The Tensor Track VIII: Stochastic Analysis

V. Rivasseau
Université Paris-Saclay, CNRS/IN2P3
IJCLab, 91405 Orsay, France

Abstract

Assuming some familiarity with quantum field theory and with the tensor track approach that we presented in the previous series "Tensor Track I-VII", we provide, as usual, the developments in tensors models of the last two years. Then we expose the fundamental breakthrough of Martin Hairer on regularity structures and the work of Léonard Ferdinand on stochastic analysis applied to super-renormalizable tensor field theories. We conclude with the hope that this work could be extended to just-renormalizable and asymptotically free models.

keywords Quantum Gravity, Tensors Models, Stochastic Analysis.

1 Introduction

Random tensors, like matrix models, originated in theoretical physics. In the 70's the hot stuff in theoretical physics was to quantize the elementary particles like quarks and gluons. In this period matrix models had a lot of success in quantizing the strong interaction.

In the 90's the dominating theory in quantizing gravity was string theory. Random matrix models were seen at this time as a successful theory for quantizing gravity, but only in two dimensions. The inventors of random tensor models, such as Ambjorn, Gross and Sasakura, wanted to replicate the success of matrix models for dimensions three and four. But they lacked an essential tool, the $1/N$ expansion.

Let's come to 2010's. The tensor track [1]-[7] is an attempt to quantize gravity in dimensions greater than two, by combining random tensor models, discrete geometry and the renormalisation group. The tensor track lies at the crossroad of several closely related approaches to quantize gravity, most notably causal dynamical triangulations, quantum field theory on non-commutative spaces and group field theory. Random tensors share with random matrices the fact that they are a zero-dimensional world, and, as such, they are background-independent; they made no references whatsoever of any particular space-time.

Moreover, random tensors models, based on the quantum field theory of Feynman, are manageable by renormalisation group techniques. Simple just-renomalizable models even share with non-Abelian gauge theories the property of asymptotic freedom. The simplest such model is the T_5^4 model.

Random tensors are expected to play a growing role in many areas of mathematics, physics, and computer science, but communities using random tensors have grown apart, developing their own tools and results; for an up-to-date review of distinct approaches to quantum gravity exposing shared challenges and common directions, we suggest consulting [8].

2 The Tensor Track

In [1]-[4] we proposed a new way of looking at the problem of quantum gravity. Let us summarize briefly what it's all about. First we would like to say that the tensor track has its birth in extending the matrix models and their $1/N$ expansion to tensor models.

In the Hermitian matrix ensemble (GUE) perturbed by a quadratic interaction, the $1/N$ expansion is well known. The free partition function is $\int dM e^{-\frac{N}{2}\operatorname{Tr} M^2}$, where

$$dM = \prod_k dM_{kk} \prod_{i<j} d\operatorname{Re} M_{ij} d\operatorname{Im} M_{ij}, \tag{1}$$

and the expectation values of $U(N)$ invariants

$$< \operatorname{Tr} M^{p_1} \operatorname{Tr} M^{p_2} ... \operatorname{Tr} M^{p_k} > \tag{2}$$

is entirely determined by the propagator

$$C_{ij,kl} = \frac{1}{N} \delta_{il} \delta_{jk} \tag{3}$$

and by Wick's rule.

Any scalar function of a tensorial quantum field theory can be further decomposed as a big functional integral on a Gaussian measure and an interactive part. In the tensorial case this interactive part is a sum of invariants of the tensor. For example the partition function is a scalar function of N, defined by

$$Z(N) = \int d\mu(T) e^{-\sum_{Inv} S_{Inv}(T)}. \tag{4}$$

The partition function and the corresponding free energy (also a scalar function) are related by a normalized logarithm

$$F(N) = \frac{1}{N^D} \log Z(N). \tag{5}$$

The invariants themselves can be classified in terms of graphs. Of course these graphs depend upon the group symmetries of the tensor. For matrix models the expectation values of the invariants can be classified by ribbon diagrams. In the case of tensor models Figure 1 depicts a partial list of connected invariants for $\bigotimes_{i=1}^{3} U(N)$. To generalize the $1/N$ expansion to

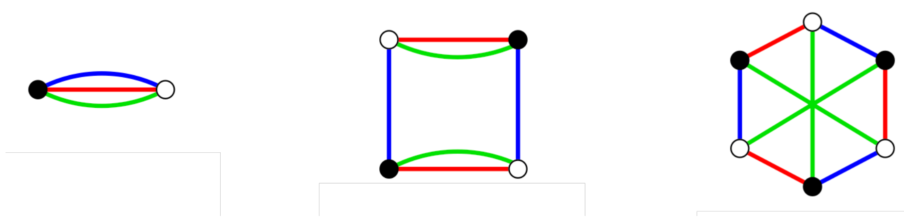


Figure 1: Examples of $U(N)$ invariants. The graphs presented in the left and center panels are *melonic*, while the one on the right panel is not.

the tensor case, the first step is to choose an invariant, for example a quartic invariant, and to normalize the $S_{Inv_0}(T)$ for that invariant:

$$S_{Inv_0}(T) \to \frac{\lambda}{N^\alpha} S_{Inv_0}(T). \tag{6}$$

Now the partition function and the corresponding free energy depend on *two variables*. In a quantum field theory the usual form of perturbation theory is to expand in power series in λ, and the perturbation is indexed by Feynman amplitudes associated to Feynman graphs. For instance the perturbation of the free energy is of the following form

$$F(\lambda, N) = \sum_G \frac{(-\lambda)^{v(G)}}{sym(G)} A(G, N). \tag{7}$$

Once this is done, the hard step is to find α such that 1/N expansion exists, i.e. such that the perturbation of the free energy is of the following form

$$F(\lambda, N) = \sum_G \frac{(-\lambda)^{v(G)}}{sym(G)} A(G, N) = \sum_{\omega \in \mathbb{N}} N^{-\omega} F_\omega(\lambda). \tag{8}$$

In the case of $\bigotimes_{i=1}^D U(N)$ and a quartic D-melonic invariant, the $1/N$ expansion is governed by the Gurău degree and $F_0(\lambda)$ is formed by all the $D+1$ melonic graphs (the 0 color being associated to the Feynman propagators [9]). The family of $D+1$ melonic graphs [10] which lead the 1/N expansion of random tensors models can perhaps be called too trivial from a topologist point of view; it corresponds to some triangulations of the sphere S_D. But, as the Gurău degree is not only purely topological, the interplay between combinatorics and topology in sub-leading terms can be amazingly complex.

Now that we have been able to identify the leading terms in the $1/N$ expansion, the second step is to be able to resum them, i.e. to explicitly compute $F_0(\lambda)$. This step has been performed for the first time by a paper by Gurău and Ryan [11]. From a probabilistic and statistical mechanical point of view, it corresponds to Aldous phase of branched polymers.

Once we have been able to compute explicitly $F_0(\lambda)$, many possibilities are open to us: modifying the symmetry of the main tensor, include the renormalization group by modifying in a specific way our propagators, a non-perturbative treatment of some simple models...

From the perspective of matrix models, to go further in the two parameters approach require a particular technique, namely double scaling. The first step of applying this technique to tensors has been done in [12, 13]. The initial papers have been followed by mixed results, some results suggest the universality of branched polymers, others pointing to the fact that some simple and natural restrictions change that universality class. But, from the perspective of quantum gravity, the main goal is to resum the sub-leading terms in order to find a more interesting phase of geometry pondered by Einstein-Hilbert action.

Let us come to the Sachdev-Ye-Kitaev (SYK) model. Discovered by Kitaev [14], it is a quartic model of N Majorana fermions coupled by a disordered tensor [15]. It is a model of condensed matter, hence it depends on time though a Hamiltonian. The disordered tensor is centered Gaussian iid

$$< J_{abcd} >= 0, \quad < J_{abcd}^2 >= \frac{\lambda^2}{N^3}, \tag{9}$$

and the Hamiltonian is simply $H = J_{abcd}\psi_a\psi_b\psi_c\psi_d$. This model posses three important properties: it is solvable at large N, there is a conformal symmetry at strong coupling, hence it can be a fixed point of the renormalization group, and, above all from quantum gravity, it is maximally chaotic in the sense of Maldacena, Shenker and Stanford [16]. Hence the SYK model, although very simple, offers a path to the main theoretical concepts of quantum gravity, such as Bekenstein-Hawking entropy and holography.

SYK became a very active field, from the early papers to nowadays. Quite naturally we devoted our common article with Nicolas Delporte to that subject and we entitled it "Holographic Tensors" [5].

At large N the Schwinger-Dyson equation for the 2-point function is closed. The conformal symmetry can be broken and the corresponding subject goes under the name of near-AdS_2/near-CFT_1 correspondence. This entails a relationship with Jackiw-Teitelboim two-dimensional quantum gravity.

Witten has found a genuine field theory model (with no disorder), in which the tensors plays a much more fundamental role [17]. In a nutshell, he discovered that his model has the same melonic limit as the tensors models pioneered by Gurău. Klebanov and Tarnopolsky [18], when combined with an earlier work of Carrozza and Tanasa [19], allows on a big simplification of the group symmetry of the main tensor, from $U(N)^{D(D-1)/2}$ to $O(N)^D$.

Unlike the initial SYK model, the field tensor models of [17, 18, 19] fit in the framework of local quantum field theory with $D = 1$. Hence there is a possibility to extend them in $D > 1$! But these models of SYK-type still are quantum mechanical and lost background-independence since they make use of a preferred time. We stumbled for a while to that particular problem, namely to restore background-independence to models of SYK-type. Zero dimensional tensor models create naturally trees or unicycles as Gromov-Hausdorff limits. If we can approximate the sub-dominant terms as matter fields living on trees or unicycle (and it's a big "if"), we shall get in this approximation an SYK-type model on a random tree or unicycle. Thermal Euclidean, which plays such a natural role in SYK models, leads us to choose unicycles rather than trees. Models of this type can be studied first by perturbative field theory techniques.

Then our main result together with Nicolas Delporte [6, 20] is that, under reasonable assumptions, the SYK model for bosons averaged on long, infrared unicycles possesses a two-point function exhibiting much the same behavior, but with a critical infrared exponent different from the one of ordinary SYK, sensitive to the spectral dimension of the underlying graph.

This can be seen as a simpler version of the well-known 2d CFT coupled to gravity (ie CFT on $\mathbb{R}^2$ but coupled to the Liouville field). The change in critical exponent is a simpler analog of the Knizhnik-Polyakov-Zamolodchikov and David-Distler-Kawai relations, which tell how critical exponents change when coupled to 2d gravity. The cycle in a unicycle can be identified to a lattice-regularized flat $U(1)$ thermal circle, and the trees decorating the unicycles are then the unidimensional analog of the bidimensional Liouville field bumps which are at the source of the modification of critical exponents. In this way, field theory on random unicycles can be seen as "gravity in one dimension" or "gravitational time".

We return to a more general problem with Ouerfelli and Tamaazousti; that of making a big (maybe too ambitious?) jump, from quantum gravity to artificial intelligence [7]. More specifically the context of that paper is the following. Tensor PCA was introduced in the pioneer work of [21] and consists in recovering a signal spike $v_0^{\otimes k}$ that has been corrupted by a noise tensor Z: $T = Z + \beta v_0^{\otimes k}$ where v_0 is a unitary vector and β the signal to noise ratio.

Matrix data analysis and principal component analysis (PCA) is mostly stated in the "quantum-mechanics" language of eigenvalues rather than in the "quantum-field theoretic" language of invariants and (Feynman) graphs. For tensors the quantum-field language is the natural one. An important task in tensor data analysis is therefore to translate the results of matrix data analysis and PCA into the quantum-field theoretic language of invariants and graphs.

An original connection have been made in [22] between tensorial data analysis and the tensor track. This connection is based on the introduction of matrices that are built out of a graph and cutting an edge. Indeed, given a graph invariant, we call "cutting an edge" the fact of not performing a sum over the index associated to this edge, which gives us a matrix. The eigenvector associated to the largest eigenvalue of this matrix can then be proven to be correlated to the signal vector v for a significant range of signal-to-noise ratio β.

Another article with Ouerfelli and Tamaazousti refers to an heuristic algorithm named SMPI [23]. This algorithm seems to be considerably better that the state of the art; more progress is expected during the coming months and years.

3 Recent work from our group

Rasvan Gurău is busy by combining random tensors and conformal field theory, and he performed recently a beautiful set of lectures entitled “From random tensors to tensor field theory” in a six-weeks program at Institut Henri Poincaré in the winter of 2023. From his recent output I have extracted three papers that I consider emblematic of his activity in Heidelberg.

With Dario Benedetti, Sabine Harribey and Davide Lettera, he consider the long-range bosonic $O(N)^3$ model where N gets large enough [24]. The model displays four large-N fixed points and the authors confirm that the F-theorem holds in this case. This result is subtle, as one of the couplings (the “tetrahedral” coupling) is imaginary, and therefore the model is non-unitary at finite N.

In the same vein, with his collaborators Jürgen Berges and Thimo Preis, he studied further the quantum field theory with global $O(N)^3$ symmetry where N is sufficiently large. He find that both asymptotic freedom and boundedness from below can be realized in this theory [26]. To uncover its scale dependence, the authors analyze the renormalization group flow for the tensor field which features two real quartic couplings, g_1 and g_2 (which correspond to the pillow and to the double-trace), and an imaginary tetrahedral coupling ig (it is here that the link with [24] is used). It turns out that all the couplings exhibit asymptotic freedom, leading to a just-renormalizable field theory with an ultraviolet limit and a strongly coupled infrared limit in four space-time Euclidean dimensions, just like QCD!

On the other hand, from his “constructive vein” and with his collaborators Dario Benedetti, Hannes Keppler and Davide Lettera, he has begun a study of Ecalle’s trans-series, again on the $O(N)$ model, but this time at small N. They recover that both the partition function $Z(g, N)$ and $W(g, N) = \log Z(g, N)$ are Borel summable functions. Then, using our constructive field theory techniques such as the loop vertex expansion [27], they prove that the trans-series expansion of the Taylor coefficients of these expansions, $Z_n(g)$ and $W_n(g)$, are different. What I find especially strong in their article is the fact that they were able to extend the Borel transform to the angle $\frac{3}{2}\pi$ and to prove that, while $W(g, N)$ displays contributions from arbitrarily many multi-instantons, $W_n(g)$ exhibits contributions of only up to n-instanton sectors.

Recently Razvan and I made a review entitled "Quantum Gravity and Random Tensors" in the context of the Poincaré Seminar [28].

Joseph Ben Geloun has defended his french "Habilitation à diriger des recherches". Along the many gems of his HdR, I choose one which is particularly spectacular. With Sanjaye Ramgoolam they have been able to give a combinatorial interpretation of the Kronecker coefficients, a problem which existed since 85 years! Kronecker coefficients are widely studied in mathematics from many points of view: symmetric polynomials, complexity theory, combinatorics... The standard mathematical approach is to think of them as the structure constants of irreducible representations of symmetric groups and it's around this approach that they've built their results [29].

Joseph Ben Geloun is a most active researcher and we would like to stressed four recent contributions of him. With Dina Andriantsiory and Mustapha Lebbah, he propose a method of clustering for 3-order tensors of different dimensions via an affinity matrix. Based on a notion of similarity between the tensor slices and the spread of information of each slice, their model builds an matrix on which they apply advanced clustering methods. The combination of all clusters delivers the desired multiway clustering. Their method and their associated algorithm, which they baptized MCAM, achieves competitive results compared with other known algorithms, both on synthetics and real datasets [33, 34, 35].

With Reiko Toriumi, he wants to escape the branched polymer phase of tensor models. They explore two just-renormalizable quartic enhanced[1] tensor field theories [31], and they obtain results which I find interesting. At all orders, both models have a constant wave function renormalization and therefore no anomalous dimension. They also analyse their RG flow, which depend of two coupling constants. They compute the perturbative β-functions of these coupling constants at one-loop. For the first model, the flow exhibits neither asymptotic freedom nor the ordinary Landau ghost of ϕ^4_4: one of the coupling stays fixed, and the other has a linear behavior in the

[1]Enhanced tensor field theories have dominant graphs that do not correspond to melonic graphs.

time scale. For the second model both couplings do not flow at all, according this one-loop approximation.

In the article with Andreas Pithis and Johannes Thüringen [32], their goal is to find a phase transition from discrete quantum-gravitational to continuum geometry starting with tensor invariants. In the so-called cyclic-melonic potential approximation of a tensorial field theory on the r-dimensional torus, Andreas Pithis and Johannes Thüringen recently showed, using functional renormalization group techniques, that no such phase transition is possible. In [32] they show how to overcome this limitation by introducing local degrees of freedom on $\mathbb{R}^d$. They find that the effective $r-1$ dimensions of the torus part dynamically vanish along the renormalization group flow while the d local dimensions persist up to small momentum scales. Consequently, for $d > 2$ they find a possibility to allow some phase transitions.

Recently Joseph Ben Geloun and Sanjaye Ramgoolam propose a theory of complexity pertaining to data analysis [30]. Their goal is to detect projectors in associative algebras, labelled by representation data. To illustrate this theory they propose three examples. One is based on a quantum algorithm based on quantum phase estimation, and they compare it to a classical algorithm based on the AdS/CFT correspondence. The other two, around the line of [29], are projectors labelled by a triple of Young diagrams, all having n boxes, or with m, n and $m+n$ boxes, and having non-vanishing Kronecker coefficients in the case of having n boxes, or having non-vanishing Littlewood-Richardson coefficients in the case of having m, n and $m+n$ boxes.

Adrian Tanasa is very active in Bordeaux. His book on combinatorial physics has been published [36] and his scientific interests often revolve around the problem of the double scaling of many models [37, 39, 38]. With Hannes Keppler, Thomas Krajewski and Thomas Muller, he is also interested in so called negative dimension theorems, or N to $-N$ dualities, relating the orthogonal and symplectic group vial the formal relation $SO(-N) \simeq Sp(N)$ [40].

Sabine Harribey is currently a postdoc in Nordita. With her, Igor Klebanov and Zimo Sun explore a new approach to boundaries and interfaces in the $O(N)$ model where they add certain localized cubic interactions. They use the technique of $1/\epsilon$ expansion and they show that the one-loop beta functions of the cubic couplings are affected by the quartic bulk interactions. For the interfaces, they find real fixed points up to the critical value $N_{\text{crit}} \approx 7$, while for $N > 4$ there are IR stable fixed points with purely imaginary values of the cubic couplings [41]. Recently with Dario Benedetti, Razvan Gurau and Davide Lettera, she publishes an article on finite-size versus finite-temperature effets in the O(N) model [42]. In it they consider the classical model, which is conformally invariant at criticality, and they introduce one compact spatial direction. They show that the finite size dynamically induces an effective mass and they compute the one-point functions for bilinear primary operators with arbitrary spin and twist. Second, they study the quantum model, mapped to a Euclidean anisotropic field theory, local in Euclidean time and long-range in space, which the authors dub fractional Lifshitz field theory. They show that this model admits a fixed point at zero temperature,

Dario Benedetti and Valentin Bonzom have both defended their HDR. Dario Benedetti has defended his HdR in the Ecole Polytechnique [43] and Valentin Bonzom has defended his HdR in the Paris Nord university [53]. Dario Benedetti, Sylvain Carrozza, Reiko Toriumi and Guillaume Valette studied the double and triple-scaling limits of a complex multi-matrix model. Their main result is, in a double-scaling limit, to characterize the Feynman graphs of arbitrary genus, and in the triple-scaling limit, to classify all the three-edge connected dominant graphs and to prove that their critical behavior falls in the universality class of Liouville quantum gravity [44].

Sylvain Carrozza has been recruited permanently in Université de Bourgogne in Dijon. We would like to stressed one recent contributions in his domain (apart from [44]). For $p = 3$ and $p = 5$ there exist a melonic large N limit for p-irreducible tensors in the sense of Young tableaux. Sylvain Carrozza and Sabine Harribey overcome huge difficulties to solve the case $p = 5$ [45]. They demonstrate that random tensors transforming under rank-5 irreducible representations of $\mathrm{O}(N)$ can support melonic large N expansions. Their construction is based on models with sextic (5-simplex) interaction, which generalize previously studied rank-3 models with quartic (tetrahedral) interaction. Their proof relies on recursive bounds derived from a detailed combinatorial analysis of the Feynman graphs. Their results provide further evidence that the melonic limit is a universal feature of irreducible tensor models in arbitrary rank.

Luca Lionni has been recruited permanently in mathematics at Institut Camille Jordan in Lyon. With Benoit Collins and Razvan Gurău, he explore a generalization of the Harish-Chandra–Itzykson–Zuber integral to tensors and its expansion over trace-invariants of the two external tensors. This gives rise to natural generalizations of monotone double Hurwitz numbers. They find an expression of these numbers in terms of monotone simple Hurwitz numbers and they give an interpretation of their different combinatorial quantities in terms of enumeration of nodal surfaces [47]. Still with Benoit Collins and Razvan Gurău, they analyze a two-parameter class of asymptotic scalings when N, the size of the tensors, is large enough, uncovering several non-trivial asymptotic regimes. This study is relevant for analyzing the entanglement properties of multipartite quantum systems [48].

With Timothy Budd, his research is centered in random triangulations of manifolds, a goal which is central to the random geometry approach to quantum gravity. In case of the 3-sphere the pursuit is held back by serious challenges, including the wide open problem of enumerating triangulations. First, they identify a restricted family of triangulations, of which the enumeration appears less daunting. Then they prove that these are in bijection with a combinatorial family of triples of plane trees satisfying restrictions. An important ingredient is a reconstruction of the triangulations from triples of trees that results in a subset of the so-called locally constructible triangulations. Finally, several exponential enumerative bounds are deduced from the triples of trees and some simulation results are presented [49].

Stephane Dartois with Camille Male and Ion Nechita studied the tensor flattenings appear naturally in quantum information when one produces a density matrix by partially tracing the degrees of freedom of a pure quantum state. In their paper, they study the joint distribution of the flattenings of large random tensors under mild assumptions, in the sense of free probability theory. They show the convergence toward an operator-valued circular system with amalgamation on permutation group algebras for which we describe the covariance structure. As an application they describe the law of large random density matrix of bosonic quantum states [51].

Nicolas Delporte has defended his PhD [52]. With Dario Benedetti he revisit the Amit-Roginsky (AR) model in the light of recent studies on SYK and tensor models. It is a model of N scalar fields transforming in an N-dimensional irreducible representation of $SO(3)$. The most relevant (in renormalization group sense) invariant interaction is cubic in the fields and mediated by a Wigner $3jm$ symbol. The latter can be viewed as a particular rank-3 tensor coupling, thus highlighting the similarity to the SYK model, in which the tensor coupling is however random and of even rank. As in the SYK and tensor models, in the large-N limit the perturbative expansion is dominated by melonic diagrams. The lack of randomness, and the rapidly growing number of invariants that can be built with n fields, makes the AR model somewhat closer to tensor. In the short range version of the model, they find, for $5.74 < d < 6$, a fixed-point defining a real CFT, while for smaller d complex dimensions appear. They also introduce and study a long-range version of the model, for which the cubic interaction is marginal at large N, and they find a real and unitary CFT for any $d < 6$, both for real and imaginary coupling constant, up to a critical coupling [54].

Riccardo Martini and Reiko Toriumi give a procedure to construct trisection diagrams for closed pseudo-manifolds generated by colored tensor models without restrictions on the number of simplices in the triangulation, therefore generalizing previous works in the context of crystallizations and PL-manifolds. They further speculate on generalization of similar constructions for a class of pseudo-manifolds generated by simplicial colored tensor models [55].

The authors of [56] introduce a dually-weighted multi-matrix model that for a suitable choice of weights reproduce two-dimensional Causal Dynamical Triangulations (CDT) coupled to the Ising model. When Ising degrees of freedom are removed, this model corresponds to the 2d CDT-matrix model introduced by Dario Benedetti and Joe Henson [57]. They present exact as well as approximate results for the Gaussian averages of characters of a Hermitian matrix A and A^2 for a given representation and establish the present limitations that prevent them to solve the model analytically. This sets the stage for the formulation of more sophisticated matter models coupled to two-dimensional CDT as dually weighted multi-matrix models providing a complementary view to the standard simplicial formulation of CDT-matter models.

Vincent Lahoche together with Corinne de Lacroix and Harold Erbin [58] compute the gravitational action of a free massive Majorana fermion coupled to two-dimensional gravity on compact Riemann surfaces of arbitrary genus. The structure is similar to the case of the massive scalar. The small-mass expansion of the gravitational yields the Liouville action at zeroth order, and they can identify the Mabuchi action at first order. While the massive Majorana action is a conformal deformation of the massless Majorana CFT, they find an action different from the one given by the David-Distler-Kawai (DDK) ansatz.

Bio Wahabou Kpera, Vincent Lahoche and Dine Samary want to study the probability laws associated with random tensors or tensor field theories. Their approach is to the quantize through a Langevin type equation. The method they propose use the self averaging property of the tensorial invariants in the large N limit. In this regime, the dynamics is governed by the melonic sector. Their work focuses on the cyclic (i.e. non-branching) melonic sector, and they study the way that the system returns to the equilibrium regime. In particular, they provide a general formula for the transition temperature between these regimes. Numerical simulations are made to support the theoretical analysis [59]. Also they explored with Seke Yerima the Ward-Takahashi identities in tensorial group field theories. Ward's identities result from a expansion around the identity, and it is expected that a first-order expansion is indeed sufficient. They show that it doesn't occur for a complex tensor theory model with a kinetic term involving a Laplacian [60].

4 Blitz Review of Stochastic Analysis

Stochastic analysis has been revolutionized by the work of Martin Hairer on regularity structures [61], which provides a framework for studying a large class of stochastic partial differential equations arising from quantum field theory. This framework covers the Kardar–Parisi–Zhang equation, the Φ^4_3 equation and the parabolic Anderson model, all of which require renormalization in order to have a well-defined notion of solution. Among the fashionable follows-up we shall cite the dynamical approach of Barashkov and Gubinelli [62] and the variational approach of Gubinelli and Hofmanová [63].

Regularity structures and associated models are a way of solving a stochastic equation by choosing objects replacing the polynomials of the Taylor expansion with non-polynomial coefficients of the increase h between x and $x + h$. This generalized expansion is based on non-integer powers and even negatives ones. In doing so, we perform intelligently a renormalization which depends only the form of the stochastic equation, not the constants involved or the stochastic noise.

This point of view follows the footsteps of the fathers of differential calculus, where $\frac{df}{dx}$ can make sense without having to make sense of df and dx separately, and the point of view of distributions, where $\int \phi(x)\delta(x)dx$ makes sense for a test function ϕ without having to make sense of $\delta(x)$, the Dirac "function". But the later developments overcome the main obstacle of the latter theory, the multiplication of distributions!

Like Wilson's, Hairer's point of view is susceptible to many generalizations. Let's start with a trivial example:

$$A_\epsilon = \{[x_\epsilon(t), y_\epsilon(t)]\, , t \in \mathbb{R}\}, \qquad x_\epsilon(t) = \epsilon t + \frac{1}{\epsilon}, \quad y_\epsilon(t) = \epsilon \cos(t).$$

At t fixed and $\epsilon \to 0$, $[x_\epsilon(t), y_\epsilon(t)] \to [\infty, 0]$, does not converge. If we perform a reparametrization $t \mapsto t/\epsilon - 1/\epsilon^2$, the solution reparametrised

$$[\hat{x}_\epsilon(t) = t\ , \hat{y}_\epsilon(t) = \epsilon \cos\left(\frac{t}{\epsilon} - \frac{1}{\epsilon^2}\right)] \to A_0 = \mathbb{R} \times \{0\}$$

indeed converges. In this analogy (x_ϵ, y_ϵ) plays the role of the 'bare' solution ϕ_ϵ, while $(\hat{x}_\epsilon, \hat{y}_\epsilon)$ plays the role of the renormalized solution.

Let us come the heart of Hairer's formalism.

Definition 1 *A regularity structure is a triple $\mathcal{T} = (A, T, G)$ consisting of:*

- *a subset A (index set) of $\mathbb{R}$ that is bounded from below and has no accumulation points;*
- *the model space: a graded vector space $T = \oplus_{\alpha \in A} T_\alpha$, where each T_α is a Banach space;*
- *and the structure group: a group G of continuous linear operators $\Gamma \colon T \to T$ such that, for each $\alpha \in A$ and each $\tau \in T_\alpha$, we have*

$$(\Gamma - 1)\tau \in \oplus_{\beta < \alpha} T_\beta. \tag{10}$$

Moreover $\Gamma 1 = 1$ for each $\Gamma \in G$.

Let us use the notation X^k for $X_1^{k_1} \cdots X_d^{k_d}$ for any multi-index k. The polynomial canonical model we're trying to generalize is then given by

- $A = \mathbb{N}$,
- T_k the linear space generated by monomials of degree k,
- a group of structure G (in this case $\mathbb{R}^d$ endowed with addition) which acts on T via $\Gamma_h X^k = (X - h)^k$, $h \in \mathbb{R}^d$.

However, it's not the complete algebraic structure that describes Taylor developments. An essential element is that a development around a certain point x_0 can be developed around any other point x_1 by the formula

$$(x - x_0)^m = \sum_{k+\ell=m} \binom{m}{k} (x_1 - x_0)^k \cdot (x - x_1)^\ell\ .$$

Such a redevelopment application Γ_{st} has the property that

$$\{\Gamma_{st}\tau - \tau\} \in \bigoplus_{\beta<\alpha} T_\beta =: T_{<\alpha}.$$

In other words, when redeveloping a homogeneous monomial around a different point, the leading-order coefficient remains the same, but lower-order monomials may appear. This is the meaning of the condition (10).

With the algebraic skeleton so defined, we move on to the associated analytical flesh. A further key notion is that of a *model*, which is a way of associating to any $\tau \in T$ and $x_0 \in \mathbb{R}^d$ a "Taylor polynomial" based at x_0 and represented by τ, subject to some consistency requirements. Concretely it consisting of a *family of applications*

$$\Pi\colon \mathbb{R}^d \to \mathcal{L}\big(T, \mathcal{S}'(\mathbb{R}^d)\big); \qquad \Gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to G$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the space of distributions (not necessarily temperate) on $\mathbb{R}^d$. In the polynomial natural canonical model this family of applications is

$$\big(\Pi_x X^k\big)(y) = (y-x)^k, \quad \Gamma_{xy} = \Gamma_{y-x}, \quad \Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}, \quad \Pi_x\Gamma_{xy} = \Pi_y.$$

A first difficulty arises. We want to allow τ elements in T_α to represent distributions and not just functions that cancel each other out to order α around a point. But then we cannot evaluate them point by point.

A second difficulty lies in the notion to "cancel to the order of α" when α is negative. It can only say tend to infinity slowly enough as a function of α. We therefore need a extended notion of "cancelling to the order of α". We will achieve this by controlling the size of our distributions in a small region around the given point x_0. Consider a test function ϕ and define

$$\phi_x^\lambda(y) := \lambda^{-d}\phi\big(\lambda^{-1}(y-x)\big) \ .$$

Let r be a natural number and $\mathcal{B}_r$ the set of test functions $\phi\colon \mathbb{R}^d \to \mathbb{R}$ with support in the ball $B(0,1)$ such as $\|\phi\|_{\mathcal{C}^r} \leq 1$. Combining intelligently everything, we can define at last the notion of models:

Definition 2 *Given a regularity structure $\mathfrak{T}$ and an integer $d \geq 1$, a* model *for $\mathfrak{T}$ over $\mathbb{R}^d$ consists applications*

$$\begin{array}{rclcrcl} \Pi\colon \mathbb{R}^d & \to & \mathcal{L}\big(T, \mathcal{S}'(\mathbb{R}^d)\big) & \qquad & \Gamma\colon \mathbb{R}^d \times \mathbb{R}^d & \to & G \\ x & \mapsto & \Pi_x & & (x,y) & \mapsto & \Gamma_{xy} \end{array}$$

so that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x\Gamma_{xy} = \Pi_y$ (Chen relations). Moreover, given $r > |\inf A|$, for any compact set $\mathfrak{K} \subset \mathbb{R}^d$ and constant $\gamma > 0$ there exists a constant C such that the bounds

$$\big|\big(\Pi_x\tau\big)(\phi_x^\lambda)\big| \le C\lambda^{|\tau|}\|\tau\|_\alpha\ , \qquad \|\Gamma_{xy}\tau\|_\beta \le C|x-y|^{\alpha-\beta}\|\tau\|_\alpha\ ,$$

are satisfied uniformly over $\phi \in \mathcal{B}_r$, $(x,y) \in \mathfrak{K}$, $\lambda \in (0,1]$, $\tau \in T_\alpha$, with $\alpha \le \gamma$ and $\beta < \alpha$.

Let us make a few remarks.

- Chen's relations $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x\Gamma_{xy} = \Pi_y$ are natural.

- The first bound indicates precisely what we mean when we say that $\tau \in T_\alpha$ represents a term of the order of α.

- The second bound is also very natural: it indicates that when developing a monomer of order α around a new point at a distance h from the old one, the coefficient in front of monomials of order β is at most of order $h^{\alpha-\beta}$.

- In many interesting cases, it's natural to scale the different directions of $\mathbb{R}^d$ in a different way. For example to construct solutions for stochastic parabolic PDEs, where the time direction "counts double", we define ϕ_x^λ so that the ith direction is dilated by λ^{s_i}, where $s = \{s_i, i = 1, \cdots, d\}$. In this case, the corresponding scaled distance is $|x|_s = \sum_i |x_i|^{1/s_i}$.

- For a given structure and regularity pattern, the distribution represented by f should be $\mathcal{R}f(x) := [\Pi_x f](x)$. However, this definition is not correct since $\Pi_x f(x)$ is in general a distribution, and therefore may not be evaluated in x! In addition the models must be subject to a "reconstruction theorem" which says that in a unique way the distribution is reconstructible by its generalized Taylor expansion. And this theorem cannot be obtained in a *linear* way.

Hairer's theory solves this difficulty by defining another space that Hairer calls D^γ, endowed with a topology metrizable but not normable which reflects the non-linearity of the problem.

Definition 3 *Consider a regularity structure* $\mathfrak{T}$ *equipped with a model* (Π, Γ) *defined on* $\mathbb{R}^d$*. The space* $\mathcal{D}^\gamma = \mathcal{D}^\gamma(\mathfrak{T}, \Gamma)$ *is given by the set of functions* $f \colon \mathbb{R}^d \to \bigoplus_{\alpha<\gamma} T_\alpha$ *such that, for each compact set* $\mathfrak{K}$ *and each* $\alpha < \gamma$*, there exists a constant* C *with*

$$\|f(x) - \Gamma_{xy} f(y)\|_\alpha \leq C|x - y|^{\gamma-\alpha} \tag{11}$$

uniformly over $x, y \in \mathfrak{K}$*.*

Let us comment on this definition. $\mathfrak{M} \ltimes \mathcal{D}^\gamma := \{(\Pi, \Gamma, F)\}$ is a metrizable space but not a normed space. The distance between (Π, Γ, f) and $(\bar{\Pi}, \bar{\Gamma}, \bar{f})$ is given by $\inf \rho$ such that

$$\begin{aligned} \|f(x) - \bar{f}(x) - \Gamma_{xy} f(y) + \bar{\Gamma}_{xy}\bar{f}(y)\|_\alpha &\leq \rho|x - y|^{\gamma-\alpha}, \\ |\big(\Pi_x \tau - \bar{\Pi}_x \tau\big)(\phi_x^\lambda)| &\leq \rho \lambda^\alpha \|\tau\|, \\ \|\Gamma_{xy}\tau - \bar{\Gamma}_{xy}\tau\|_\beta &\leq \rho|x - y|^{\alpha-\beta}\|\tau\| \ , \end{aligned}$$

uniformly for x, y in a compact set. So we can formulate the reconstruction theorem:

Theorem 1 *Let* $\mathfrak{T}$ *be a regularity structure as above and let* (Π, Γ) *be a model for* $\mathfrak{T}$ *over* $\mathbb{R}^d$*. There is a unique linear map* $\mathcal{R} \colon \mathcal{D}^\gamma \to \mathcal{S}'(\mathbb{R}^d)$ *such that*

$$\big|\big(\mathcal{R} f - \Pi_x f(x)\big)(\phi_x^\lambda)\big| \lesssim \lambda^\gamma \ , \tag{12}$$

uniformly over $\phi \in \mathcal{B}_r$ *and* λ *and locally uniformly over* x*.*

Uniqueness is much easier than existence. The existence of such a function appealed in a crucial way to the existence of a wavelet basis consisting of C_r functions with compact support, which was demonstrated in 1988 by Ingrid Daubechies.

We take as a dynamic point of view the example of Φ^4 which stochastic equation is

$$\partial_t \Phi = \Delta\Phi - \Phi^3 + \xi \ , \tag{13}$$

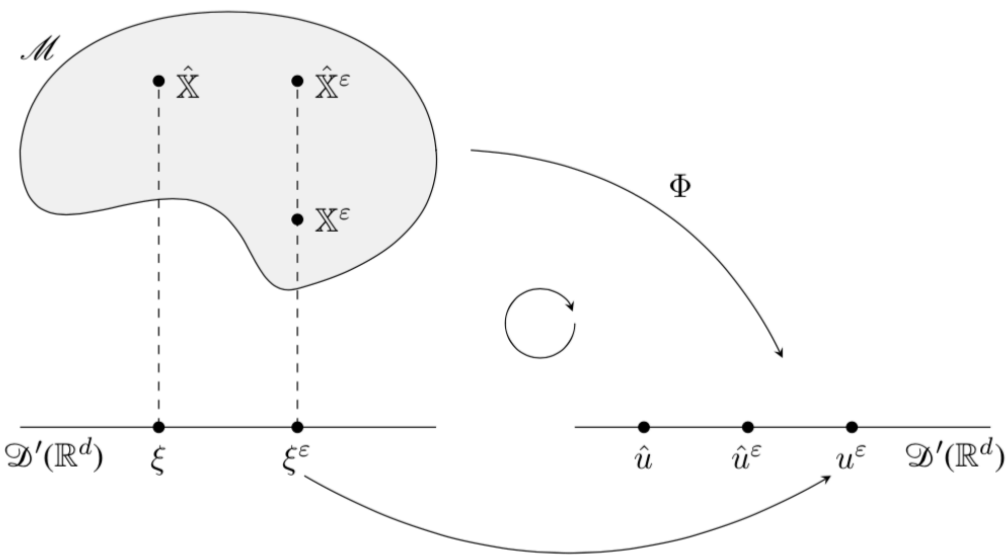
- We place ourselves in dimension 3 on a torus T^3 in direct space,
- ξ is a stochastic variable, for example Gaussian with covariance

$$\mathbf{E}\xi(t, x)\xi(s, y) = \delta(t - s)\delta(y - x) \ . \tag{14}$$

- ρ_ϵ is a "mollifier" smooth with compact support which tends when $\epsilon \to 0$ to $\delta(t-s)\delta(y-x)$,
- the *smoothed* stochastic equation u_ϵ for $\xi_\epsilon = \rho_\epsilon * \xi$,

$$\partial_t u_\epsilon = \Delta u_\epsilon + C_\epsilon u_\epsilon - u_\epsilon^3 + \xi_\epsilon \ , \tag{15}$$

  should tend to the sought solution when $\epsilon \to 0$.



In this figure, we show the factorization of the application $\xi^\epsilon \mapsto u^\epsilon$ into $\xi^\epsilon \mapsto \mathbb{X}^\epsilon \mapsto \Phi(\mathbb{X}^\epsilon) = u^\epsilon$. We can also see that in the space of $\mathfrak{M}$ models we have several antecedents of $\xi^\epsilon \in \mathcal{S}'(\mathbb{R}^d)$, for example the canonical model $\mathbb{X}^\epsilon$ and the renormalized model $\hat{\mathbb{X}}^\epsilon$; it's the latter that converges to a $\hat{\mathbb{X}}$ model, thus offering a bearing on ξ. Note that $\hat{u}^\epsilon = \Phi(\hat{\mathbb{X}}^\epsilon)$ and $\hat{u} = \Phi(\hat{\mathbb{X}})$.

The BPHZ solution goes through the following steps

- *Algebraic step*: Construction of the space of models $(\mathfrak{M}, \mathrm{d})$ and renormalization of the canonical model $\mathfrak{M} \ni \mathbb{X}^\epsilon \mapsto \hat{\mathbb{X}}^\epsilon \in \mathfrak{M}$ [64].
- *Analytical step*: Continuity of application $\mathfrak{M} \to \mathcal{S}'(\mathbb{R}^d)$ [61].

- *Probabilistic step*: Convergence in the sense of probabilities from the renormalized model $\hat{\mathbb{X}}^\epsilon$ to $\hat{\mathbb{X}} \in (\mathfrak{M}, \mathrm{d})$ [65].

- *Second algebraic step*: Identification of the final application $\Phi(\hat{\mathbb{X}}^\epsilon)$ with the classical solution with local counterterms at the BPHZ [66].

5 The work of Léonard Ferdinand

In this section we summarize briefly the articles of Léonard Ferdinand, a PhD of mine.

Léonard Ferdinand, Razvan Gurău, Carlos Perez-Sanchez and Fabien Vignes-Tourneret consider a quartic $O(N)$-vector model [50]. Using the loop vertex expansion, they prove the Borel summability in $1/N$ for the cumulants (including the free energy, which one considers the cumulant of zero order). The Borel summability holds uniformly in the coupling constant, as long as the latter belongs to a cardioid domain of the complex plane. Among their toolbox, let's remark that they use ciliated trees in a relatively new sense.

Then we summarize briefly the articles of Ismail Bailleul, Nguyen Viet Dang, Léonard Ferdinand and Tat Dat Tô [67, 68]. These authors deal with the Φ^4_3 measure, but what is more original, on a arbitrary 3-dimensional compact Riemannian manifold without boundary. They prove the nontriviality and the local covariance under Riemannian isometries of the corresponding measure. This answers a longstanding open problem of constructive quantum field theory on curved backgrounds in dimension 3. To control analytically several Feynman diagrams appearing in the construction of a number of random fields, they introduce a novel approach of renormalization using microlocal and harmonic analysis. This allows them to obtain a renormalized equation which involves some universal constants independent of the manifold.

In a companion paper [68], they develop in a self-contained way all the tools from paradifferential and microlocal analysis that they use in [67], setting a number of analytic and probabilistic objects. In [69] the authors argue that the spectrally cut-off Gaussian free field Φ_Λ on a compact riemannian manifold or on $\mathbb{R}^n$ cannot satisfy the spatial Markov property.

Finally we want to summarize the article of Ajay Chandra and Léonard Ferdinand [70]. These authors present two different arguments using stochastic analysis to construct super-renormalizable tensor field theories, namely the T_3^4 and T_4^4 models. The first approach is the construction of a Langevin dynamic [61, 63] combined with a PDE energy estimate while the second is an application of the variational approach of Barashkov and Gubinelli [62]. By leveraging the melonic structure of divergences, regularising properties of non-local products, and controlling certain random operators, they demonstrate that for tensor field theories these arguments can be significantly simplified in comparison to what is required for Φ_3^4 model.

In their most recent article [71] Ajay Chandra and Léonard Ferdinand show that the flow approach of Duch [Duc21] can be adapted to prove local well-posedness for the generalised KPZ equation. The key step is to extend the flow approach so that it can accommodate semi-linear equations involving smooth functions of the solution instead of only polynomials - this is accomplished by introducing coordinates for the flow built out of the elementary differentials associated to the equation.

6 Conclusion

This review is a modest step in the direction of bring closer the different people working on random tensors and stochastic analysis and it suggests research in many directions, among which:

- our next goal is T_5^4 which is just renormalizable and asymptotically free. We are reasonably confident that it can be solvable soon among the strategy and the tactics defined by [71, 72],
- for the future one should further develop the model of Razvan Gurau and collaborators [26] in the direction of constructive field theory. This promising model lies in the class of tensor field theory with imaginary tetrahedral coupling, and, what is more important for the physics, it is asymptotically free and in dimension four,
- in the more distant future, the common goal of all the people working on field theory with stochastic analysis (including Martin Hairer himself [73]) is to tackle *gauge theories* in the spirit of Parisi-Wu [74].

7 Appendix: Besov spaces

In mathematics, the Besov space $B^s_{p,q}(\mathbf{R})$ is a complete quasinormed space which is a Banach space when $1 \leq p, q \leq \infty$. These spaces serve to generalize more elementary function spaces such as Sobolev spaces and are effective at measuring regularity properties of functions.

Let $\Delta_h f(x) = f(x-h) - f(x)$ and define the modulus of continuity by

$$\omega_p^2(f,t) = \sup_{|h| \leq t} \left\| \Delta_h^2 f \right\|_p . \tag{16}$$

Let n be a non-negative integer and define: $s = n + \alpha$ with $0 < \alpha \leq 1$. The Besov space $B^s_{p,q}(\mathbf{R})$ contains all functions f such that

$$f \in W^{n,p}(\mathbf{R}), \qquad \int_0^\infty \left| \frac{\omega_p^2\left(f^{(n)}, t\right)}{t^\alpha} \right|^q \frac{dt}{t} < \infty. \tag{17}$$

The Besov space $B^s_{p,q}(\mathbf{R})$ is equipped with the norm

$$\|f\|_{B^s_{p,q}(\mathbf{R})} = \left(\|f\|^q_{W^{n,p}(\mathbf{R})} + \int_0^\infty \left| \frac{\omega_p^2\left(f^{(n)}, t\right)}{t^\alpha} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} . \tag{18}$$

The Besov spaces $B^s_{2,2}(\mathbf{R})$ coincide with the more classical Sobolev spaces $H^s(\mathbf{R})$.

Next, on a variety such as $B^s_{p,q}(\mathbb{T}^d)$, the authors of [67] want to define a number of operators on functions spaces by using local charts. For that task they import some known regularity properties of the corresponding objects from the flat to the curved setting. They denote as usual by $B^\gamma_{p,q}(\mathbb{T}^d)$ the Besov spaces over $\mathbb{T}^d$ and by $C^\gamma(\mathbb{T}^d)$ the Besov-Hölder space $B^\gamma_{\infty,\infty}(\mathbb{T}^d)$, with associated norm denoted by $\|\cdot\|_{C^\gamma}$. And so they define and use of the Besov spaces in their analysis.

References

[1] Rivasseau, V. (2011). Quantum Gravity and Renormalization: The Tensor Track. AIP Conf. Proc. 1444, 18.

[2] Rivasseau, V. (2013). The tensor track: an update. In Symmetries and Groups in Contemporary Physics (pp. 63-74).

[3] Rivasseau, V. (2014). The tensor track III. Fortschritte der Physik, 62(2), 81-107.

[4] Rivasseau, V. (2016). The tensor track IV. PoS Corfou 2015, 106, arXiv preprint arXiv:1604.07860.

[5] Delporte, N., & Rivasseau, V. (2018). The tensor track V: holographic tensors. arXiv preprint arXiv:1804.11101.

[6] Delporte, N., & Rivasseau, V. (2020). The tensor track VI: Field theory on random trees and SYK on random unicyclic graphs. arXiv preprint arXiv:2004.13744.

[7] Ouerfelli, M., Rivasseau, V., & Tamaazousti, M. (2022). The tensor track VII: From quantum gravity to artificial intelligence. arXiv preprint arXiv:2205.10326.

[8] de Boer, J., Dittrich, B., Eichhorn, A., Giddings, S. B., Gielen, S., Liberati, S., Livine, E. Daniele Oriti, D., Papadodimasi, K., Pereira, A., Sakellariadou, M., Suryal, S., & Verlinde, H. (2022). Frontiers of Quantum Gravity: shared challenges, converging directions. arXiv preprint arXiv:2207.10618.

[9] Bonzom, V., Gurău, R., & Rivasseau, V. (2012). Random tensor models in the large N limit: Uncoloring the colored tensor models. Physical Review D, 85(8), 084037.

[10] Bonzom, V., Gurău, R., Riello, A., & Rivasseau, V. (2011). Critical behavior of colored tensor models in the large N limit. Nuclear Physics B, 853(1), 174-195.

[11] Gurău, R., & Ryan, J. P. (2014, November). Melons are branched polymers. In Annales Henri Poincare (Vol. 15, pp. 2085-2131). Springer Basel.

[12] Dartois, S., Gurău, R., & Rivasseau, V. (2013). Double scaling in tensor models with a quartic interaction. Journal of High Energy Physics, 2013(9), 1-33.

[13] Gurău, R. G., & Schaeffer, G. (2016). Regular colored graphs of positive degree. Annales de l'Institut Henri Poincaré D, 3(3), 257-320.

[14] Kitaev, A. (2015). A simple model of quantum holography (part 2). Entanglement in Strongly-Correlated Quantum Matter, 38.

[15] Maldacena, J., & Stanford, D. (2016). Remarks on the sachdev-ye-kitaev model. Physical Review D, 94(10), 106002.

[16] Maldacena, J., Shenker, S. H., & Stanford, D. (2016). A bound on chaos. Journal of High Energy Physics, 2016(8), 1-17.

[17] Witten, E. (2019). An SYK-like model without disorder. Journal of Physics A: Mathematical and Theoretical, 52(47), 474002. [arXiv:1610.09758 [hep-th]].

[18] I. Klebanov and G. Tarnopolsky, "Uncolored Random Tensors, Melon Diagrams, and the SYK Models," Phys. Rev. **D95** (2017) 046004.

[19] S. Carrozza and A. Tanasa, "$O(N)$ Random Tensor Models," Lett. Math. Phys. **106**, no. 11, 1531 (2016).

[20] Delporte, N., & Rivasseau, V. (2021). Perturbative quantum field theory on random trees. Communications in Mathematical Physics, 381, 857-887.

[21] Richard, E., & Montanari, A. (2014). A statistical model for tensor PCA. Advances in neural information processing systems, 27.

[22] Ouerfelli, M., Tamaazousti, M., & Rivasseau, V. (2022, June). Random tensor theory for tensor decomposition. In Proceedings of the AAAI Conference on Artificial Intelligence (Vol. 36, No. 7, pp. 7913-7921).

[23] Ouerfelli, M., Tamaazousti, M., & Rivasseau, V. (2023). Selective multiple power iteration: from tensor PCA to gradient-based exploration of landscapes. The European Physical Journal Special Topics, 1-16.

[24] Benedetti, D., Gurău, R., Harribey, S., & Lettera, D. (2022). The F-theorem in the melonic limit. Journal of High Energy Physics, 2022(2), 1-58.

[25] Benedetti, D., Gurău, R., Keppler, H., & Lettera, D. (2022). The small-N series in the zero-dimensional $O(N)$ model: constructive expansions and transseries. arXiv preprint arXiv:2210.14776.

[26] Berges, J., Gurău, R., & Preis, T. (2023). Asymptotic freedom in a strongly interacting scalar quantum field theory in four Euclidean dimensions. Physical Review D, 108(1), 016019.

[27] Rivasseau, V. (2007). Constructive matrix theory. Journal of High Energy Physics, 2007(09), 008.

[28] Gurău, R., & Rivasseau, V. (2024). Quantum Gravity and Random Tensors. arXiv preprint arXiv:2401.13510.

[29] Geloun, J. B., & Ramgoolam, S. (2020). Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients. arXiv preprint arXiv:2010.04054.

[30] Geloun, J. B., & Ramgoolam, S. (2023). The quantum detection of projectors in finite-dimensional algebras and holography. Journal of High Energy Physics, 2023(5), 1-38.

[31] Geloun, J. B., & Toriumi, R. (2023). Beta-functions of enhanced quartic tensor field theories. arXiv preprint arXiv:2303.09829.

[32] Geloun, J. B., Pithis, A. G., & Thürigen, J. (2023). QFT with Tensorial and Local Degrees of Freedom: Phase Structure from Functional Renormalization. arXiv preprint arXiv:2305.06136.

[33] Andriantsiory, D. F., Geloun, J. B., & Lebbah, M. (2021, December). Multi-slice clustering for 3-order tensor. In 2021 20th IEEE International Conference on Machine Learning and Applications (ICMLA) (pp. 173-178). IEEE.

[34] Andriantsiory, D. F., Geloun, J. B., & Lebbah, M. (2023). Multiway clustering of 3-order tensor via affinity matrix. arXiv preprint arXiv:2303.07757.

[35] Andriantsiory, D. F., Geloun, J. B., & Lebbah, M. (2023). DB-SCAN of Multi-Slice Clustering for three-order Tensor. arXiv preprint arXiv:2303.07768.

[36] Tanasa, A. (2021). Combinatorial Physics: Combinatorics, Quantum Field Theory, and Quantum Gravity Models. Oxford University Press.

[37] Bonzom, V., Nador, V., & Tanasa, A. (2022). Double scaling limit for the O(N) 3-invariant tensor model. Journal of Physics A: Mathematical and Theoretical, 55(13), 135201.

[38] Krajewski, T., Muller, T., & Tanasa, A. (2023). Double scaling limit of the prismatic tensor model. Journal of Physics A: Mathematical and Theoretical, 56(23), 235401.

[39] Bonzom, V., Nador, V., & Tanasa, A. (2023). Double scaling limit of multi-matrix models at large D. Journal of Physics A: Mathematical and Theoretical, 56(7), 075201.

[40] Keppler, H., Krajewski, T., Muller, T., & Tanasa, A. (2023). Duality of O(N) and Sp(N) random tensor models: tensors with symmetries. arXiv preprint arXiv:2307.01527.

[41] Harribey, S., Klebanov, I. R., & Sun, Z. (2023). Boundaries and Interfaces with Localized Cubic Interactions in the $O(N)$ Model. arXiv preprint arXiv:2307.00072.

[42] Benedetti, D., Gurau, R., Harribey, S., & Lettera, D. (2023). Finite-size versus finite-temperature effects in the critical long-range $O(N)$ model. arXiv preprint arXiv:2311.04607.

[43] Benedetti, D. (2023). The Melonic Large-N Limit in Quantum Field Theory (Doctoral dissertation, Institut Polytechnique de Paris).

[44] Benedetti, D., Carrozza, S., Toriumi, R., & Valette, G. (2022). Multiple scaling limits of $\mathrm{U}(N)^2 \times \mathrm{O}(D)$ multi-matrix models. Annales de l'Institut Henri Poincaré D, 9(2), 367-433.

[45] Carrozza, S., & Harribey, S. (2022). Melonic large N limit of 5-index irreducible random tensors. Communications in Mathematical Physics, 390(3), 1219-1270.

[46] Carrozza, S., & Höhn, P. A. (2022). Edge modes as reference frames and boundary actions from post-selection. Journal of High Energy Physics, 2022(2), 1-94.

[47] Collins, B., Gurău, R. G., & Lionni, L. (2023). The tensor Harish-Chandra–Itzykson–Zuber integral I: Weingarten calculus and a generalization of monotone Hurwitz numbers. Journal of the European Mathematical Society.

[48] Collins, B., Gurău, R., & Lionni, L. (2023). The tensor Harish-Chandra–Itzykson–Zuber integral II: detecting entanglement in large quantum systems. Communications in Mathematical Physics, 1-48.

[49] Budd, T., & Lionni, L. (2022). A family of triangulated 3-spheres constructed from trees. arXiv preprint arXiv:2203.16105.

[50] Ferdinand, L., Gurău, R., Perez-Sanchez, C. I., & Vignes-Tourneret, F. (2023, July). Borel summability of the 1/N expansion in quartic O(N)-vector models. In Annales Henri Poincaré (pp. 1-28). Cham: Springer International Publishing.

[51] Dartois, S., Male, C., & Nechita, I. (2023). The $\mathfrak{S}_k$-circular limit of random tensor flattenings. arXiv preprint arXiv:2307.11439.

[52] Delporte, N. (2020). Tensor Field Theories: Renormalization and Random Geometry. arXiv preprint arXiv:2010.07819.

[53] Bonzom, V. (2022). Some structural and enumerative aspects of discrete surfaces and PL-manifolds. arXiv preprint arXiv:2212.12200.

[54] Benedetti, D., & Delporte, N. (2021). Remarks on a melonic field theory with cubic interaction. Journal of High Energy Physics, 2021(4), 1-30.

[55] Martini, R., & Toriumi, R. (2023). Trisections in colored tensor models. Annales de l'Institut Henri Poincaré D.

[56] Abranches, J. L., Pereira, A. D., & Toriumi, R. (2023). Dually weighted multi-matrix models as a path to causal gravity-matter systems. arXiv preprint arXiv:2310.13503.

[57] Benedetti, D., & Henson, J. (2009). Imposing causality on a matrix model. Physics Letters B, 678(2), 222-226.

[58] de Lacroix, C., Erbin, H., & Lahoche, V. (2024). Gravitational action for a massive Majorana fermion in 2d quantum gravity. Journal of High Energy Physics, 2024(1), 1-40.

[59] Wahabou Kpera, B., Lahoche, V., & Ousmane Samary, D. (2023). Stochastic melonic kinetics with random initial conditions. arXiv e-prints, arXiv-2302.

[60] Kpera, B. W., Lahoche, V., Samary, D. O., & Yerima, S. F. Z. (2023). Anomalous higher order Ward identities in tensorial group field theories without closure constraint. arXiv preprint arXiv:2307.12446.

[61] Hairer, M. (2014). A theory of regularity structures. Inventiones mathematicae, 198(2), 269-504.

[62] Barashkov, N., & Gubinelli, M. (2020). A variational method for Φ_3^4. arXiv preprint arXiv:1805.10814.

[63] Gubinelli, M., & Hofmanová, M. (2021). A PDE construction of the Euclidean Φ_3^4 quantum field theory. Communications in Mathematical Physics, 384(1), 1-75

[64] Bruned, Y., Hairer, M., & Zambotti, L. (2019). Algebraic renormalisation of regularity structures. Inventiones mathematicae, 215, 1039-1156.

[65] Chandra, A., & Hairer, M. (2016). An analytic BPHZ theorem for regularity structures. arXiv preprint arXiv:1612.08138.

[66] Bruned, Y., Chandra, A., Chevyrev, I., & Hairer, M. (2020). Renormalising SPDEs in regularity structures. Journal of the European Mathematical Society, 23(3), 869-947.

[67] Bailleul, I., Dang, N.V., Ferdinand L., & Tô T.D., Φ_3^4 measures on compact Riemannian 3-manifolds. arXiv preprint arXiv:2304.10185.

[68] Bailleul, I., Dang, N.V., Ferdinand L., & Tô T.D., Global harmonic analysis for Φ_3^4 on closed Riemannian manifolds. arXiv preprint arXiv:2306.07757.

[69] Bailleul, I., Dang, N. V., Ferdinand, L., Leclerc, G., & Lin, J. (2023). Spectrally cut-off GFF, regularized Φ^4 measure, and reflection positivity. arXiv preprint arXiv:2312.15511.

[70] Chandra, A., & Ferdinand, L. (2023). A Stochastic Analysis Approach to Tensor Field Theories. arXiv preprint arXiv:2306.05305.

[71] Chandra, A., & Ferdinand, L. (2024). A flow approach to the generalized KPZ equation. arXiv preprint arXiv:2402.03101.

[72] Rivasseau, V., & Vignes-Tourneret, F. (2021). Can we make sense out of" Tensor Field Theory"?. SciPost Physics Core, 4(4), 029.

[73] Chandra, A., Chevyrev, I., Hairer, M., & Shen, H. (2022). Stochastic quantisation of Yang-Mills-Higgs in 3D. arXiv preprint arXiv:2201.03487.

[74] Parisi, G., & Wu, Y. S. (1981). Perturbation theory without gauge fixing. Sci. sin, 24(4), 483-496.