

NONASSOCIATIVE CYCLIC ALGEBRAS AND THE SEMIASSOCIATIVE BRAUER MONOID

S. PUMPLÜN

ABSTRACT. We look at classes of semiassociative algebras, with an emphasis on those that canonically generalize associative (generalized) cyclic algebras, and at their behaviour in the semiassociative Brauer monoid defined by Blachar, Haile, Matri, Rein, and Vishne. A possible way to generalize this monoid in characteristic p that includes nonassociative differential algebras is briefly considered.

INTRODUCTION

Recently, semiassociative algebras and the semiassociative Brauer monoid denoted $Br^{sa}(F)$ were introduced by Blachar, Haile, Matri, Rein, and Vishne [4] as canonical generalizations of associative central simple algebras and their Brauer group. Semiassociative algebras A over a field F are F -central and are characterized by having an étale algebra E contained in their nucleus, such that A is cyclic and faithful as an $E \otimes_F E$ -module via the action $(e \otimes e')a = eae'$ for all $a \in A$, $e, e' \in E$. This definition makes it possible to use classical Brauer factor sets [12, Chapter 2] when developing the theory, and guarantees that the algebras are forms of skew matrix algebras, which are defined and investigated in depth [4].

Together with the tensor product, equivalence classes of semiassociative algebras over F form a monoid that contains the classical Brauer group as its unique maximal subgroup. The skew matrix algebras now play the role of the classical matrices in the Brauer group. In particular, a semiassociative algebra is called split if it is isomorphic to a skew matrix algebra. The authors state that “the key example for semiassociative algebras are skew matrices” [4].

In this paper we will look at another important example of semiassociative algebras; the nonassociative (generalized) cyclic algebras (and their opposite algebras). It is well known that (generalized) cyclic algebras play a prominent role in the structure theory of classical central simple algebras. Here, we look at the role nonassociative (generalized) cyclic algebras play in the structure theory of semiassociative simple algebras. These algebras are canonical generalizations of associative cyclic algebras (respectively, of the generalized associative cyclic algebras introduced by Jacobson [12]) over F .

If F has a cyclic Galois field extension K/F of degree n , then there exist nonassociative cyclic algebras that have K as their nucleus, and these are semiassociative algebras of degree

2010 *Mathematics Subject Classification.* Primary: 17A35; Secondary: 17A60, 17A36.

Key words and phrases. Skew polynomial ring, semiassociative algebra, cyclic algebra, differential algebra, Menichetti algebra, nonassociative algebra, semiassociative Brauer monoid.

n . More generally, for any field F with cyclic Galois field extensions, there exist nonassociative generalized cyclic algebras. These have a central simple algebra in their nucleus with center a separable field extension of F and are semiassociative algebras as well. Both these types of semiassociative algebras are not semicentral, and thus in particular not homogeneous. They have infinite order in $Br^{sa}(F)$, even when they are not division algebras. Nonassociative (generalized) cyclic algebras also nicely show that the splitting behaviour of an algebra in $Br^{sa}(F)$ (i.e., whether or not it will be split or split under a field extension) only depends on its nucleus.

Cyclic algebras $(K/F, \sigma, d)$ of degree n that are not associative are division algebras in many cases, e.g. for all prime n . Some of the tensor products we consider yield examples of semiassociative algebras which again are division algebras, and are nonassociative generalized cyclic algebras.

Nonassociative (generalized) cyclic algebras already appeared in space-time block coding [26, 25, 19, 20], and in (f, σ, δ) -codes [22]. In this paper we generalize their definition which previously usually employed skew polynomials in $D[t; \sigma]$ over division rings D , and drop the assumption that D has no zero divisors. Generalized Menichetti algebras [17, 27] can be seen as generalizations of both crossed products and nonassociative cyclic algebras, and make up the second class of semiassociative algebras we present.

We finish by suggesting possible generalizations of the semiassociative Brauer monoid, which allow us to include nonassociative (generalized) differential extensions as classes of algebras in the monoid, if the characteristic of F is prime. Our motivation is that the associative differential algebras play an important role in the Brauer group, so it seems natural to try find a way to include them in some semiassociative Brauer monoid definition.

The structure of the paper is as follows: we collect the basic results needed in Section 1. In Section 2, we generalize the definition of nonassociative cyclic algebras $(K/F, \sigma, d)$ to include the case that K/F is an étale extension, and the definition of generalized nonassociative cyclic algebras (D, σ, d) to include the case that the algebra D employed in the construction with the skew polynomial $t^m - d \in D[t; \sigma]$ has zero divisors, collecting and generalizing several previously achieved results, including the explicit computation of the right nucleus for these algebras. We observe that the opposite algebra of a nonassociative cyclic algebra is semiassociative, but need not be a nonassociative cyclic algebra again (Corollary 2). In Section 3, we look at the tensor product of a central simple algebra and a nonassociative cyclic algebra. We investigate the behaviour of nonassociative (generalized) cyclic algebras in the Brauer monoid $Br^{sa}(F)$ in Section 4, and briefly look at $Br^{sa}(\mathbb{R})$ and $Br^{sa}(\mathbb{F}_q)$.

When F is a field of prime characteristic p , the definition of $Br^{sa}(F)$ may benefit from a generalization that includes a class of algebras that generalize algebras that are associative differential extensions [12]: All associative central division algebras over a field F of characteristic zero can be constructed using differential polynomials (Amitsur [2], and later [11], [12, Sections 1.5, 1.8, 1.9]). The construction method is an analogue to the well-known crossed product construction, except that instead of algebraic splitting fields it uses splitting fields K , where F is algebraically closed in K . For p -algebras over base fields of characteristic $p > 0$ the construction employs differential polynomial rings $D[t; \delta]$ (where D is a

central division algebra over C), factoring out a two-sided ideal generated by some suitable $f \in D[t; \delta]$. This construction was generalized to the nonassociative setting in [23]. We briefly consider these algebras in characteristic p and the pros and cons to include them in potential generalizations of $Br^{sa}(F)$ in the last two sections.

1. PRELIMINARIES

1.1. Nonassociative algebras. Let F be a field. An F -vector space A is an *algebra* over F , if there exists an F -bilinear map $A \times A \rightarrow A$, $(x, y) \mapsto x \cdot y$, usually denoted by juxtaposition, the *multiplication* of A . An algebra A is called *unital* if there is an element in A , denoted by 1 , such that $1x = x1 = x$ for all $x \in A$. We only consider unital algebras. The *center* of A is $C(A) = \{x \in \text{Nuc}(A) \mid xy = yx \text{ for all } y \in A\}$ and A is called *(F-)central* if it has center F .

Associativity in A is measured by the *left nucleus* $\text{Nuc}_l(A) = \{x \in A \mid [x, A, A] = 0\}$, the *middle nucleus* $\text{Nuc}_m(A) = \{x \in A \mid [A, x, A] = 0\}$ and the *right nucleus* $\text{Nuc}_r(A) = \{x \in A \mid [A, A, x] = 0\}$ of A , where $[x, y, z] = (xy)z - x(yz)$ is the *associator*. $\text{Nuc}_l(A)$, $\text{Nuc}_m(A)$, and $\text{Nuc}_r(A)$ are associative subalgebras of A , and their intersection $\text{Nuc}(A) = \{x \in A \mid [x, A, A] = [A, x, A] = [A, A, x] = 0\}$ is the *nucleus* of A . $\text{Nuc}(A)$ is an associative subalgebra of A containing $F1$ and $x(yz) = (xy)z$ whenever one of the elements x, y, z is in $\text{Nuc}(A)$. Multiplication on both sides make A into a bimodule over its nucleus. Moreover, for every subalgebra N of the nucleus the N -bimodule structure of A can be viewed as a left module structure over the ring $N^e = N \otimes_F N^{\text{op}}$.

An algebra $A \neq 0$ is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a , $L_a(x) = ax$, and the right multiplication with a , $R_a(x) = xa$, are bijective. If A has finite dimension over F , A is a division algebra if and only if A has no zero divisors [29, pp. 15, 16].

An *étale* algebra over F is a finite direct product of finite separable field extensions of F .

1.2. Semiassociative algebras. (cf. [4])

A finite dimensional nonassociative F -central algebra A is called *semiassociative* if its nucleus has an étale F -subalgebra E , such that A is cyclic and faithful over $E \otimes_F E$ via the action $(e \otimes e')a = eae'$ for all $a \in A$, $e, e' \in E$. The dimension of a semiassociative algebra A is a square [4, Corollary 3.4] and the root of the dimension of A is called the *degree* of A . If A is semiassociative of degree n , then any n -dimensional étale subalgebra E of $\text{Nuc}(A)$ is a maximal commutative subalgebra of A [4, Corollary 7.3].

If A is a nonassociative algebra containing an étale subalgebra E in its nucleus, then any two of the following conditions imply the third: A is faithful over $E \otimes E^{\text{op}}$, A is cyclic over $E \otimes E^{\text{op}}$, and $\dim A = (\dim E)^2$ [4, Remark 3.3].

Every associative central simple algebra of degree n has a maximal étale subalgebra E of dimension n and is semiassociative. We call A *E-semiassociative* if E is an étale F -subalgebra of its nucleus, such that A is cyclic and faithful over $E^e = E \otimes_F E$. The nucleus of a nonassociative algebra may contain more than one étale subalgebra of the same dimension. However, if A is a semiassociative algebra with respect to one étale subalgebra of its nucleus, then it is semiassociative with respect to all étale subalgebras of its nucleus

[4, Proposition 3.6]. A scalar extension of a semiassociative algebra is semiassociative [4, Proposition 12.1].

A tensor c_{ijk} of degree n of $n \times n \times n$ scalars in F is called a *skew set c of degree n* . A skew set c is called *reduced* if $c_{iij} = c_{jii} = 1$ for all i, j . Let c be such a reduced skew set. Then the *skew matrix algebra* $M_n(F; c)$ is the F -vector space with basis the matrix units e_{ij} and multiplication given by $e_{ij}e_{kl} = \delta_{jk}c_{ijl}e_{il}$. Note that $M_n(F) = M_n(F; 1)$.

A semiassociative algebra A is called *split*, if it is a skew matrix algebra.

A field extension K/F *splits* a semiassociative algebra A , if $A_K = A \otimes_F K$ is split. A semiassociative algebra of degree n is split if and only if F^n is a unital subalgebra of the nucleus [4, Proposition 7.2]. Let A be an E -semiassociative algebra with $E = \text{Nuc}(A)$. Then a field extension K/F splits A if and only if it splits E [4, Corollary 7.5]. If K is a field that splits an étale subalgebra in the nucleus of an n -dimensional semiassociative algebra A of degree n , and F is an infinite field, then K splits A [4, Theorem 7.1].

If A is semiassociative of degree n , then any n -dimensional étale subalgebra E of $\text{Nuc}(A)$ is a maximal commutative subalgebra of A [4, Corollary 7.3].

Let $J(\text{Nuc}(A))$ denote the radical of the associative algebra $\text{Nuc}(A)$. For a semiassociative algebra A , the simple components of the semisimple quotient $\sigma(A) = \text{Nuc}(A)/J(\text{Nuc}(A))$ are called the *atoms* of A . A semiassociative algebra over F is called *semicentral*, if all of its atoms are F -central [4, Definition 16.1]. A semiassociative algebra is *homogeneous* if it is semicentral, and the atoms are all Brauer equivalent to each other [4, Definition 17.2].

Two semiassociative algebras A and B over F are called *Brauer equivalent*, if there exist skew matrix algebras $M_n(F; c)$ and $M_m(F; c')$ such that $A \otimes_F M_n(F; c) \cong B \otimes_F M_m(F; c')$. The *semiassociative Brauer monoid* $Br^{sa}(F)$ is the set of equivalence classes with respect to Brauer equivalence, with product $[A]^{sa}[B]^{sa} = [A \otimes_F B]^{sa}$ and unit element $[F]^{sa}$. If A is a homogeneous semiassociative algebra, and D the (associative) underlying division algebra of its atoms, then there is a decomposition $A \cong D \otimes_F M$, where M is a skew matrix algebra and D is the unique member of minimal degree in the class $[D]^{sa} \in Br^{sa}(F)$ [4, Proposition 18.2, Corollary 18.3].

1.3. Nonassociative algebras obtained from skew polynomial rings. (for details, cf. [22])

Let S be a unital associative noncommutative ring, $\sigma \in \text{Aut}(S)$, and $\delta : S \rightarrow S$ a σ -*derivation*, i.e. an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in S$. The *skew polynomial ring* $R = S[t; \sigma, \delta]$ is the set of skew polynomials $a_0 + a_1t + \cdots + a_nt^n$ with $a_i \in S$, where addition is defined term-wise and multiplication by $ta = \sigma(a)t + \delta(a)$ for all $a \in S$. We write $S[t; \sigma] = S[t; \sigma, 0]$ and $S[t; \delta] = S[t; id, \delta]$.

For $f(t) = a_0 + a_1t + \cdots + a_nt^n$ with $a_n \neq 0$ define $\deg(f) = n$ and $\deg(0) = -\infty$. Then $\deg(gh) \leq \deg(g) + \deg(h)$ for $f, g \in S[t]$ (with equality if h or g have an invertible leading coefficient, or if S is a division ring). An element $f \in R$ is *irreducible* in R if it is not a unit and it has no proper factors, i.e. if there do not exist $g, h \in R$ with $\deg(g), \deg(h) < \deg(f)$ such that $f = gh$. We call $f \in R$ *right-invariant* polynomial, if $fR \subset Rf$. If f is right invariant then Rf is a two-sided ideal.

Let $f \in R$ have degree m and an invertible leading coefficient. Then for all $g(t) \in R$ of degree $l \geq m$, there exist uniquely determined $r, q \in R$ with $\deg(r) < \deg(f)$, such that $g(t) = q(t)f(t) + r(t)$. This generalizes the right division algorithm in R that is well-known when S is a division ring [12, p. 6].

From now on we assume that $f \in R = S[t; \sigma, \delta]$ is monic of degree m . Let $\text{mod}_r f$ denote the remainder of right division by f . Since the remainder is uniquely determined, the skew polynomials of degree less than m canonically represent the elements of the left $S[t; \sigma, \delta]$ -module $S[t; \sigma, \delta]/S[t; \sigma, \delta]f$.

The additive group $\{g \in R \mid \deg(g) < m\}$, together with the multiplication $g \circ h = gh \text{ mod}_r f$ for all $g, h \in R$ of degree less than m , is a unital nonassociative algebra over $S_0 = \{a \in S \mid ah = ha \text{ for all } h \in S_f\}$, denoted by S_f or R/Rf . The construction for S a division algebra goes back to [18], and these algebras are called *Petit algebras*. S_0 is a commutative subring of S , and if S is a division algebra, it is a subfield of S . S_f is associative if and only if f is two-sided. If S_f is not associative then $S \subset \text{Nuc}_l(S_f)$, $S \subset \text{Nuc}_m(S_f)$ (if S is a division ring, the inclusions become equalities), and the eigenspace of f is the right nucleus: $\text{Nuc}_r(S_f) = \{g \in R \mid \deg(g) < m \text{ and } fg \in Rf\}$.

If $f \in S[t; \sigma, \delta]$ is reducible then S_f contains zero divisors. If S is a division ring, then S_f has no zero divisors if and only if f is irreducible.

If Rf is a two-sided ideal in R (i.e. f is *(right)-invariant*) then S_f is the associative quotient algebra obtained by factoring out the ideal generated by a two-sided $f \in S[t; \sigma, \delta]$.

For all $g \in R$ of degree $s \geq m$, there also exist uniquely determined $r, q \in R$ with $\deg(r) < \deg(f)$, such that $g(t) = f(t)q(t) + r(t)$. Let $\text{mod}_l f$ denote the remainder of left division by f . Then the additive group $\{g \in R \mid \deg(g) < m\}$ together with the multiplication $g \diamond h = gh \text{ mod}_l f$ defined for all $g, h \in R$ of degree less than m , is also a unital nonassociative algebra ${}_f S$ over S_0 denoted by R/fR . Moreover, the canonical anti-automorphism

$$\psi : S[t; \sigma, \delta] \rightarrow S^{op}[t; \sigma^{-1}, -\delta \circ \sigma^{-1}], \quad \psi\left(\sum_{k=0}^n a_k t^k\right) = \sum_{k=0}^n \left(\sum_{i=0}^k \Delta_{n,i}(a_k)\right) t^k$$

induces an anti-automorphism between the algebras $S_f = S[t; \sigma, \delta]/S[t; \sigma, \delta]f$ and ${}_{\psi(f)} S = S^{op}[t; \sigma^{-1}, -\delta \circ \sigma^{-1}]/\psi(f)S^{op}[t; \sigma^{-1}, -\delta \circ \sigma^{-1}]$, so that $S_f^{op} = {}_{\psi(f)} S$. Here, $\Delta_{n,j}$ is defined recursively via $\Delta_{n,j} = \delta(\Delta_{n-1,j}) + \sigma(\Delta_{n-1,j-1})$, with $\Delta_{0,0} = id_S$, $\Delta_{1,0} = \delta$, $\Delta_{1,1} = \sigma$ and so $\Delta_{n,j}$ is the sum of all polynomials in σ and δ of degree j in σ and degree $n-j$ in δ [12, p. 2]. If $\delta = 0$, then $\Delta_{n,j} = \sigma^n$.

2. NONASSOCIATIVE (GENERALIZED) CYCLIC ALGEBRAS AND GENERALIZED MENICHETTI ALGEBRAS

2.1. Nonassociative cyclic algebras. The equivalence class of a homogeneous semiassociative algebra in $Br^{sa}(F)$ is represented by a unique central associative division algebra [4]. However, we will see now that the question whether an algebra is a division algebra or not is much less important in $Br^{sa}(F)$.

Nonassociative cyclic algebras of degree n are canonical generalizations of associative cyclic algebras of degree n and were first introduced over finite fields by Sandler [28]. Indeed, nonassociative quaternion algebras (where $n = 2$) were the first known example of a

nonassociative division algebra [9]. Over arbitrary fields they were investigated by Steele [30, 31] (Steele studied the opposite algebras of the nonassociative algebras we define here, but used our notation).

Definition 1. Let K be an étale algebra of dimension n over F and $\sigma \in \text{Aut}_F(K)$ of order n . Let $f(t) = t^n - d \in K[t; \sigma]$ with $d \in K^\times$. The F -central algebra $S_f = K[t; \sigma]/K[t; \sigma]f$ is called a *nonassociative cyclic algebra* over F and denoted by $(K/F, \sigma, d)$.

This definition generalizes the one used in most papers, where K is a cyclic Galois field extension of degree n (for an earlier generalization, cf. [22]).

If K/F is a cyclic Galois field extension of degree n with Galois group $\text{Gal}(K/F) = \langle \sigma \rangle$, then $(K/F, \sigma, d)$ is a classical associative cyclic algebra over F of degree n if $d \in F^\times$, and a nonassociative cyclic algebra as defined in [6, 28, 30], if $d \in K \setminus F$. We note that with our more general definition, we now have for instance that $(K/F, \sigma, d) \otimes_F K \cong (K \otimes_F K, \sigma, d)$ with σ denoting the canonical extension $\sigma \otimes \text{id}$ of σ to $K \otimes_F K$. For more details and proofs in the case that K is a division algebra, cf. [6, 30].

If K is a cyclic Galois field extension of F and $d \in K \setminus F$, then $\text{Nuc}((K/F, \sigma, d)) = K$. The proof that $K \subset \text{Nuc}_r((K/F, \sigma, d))$ when K is étale is analogous to the case where K is a field and implies that $K \subset \text{Nuc}((K/F, \sigma, d))$ holds in our general definition, too.

The easiest example of a nonassociative cyclic algebra is a nonassociative quaternion division algebra $(K/F, \sigma, d)$, where K/F is a quadratic field extension, and $d \in K \setminus F$. This is, up to isomorphism, also the only simple K -semiassociative division algebra of degree 2 that is not associative [32]. This algebra and the simple skew matrix algebra that represents it when it splits over the field extension K are presented in [32].

Theorem 1. Let K/F be a cyclic Galois field extension of degree n with Galois group $\text{Gal}(K/F) = \langle \sigma \rangle$ and $d \in K \setminus F$. Let $H = \{\tau \in G \mid \tau(d) = d\}$. Then $H = \langle \sigma^s \rangle$ for some integer s such that $n = sr$ and

$$\text{Nuc}_r((K/F, \sigma, d)) = (K/E, \sigma^s, d)$$

is a cyclic algebra of degree r over $E = \text{Fix}(\sigma^s)$, where $[E : F] = |H|$. In particular, if n is prime then $\text{Nuc}_r((K/F, \sigma, d)) = K$.

Proof. By [31, Proposition 3.2.3], $\text{Nuc}_r((K/F, \sigma, d)) = K \oplus Kt^s \oplus \dots \oplus Kt^{(r-1)s}$. By [30, Theorem 5.1], the linear subspace $K \oplus Kt^s \oplus \dots \oplus Kt^{(r-1)s}$ is the cyclic subalgebra $(K/E, \sigma^s, d)$ of degree r over $E = \text{Fix}(\sigma^s)$, where $[E : F] = |H|$ and $d \in E$. \square

Corollary 2. Let K/F be a cyclic Galois field extension of degree n with Galois group $\text{Gal}(K/F) = \langle \sigma \rangle$ and $d \in K \setminus F$. Let $H = \{\tau \in G \mid \tau(d) = d\} = \langle \sigma^s \rangle$ for some integer s with $1 < s < n$. Then $(K/F, \sigma, d)^{\text{op}}$ is not isomorphic to a nonassociative cyclic algebra.

Proof. If $1 < s < n$ then $\text{Nuc}_r((K/F, \sigma, d)) = (K/F, \sigma^s, d)$ and E is a proper intermediate field of K/F . Thus $\text{Nuc}_l((K/F, \sigma, d)^{\text{op}}) = (K/F, \sigma^s, d)$ is unequal to $K = \text{Nuc}_l((K/F, \sigma, d'))$ for any nonassociative cyclic algebra $(K/F, \sigma, d')$. (Note that the middle nucleus of a nonassociative cyclic algebra is K , so if A is a nonassociative cyclic algebra isomorphic to $(K/F, \sigma, d)^{\text{op}}$, A must involve the same field extension K/F , up to isomorphism.) \square

Nonassociative cyclic algebras (and analogously, their opposite algebras) are important examples of semiassociative (division) algebras that are not semicentral, thus in particular not homogeneous:

Proposition 3. (i) *Every nonassociative cyclic algebra $(K/F, \sigma, d)$ over F is K -semiassociative of degree n and thus semiassociative.*

(ii) *$(K/F, \sigma, d)$ is split if and only if $K = F^n$.*

(iii) *$(K/F, \sigma, d) \otimes_F K$ splits.*

(iv) [30] *Let K/F be a cyclic field extension of degree n . Then $(K/F, \sigma, d)$ is a division algebra for all $d \in K \setminus F$, such that $1, d, \dots, d^{n-1}$ are linearly independent over F . If K/F has prime degree then $(K/F, \sigma, d)$ is a division algebra for all $d \in K \setminus F$.*

(v) *Let K/F be a cyclic field extension, then for all $d \in K \setminus F$, $(K/F, \sigma, d)$ is not semicentral.*

The proof of (i), (ii), (iii) is straightforward employing results from [4] listed in Section 1.2. For $n = 2$, (i) was already pointed out in [4]. (v) is clear because $K = \text{Nuc}((K/F, \sigma, d))$.

Example 4. The split nonassociative quaternion algebra defined in [32] is simple, has nucleus $F \times F$, the basis e_1, e_2, e_3, e_4 , with $e_1 = (1, 0)$, $e_2 = (0, 1)$ and unit element $e_1 + e_2$, and its multiplication is given by $e_1 e_1 = e_1$, $e_1 e_2 = 0 = e_2 e_1$, $e_1 e_3 = e_3$, $e_1 e_4 = 0$, $e_2 e_2 = e_2$, $e_2 e_3 = 0$, $e_2 e_4 = e_4$, $e_3 e_1 = 0$, $e_3 e_2 = e_3$, $e_3 e_4 = e_1$, $e_4 e_1 = e_4$, $e_4 e_2 = 0$, $e_4 e_3 = \lambda e_2$, $e_4 e_4 = 0$, with $\lambda \neq 0, 1$. This is the skew matrix algebra $M_2(F; c)$ of degree 2 with the reduced tensor given by $c_{212} = \lambda$, $c_{121} = 1$, where $e_{11} = e_1, e_{22} = e_2, e_3 = e_{12}, e_4 = e_{21}$. As already noted in [32], $M_2(F; c) = M_2(F)$ when $\lambda = 1$, and for a nonassociative quaternion algebra $(K/F, \sigma, d)$ we have $(K/F, \sigma, d) \otimes_F K \cong M_2(F; c)$ with $\lambda = \sigma(d)/d$ in the reduced tensor c . This is a skew matrix algebra of the type mentioned in [4, Example 6.12 (3)].

Remark 5. Let K be an étale algebra of dimension n over F and $\sigma \in \text{Aut}_F(K)$ of order n . Let $f(t) = t^n \in K[t; \sigma]$. Then $S_f = K[t; \sigma]/K[t; \sigma]f$ is an associative algebra over F which is semiassociative but not simple; the semisimple quotient is K . Abusing notation we denote it by $(K/F, \sigma, 0)$ (cf. [4, Remark 3.8] for $n = 2$). It can be viewed as a generalization of the dual quaternion algebra $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{D}$ used in physics, where $\mathbb{D} = \mathbb{R}[t]/(t^2)$ are the dual numbers.

2.2. Nonassociative generalized cyclic algebras. (for details on the case that B is a division algebra, cf. [6])

Let B be a central simple algebra over C (i.e., C -central) of degree n , and $\sigma \in \text{Aut}(B)$ such that $\sigma|_C$ has finite order m and for $F = \text{Fix}(\sigma) \cap C$ assume that C/F is a cyclic Galois field extension of degree m with $\text{Gal}(C/F) = \langle \sigma|_C \rangle$. (This last assumption is automatically satisfied, if B is a division algebra.)

Definition 2. Let $f(t) = t^m - d \in B[t; \sigma]$, $d \in B^\times$. We call $(B, \sigma, d) = B[t; \sigma]/B[t; \sigma]f$ a *nonassociative generalized cyclic algebra* over F .

This definition generalizes the definition of both a nonassociative and an associative generalized cyclic algebra in [6] (see [12, p. 19] for the associative case), which also assumed that B is a division algebra.

The algebra (B, σ, d) has dimension $m^2 n^2$ over F and is F -central. If $d \in F^\times$ and B is a division algebra, then (B, σ, d) is a classical associative generalized cyclic algebra over F of degree mn . Indeed, (B, σ, d) is associative if and only if $d \in F$.

We have $B \subset \text{Nuc}_l((B, \sigma, d)) = \text{Nuc}_m((B, \sigma, d))$ with equality when B is a division algebra. Moreover, if $d \in C \setminus F$ then $B \subset \text{Nuc}_r((B, \sigma, d))$, i.e. $B \subset \text{Nuc}((B, \sigma, d))$ with equalities when B is a division algebra [7].

In particular, if $B = C$, C/F is a cyclic Galois extension of degree m with Galois group generated by σ and $f(t) = t^m - d \in C[t; \sigma]$, we obtain the nonassociative cyclic algebra $(C/F, \sigma, d)$.

Lemma 6. *Let K be a maximal étale subalgebra of B of dimension n .*

- (i) [4, Remark 3.3] *(B, σ, d) has the étale algebra K/F of dimension mn in its nucleus.*
- (ii) *For all $d \in C^\times$, (B, σ, d) is a K -semiassociative algebra over F of degree mn . (B, σ, d) is not semicentral.*

Proof. (ii) If $d \in F$ then (B, σ, d) is an associative central simple algebra over F and trivially semiassociative. For all $d \in C$, $d \notin F$ we have $B \subset \text{Nuc}((B, \sigma, d))$ and the étale algebra K/F of degree mn lies in $\text{Nuc}((B, \sigma, d))$. The rest is a straightforward calculation as well: (B, σ, d) is a faithful K^e -module, thus cyclic as a K^e -module. In particular, since $B \subset \text{Nuc}((B, \sigma, d))$ is an C -central simple algebra, (B, σ, d) is not semicentral. \square

Theorem 7. *Let $B = D$ be a division algebra and $d \in F \setminus F$. Then C/F is a cyclic Galois field extension of degree m with Galois group $G = \langle \sigma|_C \rangle$. Write σ for $\sigma|_C$ for ease of notation. Let $H = \{\tau \in G \mid \tau(d) = d\}$. Then $H = \langle \sigma^s \rangle$ for some integer s such that $m = sr$ and*

$$\text{Nuc}_r((D, \sigma, d)) = D[t; \sigma^s]/(t^m - d).$$

In particular, if m is prime then $\text{Nuc}_r((D, \sigma, d)) = D$.

Proof. By an analogous proof as the one for [31, Proposition 3.2.3], it is easy to see that $\text{Nuc}_r((D, \sigma, d)) = D \oplus Dt^s \oplus \dots \oplus Dt^{(r-1)s}$ as a left D -module. By an analogous proof as the one for [30, Theorem 5.1], the linear subspace $D \oplus Dt^s \oplus \dots \oplus Dt^{(r-1)s}$ together with the inherited algebra multiplication, is the associative subalgebra $D[t; \sigma^s]/(t^m - d)$ of dimension $n^2[C : F]r = n^2mr$. Here, $d \in E = \text{Fix}(\sigma^s)$, and $[E : F] = |H| = r$. \square

Corollary 8. *Let $B = D$ be a division algebra and suppose that $d \in C \setminus F$, $G = \langle \sigma|_C \rangle$, and $H = \{\tau \in G \mid \tau(d) = d\} = \langle \sigma^s \rangle$, so that s is an integer with $1 < s < n$. Then $(D, \sigma, d)^{op}$ is not isomorphic to a nonassociative generalized cyclic algebra.*

This is trivial now as the right nucleus is not D in this case. Note that $(D, \sigma, d)^{op}$ is semiassociative, however.

If D is a division algebra of degree n over C , then (D, σ, d) is a division algebra over F_0 if and only if $t^m - d \in D[t; \sigma]$ is irreducible [18, (7)]. We know that $t^2 - d \in D[t; \sigma]$ is irreducible if and only if $\sigma(z)z \neq d$ for all $z \in D$, $t^3 - d \in D[t; \sigma]$ is irreducible if and only if $d \neq \sigma^2(z)\sigma(z)z$ for all $z \in D$, and $t^4 - d \in D[t; \sigma]$ is irreducible if and only if $\sigma^2(y)\sigma(y)y + \sigma^2(x)y + \sigma^2(y)\sigma(x) \neq 0$ or $\sigma^2(x)x + \sigma^2(y)\sigma(y)x \neq d$ for all $x, y \in D$ (cf. [18, 24], and [5, Theorem 3.19], [7]). More generally, if F contains a primitive m th root of

unity and m is prime then $t^m - d \in D[t; \sigma]$ is irreducible if and only if $d \neq \sigma^{m-1}(z) \cdots \sigma(z)z$ for all $z \in D$ ([5, Theorem 3.11], see also [24, Theorem 6]), which generalizes the equivalent condition in the associative setup.

2.3. Menichetti algebras. [17, 27]

Let K/F be a Galois field extension of F of degree m with $\text{Gal}(K/F) = \{\tau_0, \dots, \tau_{m-1}\}$. Let $k_i \in K^\times$, $i \in 0, \dots, m-1$, and let

$$c_{i,j} = k_0^{-1} k_1^{-1} \cdots k_{j-1}^{-1} k_i k_{i+1} \cdots k_{i+j-1}$$

for all $i, j \in \mathbb{Z}_m$. Let z_0, \dots, z^{m-1} be an F -basis of K^m and define a multiplication on K^m via

$$(az_i) \cdot (bu_j) = \tau_j(a)b(u_i \cdot u_j), \quad z_i \cdot z_0 = z_0 \cdot z_i = z_i \text{ for all } i \in \mathbb{Z}_m,$$

$$z_i \cdot z_j = c_{ji} z_{i+j} \text{ for all } i \in \mathbb{Z}_m \setminus \{0\}$$

for all $a, b \in K$. Then K^m is a nonassociative unital algebra over F of dimension m^2 we denote $(K/F, k_0, \dots, k_{m-1})$, and call a *Menichetti algebra*, as the construction generalizes [17]. Define

$$M(x_0, \dots, x_{m-1}) = \begin{bmatrix} x_0 & c_{m-1,1}\tau_1(x_{m-1}) & \cdots & c_{1,m-1}\tau_{m-1}(x_1) \\ x_1 & \tau_1(x_0) & \cdots & c_{2,m-1}\tau_{m-1}(x_2) \\ x_2 & c_{1,1}\tau_1(x_1) & \tau_2(x_0) & c_{3,m-1}\tau_{m-1}(x_3) \\ \cdots & \cdots & \cdots & \cdots \\ x_{m-2} & c_{m-3,1}\tau_1(x_{m-3}) & \cdots & c_{m-1,m-1}\tau_{m-1}(x_{m-1}) \\ x_{m-1} & c_{m-2,1}\tau_1(x_{m-2}) & \cdots & \tau_{m-1}(x_0) \end{bmatrix}$$

and identify $x_0 z_0 + \cdots + x_{m-1} z_{m-1}$ with (x_0, \dots, x_{m-1}) , $x_i \in K$, then

$$(x_0, \dots, x_{m-1}) \cdot (y_0, \dots, y_{m-1}) = M(x_0, \dots, x_{m-1})(y_0, \dots, y_{m-1})^t.$$

It is easy to see that $K \subset \text{Nuc}((K/F, k_0, \dots, k_{m-1}))$ and that $(K/F, k_0, \dots, k_{m-1})$ is a semi-associative algebra over F . The algebras $(K/F, k_0, \dots, k_{m-1})^{op}$ generalize nonassociative cyclic algebras $(K/F, \sigma, d)$ [27].

2.4. Generalized Menichetti algebras. [27]

Let D be a central simple algebra over C of degree n . Let $\sigma \in \text{Aut}(D)$ such that $\sigma|_C$ has finite order m , and put $F = \text{Fix}(\sigma) \cap C$. Assume that C/F is a cyclic Galois field extension of degree m with $\text{Gal}(C/F) = \langle \sigma|_C \rangle$ (this is automatically satisfied, if D is a division algebra). Let $k_i \in C^\times$, $i \in 0, \dots, m-1$, and

$$c_{i,j} = k_0^{-1} k_1^{-1} \cdots k_{j-1}^{-1} k_i k_{i+1} \cdots k_{i+j-1}$$

for all $i, j \in \mathbb{Z}_m$. Let z_0, \dots, z^{m-1} be a C -basis of D^m and define a multiplication on D^m via

$$(az_i) \cdot (bz_j) = \sigma^j(a)b(z_i \cdot z_j),$$

$$z_i \cdot z_0 = z_0 \cdot z_i = z_i \text{ for all } i \in \mathbb{Z}_m, \quad z_i \cdot z_j = c_{ji} z_{i+j} \text{ for all } i \in \mathbb{Z}_m \setminus \{0\}$$

for all $a, b \in D$. This yields a nonassociative unital algebra over F of dimension $n^2 m^2$ that we denote by $(D, \sigma, k_0, \dots, k_{m-1})$ and call a *generalized Menichetti algebra of degree mn* . Its multiplication can be written as

$$(x_0, \dots, x_{m-1}) \cdot (y_0, \dots, y_{m-1}) = M(x_0, \dots, x_{m-1})(y_0, \dots, y_{m-1})^t$$

for all $x_i, y_i \in D$, with

$$M(x_0, \dots, x_{m-1}) = \begin{bmatrix} x_0 & c_{m-1,1}\sigma(x_{m-1}) & \dots & c_{1,m-1}\sigma^{m-1}(x_1) \\ x_1 & \sigma(x_0) & \dots & c_{2,m-1}\sigma^{m-1}(x_2) \\ x_2 & c_{1,1}\sigma(x_1) & \sigma^2(x_0) & c_{3,m-1}\sigma^{m-1}(x_3) \\ \dots & \dots & \dots & \dots \\ x_{m-2} & c_{m-3,1}\sigma(x_{m-3}) & \dots & c_{m-1,m-1}\sigma^{m-1}(x_{m-1}) \\ x_{m-1} & c_{m-2,1}\sigma(x_{m-2}) & \dots & \sigma^{m-1}(x_0) \end{bmatrix}.$$

Generalized Menichetti algebras $(D, \sigma, k_0, \dots, k_{m-1})^{op}$ can be seen as generalizations of the generalized cyclic algebras (D^{op}, σ, d) for $d \in C$.

Let E be a maximal étale subalgebra of D of dimension n . Then $(D, \sigma, k_0, \dots, k_{m-1})$ has the étale algebra E/F of dimension mn in its nucleus. $(D, \sigma, k_0, \dots, k_{m-1})$ is a faithful E^e -module, is cyclic as a E^e -module [4, Remark 3.3] and thus a semiassociative algebra [4].

Let now K/C be a cyclic field extension of degree m with $\text{Gal}(K/C) = \langle \sigma \rangle$. Let D_0 be a central simple algebra over C of degree n , and put $D = D_0 \otimes_C K$. Let $\tilde{\sigma}$ be the unique extension of σ to D such that $\tilde{\sigma}|_{D_0} = id_{D_0}$. Then it is straightforward to check that

$$A = D_0 \otimes_F (K/F, k_0, \dots, k_{m-1}) \cong (D_0 \otimes_C K, \tilde{\sigma}, k_0, \dots, k_{m-1})$$

is a nonassociative generalized Menichetti algebra over F of degree mn . Since $\text{Nuc}(A) = D_0 \otimes \text{Nuc}((K/F, k_0, \dots, k_{m-1}))$, we have $D \subset \text{Nuc}(A)$. A is a semiassociative algebra.

3. THE TENSOR PRODUCT OF AN ASSOCIATIVE ALGEBRA AND A NONASSOCIATIVE CYCLIC ALGEBRA

The tensor product of two semiassociative algebras is again a semiassociative algebra. The special case of tensoring an associative central division algebra and a nonassociative cyclic division algebra appeared already when constructing space time block codes [26, 19, 20, 16, 24]. The resulting algebra is a generalized nonassociative cyclic algebra (cf. [12, p. 36] for the associative setup, where D is assumed to be a division algebra, but the proof goes through verbatim if not):

Let E/F be a cyclic field extension of degree m with $\text{Gal}(E/F) = \langle \tau \rangle$. Let D_0 be a central simple algebra over F of degree n , and put $D = D_0 \otimes_F E$. Let $D_1 = (E/F, \tau, d)$ be a nonassociative cyclic algebra over F of degree m (i.e. $c \in F^\times$ and $d \in E^\times$), and let $\tilde{\tau}$ be the unique extension of τ to D such that $\tilde{\tau}|_{D_0} = id_{D_0}$. Then

$$A = D_0 \otimes_F (E/F, \tau, d) \cong (D_0 \otimes_F E)[t, \tilde{\tau}]/(D_0 \otimes_F E)[t, \tilde{\tau}](t^m - d) = (D, \tilde{\tau}, d)$$

[21] and is a nonassociative generalized cyclic algebra over F of degree mn . Now $(E/F, \tau, d)$ is associative if and only if $d \in F^\times$, so assume that $d \in E \setminus F$. Then $\text{Nuc}((D, \tilde{\tau}, d)) = D_0 \otimes_F E = D$ is a normal algebra over F and $D_0 \otimes_F (E/F, \tau, d)$ is a semiassociative algebra over F of degree mn that is not semicentral.

For any maximal étale subalgebra L in D_0 , $K = L \otimes_F E \subset \text{Nuc}(A)$ is a maximal étale subalgebra of A of degree mn over F .

Proposition 9. *Let $H = \{\gamma \in \text{Gal}(E/F) \mid \gamma(d) = d\} = \langle \tau^s \rangle$ for some integer s such that $m = sr$.*

(i) *Let $M = \text{Fix}(\tau^s)$. Then*

$$\text{Nuc}_r((D, \tilde{\tau}, d)) = D_0 \otimes_F (K/M, \sigma^s, d) \cong D \oplus Dt^s \oplus \cdots \oplus Dt^{s(r-1)}$$

with $(K/M, \sigma^s, d)$ a cyclic algebra of degree r over M , where $[M : F] = |H|$. If m is prime then $\text{Nuc}_r((D, \tilde{\tau}, d)) = D$.

(ii) *If $1 < s < m$, then $(D, \tilde{\tau}, d)^{op}$ is not a generalized nonassociative cyclic algebra.*

Proof. (i) We have $\text{Nuc}_r((D, \tilde{\tau}, d)) = D_0 \otimes_F \text{Nuc}_r((E/F, \tau, d))$, where $\text{Nuc}_r((E/F, \tau, d))$ is the cyclic subalgebra $(E/M, \tau^s, d)$ of degree r over $M = \text{Fix}(\tau^s)$. Here, $\text{Nuc}_r((K/F, \tau, d)) = (K/M, \sigma^s, d)$ is a cyclic algebra of degree r over $M = \text{Fix}(\tau^s)$, and $[M : F] = |H|$. In particular, if m is prime then $\text{Nuc}_r((K/F, \tau, d)) = D_0 \otimes_F E = D$.

(ii) The proof is straightforward, we just compare the left nuclei. \square

Remark 10. (i) Let K be an étale algebra of dimension n over F . Note that the associative algebra $D_0 \otimes_F (K/F, \sigma, 0)$ also is a semiassociative algebra over F_0 . It is not simple, the semisimple quotient is D . Abusing notation we denote it by $(D_0 \otimes_F K, \sigma, 0)$ (cf. [4, Remark 3.8] for $n = 2$).

(ii) The nucleus of $(D, \tilde{\tau}, d) \otimes_F (D, \tilde{\tau}, d)^{op}$ is $M_{n^2 m^2}(K)$, so this algebra is not split.

In the following, let L/F be a cyclic Galois field extension of degree n with $\text{Gal}(L/F) = \langle \sigma \rangle$. Let $K = L \otimes_F E$, then σ and τ canonically extend to K .

Let now $D_0 = (L/F, \sigma, c)$ be an associative cyclic algebra over F and $D_1 = (E/F, \tau, d)$ be a nonassociative cyclic algebra over F , i.e. $c \in F^\times$. Then $D = (L/F, \sigma, c) \otimes_F E$ is a central simple algebra over E of degree n and K/E is a maximal étale subalgebra of D of degree n (i.e., $D \cong (K/E, \sigma, c)$ using our new generalized definition of a cyclic algebra). Then

$$A = (L/F, \sigma, c) \otimes_F (E/F, \tau, d)$$

is a semiassociative algebra over F of degree mn , and if $d \in E \setminus F$ then $\text{Nuc}(A) = D_0 \otimes_F E = D$ and $K = L \otimes_F E \subset \text{Nuc}(A)$ is a maximal commutative subalgebra of A that is an étale algebra of degree mn over F (if $d \in F^\times$ then $(E/F, \tau, d)$ is associative).

In this case $\tilde{\tau}$ is the unique L -linear automorphism of D such that $\tilde{\tau}|_K = \tau$, i.e. $\tilde{\tau}(x) = \tau(x_0) + \tau(x_1)t + \tau(x_2)t^2 + \cdots + \tau(x_{n-1})t^{n-1}$ for $x = x_0 + x_1t + x_2t^2 + \cdots + x_{n-1}t^{n-1} \in D$ ($x_i \in K$, $1 \leq i \leq n$). So here $\tilde{\tau}|_E$ has order m and $\text{Fix}(\tilde{\tau}) = F$.

Corollary 11. *The algebra $(L/F, \sigma, c) \otimes_F (E/F, \tau, d) \cong (D, \tilde{\tau}, d)$ is a generalized nonassociative cyclic algebra over F if $d \in F^\times$. It is associative if $d \in F$, and has nucleus D if $d \in E \setminus F$. It is K -semiassociative of degree mn and, if it is not associative, it is not semicentral.*

Remark 12. The proof of this result is a straightforward generalization of the proof [24, Theorem 11] which assumed that D is a division algebra, and that L and E are linearly disjoint over F , so that K is a field: the proof that $(L/F, \sigma, c) \otimes_F (E/F, \tau, d) \cong$

$(D, \tilde{\tau}, d)D[t; \tilde{\tau}]/D[t; \tilde{\tau}](t^m - d)$ goes through verbatim in our more general setting, as the whole theory does not depend on these two assumptions (it was originally developed for space-time block codes which are built from division algebras). Since we look at the opposite cyclic algebras than the one employed throughout [24, Theorem 11], $\tilde{\tau}^{-1}$ in [24, Theorem 11] in our setup becomes $\tilde{\tau}$. If $d \in E \setminus F$ the algebra has as the nucleus the central simple algebra $D = (L/F, \sigma, c) \otimes_F E$, and K/F is a maximal étale subalgebra of the nucleus of degree mn .

Let $(E/F, \tau, d)$ be a cyclic associative division algebra of prime degree m . Suppose that B_0 is a central associative algebra over F such that $B = B_0 \otimes_F E$ is a division algebra. By a classical result by Jacobson, the tensor product $B_0 \otimes_F (E/F, \tau, d)$ is a division algebra if and only if $d \neq \tilde{\tau}^{m-1}(z) \cdots \tilde{\tau}(z)z$ for all $z \in B$ ([12, Theorem 1.9.8], see also [1, Theorem 12, Ch. XI]).

This result can be generalized to the tensor product of a cyclic and a nonassociative cyclic algebra, if the base field contains a suitable root of unity [24]. We now put the main results from [24] into the context of semiassociative algebras, adjusting them where needed (some of the algebras studied in [24, Section 3] are the opposite algebras of ours).

The generalization of Jacobson's condition is a necessary condition for $d \in E^\times$ in our general nonassociative case as well:

Proposition 13. [24, Proposition 20] *Let $D_0 = (L/F, \sigma, c)$ be an associative cyclic algebra of degree n over F , such that $D = D_0 \otimes_F E$ is a division algebra. If $D_0 \otimes_F (E/F, \tau, d)$ is a division algebra then $d \neq z\tilde{\tau}(z) \cdots \tilde{\tau}^{m-1}(z)$ for all $z \in D$.*

From now on, let L and E be linearly disjoint over F , then $K = L \otimes_F E = L \cdot E$ is the composite of L and E over F with Galois group $\text{Gal}(K/F) = \langle \sigma \rangle \times \langle \tau \rangle$, and K/E is a maximal separable subfield of $D = (L/F, \sigma, c) \otimes_F E$ of degree n .

Theorem 14. ([24, Theorems 14, 15, 16]) *Suppose that $D = (L/F, \sigma, c) \otimes_F E$ is a division algebra, m is prime and in case $m \neq 2, 3$, additionally that F contains a primitive m th root of unity.*

(i) *$(L/F, \sigma, c) \otimes_F (E/F, \tau, d)$ is a semiassociative division algebra if and only if $d \neq z\tilde{\tau}(z) \cdots \tilde{\tau}^{m-1}(z)$ for all $z \in D$, if and only if $t^m - d \in D[t; \tilde{\tau}]$ is irreducible.*

(ii) *If $\tau(d^n) \neq d^n$ then $(L/F, \sigma, c) \otimes_F (E/F, \tau, d)$ is a K -semiassociative division algebra of degree mn with nucleus D .*

(iii) *If $d \in E$ such that $d^n \notin N_{D/F}(D^\times)$, then $(L/F, \sigma, c) \otimes_F (E/F, \tau, d)$ is a K -semiassociative division algebra of degree mn with nucleus D . In particular, for all $d \in E \setminus F$ with $d^n \notin F$, $(L/F, \sigma, c) \otimes_F (E/F, \tau, d)$ is a K -semiassociative division algebra of degree mn .*

Theorem 15. [24, Theorem 17] *Let F be of characteristic not 2. Let $(a, c)_F$ be a quaternion algebra over F which is a division algebra over $E = F(\sqrt{b})$, and $(F(\sqrt{b})/F, \tau, d)$ a nonassociative quaternion algebra over F . Then $(a, c)_F \otimes_F (F(\sqrt{b})/F, \tau, d)$ is a semiassociative division algebra over F of degree 4 with nucleus $(a, c)_{F(\sqrt{b})}$.*

More generally, let B be a central simple algebra over C of degree n , and $\sigma \in \text{Aut}(B)$ such that $\sigma|_C$ has finite order m , $F = \text{Fix}(\sigma) \cap C$ and C/F is a cyclic Galois field extension

of degree m with $\text{Gal}(C/F) = \langle \sigma|_C \rangle$ and $d \in C^\times$. Let D_0 be a central simple algebra over F of degree s , and let $\tilde{\sigma}$ be the unique extension of σ to $D_0 \otimes_{F_0} B$ such that $\tilde{\sigma}|_{D_0} = \text{id}_{D_0}$. Then $\tilde{\sigma}$ has order m over F_0 , and

$$D_0 \otimes_F (B, \sigma, d) \cong (D_0 \otimes B, \tilde{\sigma}, d)$$

with $(D_0 \otimes_F B, \tilde{\sigma}, d) = (D_0 \otimes_F B)[t; \tilde{\sigma}] / (D_0 \otimes_F B)[t; \tilde{\sigma}](t^m - d)$. We get a generalized nonassociative cyclic algebra of degree mns with $D_0 \otimes_F B$ contained in its nucleus.

4. THE SEMIASSOCIATIVE BRAUER MONOID

4.1. The classes in $Br^{sa}(F)$ that contain the homogeneous semiassociative algebras are determined by the Brauer group and are of the kind $[B]^{sa}$ with B an associative central simple algebra over F . In particular, if D is an associative F -central division algebra, then $[D]^{sa}$ is the unique element of minimal degree in the class $[D]^{sa} \in Br^{sa}(F)$ which contains the homogeneous semiassociative algebras of the kind $D \otimes_F M$, where M is a skew matrix algebra [4, Example 14.5, Corollary 18.3]. Moreover, if F is a field with nontrivial Brauer group, then $Br^{sa}(F)$ has elements $[A]^{sa}$ of infinite order [4, Corollary 20.4]. From the proof of [4, Corollary 20.4], it is clear that these elements are constructed by finding semiassociative algebras A , such that $\sigma(A) = F \oplus B$, where B is a central division algebra over F of index p , so the similarity class $[A]^{sa}$ contains elements that are all semicentral (although the algebras are not explicitly constructed there).

We now collect some observations on elements in the semiassociative Brauer monoid $Br^{sa}(F)$. In particular, $Br^{sa}(F)$ can be nontrivial even if the classical Brauer group is trivial, as we can easily conclude from our previous results:

Proposition 16. (i) *Let F be a field that has a cyclic Galois field extension K/F of degree n , $\text{Gal}(K/F) = \langle \sigma \rangle$. Then $[F]^{sa} \neq [(K/F, \sigma, d)]^{sa}$ for all $d \in K \setminus F$ and so $Br^{sa}(F)$ is nontrivial. Moreover, $[(K/F, \sigma, d)]^{sa}$ has infinite order in $Br^{sa}(F)$, i.e. the powers of these element are distinct.*

(ii) *Let F be a field that has a Galois field extension K/F of degree n , and $(K/F, k_0, \dots, k_{m-1})$ be any Menichetti algebra that is not associative. Then $[F]^{sa} \neq [(K/F, k_0, \dots, k_{m-1})]^{sa}$ and so $Br^{sa}(F)$ is nontrivial, and $[(K/F, k_0, \dots, k_{m-1})]^{sa}$ has infinite order in $Br^{sa}(F)$.*

Proof. A semiassociative algebra over F of degree kn is split if and only if F^{kn} is contained in its nucleus as a unital subalgebra.

(i) Now $A = (K/F, \sigma, d) \otimes_F \cdots \otimes_F (K/F, \sigma, d)$ (k -times) has degree kn and nucleus $K \otimes_F K \otimes_F \cdots \otimes_F K$ (k -times). If K/F is a cyclic field extension of degree n with Galois group G then $K \otimes_F K \otimes_F \cdots \otimes_F K \cong \prod_{G^{k-1}} K$, where the index set G^{k-1} is the $(k-1)$ -fold product of G . So clearly the étale algebra F^{nk-n} is a unital subalgebra of the nucleus of A , but F^{kn} is not.

(ii) $(K/F, k_0, \dots, k_{m-1}) \otimes_F \cdots \otimes_F (K/F, k_0, \dots, k_{m-1})$ has degree kn and nucleus $K \otimes_F K \otimes_F \cdots \otimes_F K$ (k -times), so the assertion follows as in (i). \square

Since $(K/F, \sigma, d)$ is not semicentral for all $d \in K \setminus F$, it does not lie in the similarity class of any F -central simple algebra B in $Br^{sa}(F)$, and if n is prime (or if $1, d, \dots, d^{n-1}$ are

linearly independent over F), then $(K/F, \sigma, d)$ is always a division algebra, thus is a division algebra of smallest degree in $[(K/F, \sigma, d)]^{sa}$.

Lemma 17. *Let K/F be a field extension of degree m and D be a central simple algebra over F of degree n . Then*

$$[D]^{sa}[(K/F, \sigma, d)]^{sa} = [(D \otimes_F K, \tilde{\sigma}, d)]^{sa}$$

for all $d \in K$. In particular, for all $d \in K \setminus F$ we have

$$[(K/F, \sigma, d)]^{sa} = [(M_n(K), \tilde{\sigma}, d)]^{sa} \neq [F]^{sa}$$

and

$$[D]^{sa}[(K/F, \sigma, d)]^{sa} = [(M_n(K), \tilde{\sigma}, d)]^{sa} = [(K/F, \sigma, d)]^{sa}$$

if K is a splitting field of D .

Moreover, for a generalized nonassociative cyclic algebra (B, σ, d) over F , we have

$$[D]^{sa}[(B, \sigma, d)]^{sa} = [(D \otimes_F B, \tilde{\sigma}, d)]^{sa},$$

where $\tilde{\sigma}$ is the unique extension of σ to $D \otimes_F B$ such that $\tilde{\sigma}|_D = id_D$.

For a Menichetti algebra $(K/F, k_0, \dots, k_{m-1})$ over F , we have analogously

$$[D]^{sa}[(K/F, k_0, \dots, k_{m-1})]^{sa} = [(D \otimes_F K, \tilde{\sigma}, k_0, \dots, k_{m-1})]^{sa}$$

and for a generalized Menichetti algebra $(B, \sigma, k_0, \dots, k_{m-1})$ over F , we have

$$[D]^{sa}[(B, \sigma, k_0, \dots, k_{m-1})]^{sa} = [(D \otimes_F B, \tilde{\sigma}, k_0, \dots, k_{m-1})]^{sa}$$

for all $k_i \in F$, where $\tilde{\sigma}$ is the unique extension of σ to $D \otimes_F B$ such that $\tilde{\sigma}|_D = id_D$.

Proof. Since $D \otimes_F (K/F, \sigma, d) \cong (D \otimes_F K, \tilde{\sigma}, d)$ is a nonassociative generalized cyclic algebra over F of degree nm , we obtain

$$[D]^{sa}[(K/F, \sigma, d)]^{sa} = [(D \otimes_F K, \tilde{\sigma}, d)]^{sa} = [M_2(F)]^{sa}[(K/F, \sigma, d)]^{sa} = [(K/F, \sigma, d)]^{sa}$$

for all $d \in K \setminus F$. In particular, $M_n(F) \otimes_F (K/F, \sigma, d) \cong (M_n(K), \tilde{\sigma}, d)$ has nucleus $M_n(K)$. The maximal étale F -algebra in its nucleus is K^n . This yields the assertion that $[F]^{sa} \neq [(M_n(K), \tilde{\sigma}, d)]^{sa} = [M_n(F)]^{sa}[(K/F, \sigma, d)]^{sa} = [(K/F, \sigma, d)]^{sa}$. The rest is clear. \square

Theorem 18. *Let A and A' be two semiassociative algebras over F .*

(i) *Let $\text{Nuc}(A) = K$ and $\text{Nuc}(A') = L$ be two field extensions of F . If $[A]^{sa} = [A']^{sa} \in Br^{sa}(F)$ then $K \cong L$.*

(ii) *Let A and A' have a simple nucleus N , respectively N' , where N is an E -central simple algebra and N' is an E' -central simple algebra, with E and E' some separable field extensions of F . If $[A]^{sa} = [A']^{sa} \in Br^{sa}(F)$ then $E \cong E'$ and $[N] = [N'] \in Br(E)$.*

Proof. Since $A \sim A'$ we have $A \otimes_F M_n(F; c) \cong A' \otimes_F M_s(F; c')$ for suitable skewed matrix algebras $M_n(F; c)$, $M_s(F; c')$. From $\sigma(A \otimes_F M_n(F; c)) \cong \sigma(A' \otimes_F M_s(F; c'))$ it follows that $\sigma(A) \otimes_F \sigma(M_n(F; c)) \cong \sigma(A') \otimes_F \sigma(M_s(F; c'))$ by [4, Proposition 13.5]. Now $\sigma(M_n(F; c))$ and $\sigma(M_s(F; c'))$ are sums of matrix algebras over F whose degrees sum up to n , respectively to s : $\sigma(M_n(F; c)) \cong M_{n_1}(F) \oplus \dots \oplus M_{n_r}(F)$, respectively $\sigma(M_s(F; c')) \cong M_{s_1}(F) \oplus \dots \oplus M_{s_j}(F)$.

(i) Since K and L are fields we have $\text{Nuc}(A) = K = \sigma(K)$ and $\text{Nuc}(A') = L = \sigma(L)$. We

obtain $M_{n_1}(K) \oplus \cdots \oplus M_{n_r}(K) \cong M_{s_1}(L) \oplus \cdots \oplus M_{s_j}(L)$. These decompositions are unique up to permutations of summands, so $r = j$ and $K \cong L$.

(ii) Here, $J(\text{Nuc}(A)) = J(\text{Nuc}(A')) = 0$ and so $N = \text{Nuc}(A) = \sigma(A)$ and $N' = \text{Nuc}(A') = \sigma(A')$ and the above argument yields $M_{n_1}(N) \oplus \cdots \oplus M_{n_r}(N) \cong M_{s_1}(N') \oplus \cdots \oplus M_{s_j}(N')$. These decompositions are unique up to permutations of summands, so $r = j$ and $M_{n_1}(N) \cong M_{n_t}(N')$ for some t , where N is an E -central simple algebra and N' is an E' -central simple algebra, with E and E' some separable field extensions of F . This implies that $E \cong E'$ as both algebras must have the same center. Moreover, then $[N] = [N'] \in \text{Br}(E)$. \square

In particular, if K/F and L/F are two cyclic field extensions and $[(K/F, \sigma, d)]^{sa} = [(L/F, \tau, d')]^{sa}$ then $K = L$. It is an open and seemingly non-trivial problem, if two non-isomorphic cyclic algebras $(K/F, \sigma, d)$ and $(K/F, \sigma, d')$ which are both not associative, can lie in the same similarity class in $\text{Br}^{sa}(F)$.

Corollary 19. (i) Let K/F and L/F be two cyclic field extensions and $(K/F, \sigma, d)$, $(L/F, \sigma', d')$ be two nonassociative cyclic algebras. If K and L are not isomorphic then $[(K/F, \sigma, d)]^{sa} \neq [(L/F, \sigma', d')]^{sa}$ in $\text{Br}^{sa}(F)$.

(ii) Let $\text{Nuc}(A) = K$ be a field extension of degree n and $\text{Nuc}(A') = D$ an F -central algebra of degree $m \geq 2$. Then $[A]^{sa} \neq [A']^{sa}$ in $\text{Br}^{sa}(F)$.

(iii) Let D_0, D'_0 be two central simple algebras, and $A = D_0 \otimes_F (E/F, \tau, d) \cong (D, \tilde{\tau}, d)$ and $B = D'_0 \otimes_F (E'/F, \tau', d') \cong (D', \tilde{\tau}', d')$ with $E/F, E'/F$ two separable field extensions and $d \in E \setminus F, d' \in E' \setminus F$. If $[A]^{sa} = [A']^{sa} \in \text{Br}^{sa}(F)$ then $E \cong E'$ and $[D_0 \otimes_F E] = [D'_0 \otimes_F E'] \in \text{Br}(E)$.

Proof. (i) is clear.

(ii) The analogous argument as in Theorem 18 (i) and (ii) shows that if $[A]^{sa} = [A']^{sa} \in \text{Br}^{sa}$, then $M_{n_1}(K) \oplus \cdots \oplus M_{n_r}(K) \cong M_{s_1}(D) \oplus \cdots \oplus M_{s_j}(D)$ implies $M_{n_1}(K) \cong M_{n_t b}(D_0)$ for some b, t and some F -central division algebra D_0 , a contradiction.

(iii) Since $d \in E \setminus F$ and $d' \in E' \setminus F$, we have $\text{Nuc}((D, \tilde{\tau}, d)) = D_0 \otimes_F E = D$, and $\text{Nuc}((D', \tilde{\tau}', d')) = D'_0 \otimes_F E' = D'$, therefore $[(D, \tilde{\tau}, d)]^{sa} = [(D', \tilde{\tau}', d')]^{sa}$ implies $[D_0 \otimes_F E] = [D'_0 \otimes_F E'] \in \text{Br}(E)$ by Theorem 18 (iii). \square

Proposition 20. Let $(K_i/F, \sigma_i, d_i)$ be nonassociative cyclic algebras of degree n_i which are all not associative (i.e., $d_i \in K \setminus F$), $i = 1, \dots, r$, and let

$$A = (K_1/F, \sigma_1, d_1) \otimes_F \cdots \otimes_F (K_r/F, \sigma_r, d_r)$$

be their tensor product (which is a semiassociative algebra of degree $n_1 \cdots n_r$).

(i) The nucleus of A is the étale algebra $E = K_1 \otimes_F \cdots \otimes_F K_r$.

(ii) A is split if and only if E is a split étale algebra.

(iii) If K_1, \dots, K_r are linearly disjoint field extensions over F (e.g. all of different prime degrees) then A is a semiassociative algebra of degree $n_1 \cdots n_r$ with nucleus the field extension E/F of degree $n_1 \cdots n_r$. In particular, A is not semicentral.

(iv) If A is a division algebra then K_1, \dots, K_r are linearly disjoint field extensions over F and E/F is a field extension of degree $n_1 \cdots n_r$. In particular, A is not semicentral.

The proof is trivial, employing previously mentioned results from [4].

Mirroring the classical setup, the semiassociative Brauer monoid of an algebraically closed field is trivial, e.g. $Br_{sa}(\mathbb{C}) = 1$ [4, Example 14.5], so any semiassociative algebra over \mathbb{C} splits.

4.2. $Br^{sa}(\mathbb{R})$. We know that $Br(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\}$ is a cyclic group of order 2; and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$. Therefore the two classes in $Br^{sa}(\mathbb{R})$ that contain the homogeneous semiassociative algebras are $[\mathbb{R}]^{sa}$ and $[\mathbb{H}]^{sa}$.

Up to isomorphism, every nonassociative simple algebra of dimension four with \mathbb{C} as its nucleus is a nonassociative quaternion algebra [32] (note that $(\mathbb{C}/\mathbb{R}, -, 0)$ is semiassociative, even associative, but not simple). For every $a \in \mathbb{C} \setminus \mathbb{R}$, the nonassociative quaternion algebra $(\mathbb{C}/\mathbb{R}, -, a)$ is a semiassociative division algebra over \mathbb{R} of degree two that is not semicentral, and $[(\mathbb{C}/\mathbb{R}, -, a)]^{sa}$ has infinite order in $Br^{sa}(\mathbb{R})$. The class $[(\mathbb{C}/\mathbb{R}, -, a)]^{sa}$ thus contains algebras that are not semicentral and $(\mathbb{C}/\mathbb{R}, -, a)$ is a division algebra of smallest degree in $[(\mathbb{C}/\mathbb{R}, -, a)]^{sa}$. For $a, b \in \mathbb{C} \setminus \mathbb{R}$, we have $(\mathbb{C}/\mathbb{R}, -, a) \cong (\mathbb{C}/\mathbb{R}, -, b)$ if and only if there is $x \in \mathbb{R}$ such that either $a = x^2b$ or $\bar{a} = x^2b$ [32]. It is not clear, however, if two nonisomorphic quaternion division algebras can lie in the same similarity class in $Br^{sa}(\mathbb{R})$.

Furthermore, for all $d \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\begin{aligned} [\mathbb{H}]^{sa}[(\mathbb{C}/\mathbb{R}, -, d)]^{sa} &= [(M_2(\mathbb{C}), \tilde{-}, d)]^{sa} \neq [\mathbb{R}]^{sa}, \\ [M_n(\mathbb{R})]^{sa}[(\mathbb{C}/\mathbb{R}, -, d)]^{sa} &= [M_n(\mathbb{C}), \tilde{-}, d]^{sa}, \end{aligned}$$

so that

$$[\mathbb{H}]^{sa}[(\mathbb{C}/\mathbb{R}, -, d)]^{sa} = [(M_2(\mathbb{C}), \tilde{-}, d)]^{sa} = [M_2(\mathbb{R})]^{sa}[(\mathbb{C}/\mathbb{R}, -, d)]^{sa} = [(\mathbb{C}/\mathbb{R}, -, d)]^{sa}.$$

4.3. $Br^{sa}(\mathbb{F}_q)$. The Brauer group $Br(\mathbb{F}_q)$ is trivial. Therefore the only class in $Br^{sa}(\mathbb{F}_q)$ that contains homogeneous semiassociative algebras is the trivial class $[\mathbb{F}_q]^{sa} = [M_n(\mathbb{F}_q; c)]^{sa}$ with $M_n(\mathbb{F}_q; c)$ a skew matrix algebra over F . The semiassociative Brauer monoid $Br^{sa}(\mathbb{F}_q)$ is not trivial: For each finite field extension K/\mathbb{F}_q of degree n , there exist simple nonassociative cyclic algebras $(K/\mathbb{F}_q, \sigma, a)$ of degree n , with nucleus K if $a \in K \setminus \mathbb{F}_q$. Two such algebras $(K/\mathbb{F}_q, \sigma, a)$ and $(K'/\mathbb{F}_q, \sigma', a')$ will automatically be nonisomorphic for two nonisomorphic field extensions K and K' , and are not semisimple.

Again, $[M_n(\mathbb{F}_q)]^{sa}[(K/\mathbb{F}_q, \sigma, d)]^{sa} = [(M_n(K), \tilde{\sigma}, d)]^{sa}$, i.e. $[(K/\mathbb{F}_q, \sigma, d)]^{sa} = [(M_n(K), \tilde{\sigma}, d)]^{sa}$.

There exist other large classes of semifields, e.g. Menichetti algebras, to name just one, that are all semiassociative.

5. ALGEBRAS THAT ARE NOT SEMIASSOCIATIVE

Semiassociative algebras over F can be defined in terms of simple subalgebras of the nucleus whose center is separable over F [4, Section 5]. This avoids problems when tensoring these algebras, as the tensor product of two field extensions that are not both separable may be a very “unpleasant” algebra, but excludes algebras with nuclei that are simple subalgebras, and whose centers are (purely) inseparable over F , so creates restrictions when $\text{char}(F) = p$ is prime. An example of such algebras that are excluded in the current set-up are nonassociative algebras of square dimension that are a canonical generalization of cyclic

p -algebras, and of Amitsur's differential algebras ([2, 3, 11], [12, Sections 1.5, 1.8, 1.9]). Their nucleus is a purely inseparable field extension of F :

5.1. Nonassociative differential extensions of a field. Let K be a field of characteristic p together with an algebraic derivation $\delta : K \rightarrow K$ of K of degree p with minimum polynomial $g(t) = t^{p^e} + a_1 t^{p^{e-1}} + \cdots + a_e t \in F[t]$, where $F = \text{Const}(\delta) = \{a \in K \mid \delta(a) = 0\}$. Put $R = K[t; \delta]$. Then K/F is a purely inseparable extension of exponent one, and $[K : F] = p$. Let $f(t) = g(t) - d \in K[t; \delta]$, then the nonassociative F -central algebra

$$(K, \delta, d) = K[t; \delta]/K[t; \delta]f$$

has dimension p^{2e} and is called a (*nonassociative*) *differential extension* of K , or a (*nonassociative*) *differential algebra*. (K, δ, d) is a division algebra if and only if $f \in K[t; \delta]$ is irreducible. (K, δ, d) is associative if and only if $d \in F$.

If $f(t) \in F[t]$ then (K, δ, d) is an associative central simple algebra over F , and K is a maximal subfield of (K, δ, d) of dimension p [12, p. 23]. If $f(t) \in F[t]$ is irreducible and $g(t) = t^p - t$, then (K, δ, d) contains the cyclic separable field extension $F[t]/(t^p - t - d)$ of degree p , and is a cyclic p -algebra.

For $d \in K \setminus F$, (K, δ, d) is not semiassociative as its nucleus is the purely inseparable field extension K/F of degree p [23]. The K -algebra $(K, \delta, d) \otimes_F K$ contains the simple truncated polynomial algebra $K \otimes_F K$ in its nucleus, and is called a *split differential extension*.

5.2. Nonassociative differential extensions of a division algebra. There are classes of algebras over F that have a central simple algebra D over a field C as their nucleus, but the field extension C/F is purely inseparable of degree p , so the center C of D is purely inseparable over F [23]:

Let C be a field of characteristic p and D be a central simple algebra over C of degree n ($D = C$ is allowed and brings us back to the setup of the previous subsection). Let δ be a derivation of D , such that $\delta|_C$ is algebraic with minimum polynomial $g(t) = t^{p^e} + a_1 t^{p^{e-1}} + \cdots + a_e t \in F[t]$ and let $F = \text{Const}(\delta)$.

Assume that $g(\delta) = id_{d_0}$ is an inner derivation of D and that there exists $d_0 \in F$ so that $\delta(d_0) = 0$ (this is always possible if D is a division algebra [12, Lemma 1.5.3]). The center of $R = D[t; \delta]$ is $F[z]$ with $z = g(t) - d_0$.

For all $f(t) = g(t) - d \in D[t; \delta]$, the nonassociative unital F -algebra

$$(D, \delta, d) = S_f = D[t; \delta]/D[t; \delta]f(t)$$

has dimension $p^{2e}n^2$ over F and is called a *nonassociative generalized differential algebra/extension*. (Note that Amitsur's associative differential extensions of division rings D were generalized to simple rings D already in [14].) (D, δ, d) is an associative algebra if and only if $d \in F$ [23, Theorem 20]. For $d \in F$, (D, δ, d) is a central simple associative algebra over F (cf. [12, p. 23] if D is a division algebra.) If $f(t) = g(t) - d \in F[t]$ is irreducible, then (D, δ, d) contains the field extension $F[t]/(g(t) - d)$ as a subfield.

If $d \in C \setminus F$, then the differential algebra $(C, \delta|_C, d)$ is a subalgebra of (D, δ, d) of dimension p^{2e} .

We now correct part of [23, Lemma 19]:

Lemma 21. *For $d \in C \setminus F$, $D \subset \text{Nuc}_r(D, \delta, d)$, thus $D \subset \text{Nuc}((D, \delta, d))$. If D is a division ring, the inclusions become equalities.*

Proof. Since g is semi-invariant and monic of minimal degree, we have $g(t)a = ag(t)$ for all $a \in D$ ([15, (2.1), p. 3], this also holds when D is only simple and not a division algebra, since the polynomial is monic). Since $d \in C \setminus F$, we get $f(t)a = ag(t) - ad = af(t)$ for all $a \in D$ and so in turn f is semi-invariant as well. Hence the right nucleus of (K, δ, d) contains D [23, Proposition 3]. \square

Thus every maximal étale subalgebra N of D also lies in the nucleus and has dimension $p^e n$ as algebra over F .

As an algebra over F , N is the product of finite dimensional field extensions that are each of the type N_i/F , where we have a tower of field extensions $F \subset C \subset N_i$, such that N_i/C is separable of degree n and C/F purely inseparable of exponent one. This means we can write every N_i as a tensor product $N_i = S_i \otimes_F C$, where S_i is the maximal separable subfield of N_i/F [12, p. 32], and obtain that

$$N = N_1 \times \cdots \times N_r = (S_1 \otimes_F C) \times \cdots \times (S_r \otimes_F C) = (S_1 \times \cdots \times S_r) \otimes_F C$$

is an étale algebra $S_1 \times \cdots \times S_r$ over F tensored over F with the purely inseparable field extension C/F of exponent one.

Remark 22. It is well-known that $V : C \rightarrow F$, $V(a) = V_p(a) - a$ is a homomorphism between the additive groups C and F [13]. When D is a division algebra then (D, δ, d) is a division algebra if and only if f is irreducible, if and only if $d \neq V_p(z) - z$ for all $z \in D$, if and only if $d \neq (t - z)^p - t^p - z$ for all $z \in D$.

Let F have characteristic 3, and δ have minimum polynomial $g(t) = t^3 - ct \in F[t]$. Then for $f(t) = t^3 - ct - d \in C[t; \delta]$, (D, δ, d) is a unital algebra over F of dimension 9, and a division algebra if and only if $V_3(z) - cz \neq d$ and $V_3(z) - zc - d + \delta(c) \neq 0$ for all $z \in D$ [23, Theorem 23].

Let F be a field of characteristic p and D be a central simple algebra over F of degree n . Let K/F be a purely inseparable extension of exponent one such that $[K : F] = p$. Let δ be a derivation on K with $F = \text{Const}(\delta)$, such that δ is an algebraic derivation of degree p with minimum polynomial $g(t) = t^p - t \in F[t]$ of degree p . Let δ be the extension of δ to D_K such that $\delta|_D = 0$. Then $(K, \delta, d) \otimes_F D \cong (D_K, \delta, d)$ is an algebra of dimension $n^2 p^2$ over F .

Moreover, if $D_K = D \otimes_F K$ is a division algebra and (K, δ, d) is a division algebra over F , then $(K, \delta, d) \otimes_F D \cong (D_K, \delta, d)$ is a division algebra if and only if $f(t) = g(t) - d$ is irreducible in $D_K[t; \delta]$. In particular for $g(t) = t^p - t$ this is the case, if and only if $d \neq V_p(z) - z$ for all $z \in D_K$, if and only if $d \neq (t - z)^p - t^p - z$ for all $z \in D_K$ [23].

6. OUTLOOK

While there are excellent reasons to use the existing definition of $Br^{sa}(F)$ (it is the broadest possible one if we want to use Brauer factor sets), we believe it makes sense to discuss

(i) a possible refinement of the semiassociative Brauer monoid to include only simple semiassociative algebras, and (ii) a possible generalization of $Br^{sa}(F)$ that allows us to include nonassociative differential algebras, if the base field F is not perfect and has characteristic p :

(i) The simple semiassociative algebras form a submonoid of $Br^{sa}(F)$ that still contains $Br(F)$ as unique maximal subgroup. If we only consider the simple semiassociative algebras we exclude pathological cases like the associative algebras $(K/F, \sigma, 0)$ or $(D, \sigma, 0)$.

(ii) Suppose that we want to include generalized differential extensions in the definition of the Brauer monoid. Here is one possible way to proceed: Let A be an F -central nonassociative algebra over F of dimension l^2 , $\text{char}(F) = p$. We call A a *generalized semisassociative algebra*, if its nucleus contains a tensor product $N = N_1 \otimes_F \cdots \otimes_F N_s$ of finite field extensions N_i/F such that $\dim_F N = l$, with N_i either separable or purely inseparable of exponent one and N_i/F primitive of the kind $N_i = F[x]$ for $x_i^p = a \in F$. The root l of the dimension of A is called the *degree* of A .

If all N_i/F are separable then N is an étale algebra over F , so let us additionally require that A is cyclic and faithful as $N \otimes N$ -module in this case, then A is semiassociative.

Two generalized semiassociative algebras A and B over F are called *Brauer equivalent*, if there exist skew matrix algebras $M_n(F; c)$ and $M_m(F; c')$ such that $A \otimes_F M_n(F; c) \cong B \otimes_F M_m(F; c')$. This is an equivalence relation, as [4, Remark 6.9] still holds. We denote the equivalence class of a generalized semiassociative algebra A by $[A]^{gsa}$ and the monoid of equivalence classes by $Br^{gsa}(F)$.

Note that every finite purely inseparable field extension of exponent one is a tensor product of primitive extensions $F[x_1] \otimes \cdots \otimes F[x_r]$, where $x_i^p - a_i = 0$. Also note that if N is purely inseparable of exponent one, then $N \otimes N$ is a truncated polynomial algebra that is isomorphic to $F[G]$ for a finite abelian p -group G [8]. This means that N is always the tensor product of an étale algebra over F (the “separable part”) and another algebra (the “inseparable part”). This inseparable part is either a purely inseparable field extension of F of exponent one, an F -algebra $F[G]$, or a tensor product of an F -algebra $F[G]$ and a purely inseparable field extension of F of exponent one.

Let A be a generalized semiassociative algebra of degree n with $N \subset \text{Nuc}(A)$, $N = N_1 \otimes_F \cdots \otimes_F N_s$ of dimension n over F , with finite field extensions N_i/F where N_i either separable or purely inseparable and primitive of exponent one. Then A is called *split*, if $N \cong E \otimes_F F[G]$, where E/F is a split étale algebra, and $F[G]$ is a simple truncated polynomial algebra (G an abelian p -group). We allow here that $E = F$ or $F[G] = F$. A finite-dimensional field extension E/F *splits* A , if $N \otimes_F E \cong S \otimes_E E[G]$ for a suitable abelian p group G , and an étale algebra S over E . We also note that if $N = N_1 \otimes_F N_2$ is the tensor product of a separable and a purely inseparable extension of exponent one, then N/F is a finite field extension, as N_1 and N_2 are linearly disjoint over F .

Let A be a generalized semiassociative algebra with $K = \text{Nuc}(A)$ a purely inseparable field extension of exponent one. Then a finite-dimensional field extension E/F splits A , if $K \otimes_F E$ is a simple truncated polynomial algebra, which is the case if and only if $F \subset K \subset E$ is an intermediate field. In particular, K splits (K, δ, d) . For the nonassociative generalized

differential algebra (D, δ, d) we know that if K is a purely inseparable splitting field of the C -central simple algebra D , then $(D, \delta, d) \otimes_F K \cong (M_n(K), \delta, d)$ is a generalized semiassociative algebra over K whose nucleus contains the K -algebra $K \otimes_F K \cong F[G]$, but the algebra is not split, as the dimension of $F[G]$ is too small.

Moreover, for every central simple algebra D over F , we have $D \otimes_F (K, \delta, d) \cong (D_K, \delta, d)$, so if K is a purely inseparable splitting field of D , then $D \otimes_F (K, \delta, d) \cong (M_n(K), \delta, d)$, which is, however, not a split algebra over F .

Let D be a p -algebra of degree p^s with maximal separable splitting field E and purely inseparable simple splitting field L of degree $p^f \leq p^e$ or degree p^s (so D is cyclic). Then $D \otimes_F (K, \delta, d) \cong (D \otimes_F K, \delta, d)$ contains the field extension $E \otimes_F K \subset D \otimes_F K$ of degree $p^s p = p^{e+1}$ and the algebra $L \otimes_F K \subset D \otimes_F K$ in its nucleus. Here, $L \otimes_F K$ is either a finite purely inseparable field extension of exponent one, or - if $L = K$ - an algebra $F(G)$. In the later case, we get $D \otimes_F (K, \delta, d) \cong (M_{p^s}(K), \delta, d)$.

Alternatively, we could define $Br^{gsa}(F)$ as the submonoid of the above generalized one that is generated by $Br^{sa}(F)$ and the algebras (D, δ, d) and (K, δ, d) , and call the resulting algebras *generalized semiassociative algebras*.

In either case, we obtain that

$$[(K, \delta, d)]^{gsa}[D]^{gsa} = [(D_K, \delta, d)]^{gsa},$$

in particular $[(K, \delta, d)]^{gsa} = [(K, \delta, d)]^{gsa}[M_n(F)]^{gsa} = [(M_n(K), \delta, d)]^{gsa}$. Furthermore, if D is a p -algebra of degree p^s and K is a finite-dimensional purely inseparable splitting field of D then

$$[(K, \delta, d)]^{gsa}[D]^{gsa} = [(M_{p^s}(K), \delta, d)]^{gsa}.$$

It would be interesting to explore other relations.

Acknowledgement: Parts of this paper were completed while the author was a visitor at the University of Ottawa. She would like to thank the Department of Mathematics and Statistics for its hospitality.

REFERENCES

- [1] A. A. Albert, *Structure of algebras*. Vol. 24, AMS 1939.
- [2] A. S. Amitsur, *Differential polynomials and division algebras*. Annals of Mathematics, Vol. 59 (2) (1954) 245-278.
- [3] A. S. Amitsur, *Non-commutative cyclic fields*. Duke Math. J. 21 (1954), 87-105.
- [4] G. Blachar, D. Haile, E. Matri, E. Rein, U. Vishne, *Semiassociative algebras over a field*. Preprint, 2023. <https://arxiv.org/pdf/2308.14653.pdf>
- [5] C. Brown, *Petit's algebras and their automorphisms*. PhD Thesis, University of Nottingham, 2018. <https://eprints.nottingham.ac.uk/49613/>
- [6] C. Brown, S. Pumplün, *Nonassociative cyclic extensions of fields and central simple algebras*. J. Pure Applied Algebra 223 (6) (2019), 2401-2412
- [7] C. Brown, S. Pumplün, *How a nonassociative algebra reflects the properties of a skew polynomial*. Glasgow Mathematical Journal (Nov. 2019) <https://doi.org/10.1017/S0017089519000478>
- [8] L. N. Childs, *Purely inseparable field extensions*. Last accessed 04/03/2024. https://pi.math.cornell.edu/~dkmiller/galmod/Childs_purely-inseparable.pdf

- [9] L.E. Dickson, *Linear algebras in which division is always uniquely possible*, Trans. AMS 7 (3) (1906), 370-390.
- [10] J. Ducoat, F. Oggier, *On skew polynomial codes and lattices from quotients of cyclic division algebras*. Adv. Math. Comm. 10 (1) 2016, 79-94.
- [11] Hoechsmann, Klaus, *Simple algebras and derivations*. Trans. Amer. Math. Soc. 108 (1963), 1-12.
- [12] N. Jacobson, "Finite-dimensional division algebras over fields." Springer Verlag, Berlin-Heidelberg-New York, 1996.
- [13] N. Jacobson, *Abstract derivation and Lie algebras*. Trans. Amer. Math. Soc. 42 (2) (1937), 206-224.
- [14] K. Kishimoto, *On cyclic extensions of simple rings*. J. Fac. Sci. Hokkaido Univ. Ser. I 19 (1966), 74-85.
- [15] T. Y. Lam, K. H. Leung, A. Leroy, J. Matczuk, *Invariant and semi-invariant polynomials in skew polynomial rings*. Ring theory 1989 (Ramat Gan and Jerusalem, 1988/1989), 247-261, Israel Math. Conf. Proc., 1, Weizmann, Jerusalem, 1989
- [16] N. Markin, F. Oggier, *Iterated space-time code constructions from cyclic algebras*, IEEE Trans. Inform. Theory (9) 59, September 2013, 5966-5979.
- [17] G. Menichetti, *Sopra una classe di quasicorpi distributivi di ordine finito*. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 59 (5) (1975), 339-348.
- [18] J.-C. Petit, *Sur certains quasi-corps généralisant un type d'anneau-quotient*. Séminaire Dubriel. Algèbre et théorie des nombres 20 (1966-67), 1-18.
- [19] S. Pumplün, *How to obtain division algebras used for fast decodable space-time block codes*. Adv. in Math. Comm. 8 (3) (2014), 323 - 342.
- [20] S. Pumplün, A. Steele, *The nonassociative algebras used to build fast-decodable space-time block codes*, Adv. Math. Comm. 9 (4) (2015), 449-469.
- [21] S. Pumplün, *Tensor products of nonassociative cyclic algebras*. J. Algebra 451 (2016), 145-165.
- [22] S. Pumplün, *Finite nonassociative algebras obtained from skew polynomials and possible applications to (f, σ, δ) -codes* Advances in Mathematics of Communications 11 (3) (2017), 615-634.
- [23] S. Pumplün, *Nonassociative differential extensions of characteristic p* . Results Math. 72 (2017), 245-262.
- [24] S. Pumplün, *Algebras whose right nucleus is a central simple algebra*. Journal of Pure and Applied Algebra 222 (9), September 2018, 2773-2783.
- [25] S. Pumplün, A. Steele, *Fast-decodable MIMO codes from nonassociative algebras*. Int. J. of Information and Coding Theory (IJICOT) 3 (1) (2015), 15-38.
- [26] S. Pumplün, T. Unger, *Space-time block codes from nonassociative division algebras*, Adv. Math. Comm. 5 (3) (2011), 609-629.
- [27] S. Pumplün, *Generalized Menichetti algebras*. Preprint 2024, in preparation.
- [28] R. Sandler, *Autotopism groups of some finite non-associative algebras*. American Journal of Mathematics 84 (1962), 239-264.
- [29] R. D. Schafer, "An Introduction to Nonassociative Algebras." Dover Publ. Inc., New York, 1995.
- [30] A. Steele, *Nonassociative cyclic algebras*. Israel Journal of Mathematics 200 (1) (2014), 361-387.
- [31] A. Steele, *Some new classes of algebras*. PhD Thesis, University of Nottingham 2013. <http://eprints.nottingham.ac.uk/13934/1/PhdthesisFinal.pdf>
- [32] Waterhouse, W.C., *Nonassociative quaternion algebras*. Algebras, Groups and Geometries 4 (1987), 365-378.

Email address: susanne.pumpluen@nottingham.ac.uk

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM NG7 2RD, UNITED KINGDOM