} is surjective and $\mathbf{K} := \ker(\pi_{\mathrm{der}}) \subseteq Z(\tilde{\mathbf{G}}_{\mathrm{der}})$ is finite. (For example, if $\mathbf{G}$ is simple and not of type A_n, then $\mathbf{K}$ has order $1, 2, 3$ or 4.) Since $\ker(\pi)$ is connected, we have $\pi(\tilde{\mathbf{G}}^{\tilde{F}}) = \mathbf{G}^F$. But note that we do not necessarily have $\pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}) = \mathbf{G}^F$. Firstly, by Steinberg [39, Cor. 12.6], we have

(a) $\quad \pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}) = \mathbf{G}_u^F :=$ subgroup of $\mathbf{G}^F$ generated by the unipotent elements of $\mathbf{G}^F$.

Secondly, $\mathbf{G}_u^F$ is a normal subgroup of $\mathbf{G}^F$ and there is a canonical isomorphism

(b) $$\mathbf{G}^F/\mathbf{G}_u^F = \mathbf{G}^F/\pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}) \xrightarrow{\sim} \mathbf{K}/\{z^{-1}\tilde{F}(z) \mid z \in \mathbf{K}\},$$

defined as follows (see [15, Prop. 1.4.13]): Let $g \in \mathbf{G}^F$ and $\dot{g} \in \tilde{\mathbf{G}}_{\mathrm{der}}$ be such that $\pi(\dot{g}) = g$; then $u := \dot{g}^{-1}\tilde{F}(\dot{g}) \in \mathbf{K}$ and the coset of g is mapped to the coset of u.

Remark 4.2. Let $s \in \mathbf{G}$ be semisimple and $\tilde{s} \in \tilde{\mathbf{G}}$ be such that $\pi(\tilde{s}) = s$; note that $\tilde{s}$ is also semisimple. Since $\tilde{\mathbf{G}}_{\mathrm{der}}$ is simply connected, it is known that the centraliser $C_{\tilde{\mathbf{G}}}(\tilde{s})$ is connected; see Steinberg [39, §8]. However, $C_{\mathbf{G}}(s)$ need not be connected, but the group of components $A_{\mathbf{G}}(s) = C_{\mathbf{G}}(s)/C_{\mathbf{G}}^\circ(s)$ is always isomorphic to a subgroup of $\mathbf{K} = \ker(\pi_{\mathrm{der}})$. Slightly more generally, we have:

Lemma 4.3. *Let $\tilde{g} \in \tilde{\mathbf{G}}^{\tilde{F}}$ be such that $C_{\tilde{\mathbf{G}}}(\tilde{g})$ is connected, and set $g := \pi(\tilde{g}) \in \mathbf{G}$. Then $A_{\mathbf{G}}(g)$ is isomorphic to a subgroup of $\mathbf{K}$, and $A_{\mathbf{G}}(g)^F$ is isomorphic to a subgroup of $\mathbf{K}^{\tilde{F}}$.*

Proof. This is contained in [39, 9.1]; we recall the main ingredients. First note that $\pi(C_{\tilde{\mathbf{G}}}(\tilde{g})) = \pi(C_{\tilde{\mathbf{G}}}^\circ(\tilde{g})) = C_{\mathbf{G}}^\circ(g)$; see [15, 1.3.10(e)]. Then there is a canonical isomorphism

$$\delta\colon A_{\mathbf{G}}(g) \xrightarrow{\sim} \{h^{-1}\tilde{g}^{-1}h\tilde{g} \mid h \in \tilde{\mathbf{G}}\} \cap \ker(\pi)$$

defined as follows. Let $x \in C_{\mathbf{G}}(g)$ and $\dot{x} \in \tilde{\mathbf{G}}$ be such that $\pi(\dot{x}) = x$. Then $z_{\dot{x}} := \dot{x}^{-1}\tilde{g}^{-1}\dot{x}\tilde{g} \in \ker(\pi)$ and δ sends the image of $x = \pi(\dot{x})$ in $A_{\mathbf{G}}(g)$ to $z_{\dot{x}}$; note that $z_{\dot{x}} \in \tilde{\mathbf{G}}_{\mathrm{der}} \cap \ker(\pi) = \mathbf{K}$. (The construction of δ is based on the purely group-theoretical result in [39, 4.5]; see also [15, Lemma 1.1.9]). Now one checks that $\delta \circ F = \tilde{F} \circ \delta$. Hence, $A_{\mathbf{G}}(g)^F$ is isomorphic to a subgroup of $\mathbf{K}^{\tilde{F}}$. $\square$

Remark 4.4. Let $\tilde{g} \in \tilde{\mathbf{G}}$ and $g := \pi(\tilde{g}) \in \mathbf{G}$. If $C_{\tilde{\mathbf{G}}}(\tilde{g})$ is not connected, then $\pi(C_{\tilde{\mathbf{G}}}(\tilde{g})) \supseteq \pi(C_{\tilde{\mathbf{G}}}^\circ(\tilde{g})) = C_{\mathbf{G}}^\circ(g)$ and $C_{\mathbf{G}}(g)/\pi(C_{\tilde{\mathbf{G}}}(\tilde{g}))$ is a factor group of $A_{\mathbf{G}}(g)$. By exactly the same argument as above (using [39, 4.5]), we still obtain an injective homomorphism

$$\delta\colon C_{\mathbf{G}}(g)/\pi(C_{\tilde{\mathbf{G}}}(\tilde{g})) \hookrightarrow \mathbf{K},$$

Since $\ker(\pi)$ is connected, we have $\ker(\pi) \subseteq C^\circ_{\tilde{\mathbf{G}}}(\tilde{g})$ and so $|A_{\tilde{\mathbf{G}}}(\tilde{g})| = |\pi(C_{\tilde{\mathbf{G}}}(\tilde{g})) : C^\circ_{\mathbf{G}}(g)|$.

The following technical result will be useful further on.

Lemma 4.5. *Assume that $\tilde{F}$ acts trivially on $\mathbf{K}$. Let $g_1, g_2 \in \mathbf{G}^F$ be elements that are conjugate in $\mathbf{G}$. If $g_1^{-1}g_2 \in \pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$ and $C_{\tilde{\mathbf{G}}}(\tilde{g}_1)$ is connected (where $\tilde{g}_1 \in \tilde{\mathbf{G}}$ is such that $\pi(\tilde{g}_1) = g_1$), then g_1, g_2 are already conjugate in $\mathbf{G}^F$.*

Proof. Let $\mathbf{C}$ be the $\mathbf{G}$-conjugacy class containing the elements g_1, g_2. Since $\pi(\tilde{\mathbf{G}}_{\rm der}) = \mathbf{G}$, there exists a $\tilde{\mathbf{G}}_{\rm der}$-conjugacy class $\tilde{\mathbf{C}}$ such that $\pi(\tilde{\mathbf{C}}) = \mathbf{C}$. For $i = 1, 2$ let $x_i \in \tilde{\mathbf{C}}$ be such that $\pi(x_i) = g_i$; then $u_i := x_i^{-1}\tilde{F}(x_i) \in \mathbf{K}$. Note that the elements x_1, x_2 are not necessarily fixed by $\tilde{F}$. We will slightly modify them so as to obtain elements in $\tilde{\mathbf{G}}^{\tilde{F}}$. This is done as follows. Since $\tilde{F}$ acts trivially on $\mathbf{K}$, we have a canonical isomorphism

$$\mathbf{G}^F/\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}}) \xrightarrow{\sim} \mathbf{K},$$

defined as in Remark 4.1(b); note that, under this isomorphism, the coset of g_i is sent to u_i. But, assuming that g_1, g_2 belong to the same coset mod $\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$, we must have $u_1 = u_2$. Let $y := x_1^{-1}x_2 \in \tilde{\mathbf{G}}_{\rm der}$. Since $\tilde{F}(x_i) = x_iu_i$ and $u_i \in \mathbf{K} \subseteq Z(\tilde{\mathbf{G}})$ for $i = 1, 2$, we obtain $\tilde{F}(y) = y$. Hence, $x_2 = x_1y$ where $y \in \tilde{\mathbf{G}}_{\rm der}^{\tilde{F}}$.

Since $u_1 \in \mathbf{K} \subseteq \ker(\pi)$ and $\ker(\pi)$ is connected, we can further find an element $z \in \ker(\pi)$ such that $u_1 = z\tilde{F}(z)^{-1}$. Since $\ker(\pi) \subseteq Z(\tilde{\mathbf{G}})$, we obtain $\tilde{F}(zx_1) = \tilde{F}(z)\tilde{F}(x_1) = (u_1^{-1}z)(x_1u_1) = zx_1$ and $\tilde{F}(zx_2) = \tilde{F}(zx_1y) = \tilde{F}(zx_1)y = zx_1y = zx_2$. Thus, $zx_1, zx_2 \in \tilde{\mathbf{G}}^{\tilde{F}}$ where $z \in Z(\tilde{\mathbf{G}})$. Since x_1, x_2 are conjugate in $\tilde{\mathbf{G}}_{\rm der}$, the elements zx_1 and zx_2 will be conjugate in $\tilde{\mathbf{G}}$.

Finally, since $z \in Z(\tilde{\mathbf{G}})$, $\pi(\tilde{g}_1) = g_1 = \pi(x_1)$ and $\ker(\pi) \subseteq Z(\tilde{\mathbf{G}})$, we have $C_{\tilde{\mathbf{G}}}(zx_1) = C_{\tilde{\mathbf{G}}}(x_1) = C_{\tilde{\mathbf{G}}}(\tilde{g}_1)$. By assumption, this centraliser is connected. Consequently, $zx_1 \in \tilde{\mathbf{G}}^{\tilde{F}}$ and $zx_2 \in \tilde{\mathbf{G}}^{\tilde{F}}$ are not only conjugate in $\tilde{\mathbf{G}}$ but already in $\tilde{\mathbf{G}}^{\tilde{F}}$ (see 2.5). But then $g_1 = \pi(zx_1)$ and $g_2 = \pi(zx_2)$ will be conjugate in $\mathbf{G}^F$, as claimed. $\square$

Example 4.6. Assume that $\tilde{F}$ acts trivially on $\mathbf{K}$. Let $s \in \mathbf{G}^F$ be semisimple such that $r := |A_{\mathbf{G}}(s)| = |\mathbf{K}|$; let $\mathbf{C}$ be the $\mathbf{G}$-conjugacy class of s. By Remark 4.2 and Lemma 4.3, we have $A_{\mathbf{G}}(s) \cong \mathbf{K}$ and F acts trivially on $A_{\mathbf{G}}(s)$. So $\mathbf{C}^F$ splits into precisely r conjugacy classes of $\mathbf{G}^F$; let $s_1, \ldots, s_r \in \mathbf{G}^F$ be representatives of these classes. Then $s_1, \ldots, s_r$ are also representatives for the cosets of $\mathbf{G}^F$ mod $\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$ and

(a) $\quad C_i := \mathbf{C}^F \cap s_i\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$ is the $\mathbf{G}^F$-conjugacy class of s_i, for $i = 1, \ldots, r$.

Indeed, as in the above proof, we have $r = |\mathbf{K}| = |\mathbf{G}^F/\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})|$. The elements s_i are all conjugate in $\mathbf{G}$, but no two of them are conjugate in $\mathbf{G}^F$. Hence, they are in different cosets mod $\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$. Thus, (a) holds.

We give some further applications of the above results. Let $f \in \mathrm{CF}(\mathbf{G}^F)$ be any class function and $s \in \mathbf{G}^F$ be semisimple. Then we define

$$f^{(s)} := d^{-1} \sum_{\lambda} \lambda(s)^{-1}\lambda \cdot f \in \mathrm{CF}(\mathbf{G}^F),$$

where the sum runs over all linear characters $\lambda\colon \mathbf{G}^F \to \mathbb{C}^\times$ with $\pi(\tilde{\mathbf{G}}_{\text{der}}^{\tilde{F}}) \subseteq \ker(\lambda)$, and $d \geqslant 1$ is the number of such linear characters.

Proposition 4.7. *Assume that $\tilde{F}$ acts trivially on $\mathbf{K}$. Let Σ be an F-stable conjugacy class of $\mathbf{G}$ and $f \in \mathrm{CF}(\mathbf{G}^F \mid \Sigma)$. Let $s \in \mathbf{G}^F$ be the semisimple part of an element of Σ^F, and Σ_s^F be the set of all those $g \in \Sigma^F$ such that the semisimple part of g is $\mathbf{G}^F$-conjugate to s. Note that Σ_s^F is a union of $\mathbf{G}^F$-conjugacy classes. Then, for any $g \in \mathbf{G}^F$, we have*

$$f^{(s)}(g) = \begin{cases} f(g) & \text{if } g \in \Sigma_s^F, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $g \in \mathbf{G}^F$ and write $g = tv = vt$ where $t \in \mathbf{G}^F$ is semisimple and $v \in \mathbf{G}^F$ is unipotent. By Remark 4.1(a), we have $\lambda(v) = 1$ for any λ as above, and so $\lambda(g) = \lambda(t)$. This yields the formula

$$f^{(s)}(g) = d^{-1}\Big(\sum_\lambda \lambda(s^{-1}t)\Big) f(g).$$

If $g \in \Sigma_s^F$, then s, t will be conjugate in $\mathbf{G}^F$ and so $\lambda(s^{-1}t) = 1$. In this case, we obtain $f^{(s)}(g) = f(g)$, as desired. Now let $g \in \mathbf{G}^F$ be arbitrary such that $f^{(s)}(g) \neq 0$. First of all, this implies that $g \in \Sigma^F$ and, consequently, the elements s, t are conjugate in $\mathbf{G}$. By the orthogonality relations for irreducible characters, the sum $\sum_\lambda \lambda(s^{-1}t)$ will only be non-zero if $s^{-1}t \in \pi(\tilde{\mathbf{G}}_{\text{der}}^{\tilde{F}})$. Let $\tilde{s} \in \tilde{\mathbf{G}}$ be semisimple such that $\pi(\tilde{s}) = s$; then $C_{\tilde{\mathbf{G}}}(\tilde{s})$ is connected as already mentioned in Remark 4.2. Hence, Lemma 4.5 implies that s, t are conjugate in $\mathbf{G}^F$, and so $g \in \Sigma_s^F$. □

Corollary 4.8. *In the setting of Proposition 4.7, assume that f is uniform. Then $f^{(s)}$ also is a uniform function. In particular, the indicator function $\varepsilon_{\Sigma_s^F} \in \mathrm{CF}(\mathbf{G}^F \mid \Sigma)$ is a uniform function.*

Proof. Let $\lambda\colon \mathbf{G}^F \to \mathbb{C}^\times$ be a linear character with $\pi(\tilde{\mathbf{G}}_{\text{der}}^{\tilde{F}}) \subseteq \ker(\lambda)$; then λ is p-constant (see Remark 4.1(a)). Hence, $\lambda\cdot f$ is uniform for any such λ (see Example 2.3) and, consequently, $f^{(s)}$ is uniform. Finally, as already mentioned in 2.1, the indicator function $f := \varepsilon_{\Sigma^F} \in \mathrm{CF}(\mathbf{G}^F)$ is uniform. By Proposition 4.7, the indicator function $\varepsilon_{\Sigma_s^F} \in \mathrm{CF}(\mathbf{G}^F)$ is a scalar multiple of $f^{(s)}$. Hence, $\varepsilon_{\Sigma_s^F}$ is uniform. □

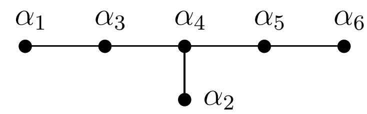
Corollary 4.9. *Assume that $\tilde{F}$ acts trivially on $\mathbf{K}$. Let C be a conjugacy class of $\mathbf{G}^F$ and $g \in C$. If $C_{\tilde{\mathbf{G}}}(\tilde{g})$ is connected (where $\tilde{g} \in \tilde{\mathbf{G}}$ is such that $\pi(\tilde{g}) = g$), then $\varepsilon_C \in \mathrm{CF}(\mathbf{G}^F)$ is a uniform function.*

Proof. Let Σ be the $\mathbf{G}$-conjugacy class of g and $s \in \mathbf{G}^F$ be the semisimple part of g; then $C \subseteq \Sigma_s^F \subseteq \Sigma^F$. By Corollary 4.8, it will be sufficient to show that $C = \Sigma_s^F$. So let $g' \in \Sigma_s^F$ be arbitrary. Conjugating g' by a suitable element of $\mathbf{G}^F$ we may assume without loss of generality that s also is the semisimple part of g'. Then $g^{-1}g'$ is unipotent and, hence, $g^{-1}g' \in \mathbf{G}_u^F = \pi(\tilde{\mathbf{G}}_{\text{der}}^{\tilde{F}})$; see Remark 4.1(a). Since $g, g' \in \Sigma_s^F \subseteq \Sigma$, the elements g, g' are conjugate in $\mathbf{G}$. So g, g' are conjugate in $\mathbf{G}^F$ by Lemma 4.5. Thus, $g' \in C$, as desired. □

5. Groups of adjoint type E_6 in characteristic $\neq 3$

Throughout this section, let $p \neq 3$ and $\mathbf{G}$ be simple, adjoint of type E_6. Let $\mathbf{T}_0 \subseteq \mathbf{G}$ be a maximal torus and $\mathbf{B} \subseteq \mathbf{G}$ be a Borel subgroup containing $\mathbf{T}_0$. Let Φ be the root system of $\mathbf{G}$, where the labelling for the simple roots is chosen as in the following diagram:

E_6



Let q be a power of p and $F_q\colon \mathbf{G} \to \mathbf{G}$ be the split Frobenius map such that $F_q(\mathbf{B}) = \mathbf{B}$ and $F_q(t) = t^q$ for all $t \in \mathbf{T}_0$. Let $\gamma\colon \mathbf{G} \to \mathbf{G}$ be the non-trivial graph automorphism such that $\gamma \circ F_q = F_q \circ \gamma$ and $\gamma(\mathbf{T}_0) = \mathbf{T}_0$, $\gamma(\mathbf{B}) = \mathbf{B}$. Then we consider the following two cases for a Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$.

(a) $F := F_q$; in this case, we have $\mathbf{G}^F = E_6(q)_{\mathrm{ad}}$ (split type).

(b) $F := F_q \circ \gamma = \gamma \circ F_q$; in this case, we have $\mathbf{G}^F = {}^2E_6(q)_{\mathrm{ad}}$ (twisted type).

Note that, in both cases, γ induces an automorphism of the finite group $\mathbf{G}^F$, which we denote by the same symbol. We fix a surjective homomorphism $\pi\colon \tilde{\mathbf{G}} \to \mathbf{G}$ as in the previous section. (An explicit root datum for $\tilde{\mathbf{G}}$ can be found in [19, §2].) Note that $\mathbf{K} := \ker(\pi_{\mathrm{der}}) = Z(\tilde{\mathbf{G}}_{\mathrm{der}})$ has order 3 in this case. Hence, if $\tilde{F}$ does not act trivially on $\mathbf{K}$, then $\mathbf{K}^{\tilde{F}} = \{1\}$ and $\mathbf{G}^F = \pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}})$; otherwise, we have $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}) \cong \mathbf{K}$. The explicit description of $Z(\tilde{\mathbf{G}}_{\mathrm{der}})$ in [15, Example 1.5.6] shows the following. The map $\tilde{F}$ acts trivially on $\mathbf{K}$ precisely when $3 \mid q-1$ in case (a), or when $3 \mid q+1$ in case (b); furthermore, if $\tilde{F}$ does not act trivially on $\mathbf{K}$, then $\tilde{F}(z) = z^{-1}$ for $z \in \mathbf{K}$.

By Mizuno [34] (see also Lübeck's tables [21]), there is a unique conjugacy class $\mathbf{C}_0$ of semisimple elements of $\mathbf{G}$ such that, for $s \in \mathbf{C}_0$, we have $|A_{\mathbf{G}}(s)| = 3 = |\mathbf{K}|$ and $C^\circ_{\mathbf{G}}(s)$ has a root system of type $A_2{+}A_2{+}A_2$. Note that the uniqueness implies that $\mathbf{C}_0$ is F-stable and $\mathbf{C}_0^{-1} = \mathbf{C}_0 = \gamma(\mathbf{C}_0)$. The aim of this section is to show how the character values $\rho(g)$ can be determined, for any $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ and any $g \in \mathbf{G}^F$ with semisimple part in $\mathbf{C}_0$.

First we deal with the case where g is not a regular element.

5.1. Let $g \in \mathbf{G}^F$ and assume that $g = su = us$ where $s \in \mathbf{C}_0^F$ and $u \in C^\circ_{\mathbf{G}}(s)^F$ is unipotent, but not regular unipotent. Let C be the $\mathbf{G}^F$-conjugacy class of g. We claim that the indicator function ε_C is a uniform function. This is seen as follows. Let $\tilde{g} \in \tilde{\mathbf{G}}^{\tilde{F}}$ be such that $\pi(\tilde{g}) = g$. Write $\tilde{g} = \tilde{s}\tilde{u}$ where $\tilde{s} \in \tilde{\mathbf{G}}^{\tilde{F}}$ is semisimple and $\tilde{u} \in \tilde{\mathbf{G}}^{\tilde{F}}$ is unipotent; here, $\tilde{\mathbf{H}} := C_{\tilde{\mathbf{G}}}(\tilde{s})$ is connected. We claim that $C_{\tilde{\mathbf{G}}}(\tilde{g}) = C_{\tilde{\mathbf{H}}}(\tilde{u})$ is connected. Note that this information can not be derived from Mizuno [34], since he classifies the conjugacy classes for E_6 of simply connected type, and he only gives the number of unipotent classes in the centraliser of a semisimple element. So some additional computations are required in order to obtain $C_{\tilde{\mathbf{H}}}(\tilde{u})$; but this is part of Michel's CHEVIE [33]. The following sequence of functions yields the required information on $C_{\tilde{\mathbf{H}}}(\tilde{u})$:

```
julia> W=rootdatum("CE6")                      # root datum of tilde{G}
julia> W.rootdec[36]
1 2 2 3 2 1                                    # 36: hightest root
```

```
julia> H=reflection_subgroup(W,[1,2,3,5,6,36]) # subgroup of type A2+A2+A2
julia> UnipotentClasses(H)                     # for any good prime p
julia> UnipotentClasses(H,2)                   # for p=2
```

The output of the last two functions shows that only the regular unipotent elements in $\tilde{\mathbf{H}}$ have a non-connected centraliser. Now Lemma 4.3 implies that $A_{\mathbf{G}}(g)^F$ is isomorphic to a subgroup of $\mathbf{K}^{\tilde{F}}$. Hence, if $\tilde{F}$ does not act trivially on $\mathbf{K}$, then $A_{\mathbf{G}}(g)^F = \{1\}$ and so ε_C is a uniform function; see Example 2.6. On the other hand, if $\tilde{F}$ acts trivially on $\mathbf{K}$, then Corollary 4.9 shows that, again, ε_C is a uniform function. Hence, in both cases, the values of $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ on $g = su$ can be computed using the formula in Example 2.2.

5.2. From now on let Σ be the F-stable $\mathbf{G}$-conjugacy class of all elements of the form $g = su$ where $s \in \mathbf{C}_0$ and $u \in C^\circ_{\mathbf{G}}(s)$ is regular unipotent. Since $\mathbf{C}_0^{-1} = \mathbf{C}_0 = \gamma(\mathbf{C}_0)$ (as noted above) and since the regular unipotent elements in a connected reductive group are all conjugate, it follows that $\Sigma^{-1} = \Sigma = \gamma(\Sigma)$. By the discussion in [11, 5.2, 5.6], we can fix a representative $g_0 \in \Sigma^F$ such that $g_0 = \pi(\tilde{g}_0)$ where $\tilde{g}_0 \in \tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}$ and $\tilde{g}_0$ is $\tilde{\mathbf{G}}^{\tilde{F}}$-conjugate to $\tilde{g}_0^{-1}$. Consequently, g_0 will be $\mathbf{G}^F$-conjugate to g_0^{-1}. By [26, Cor. 20.4] (see also [36, 4.6]), we have

$$A_{\tilde{\mathbf{G}}}(\tilde{g}_0) = 3 \quad \text{and} \quad A_{\mathbf{G}}(g_0) = C_3 \times C_3' \quad \text{where } C_3,\, C_3' \text{ are cyclic of order 3.}$$

Here, $C_3 = \langle \bar{g}_0 \rangle$ is generated by the image of g_0 in $A_{\mathbf{G}}(g_0)$; to single out a definite choice for C_3' is a bit more tricky; we will come back to this issue in Lemma 5.6 below. In any case, F acts trivially on C_3 and so either $A_{\mathbf{G}}(g_0)^F = C_3$ or $A_{\mathbf{G}}(g_0)^F = A_{\mathbf{G}}(g_0) = C_3 \times C_3'$.

The particular significance of the class Σ is as follows. By [26, Cor. 20.4] (and its proof), there are six cuspidal character sheaves on $\mathbf{G}$; they all have support given by Σ and they correspond to those linear characters $\lambda\colon A_{\mathbf{G}}(g_0) \to \mathbb{C}^\times$ that are non-trivial on C_3. For the convenience of the reader, we display the values of these six linear characters in Table 1, where a is any element in $A_{\mathbf{G}}(g_0) \setminus C_3$ and $C_3' = \langle a \rangle$.

Let A be one of the six cuspidal character sheaves on $\mathbf{G}$. Then $F^*A \cong A$ if and only if the corresponding linear character of $A_{\mathbf{G}}(g_0)$ is invariant under the action of F on $A_{\mathbf{G}}(g_0)$. Two out of the six cuspidal character sheaves are unipotent. By Shoji [36, 4.6, 4.8], they are F-invariant, regardless of whether F is the split Frobenius map or not.

If F does not act trivially on $A_{\mathbf{G}}(g_0)$, then $A_{\mathbf{G}}(g_0)^F = C_3$ and so there will only be two linear characters in Table 1 that are invariant under the action of F; these will correspond to the two unipotent cuspidal character sheaves (and these are the only F-invariant cuspidal character sheaves). On the other hand, if F acts trivially on $A_{\mathbf{G}}(g_0)$, then we will have to specify which of the six linear characters in Table 1 correspond to the two unipotent cuspidal character sheaves; this is related to the choice of the subgroup C_3' such that $A_{\mathbf{G}}(g_0) = C_3 \times C_3'$.

Proposition 5.3. *Assume that F does not act trivially on $A_{\mathbf{G}}(g_0)$ (where $g_0 \in \Sigma^F$ is $\mathbf{G}^F$-conjugate to g_0^{-1}, as above). Then $A_{\mathbf{G}}(g_0)^F = C_3$ and $C_3 = \{1, \bar{g}_0, \bar{g}_0^2\}$ is a set of representatives of the F-conjugacy classes of $A_{\mathbf{G}}(g_0)$. The characteristic functions of the*

TABLE 1. Linear characters of $A_{\mathbf{G}}(g_0)$ that are non-trivial on C_3

	1	$\bar{g}_0$	$\bar{g}_0^2$	a	$\bar{g}_0 a$	$\bar{g}_0^2 a$	a^2	$\bar{g}_0 a^2$	$\bar{g}_0^2 a^2$
ψ_1	1	θ	θ^2	1	θ	θ^2	1	θ	θ^2
$\psi_2 = \overline{\psi}_1$	1	θ^2	θ	1	θ^2	θ	1	θ^2	θ
ψ_3	1	θ	θ^2	θ	θ^2	1	θ^2	1	θ
$\psi_4 = \overline{\psi}_3$	1	θ^2	θ	θ^2	θ	1	θ	1	θ^2
ψ_5	1	θ	θ^2	θ^2	1	θ	θ	θ^2	1
$\psi_6 = \overline{\psi}_5$	1	θ^2	θ	θ	1	θ^2	θ^2	θ	1

(Here, $1 \neq \theta \in \mathbb{C}$ is a fixed third root of unity)

two (F-invariant) unipotent cuspidal character sheaves (normalised as in 3.1) are given by the following table:

	C_1	$C_{\bar{g}_0}$	$C_{\bar{g}_0^2}$
χ_1	q^3	$q^3\theta$	$q^3\theta^2$
χ_2	q^3	$q^3\theta^2$	$q^3\theta$

where $1 \neq \theta \in \mathbb{C}$ is a fixed third root of unity (as in Table 1).

Proof. By Example 2.6(a), the number of F-conjugacy classes of $A_{\mathbf{G}}(g_0)$ is given by $|A_{\mathbf{G}}(g)^F| = |C_3| = 3$. Hence, it is sufficient to show that the elements of C_3 belong to different F-conjugacy classes. To see this, let $\psi\colon A_{\mathbf{G}}(g_0) \to \mathbb{C}^\times$ be a linear character corresponding to one of the two unipotent cuspidal character sheaves; since the latter is F-invariant (as noted above), the character ψ must be constant on the F-conjugacy classes of $A_{\mathbf{G}}(g_0)$. (This immediately follows from the fact that ψ is a group homomorphism that is invariant under the action of F.) Now we observe that each of the six linear characters in Table 1 has pairwise different values on $1, \bar{g}_0, \bar{g}_0^2$. In particular, the same must be true for ψ. Hence, the elements $1, \bar{g}_0, \bar{g}_0^2$ must belong to different F-conjugacy classes. Finally, the values of χ_1, χ_2 are obtained by the formula in 3.1(b); note that $\dim \mathbf{G} - \dim \Sigma = 6$ since Σ is a conjugacy class of regular elements in $\mathbf{G}$. ☐

Lemma 5.4. *The map $\tilde{F}$ acts trivially on $\mathbf{K}$ if and only if F acts trivially on $A_{\mathbf{G}}(g_0)$.*

Proof. To see this, we consider the injective homomorphism $C_{\mathbf{G}}(g_0)/\pi(C_{\tilde{\mathbf{G}}}(\tilde{g}_0)) \hookrightarrow \mathbf{K}$ in Remark 4.4. By its construction, it is compatible with the action of F on the left side and the action of $\tilde{F}$ on the right side. Furthermore, $3 = |A_{\tilde{\mathbf{G}}}(\tilde{g}_0)| = |\pi(C_{\tilde{\mathbf{G}}}(\tilde{g}_0)) : C_{\mathbf{G}}^\circ(g_0)|$ and so $C_{\mathbf{G}}(g_0)/\pi(C_{\tilde{\mathbf{G}}}(\tilde{g}_0)) \neq \{1\}$. Hence, if $\tilde{F}$ does not act trivially on $\mathbf{K}$, then F will not act trivially on $A_{\mathbf{G}}(g_0)$ either. Conversely, assume that $\tilde{F}$ acts trivially on $\mathbf{K}$. Then, since $|A_{\mathbf{G}}(s_0)| = 3 = |\mathbf{K}|$, the set $\mathbf{C}_0^F$ will split into three conjugacy classes in $\mathbf{G}^F$; see Example 4.6. Now assume, if possible, that F does not act trivially on $A_{\mathbf{G}}(g_0)$. Then, by Proposition 5.3, we have $\Sigma^F = C_1 \cup C_{\bar{g}_0} \cup C_{\bar{g}_0^2}$. So Lemma 2.8 shows that the semisimple parts of all elements in Σ^F form one $\mathbf{G}^F$-conjugacy class — contradiction, since $\mathbf{C}_0^F$ precisely is that set of semisimple parts of elements in Σ^F; see Remark 2.9. ☐

5.5. Assume now that F acts trivially on $A_{\mathbf{G}}(g_0)$, where $g_0 \in \Sigma^F$ is $\mathbf{G}^F$-conjugate to g_0^{-1} (see 5.2). Let $C_3 = \langle \bar{g}_0 \rangle$ and choose any element $a \in A_{\mathbf{G}}(g_0) \setminus C_3$. Then, setting $C_3' = \langle a \rangle$, we have $A_{\mathbf{G}}(g_0) = C_3 \times C_3' = \langle \bar{g}_0 \rangle \times \langle a \rangle$ and Σ^F splits into 9 conjugacy classes in $\mathbf{G}^F$:

(a) $$\Sigma^F = C_1 \cup C_{\bar{g}_0} \cup C_{\bar{g}_0^2} \cup C_a \cup C_{\bar{g}_0 a} \cup C_{\bar{g}_0^2 a} \cup C_{a^2} \cup C_{\bar{g}_0 a^2} \cup C_{\bar{g}_0^2 a^2}$$

where $g_0 \in C_1 = C_1^{-1}$. By Lemma 5.4, $\tilde{F}$ also acts trivially on $\mathbf{K}$. So, since $|A_{\mathbf{G}}(s_0)| = 3 = |\mathbf{K}|$, the set $\mathbf{C}_0^F$ splits into 3 conjugacy classes of (semisimple elements of) $\mathbf{G}^F$, and all these elements arise as semisimple parts of elements of Σ^F (see Remark 2.9 and Example 4.6). Let s_0, s_1, s_2 be representatives of these classes in $\mathbf{C}_0^F$. Taking into account Lemma 2.8, we see that the notation can be fixed such that, for $i = 0, 1, 2$, the semisimple parts of the elements in the subset

(b) $$\Sigma_i^F := C_{a^i} \cup C_{\bar{g}_0 a^i} \cup C_{\bar{g}_0^2 a^i} \subseteq \Sigma^F$$

are $\mathbf{G}^F$-conjugate to s_i; thus, $\Sigma_i^F = \Sigma_{s_i}^F$ with the notation of Proposition 4.7. Since s_0, s_1, s_2 are also representatives of the 3 cosets of $\pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}})$ in $\mathbf{G}^F$ (see again Example 4.6), and since $g_0 \in \pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}})$, we can further fix the notation such that $s_0 \in \pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}})$. Consequently, we have

(c) $$\Sigma_0^F \subseteq \pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}), \qquad \Sigma_1^F \subseteq s_1\pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}), \qquad \Sigma_2^F \subseteq s_2\pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}).$$

With these preparations, we can now state:

Proposition 5.6. *Assume that F acts trivially on $A_{\mathbf{G}}(g_0)$, where $g_0 \in \Sigma^F$ is such that g_0^{-1} is $\mathbf{G}^F$-conjugate to g_0; also recall that $\gamma(\Sigma) = \Sigma$ (see 5.2, 5.5). Then an element $a \in A_{\mathbf{G}}(g_0) \setminus C_3$ can be chosen such that the action of $\gamma \in \mathrm{Aut}(\mathbf{G}^F)$ on Σ^F is given by*

$$\gamma(C_{\bar{g}_0^i}) = C_{\bar{g}_0^i} \qquad \textit{and} \qquad \gamma(C_{\bar{g}_0^i a}) = C_{\bar{g}_0^i a^2} \qquad \textit{for } i = 0, 1, 2.$$

For any such choice of $a \in A_{\mathbf{G}}(g_0) \setminus C_3$, the linear characters $\lambda \colon A_{\mathbf{G}}(g_0) \to \mathbb{C}^\times$ corresponding to the two (F-invariant) unipotent cuspidal character sheaves on $\mathbf{G}$ are precisely those that are non-trivial on C_3 and trivial on $C_3' = \langle a \rangle$ (see Table 1).

Proof. Since $\gamma(\Sigma) = \Sigma$, and since γ and F commute with each other, the automorphism γ will permute the $\mathbf{G}^F$-conjugacy classes inside Σ^F. Let A_1, A_2 be the two unipotent cuspidal character sheaves on $\mathbf{G}$ and $\chi_1, \chi_2 \in \mathrm{CF}(\mathbf{G}^F \mid \Sigma)$ the corresponding characteristic functions, normalised as in 3.1(b). By the main result of Shoji [35], [36], the functions χ_1, χ_2 are linear combinations of unipotent characters. The form of these linear combinations shows that $\chi_2 = \overline{\chi}_1$ (complex conjugate); see also [11, 5.4, 5.6]. Thus, A_1, A_2 correspond to complex conjugate linear characters of $A_{\mathbf{G}}(g_0)$ in Table 1, which must be either $\{\psi_1, \psi_2\}$ or $\{\psi_3, \psi_4\}$ or $\{\psi_5, \psi_6\}$. But that table also shows that, replacing an initially chosen element $a \in A_{\mathbf{G}}(g_0) \setminus C_3$ by $\bar{g}_0^i a$ for a suitable $i \in \{0, 1, 2\}$ if necessary, we may assume without loss of generality that the two linear characters are ψ_1, ψ_2; note that these are trivial on C_3'. The values of χ_1, χ_2 are now given as follows.

	C_1	$C_{\bar{g}_0}$	$C_{\bar{g}_0^2}$	C_a	$C_{\bar{g}_0 a}$	$C_{\bar{g}_0^2 a}$	C_{a^2}	$C_{\bar{g}_0 a^2}$	$C_{\bar{g}_0^2 a^2}$
χ_1	q^3	$q^3\theta$	$q^3\theta^2$	q^3	$q^3\theta$	$q^3\theta^2$	q^3	$q^3\theta$	$q^3\theta^2$
$\chi_2 = \overline{\chi}_1$	q^3	$q^3\theta^2$	$q^3\theta$	q^3	$q^3\theta^2$	$q^3\theta$	q^3	$q^3\theta^2$	$q^3\theta$

(Note again that $\dim \mathbf{G} - \dim \Sigma = 6$). Now it is known that all unipotent characters of $\mathbf{G}^F$ are invariant under $\gamma \in \mathrm{Aut}(\mathbf{G}^F)$; see [15, Theorem 4.5.11]. Hence, we must also have

$$\chi_1(g) = \chi_1(\gamma(g)) \qquad \text{for all } g \in \mathbf{G}^F.$$

Now γ leaves $\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$ invariant. Hence, since $\Sigma_0^F = C_1 \cup C_{\bar{g}_0} \cup C_{\bar{g}_0^2} \subseteq \pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$ (see 5.5(c)), we have $\gamma(\Sigma_0^F) = \Sigma_0^F$. Looking at the values of χ_1, and taking into account the invariance of χ_1 under γ, we conclude that $\gamma(C_{\bar{g}_0^i}) = C_{\bar{g}_0^i}$ for $i = 0,1,2$. By a similar argument, $\gamma(\Sigma_1^F)$ must be equal to Σ_1^F or Σ_2^F; consequently (again by looking at the values of χ_1), we have

$$\gamma(C_{\bar{g}_0^i a}) = C_{\bar{g}_0^i a} \quad (i = 0,1,2) \qquad \text{or} \qquad \gamma(C_{\bar{g}_0^i a}) = C_{\bar{g}_0^i a^2} \quad (i = 0,1,2).$$

Assume, if possible, that the first possibility is true, that is, $\gamma(C_{\bar{g}_0^i a}) = C_{\bar{g}_0^i a}$ for $i = 0,1,2$. Then all $\mathbf{G}^F$-conjugacy classes contained in Σ^F are fixed by γ, and so the characteristic functions of all six cuspidal character sheaves will be invariant under γ. This would imply that, if A is any of those six cuspidal character sheaves, then $\gamma^* A \cong A$, contradiction to [26, Cor. 20.4] (and its proof). So the second possibility must be true.

Thus, indeed, $a \in A_{\mathbf{G}}(g_0)$ can be chosen such that the action of γ on the $\mathbf{G}^F$-conjugacy classes contained in Σ^F is as stated. (If $F = F_q$ is split and $3 \mid q-1$, then this also follows immediately from the proof of [26, Cor. 20.4].) But then the statement concerning the linear characters corresponding to the two unipotent cuspidal character sheaves follows for any such choice of a, again by inspection of the values of χ_1 and χ_2. $\square$

5.7. Let Σ and $g_0 \in \Sigma^F$ be as in 5.2. We can now address the problem of finding a suitable basis of the subspace $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$, satisfying the requirements $(\Sigma)_1$, $(\Sigma)_2$ in Section 3. Let χ_0 be the indicator function of Σ^F; this is a uniform function and we can explicitly write it as a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$ (see 2.1 and also Remark 6.1 below.) Let χ_1, χ_2 be the characteristic functions of the two (F-invariant) unipotent cuspidal character sheaves; explicit expressions as linear combinations of unipotent characters of $\mathbf{G}^F$ can be found in [11, Prop. 5.5] (split type) and [11, Prop. 5.7] (twisted type).

(a) Assume first that F does not act trivially on $A_{\mathbf{G}}(g_0)$; then $\dim \mathrm{CF}(\mathbf{G}^F \mid \Sigma) = |A_{\mathbf{G}}(g_0)^F| = 3$. Using Proposition 5.3, a basis of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$ is given in Table 2.

(b) Now assume that F acts trivially on $A_{\mathbf{G}}(g_0)$; then $\dim \mathrm{CF}(\mathbf{G}^F \mid \Sigma) = |A_{\mathbf{G}}(g_0)^F| = 9$. Recall from 5.5 that s_0, s_1, s_2 are representatives of the $\mathbf{G}^F$-conjugacy classes contained in $\mathbf{C}_0^F$, and that these elements also form a set of representatives for the 3 cosets of $\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}})$ in $\mathbf{G}^F$, where s_0 represents the trivial coset. Hence, for $i = 0,1,2$, we have a unique linear character $\lambda_i \colon \mathbf{G}^F \to \mathbb{C}^\times$ such that $\pi(\tilde{\mathbf{G}}_{\rm der}^{\tilde{F}}) \subseteq \ker(\lambda_i)$ and

$$\lambda_i(s_0) = 1, \qquad \lambda_i(s_1) = \theta^i, \qquad \lambda_i(s_2) = \theta^{-i}.$$

We apply the construction in Proposition 4.7 to χ_0, χ_1, χ_2. This yields class functions $\chi_0^{(s_i)}$, $\chi_1^{(s_i)}$, $\chi_2^{(s_i)}$ for $i = 0, 1, 2$ which form a basis of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$; see Table 2. As already mentioned, explicit expressions for χ_1, χ_2 as linear combinations of unipotent characters of $\mathbf{G}^F$ are known. This yields explicit expressions of $\lambda_i \cdot \chi_1$ and $\lambda_i \cdot \chi_2$ as linear combinations of $\mathrm{Irr}(\mathbf{G}^F)$. (See Remark 6.8 below for more details.) Consequently, we also obtain expressions of $\chi_1^{(s_i)}$ and $\chi_2^{(s_i)}$ as linear combinations of $\mathrm{Irr}(\mathbf{G}^F)$, for $i = 0, 1, 2$. (We note that, for $i = 1, 2$, the functions $\lambda_i \cdot \chi_1$ and $\lambda_i \cdot \chi_2$ can also be identified with characteristic functions of the four non-unipotent cuspidal character sheaves on $\mathbf{G}$.)

Table 2. Bases of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$

$|A_{\mathbf{G}}(g_0)^F| = 3$:

	C_1	$C_{\bar{g}_0}$	$C_{\bar{g}_0^2}$
χ_0	1	1	1
χ_1	q^3	$q^3\theta$	$q^3\theta^2$
χ_2	q^3	$q^3\theta^2$	$q^3\theta$

$|A_{\mathbf{G}}(g_0)^F| = 9$:

	C_1	$C_{\bar{g}_0}$	$C_{\bar{g}_0^2}$	C_a	$C_{\bar{g}_0 a}$	$C_{\bar{g}_0^2 a}$	C_{a^2}	$C_{\bar{g}_0 a^2}$	$C_{\bar{g}_0^2 a^2}$
$\chi_0^{(s_0)}$	1	1	1	0	0	0	0	0	0
$\chi_1^{(s_0)}$	q^3	$q^3\theta$	$q^3\theta^2$	0	0	0	0	0	0
$\chi_2^{(s_0)}$	q^3	$q^3\theta^2$	$q^3\theta$	0	0	0	0	0	0
$\chi_0^{(s_1)}$	0	0	0	1	1	1	0	0	0
$\chi_1^{(s_1)}$	0	0	0	q^3	$q^3\theta$	$q^3\theta^2$	0	0	0
$\chi_2^{(s_1)}$	0	0	0	q^3	$q^3\theta^2$	$q^3\theta$	0	0	0
$\chi_0^{(s_2)}$	0	0	0	0	0	0	1	1	1
$\chi_1^{(s_2)}$	0	0	0	0	0	0	q^3	$q^3\theta$	$q^3\theta^2$
$\chi_2^{(s_2)}$	0	0	0	0	0	0	q^3	$q^3\theta^2$	$q^3\theta$

As discussed in Section 3, the above information about a basis of $\mathrm{CF}(\mathbf{G}^F \mid \Sigma)$ allows us to determine the values of all irreducible characters of $\mathbf{G}^F$ on elements in Σ^F.

6. Groups of adjoint type E_6 in characteristic 2

Let $\mathbf{G}$ be simple, adjoint of type E_6 and $F\colon \mathbf{G} \to \mathbf{G}$ be a Frobenius map. We shall also fix a group $\mathbf{G}^*$ dual to $\mathbf{G}$, with dual Frobenius map $F^*\colon \mathbf{G}^* \to \mathbf{G}^*$. (See [15, §1.5] for general background.) Since $\mathbf{G}$ is simple, adjoint of type E_6, the group $\mathbf{G}^*$ is simple, simply connected of type E_6.

By Lusztig's general theory, $\mathrm{Irr}(\mathbf{G}^F)$ is the disjoint union of "series" $\mathscr{E}(\mathbf{G}^F, s)$, where s runs over a set of representatives of the conjugacy classes of semisimple elements of $\mathbf{G}^{*F^*}$ (see [15, §2.6]). As a first, and crucial step, we shall now consider the unipotent characters of $\mathbf{G}^F$, which are the characters in the series $\mathscr{E}(\mathbf{G}^F, 1)$. A character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is unipotent if and only if $\langle R_{\mathbf{T},1}^{\mathbf{G}}, \rho\rangle_{\mathbf{G}^F} \neq 0$ for some F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$. There are 30 unipotent characters, both for $\mathbf{G}^F = E_6(q)$ and for $\mathbf{G}^F = {}^2E_6(q)$. Tables with their degrees (and further information) can be found in [1, p. 480/481] and [23, p. 363].

In order to explain how the values of the unipotent characters can be determined, we shall assume that the following information is available.

(A1) A parametrisation of the conjugacy classes of $\mathbf{G}^F$;
(A2) the values of $R_{\mathbf{T},1}^{\mathbf{G}}$ for all F-stable maximal tori $\mathbf{T} \subseteq \mathbf{G}$ (up to $\mathbf{G}^F$-conjugacy);
(A3) for every regular $\mathbf{L} \subsetneqq \mathbf{G}$, the values $\psi(u)$ for $\psi \in \mathscr{E}(\mathbf{L}^F, 1)$ and $u \in \mathbf{L}^F$ unipotent.

Remark 6.1. Assume that $f \in \mathrm{CF}(\mathbf{G}^F)$ is a uniform function. Using the formula in 2.1, we can express f as a linear combination of the generalised characters $R_{\mathbf{T},\theta}^{\mathbf{G}}$ where the coefficients involve the inner products $\langle f, R_{\mathbf{T},\theta}^{\mathbf{G}}\rangle_{\mathbf{T}^F}$. Now the decomposition of each $R_{\mathbf{T},\theta}^{\mathbf{G}}$ as a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$ is explicitly known by the Main Theorem 4.23 of Lusztig [23]. In order to compute $\langle f, R_{\mathbf{T},\theta}^{\mathbf{G}}\rangle_{\mathbf{G}^F}$, we need to know the values of $R_{\mathbf{T},\theta}^{\mathbf{G}}$. Hence, once **(A1)**, **(A2)** are available, we can explicitly write f as a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$.

6.2. About (A1). The conjugacy classes of semisimple elements of any finite group of Lie type can be classified by a standard procedure; see Deriziotis [4], Fleischmann–Janiszczak [7] and further references there. For $\mathbf{G}$ of type E_6, this was first done by Mizuno [34]; explicit tables for this case (and also other groups of exceptional type) are made available by Lübeck [21]. Now let $s \in \mathbf{G}^F$ be semisimple. Then one needs to classify the conjugacy classes of unipotent elements in $C_{\mathbf{G}}(s)^F$, and this is somewhat more tricky. In our case, it is convenient to pass to the group $\tilde{\mathbf{G}}$ and consider the homomorphism $\pi\colon \tilde{\mathbf{G}} \to \mathbf{G}$ with $\pi(\tilde{\mathbf{G}}^{\tilde{F}}) = \mathbf{G}^F$ (as in Section 4). In [19, §2] and [18, §3] it is reported that parametrisations of the conjugacy classes of $\tilde{\mathbf{G}}^{\tilde{F}}$ have been computed using (the older version of) CHEVIE [14]. Information about the unipotent classes in the centralisers of semisimple elements of $\tilde{\mathbf{G}}$ can now also be obtained using Michel's more recent version of CHEVIE [33], as already mentioned (and used) in 5.1; from this one can derive the conjugacy classes of $\mathbf{G}^F$. Formulae for the total number of conjugacy classes of $\mathbf{G}^F$ can be found in [7, §4] (they depend on q mod 6).

6.3. About (A2). There is a character formula which reduces the computation of the values of $R_{\mathbf{T},\theta}^{\mathbf{G}}$ (for any $\theta \in \mathrm{Irr}(\mathbf{T}^F)$) to the computation of the Green functions of $\mathbf{G}^F$, and the Green functions of subgroups of the form $C_{\mathbf{G}}^{\circ}(s)^F$ where $s \in \mathbf{G}^F$ is semisimple; see [15, Theorem 2.2.16]. The further issues in the evaluation of $R_{\mathbf{T},\theta}^{\mathbf{G}}$ are discussed in [11, §3]. If $\theta = 1$ is the trivial character, then the whole procedure simplifies drastically.

In the case of primary interest to us, that is, $p = 2$, the Green functions for $\mathbf{G}^F$ itself have been computed by Malle [32]. Nowadays, one can re-produce these results somewhat more systematically by the methods in [10] (especially for $p = 2$); these methods also apply to all subgroups $C_{\mathbf{G}}^{\circ}(s)^F$ as above. The remarks in [19, §2] and [18, §3] indicate that Frank Lübeck has already computed explicit tables with the values $R_{\mathbf{T},1}^{\mathbf{G}}(g)$ for all F-stable maximal tori $\mathbf{T} \subseteq \mathbf{G}$ and all $g \in \mathbf{G}^F$.

6.4. About (A3). First note that, since $\mathbf{G}$ has a connected center, the same is true for $\mathbf{L}$ as well. Furthermore, the possibilities for the root system of $\mathbf{L}$ are as follows. Either this is a direct sum of root systems of type A_m (for various m), or it is a root system of type D_4 or of type D_5. In the first case, we can use Example 2.2 and Remark 6.1. For the second case, explicit tables with the values of the unipotent characters (both for

$D_4(q)$ and for ${}^2D_4(q)$) are contained in the library of (the MAPLE part of) CHEVIE [14]. (Proposition 3.9 provides a proof that those tables are correct.) Finally, assume that we are in the third case, that is, $\mathbf{L}$ of type D_5. Then the information about the values of the unipotent characters of $\mathbf{L}^F$ on unipotent elements does not yet seem to be available; the required arguments are contained in an appendix due to J. Hetz (see Section 7).

Assume from now on that $p = 2$ and that the information in **(A1)**, **(A2)**, **(A3)** indeed is available. In order to proceed, we need to know the possible types of the centralisers of semisimple elements in $\mathbf{G}$.

6.5. Recall that we assume that $p = 2$. Let $s \in \mathbf{G}^F$ be semisimple and $\mathbf{H}_s := C^\circ_{\mathbf{G}}(s)$. By inspection of the list in Theorem 3.8 of Mizuno [34] (see also Lübeck's tables [21]), we see that $\mathbf{H}_s$ is either a maximal torus, or $\mathbf{H}_s$ is a regular subgroup with a root system of type

$$E_6,\ D_5,\ A_5,\ A_4{+}A_1,\ A_2{+}A_2{+}A_1,\ D_4,\ A_4,\ A_3{+}A_1,\ A_2{+}A_2,$$
$$A_2{+}A_1{+}A_1,\ A_3,\ A_2{+}A_1,\ A_1{+}A_1{+}A_1,\ A_2,\ A_1{+}A_1,\ A_1,$$

or $\mathbf{H}_s$ has a root system of type $A_2{+}A_2{+}A_2$. (Note that, in Mizuno's list, there are also subgroups with a root system of type $A_5{+}A_1$, $A_3{+}A_1{+}A_1$, $A_1{+}A_1{+}A_1{+}A_1$, but these do not occur as connected centralisers of semisimple elements when $p = 2$.) Thus, we have the following special situation for our group $\mathbf{G}$ (with $p = 2$): Either $\mathbf{H}_s = \mathbf{G}$ or $\mathbf{H}_s$ has a root system of type $A_2{+}A_2{+}A_2$, or $\mathbf{H}_s \subsetneqq \mathbf{G}$ is a regular subgroup as in 2.4.

Given the above information, we can now follow the strategy in [12, §3] to compute the values of the unipotent characters of $\mathbf{G}^F$ on all elements of $\mathbf{G}^F$. We give just some more specific comments. Let $g \in \mathbf{G}^F$ and write $g = su = us$ where $s \in \mathbf{G}^F$ is semisimple and $u \in \mathbf{G}^F$ is unipotent. As above let $\mathbf{H}_s := C^\circ_{\mathbf{G}}(s)$; then $u \in \mathbf{H}_s^F$.

Remark 6.6. Assume that $\mathbf{H}_s = \mathbf{G}$, that is, $s = 1$ and $g = u$ is unipotent. Then the values of the unipotent characters of $\mathbf{G}^F$ on g have been determined by Hetz [16, Remark 4.1.27] (split type) and [17, §8] (twisted type).

Remark 6.7. Assume that $\mathbf{H}_s \subsetneqq \mathbf{G}$ is a regular subgroup. Then we can use the information in **(A3)** and the formula in 2.4 to compute the values of the unipotent characters of $\mathbf{G}^F$ on g, exactly as explained in [12, 3.4].

Remark 6.8. Assume that $\mathbf{H}_s$ has a root system of type $A_2{+}A_2{+}A_2$. Thus, we are in the setting of Section 5. Let C be the $\mathbf{G}^F$-conjugacy class of g. For a given unipotent character $\rho \in \mathscr{E}(\mathbf{G}^F, 1)$, we then use the formula $\rho(g) = |C_{\mathbf{G}}(g)^F|\langle \rho, \varepsilon_C\rangle_{\mathbf{G}^F}$. If u is not regular unipotent in $\mathbf{H}_s$, then ε_C is uniform (see 5.1); so we can use Remark 6.1 to evaluate $\langle \rho, \varepsilon_C\rangle_{\mathbf{G}^F}$. Now assume that u is regular unipotent in $\mathbf{H}_s$; thus, $g = su \in \Sigma^F$ with Σ as in 5.2. Here, the indicator function ε_C is not uniform, but we can explicitly express ε_C as a linear combination of the functions in Table 2. Hence, it remains to determine the inner products of ρ with those functions. Firstly, χ_0 and $\chi_0^{(s_i)}$ $(i = 0, 1, 2)$ are uniform, so we can use Remark 6.1. Next, χ_1, χ_2 are (known) linear combinations of the unipotent characters of $\mathbf{G}^F$; hence, we know the inner products of ρ with χ_1 and χ_2. It remains to determine the inner products of ρ with $\chi_1^{(s_i)}$ and $\chi_2^{(s_i)}$ for $i = 0, 1, 2$. For this purpose, it is enough to consider the products $\lambda_i \cdot \chi_1$ and $\lambda_i \cdot \chi_2$, where $\lambda_i \colon \mathbf{G}^F \to \mathbb{C}^\times$ is a linear character such that $\pi(\tilde{\mathbf{G}}_{\mathrm{der}}^{\tilde{F}}) \subseteq \ker(\lambda_i)$, as in 5.7(b). We claim that

$$\langle \rho, \lambda_i \cdot \chi_1 \rangle_{\mathbf{G}^F} = \langle \rho, \lambda_i \cdot \chi_2 \rangle_{\mathbf{G}^F} = 0 \qquad \text{for } i = 1, 2.$$

This is seen as follows. By [15, Prop. 2.5.20], each linear character λ_i as above corresponds to an element $z_i \in Z(\mathbf{G}^*)^{F^*}$; furthermore, multiplication with λ_i defines a bijection $\mathscr{E}(\mathbf{G}^F, 1) \xrightarrow{\sim} \mathscr{E}(\mathbf{G}^F, z_i)$. (This follows from [15, Prop. 2.5.21].) Now assume that $i \in \{1, 2\}$. Then λ_i is non-trivial and, hence, $z_i \neq 1$. Consequently, $\lambda_i \cdot \chi_1$ and $\lambda_i \cdot \chi_2$ are linear combinations of characters in the series $\mathscr{E}(\mathbf{G}^F, z_i) \neq \mathscr{E}(\mathbf{G}^F, 1)$, and so the inner product of our unipotent character ρ with $\lambda_i \cdot \chi_1$ and with $\lambda_i \cdot \chi_2$ must be 0.

Remark 6.9. In order to determine the values of the remaining, non-unipotent characters of $\mathbf{G}^F$, one can proceed similarly as in [12, §4], assuming that the following information is available.

- **(A5)** A parametrisation of $\mathfrak{X}(\mathbf{G}, F)$, modulo the natural action of $\mathbf{G}^F$ by conjugation;
- **(A6)** parametrisations of the semisimple conjugacy classes of ${\mathbf{G}^*}^{F^*}$;
- **(A7)** tables with the values $R_{\mathbf{T},\theta}^{\mathbf{G}}(g)$ for all $(\mathbf{T}, \theta) \in \mathfrak{X}(\mathbf{G}, F)$ and all $g \in \mathbf{G}^F$;

In any case, as already mentioned in the introduction, a lot of further — mainly computational — work is required in order to produce actual tables of character values for $\mathbf{G}^F$ but, at least, we hope to have given a clear strategy of how to obtain those tables.

7. Appendix. Groups of type D_5 in characteristic 2 (by J. Hetz)

Let $p = 2$ and $\mathbf{G}$ be simple of type D_5, so that we either have $\mathbf{G}^F = D_5(q)$ (split type) or $\mathbf{G}^F = {}^2D_5(q)$ (twisted type). The aim of this appendix is to determine the values of the unipotent characters on the unipotent elements of $\mathbf{G}^F$. (This information was needed in Section 6.) First of all, by [20, Table 8.6a (p. 127)], there are 16 unipotent classes in $\mathbf{G}$ and 20 unipotent classes in $\mathbf{G}^F$ (both for the split type and the twisted type). Hence, we have $\dim \mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F) = 20$. By Proposition 3.6, there are also 20 unipotent characters of $\mathbf{G}^F$. So the issue is to determine a certain (20×20)-array of character values. There are 18 generalised characters $R_{\mathbf{T},1}^{\mathbf{G}}$ (again both for the split type and the twisted type), and we obtain their decompositions into unipotent characters as explained in 3.8. The values of their restrictions to the unipotent elements (that is, the Green functions) are known and can be computed using the methods in [10]. In view of Proposition 3.6, it remains to define suitable linear combinations of unipotent characters to obtain two further class functions $f_1, f_2 \in \mathrm{CF}(\mathbf{G}^F)$ which are orthogonal to all $R_{\mathbf{T},1}^{\mathbf{G}}$, and then compute the values of f_1, f_2 on unipotent elements.

7.1. (a) Assume that $\mathbf{G}^F = D_5(q)$ (split type). The unipotent characters of $\mathbf{G}^F$ are parametrised by symbols of rank 5 and defect $d \equiv 0 \bmod 4$ (cf. 3.8; there is no symbol of the form (S,S) in the present situation). We define two class functions $f_1, f_2 \in \mathrm{CF}(\mathbf{G}^F)$ by

$$f_1 := \tfrac{1}{2}\big(\rho_{(02,14)} - \rho_{(12,04)} - \rho_{(01,24)} + \rho_{(0124,-)}\big),$$
$$f_2 := \tfrac{1}{2}\big(\rho_{(013,124)} - \rho_{(123,014)} - \rho_{(012,134)} + \rho_{(01234,1)}\big).$$

As in 3.8, one sees that $\langle f_i, R_{\mathbf{T},1}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = 0$ for $i = 1,2$ and for all $\mathbf{T}$; also note that $\langle f_1, f_2\rangle_{\mathbf{G}^F} = 0$. Hence, every unipotent character of $\mathbf{G}^F$ is uniquely a linear combination of f_1, f_2 and the various $R_{\mathbf{T},1}^{\mathbf{G}}$.

(b) Assume that $\mathbf{G}^F = {}^2D_5(q)$ (twisted type). The unipotent characters of $\mathbf{G}^F$ are parametrised by symbols of rank 5 and defect $d \equiv 2 \bmod 4$. Here, we define two class functions $f_1, f_2 \in \mathrm{CF}(\mathbf{G}^F)$ by

$$f_1 := \tfrac{1}{2}\big(\rho_{(014,2)} - \rho_{(012,4)} - \rho_{(024,1)} + \rho_{(124,0)}\big),$$
$$f_2 := \tfrac{1}{2}\big(\rho_{(0124,13)} - \rho_{(0123,14)} - \rho_{(0134,12)} + \rho_{(1234,01)}\big).$$

As before, we check that $\langle f_1, f_2\rangle_{\mathbf{G}^F} = 0$ and $\langle f_i, R_{\mathbf{T},1}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = 0$ for $i = 1,2$ and for all $\mathbf{T}$. So again, every unipotent character of $\mathbf{G}^F$ is uniquely a linear combination of f_1, f_2 and the various $R_{\mathbf{T},1}^{\mathbf{G}}$.

The functions f_i $(i = 1,2)$ are examples of so-called (unipotent) "almost characters" as defined in [23, 4.24], that is, they arise from the unipotent characters by applying Lusztig's non-abelian Fourier transform which we referred to in 3.7. Hence, by [35, Theorem 3.2], the f_i are in fact characteristic functions of character sheaves. (In the split case where $\mathbf{G}^F = D_5(q)$, it would be sufficient to refer to [25, Cor. 14.14] for our purposes; see 7.3 below.) The values of the f_i on unipotent elements of $\mathbf{G}^F$ can thus be evaluated up to multiplication with a root of unity using the generalised Springer correspondence and the algorithm in [26, §24] which we will now briefly describe, using a notation similar to the one in [37, 1.1–1.3]. (Both the generalised Springer correspondence and Lusztig's algorithm can be accessed through Michel's CHEVIE [33].)

7.2. Let $\mathscr{N}_{\mathbf{G}}$ be the set of all pairs $(\mathscr{O}, \varsigma)$ where $\mathscr{O} \subseteq \mathbf{G}$ is a unipotent class and, for a fixed representative $u_{\mathscr{O}} \in \mathscr{O}$, ς is an irreducible character of the group $A_{\mathbf{G}}(u_{\mathscr{O}})$. Since any unipotent class of $\mathbf{G}$ is F-stable, we may (and will) assume that $u_{\mathscr{O}} \in \mathscr{O}^F$. Moreover, the group $A_{\mathbf{G}}(u_{\mathscr{O}})$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for any unipotent class $\mathscr{O} \subseteq \mathbf{G}$; we can therefore denote the irreducible characters of $A_{\mathbf{G}}(u_{\mathscr{O}}) = \langle \overline{u}_{\mathscr{O}}\rangle$ by their values at $\overline{u}_{\mathscr{O}}$. Regarding the unipotent classes of $\mathbf{G}$ themselves, we denote them by the partition of 10 giving the sizes of the Jordan blocks.

Each element $(\mathscr{O}, \varsigma) \in \mathscr{N}_{\mathbf{G}}$ parametrises an F-invariant character sheaf on $\mathbf{G}$, to which Lusztig [26, 24.2] associates a function $X_{(\mathscr{O},\varsigma)} \in \mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)$. From the generalised Springer correspondence (defined in [24]), it follows that the character sheaf corresponding to f_1 is parametrised by $(82, -1)$ and the character sheaf corresponding to f_2 by $(6211, -1)$. (We have $A_{\mathbf{G}}(u) \cong \mathbb{Z}/2\mathbb{Z}$ for u an element of any of the classes labelled by 82 and 6211.) In general, the functions $X_{(\mathscr{O},\varsigma)}$ are only defined up to multiplication with a root of unity. As for the pairs $(82,-1)$ and $(6211,-1)$, we can specify the associated functions by setting

$$X_{(82,-1)}(u) := q^{-2} f_1(u),$$
$$X_{(6211,-1)}(u) := q^{-4} f_2(u),$$

for $u \in \mathbf{G}^F$ unipotent. Having fixed such normalisations of these "X-functions", Lusztig's algorithm expresses them as explicit linear combinations of "Y-functions", which are non-zero only at elements of a single unipotent class of $\mathbf{G}$ and which are easy to compute up to multiplication with a root of unity: For any $(\mathscr{O},\varsigma) \in \mathscr{N}_{\mathbf{G}}$, there is a function $Y_{(\mathscr{O},\varsigma)} \in \mathrm{CF}_{\mathrm{uni}}(\mathbf{G}^F)$ given by $Y_{(\mathscr{O},\varsigma)} = \gamma_{(\mathscr{O},\varsigma)} Y^0_{(\mathscr{O},\varsigma)}$ for a root of unity $\gamma_{(\mathscr{O},\varsigma)} \in \mathbb{C}$, where

$$Y^0_{(\mathscr{O},\varsigma)}(u) := \begin{cases} \varsigma(a) & \text{if } u \text{ is } \mathbf{G}^F\text{-conjugate to } (u_{\mathscr{O}})_a \text{ with } a \in A_{\mathbf{G}}(u_{\mathscr{O}}), \\ 0 & \text{if } u \notin \mathscr{O}^F. \end{cases}$$

(Note that $Y^0_{(\mathscr{O},\varsigma)}$ and thus $\gamma_{(\mathscr{O},\varsigma)}$ depend upon the choice of $u_{\mathscr{O}} \in \mathscr{O}^F$.) Regarding the functions $X_{(82,-1)}$ and $X_{(6211,-1)}$ defined above, we then get

$$f_1(u) = q^2 X_{(82,-1)}(u) = q^2 Y_{(82,-1)}(u) + q^3 Y_{(6211,-1)}(u),$$
$$f_2(u) = q^4 X_{(6211,-1)}(u) = q^4 Y_{(6211,-1)}(u),$$

for $u \in \mathbf{G}^F$ unipotent. In particular, we already know that the restriction of the f_i to unipotent elements can only be non-zero at the classes labelled by 82 and 6211. In order to compute these missing values, it remains to determine the two roots of unity $\gamma_{(82,-1)}$ and $\gamma_{(6211,-1)}$.

7.3. Let $\mathscr{O} \subseteq \mathbf{G}$ be an F-stable unipotent class of $\mathbf{G}$ such that $\mathscr{O}^F$ splits into the unipotent classes $C_1, \ldots, C_r$ of $\mathbf{G}^F$, and let the further notation be as in 3.7. Then we have the identity

$$(*) \qquad \frac{|\mathbf{B}^F w \mathbf{B}^F \cap C_i| \cdot |C_i|}{|\mathbf{G}^F : \mathbf{B}^F|} = \sum_{\phi \in \mathrm{Irr}(\mathbf{W}^F)} \phi_q(T_w)[\phi](u) \qquad (1 \leqslant i \leqslant r,\ w \in \mathbf{W}^F,\ u \in C_i)$$

where ϕ_q is the irreducible character of the Iwahori–Hecke algebra $\mathscr{H}$ associated with $\mathbf{W}^F$ (and a certain weight function depending on F), and $[\phi]$ is the unipotent character of $\mathbf{G}^F$ corresponding to $\phi \in \mathrm{Irr}(\mathbf{W}^F)$. (See, for example, [2, §11D].) The values $\phi_q(T_w)$ are explicitly known via CHEVIE [14]. Furthermore, by the discussion in 7.1, each $[\phi]$ can be written as

$$[\phi] = f_{\mathrm{unif}}(\phi) + a_1(\phi) f_1 + a_2(\phi) f_2,$$

where $f_{\mathrm{unif}}(\phi) \in \mathrm{CF}(\mathbf{G}^F)$ is a uniform function (whose values on unipotent elements can thus be computed) and the coefficients $a_1(\phi), a_2(\phi) \in \mathbb{C}$ are explicitly known. We will evaluate $(*)$ with F-stable elements u in one of the two classes 82 and 6211 (and with suitably chosen $w \in \mathbf{W}^F$). As already noted in 7.2, we have $A_{\mathbf{G}}(u) \cong \mathbb{Z}/2\mathbb{Z}$ for any such u; in particular, both of the unipotent classes 82 and 6211 of $\mathbf{G}$ split into two $\mathbf{G}^F$-classes.

7.4. The regular unipotent class. Let us first consider the values of f_1 and f_2 at elements of the regular unipotent class 82 of $\mathbf{G}$. Let $u \in \mathbf{G}^F$ be an element of this class. As already observed in 7.2, we have $f_2(u) = 0$, so it remains to deal with $f_1(u)$.

(a) Assume that $\mathbf{G}^F = D_5(q)$. For $\phi \in \mathrm{Irr}(\mathbf{W})$, we get (with the notation of 7.2 and 7.3)

$$[\phi](u) = f_{\mathrm{unif}}(\phi)(u) + a_1(\phi) q^2 Y_{(82,-1)}(u).$$

We have $f_{\mathrm{unif}}(\phi)(u) = 0$ unless $\phi = 1_{\mathbf{W}}$ is the trivial character of $\mathbf{W}$, in which case $f_{\mathrm{unif}}(1_{\mathbf{W}})(u) = 1$. For $w \in \mathbf{W}$, the sum on the right hand side of $(*)$ in 7.3 then evaluates to

$$\sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \phi_q(T_w)[\phi](u) = (1_{\mathbf{W}})_q(T_w) + \tfrac{1}{2} q^2 Y_{(82,-1)}(u)\big((1.31)_q(T_w) - (11.3)_q(T_w) - (.32)_q(T_w)\big).$$

Taking $w = w_{\mathrm{c}} := s_1 s_2 s_3 s_4 s_5$ to be a Coxeter element of $\mathbf{W}$ (where for $1 \leqslant i \leqslant 5$, s_i denotes the reflection in the root α_i with respect to the diagram for D_5 printed in Section 3), we have

$$(1_{\mathbf{W}})_q(T_{w_{\mathrm{c}}}) = q^5, \quad (1.31)_q(T_{w_{\mathrm{c}}}) = -(.32)_q(T_{w_{\mathrm{c}}}) = q^3 \quad \text{and} \quad (11.3)_q(T_{w_{\mathrm{c}}}) = 0.$$

Hence, denoting by $C \subseteq \mathbf{G}^F$ the conjugacy class containing u, the equation $(*)$ in 7.3 reads

$$\frac{|\mathbf{B}^F w_{\mathrm{c}} \mathbf{B}^F \cap C| \cdot |C|}{|\mathbf{G}^F : \mathbf{B}^F|} = q^5(1 + Y_{(82,-1)}(u)) = q^5(1 + \gamma_{(82,-1)} Y^0_{(82,-1)}(u)).$$

Since the left hand side is in particular a real number (and $Y^0_{(82,-1)}(u) \in \{\pm 1\}$), the root of unity $\gamma_{(82,-1)}$ can only be ± 1. Now recall that $Y^0_{(82,-1)}$ and $\gamma_{(82,-1)}$ depend on the choice of $u_{(82)}$, which we have not yet fixed. — If we replace $u_{(82)}$ by a representative of the other $\mathbf{G}^F$-class contained in the class 82 of $\mathbf{G}$, this changes the sign of $\gamma_{(82,-1)}$. We can therefore *define* $u_{(82)}$ by the condition that $\gamma_{(82,-1)} = +1$. Note that the right hand side of the above identity then equals $2q^5$ if u is $\mathbf{G}^F$-conjugate to $u_{(82)}$, and it is 0 if u is not $\mathbf{G}^F$-conjugate to $u_{(82)}$. Hence, the $\mathbf{G}^F$-class of our chosen $u_{(82)}$ is characterised by the property that it has a non-empty intersection with $\mathbf{B}^F w_{\mathrm{c}} \mathbf{B}^F$.

(b) Assume that $\mathbf{G}^F = {}^2D_5(q)$. The argument here is very similar to the one for the split case, the only difference is that the group $\mathbf{W}^F$ is now of type B_4. Indeed, the sum $\sum_{\phi \in \mathrm{Irr}(\mathbf{W}^F)} \phi_q(T_w)[\phi](u)$ evaluates to

$$(1_{\mathbf{W}^F})_q(T_w) + \tfrac{1}{2} q^2 Y_{(82,-1)}(u)\big((2.2)_q(T_w) - (21.1)_q(T_w) - (.4)_q(T_w) + (211.)_q(T_w)\big).$$

Taking again $w = w_{\mathrm{c}} = s_1 s_2 s_3 s_4 s_5 \in \mathbf{W}^F$ (note that the automorphism of $\mathbf{W}$ induced by F interchanges s_1 and s_2, but we have $s_1 s_2 = s_2 s_1$), we get $(1_{\mathbf{W}^F})_q(T_{w_{\mathrm{c}}}) = q^5$ and

$$(211.)_q(T_{w_{\mathrm{c}}}) = -(.4)_q(T_{w_{\mathrm{c}}}) = q^3, \quad (2.2)_q(T_{w_{\mathrm{c}}}) = (21.1)_q(T_{w_{\mathrm{c}}}) = 0.$$

Hence, denoting by C the $\mathbf{G}^F$-conjugacy class of u, we obtain

$$\frac{|\mathbf{B}^F w_{\mathrm{c}} \mathbf{B}^F \cap C| \cdot |C|}{|\mathbf{G}^F : \mathbf{B}^F|} = q^5(1 + \gamma_{(82,-1)} Y^0_{(82,-1)}(u)).$$

As in (a) we see that $u_{(82)}$ can be chosen such that $\gamma_{(82,-1)} = +1$, and this determines the $\mathbf{G}^F$-class of $u_{(82)}$ (inside the regular unipotent class 82) uniquely; moreover, the $\mathbf{G}^F$-class of $u_{(82)}$ is characterised by the property that it meets $\mathbf{B}^F w_{\mathrm{c}} \mathbf{B}^F$.

7.5. The class 6211. It remains to find the values of f_1 and f_2 at elements of the unipotent class of $\mathbf{G}$ labelled by 6211. So let $u \in \mathbf{G}^F$ be an element of this class. Recall that $A_{\mathbf{G}}(u) \cong \mathbb{Z}/2\mathbb{Z}$.

(a) Assume that $\mathbf{G}^F = D_5(q)$. For $\phi \in \mathrm{Irr}(\mathbf{W})$, we have

$$[\phi](u) = f_{\mathrm{unif}}(\phi)(u) + (a_1(\phi)q^3 + a_2(\phi)q^4)Y_{(6211,-1)}(u)$$

(see 7.2, 7.3). Now the value $f_{\mathrm{unif}}(\phi)(u)$ is non-zero for $\phi \in \{1_{\mathbf{W}}, 1.4, .41\}$. In fact, we have $[\phi](u) = f_{\mathrm{unif}}(\phi)(u)$ for any of these ϕ and

$$f_{\mathrm{unif}}(1_{\mathbf{W}})(u) = 1, \quad f_{\mathrm{unif}}(1.4)(u) = q, \quad f_{\mathrm{unif}}(.41)(u) = q^2.$$

For $w \in \mathbf{W}$, we again evaluate the sum on the right hand side of $(*)$ in 7.3:

$$\begin{aligned}\sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \phi_q(T_w)[\phi](u) = {} & (1_{\mathbf{W}})_q(T_w) + (1.4)_q(T_w) \cdot q + (.41)_q(T_w) \cdot q^2 \\ & + \tfrac{1}{2} q^3 Y_{(6211,-1)}(u)((1.31)_q(T_w) - (11.3)_q(T_w) - (.32)_q(T_w)) \\ & + \tfrac{1}{2} q^4 Y_{(6211,-1)}(u)((1.211)_q(T_w) - (111.2)_q(T_w) - (.211)_q(T_w)).\end{aligned}$$

Here we take $w := s_1 s_2 s_3 s_4 \in \mathbf{W}$ and get

$$\begin{aligned}&(1_{\mathbf{W}})_q(T_w) = (.41)_q(T_w) = q^4, \quad (1.4)_q(T_w) = q^4 - q^3, \\ &(1.31)_q(T_w) = -q^3 + q^2, \quad (11.3)_q(T_w) = -q^3, \quad (.32)_q(T_w) = -q^2, \\ &(1.211)_q(T_w) = q^2 - q, \quad (111.2)_q(T_w) = -q, \quad (.221)_q(T_w) = -q^2.\end{aligned}$$

Hence, if $C \subseteq \mathbf{G}^F$ is the conjugacy class containing u, the equation $(*)$ in 7.3 reads

$$\frac{|\mathbf{B}^F w \mathbf{B}^F \cap C| \cdot |C|}{|\mathbf{G}^F : \mathbf{B}^F|} = (q^6 + q^5)(1 + \gamma_{(6211,-1)} Y^0_{(6211,-1)}(u)).$$

We can thus argue as in 7.4 to deduce that $u_{(6211)}$ can be chosen so that $\gamma_{(6211,-1)} = +1$, and this determines the $\mathbf{G}^F$-class of $u_{(6211)}$ uniquely; moreover, this $\mathbf{G}^F$-class is characterised by the property that it has a non-empty intersection with $\mathbf{B}^F w \mathbf{B}^F$.

(b) Assume that $\mathbf{G}^F = {}^2D_5(q)$. The argument is again similar to the one in (a). The sum $\sum_{\phi \in \mathrm{Irr}(\mathbf{W}^F)} \phi_q(T_w)[\phi](u)$ now evaluates to

$$\begin{aligned}&(1_{\mathbf{W}^F})_q(T_w) + (31.)_q(T_w) \cdot q + (3.1)_q(T_w) \cdot q^2 \\ &+ \tfrac{1}{2} q^3 Y_{(6211,-1)}(u)((2.2)_q(T_w) - (.4)_q(T_w) - (21.1)_q(T_w) + (211.)_q(T_w)) \\ &+ \tfrac{1}{2} q^4 Y_{(6211,-1)}(u)((1.21)_q(T_w) - (.31)_q(T_w) - (11.11)_q(T_w) + (1111.)_q(T_w)).\end{aligned}$$

Taking $w := s_1s_2s_3s_4 \in \mathbf{W}^F$, we have

$$\begin{aligned}
&(1_{\mathbf{W}^F})_q(T_w) = q^4, \quad (31.)_q(T_w) = q^4 - q^3, \quad (3.1)_q(T_w) = q^4,\\
&(2.2)_q(T_w) = 0, \quad (.4)_q(T_w) = -q^2, \quad (21.1)_q(T_w) = -q^3, \quad (211.)_q(T_w) = -q^3 + q^2,\\
&(1.21)_q(T_w) = q, \quad (.31)_q(T_w) = -q^2 + q, \quad (11.11)_q(T_w) = 0, \quad (1111.)_q(T_w) = q^2.
\end{aligned}$$

Hence, denoting by C the $\mathbf{G}^F$-conjugacy class of u, we obtain

$$\frac{|\mathbf{B}^F w\mathbf{B}^F \cap C| \cdot |C|}{|\mathbf{G}^F : \mathbf{B}^F|} = (q^6 + q^5)(1 + \gamma_{(6211,-1)} Y^0_{(6211,-1)}(u)).$$

As before we conclude that $u_{(6211)}$ can be chosen such that $\gamma_{(6211,-1)} = +1$, and this determines the $\mathbf{G}^F$-class of $u_{(6211)}$ uniquely; moreover, the $\mathbf{G}^F$-class of $u_{(6211)}$ is characterised by the property that it meets $\mathbf{B}^F w\mathbf{B}^F$.

7.6. In 7.4 and 7.5, we defined $u_{(82)}$ and $u_{(6211)}$ by requiring the signs $\gamma_{(82,-1)}$ and $\gamma_{(6211,-1)}$ to be $+1$, which we showed to be equivalent to saying that their $\mathbf{G}^F$-classes have a non-empty intersection with the F-stable points of a suitable Bruhat cell. Hence, if we are given an explicit element $u \in \mathbf{G}^F$ lying in the class 82 (respectively, 6211) of the algebraic group $\mathbf{G}$, in order to show that u is $\mathbf{G}^F$-conjugate to $u_{(82)}$ (respectively, $u_{(6211)}$), it suffices to find *one* $\mathbf{G}^F$-conjugate of u which lies in the associated Bruhat cell. This can be achieved using the criterion in [16, Lemma 3.2.24]. Indeed, let $\mathbf{U} \subseteq \mathbf{B}$ be the unipotent radical of the Borel subgroup $\mathbf{B}$ and, for $1 \leqslant i \leqslant 5$, let $\mathbf{U}_{\alpha_i} \subseteq \mathbf{U}$ be the root subgroup associated to the simple root α_i, with corresponding homomorphism $u_i := u_{\alpha_i} \colon k \to \mathbf{U}_{\alpha_i}$. Then $u_1(1)u_2(1)u_3(1)u_4(1)u_5(1)$ is an element of the regular unipotent class 82 of $\mathbf{G}$, and it also lies in $\mathbf{G}^F$ regardless of whether $\mathbf{G}^F = D_5(q)$ or $\mathbf{G}^F = {}^2D_5(q)$ (as $u_1(1)u_2(1) = u_2(1)u_1(1)$). Similarly, $u_1(1)u_2(1)u_3(1)u_4(1) \in \mathbf{G}^F$ is an element of the unipotent class 6211 of $\mathbf{G}$. In view of our definition of the $w \in \mathbf{W}^F$ associated to the classes 82 (in 7.4) and 6211 (in 7.5), it directly follows from [16, Lemma 3.2.24] that we may take

$$\begin{aligned}
u_{(82)} &= u_1(1)u_2(1)u_3(1)u_4(1)u_5(1),\\
u_{(6211)} &= u_1(1)u_2(1)u_3(1)u_4(1).
\end{aligned}$$

The values at unipotent elements of the class functions f_1 and f_2 defined in 7.1 are given by Table 3, where $u'_{(82)} \in \mathbf{G}^F$ denotes an element of the regular unipotent class 82 which is not $\mathbf{G}^F$-conjugate to $u_{(82)}$, and $u'_{(6211)} \in \mathbf{G}^F$ is an element of the unipotent class 6211 which is not $\mathbf{G}^F$-conjugate to $u_{(6211)}$.

TABLE 3. Values of f_1 and f_2 at unipotent elements for groups of type D_5

	$u_{(82)}$	$u'_{(82)}$	$u_{(6211)}$	$u'_{(6211)}$	$u \notin (82 \cup 6211)$
f_1	q^2	$-q^2$	q^3	$-q^3$	0
f_2	0	0	q^4	$-q^4$	0

Acknowledgements. I thank Jonas Hetz and Gunter Malle for a careful reading of the manuscript and a number of useful comments. I am indebted to Jonas Hetz for allowing me to include his results on groups of type D_5 in an appendix here. Thanks are also due to Jean Michel for help with his `Julia` version of the CHEVIE package. This article is a contribution to SFB-TRR 195 by the DFG (Deutsche Forschungsgemeinschaft), Project-ID 286237555.

References

[1] R. W. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters, Wiley, New York, 1985; reprinted 1993 as Wiley Classics Library Edition.

[2] C. W. Curtis and I. Reiner, Methods of Representation Theory Vol. I, Wiley, New York, 1981.

[3] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. **103** (1976), 103–161.

[4] D. I. Deriziotis, Conjugacy Classes and Centralizers of Semisimple Elements in Finite Groups of Lie Type, Lecture Notes in Mathematics at the University of Essen, vol. 11, Universität Essen, Fachbereich Mathematik, Essen, 1984.

[5] F. Digne and J. Michel, Fonctions $\mathscr{L}$ des variétés de Deligne–Lusztig et descente de Shintani, Mém. Soc. Math. France, no. 20, suppl. au Bull. S. M. F. **113**, 1985.

[6] F. Digne and J. Michel, Representations of Finite Groups of Lie Type, London Mathematical Society Student Texts, vol. 21, 2nd Edition, Cambridge University Press, 2020.

[7] P. Fleischmann and I. Janiszczak, On the computation of conjugacy classes of Chevalley groups, Appl. Algebra Engrg. Comm. Comput. **7** (1996), 221–234.

[8] M. Geck, A first guide to the character theory of finite groups of Lie type, Local Representation Theory and Simple Groups (eds. R. Kessar, G. Malle, D. Testerman), pp. 63–106, EMS Lecture Notes Series, Eur. Math. Soc., Zürich, 2018.

[9] M. Geck, On the values of unipotent characters in bad characteristic, Rend. Cont. Sem. Mat. Univ. Padova **141** (2019), 37–63.

[10] M. Geck, Computing Green functions in small characteristic (Special issue in memory of Kay Magaard), J. Algebra **561** (2020), 163–199.

[11] M. Geck, On the computation of character values for finite Chevalley groups of exceptional type, George Lusztig special issue, Pure Appl. Math. Q., to appear; preprint at `arXiv:2105.00722`.

[12] M. Geck, The character table of the finite Chevalley group $F_4(q)$ for q a power of 2, Arch. Math. **121** (2023), 669–679.

[13] M. Geck and G. Hiss. Basic sets of Brauer characters of finite groups of Lie type, J. reine und angew. Math. **418** (1991), 173–188.

[14] M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, CHEVIE — A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, Appl. Algebra Engrg. Comm. Comput. **7** (1996), 175–210.

[15] M. Geck and G. Malle, The Character Theory of Finite Groups of Lie Type: A Guided Tour, Cambridge Studies in Advanced Mathematics **187**, Cambridge University Press, Cambridge, 2020.

[16] J. Hetz, Characters and Character Sheaves of Finite Groups of Lie Type, Dissertation, University of Stuttgart, 2023; available at `http://dx.doi.org/10.18419/opus-12932`.

[17] J. Hetz, The values of unipotent characters at unipotent elements for groups of type E_8 and ${}^2E_6$; preprint at `https://arxiv.org/abs/2309.09915`.

[18] G. Hiss and F. Lübeck, The Brauer trees of the exceptional Chevalley groups of type F_4 and ${}^2E_6$, Arch. Math. **70** (1998), 16–21.

[19] G. Hiss, F. Lübeck and G. Malle, The Brauer trees of the exceptional Chevalley groups of type E_6, Manuscripta Math. **87** (1995), 131–144.

[20] M. W. Liebeck and G. M. Seitz, Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras, Math. Surveys and Monographs, vol. 180, Amer. Math. Soc., Providence, RI, 2012.

[21] F. Lübeck, Centralizers and numbers of semisimple classes in exceptional groups of Lie type, online data at `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/CentSSClasses`.

[22] G. Lusztig, Irreducible representations of finite classical groups. Invent. Math. **43** (1977), 125–175.

[23] G. Lusztig, Characters of Reductive Groups over a Finite Field, Annals Math. Studies, vol. 107, Princeton University Press, 1984.

[24] G. Lusztig, Intersection cohomology complexes on a reductive group, Invent. Math. **75**, 205–272 (1984).

[25] G. Lusztig, Character sheaves, III, Adv. in Math. **57** (1985), 266–315.

[26] G. Lusztig, Character sheaves IV, Adv. Math. **59** (1986), 1–63.

[27] G. Lusztig, Green functions and character sheaves, Ann. of Math. (2) **131** (1990), 355–408.

[28] G. Lusztig, Remarks on computing irreducible characters, J. Amer. Math. Soc. **5** (1992), 971–986.

[29] G. Lusztig, From conjugacy classes in the Weyl group to unipotent classes, Represent. Theory **15** (2011), 494–530.

[30] G. Lusztig, On the cleanness of cuspidal character sheaves, Mosc. Math. J. **12** (2012), 621–631.

[31] G. Lusztig, On induction of class functions, Represent. Theory **25** (2021), 412–421.

[32] G. Malle, Green functions for groups of type F_4 and E_6 in characteristic 2, Comm. Algebra **21** (1993), 747–798.

[33] J. Michel, The development version of the CHEVIE package of GAP3, J. Algebra **435** (2015), 308–336; see `https://github.com/jmichel7` (where also a `Julia` version is available).

[34] K. Mizuno, The conjugate classes of Chevalley groups of type E_6, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 525–563.

[35] T. Shoji, Character sheaves and almost characters of reductive groups, Adv. Math. **111** (1995), 244–313.

[36] T. Shoji, Character sheaves and almost characters of reductive groups, II, Adv. Math. **111** (1995), 314–354.

[37] T. Shoji, Generalized Green functions and unipotent classes for finite reductive groups, I, Nagoya Math. J. **184** (2006), 155–198.

[38] T. Shoji, Lusztig's conjecture for finite classical groups with even characteristic, Contemp. Math. **478** (2009), 207–236, Amer. Math. Soc. Providence, RI..

[39] R. Steinberg, Endomorphisms of Linear Algebraic Groups, Memoirs of the AMS no. 80, Amer. Math. Soc., Providence, RI, 1968.

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