

HOMOCLINIC CLASSES OF GEODESIC FLOWS ON RANK 1 MANIFOLDS

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ABSTRACT. We prove that the homoclinic class of every hyperbolic periodic orbit of a geodesic flow over a C^∞ closed rank 1 Riemannian manifold equals the unit tangent bundle. As an application, we give a proof using symbolic dynamics of the theorem of Knieper on the uniqueness of the measure of maximal entropy [Kni98] and theorems of Burns et al on the uniqueness of equilibrium states [BCFT18].

1. INTRODUCTION

Homoclinic classes are an efficient way of decomposing the dynamics of hyperbolic systems into smaller pieces. Introduced by Newhouse in [New72] as generalizations of the basic sets introduced by Smale in his spectral decomposition theorem [Sma67], homoclinic classes have been used in more complicated settings beyond uniform hyperbolicity, see e.g. [BC04].

Rodriguez Hertz et al introduced ergodic homoclinic classes of hyperbolic periodic points, and studied the uniqueness of SRB measures for surface diffeomorphisms [RHRHTU11]. Buzzi, Crovisier and Sarig introduced homoclinic classes of hyperbolic measures and proved that each class carries at most one measure of maximal entropy [BCS22]. Later, Buzzi, Crovisier and Lima studied homoclinic classes of measures for three-dimensional flows with positive speed [BCL23].

In this article, we consider homoclinic classes for geodesic flows over rank 1 manifolds, in any dimension. Let M be a C^∞ closed Riemannian manifold with nonpositive sectional curvature, and let $\varphi = \{\varphi^t\}_{t \in \mathbb{R}} : T^1M \rightarrow T^1M$ be its geodesic flow. We say that M is *rank 1* if there is a vector $x \in T^1M$ without a parallel Jacobi field perpendicular to the flow direction. This assumption implies various geometrical and dynamical properties of φ , see Section 2.1. Let $\text{HC}(\mathcal{O})$ denote the homoclinic class of the periodic orbit \mathcal{O} , see Section 2.2 for the definition.

Main Theorem. *Let φ be the geodesic flow over a C^∞ closed rank 1 Riemannian manifold. If \mathcal{O} is a hyperbolic periodic orbit, then $\text{HC}(\mathcal{O}) = T^1M$.*

Returning to the case of diffeomorphisms, Buzzi, Crovisier and Sarig proved that the constructions of countable Markov partitions of Sarig [Sar13] and Ben Ovadia [BO18] code homoclinic classes by *irreducible* countable topological Markov shifts,

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and that each homoclinic class supports at most one equilibrium state for each admissible potential, see Theorem 3.1 and Corollary 3.3 in [BCS22]. Employing the same methods, we provide applications of the Main Theorem to give proofs, for rank 1 manifolds, of the theorem of Knieper on the uniqueness of the measure of maximal entropy [Kni98] and theorems of Burns et al on the uniqueness of equilibrium states for some classes of potentials [BCFT18]. Let ψ^u denote the geometric potential of φ and let Sing be the *singular set* of φ , see Section 2.1 for the definitions.

Theorem 1.1 (Theorem A of [BCFT18]). *Let M be a C^∞ closed rank 1 Riemannian manifold, and let $\psi : T^1M \rightarrow \mathbb{R}$ be Hölder continuous or of the form $\psi = q\psi^u$ for $q \in \mathbb{R}$. If $P(\text{Sing}, \psi) < P(\psi)$, then ψ has a unique equilibrium state μ . This measure is hyperbolic and fully supported.*

Above, $P(\psi)$ and $P(\text{Sing}, \psi)$ denote the topological pressure and topological pressure restricted to Sing respectively. We note that Theorem 1.1 is not the full statement proved in [BCFT18], since we do not characterize μ as the weak- $*$ limit of hyperbolic periodic orbits. We also remark that μ is Bernoulli, due to [LLS16] in dimension two and to [CT22, ALP20] in any dimension.

For multiples of the geometric potential in surfaces, we also recover part of Theorem C of [BCFT18].

Theorem 1.2 (Theorem C of [BCFT18]). *If M is a C^∞ closed rank 1 surface, then its geodesic flow φ has a unique equilibrium state μ_q for the potential $q\psi^u$ for each $q \in (-\infty, 1)$. This measure is hyperbolic and fully supported.*

Again, we are not able to characterize μ as the weak- $*$ limit of hyperbolic periodic orbits. Finally, using the pressure gap $P(\text{Sing}, 0) < P(0)$ proved in [BCFT18, Theorem B], we are also able to recover Knieper's theorem on the uniqueness of the measure of maximal entropy.

Theorem 1.3 ([Kni98]). *If M is a C^∞ closed rank 1 Riemannian manifold, then its geodesic flow φ has a unique measure of maximal entropy. This measure is hyperbolic and fully supported.*

Let us stress that, in dimension two, the pressure gap is automatic, hence we provide a self-contained proof of Knieper's theorem in dimension two, using homoclinic classes and symbolic dynamics.

2. PRELIMINARIES

2.1. Geodesic flows in nonpositive curvature. Let M be a C^∞ closed Riemannian manifold and T^1M be its unit tangent bundle, which is also a closed Riemannian manifold. Let $\varphi = \{\varphi^t\}_{t \in \mathbb{R}} : T^1M \rightarrow T^1M$ be the geodesic flow on M , and X be the vector field generating this flow. We assume that M is a *rank 1* manifold: it has nonpositive sectional curvature and there is a vector without parallel Jacobi fields perpendicular to the flow direction.

REGULAR AND SINGULAR SETS: The *regular set* of φ is defined by

$$\text{Reg} = \{x \in T^1M : \exists t \in \mathbb{R} \text{ s.t. } K(\pi) < 0 \text{ for every plane } \pi \subset T_{\gamma_x(t)}M\},$$

where γ_x is the geodesic starting at x . The *singular set* of φ is defined by

$$\text{Sing} = T^1M \setminus \text{Reg} = \{x \in T^1M : \forall t \in \mathbb{R}, \exists \pi \subset T_{\gamma_x(t)}M \text{ plane s.t. } K(\pi) = 0\}.$$

Clearly, Reg and Sing form a partition of T^1M , with Reg open and Sing closed. Geodesic flows on rank 1 manifolds have (weak) invariant directions and manifolds at every point. More specifically, there are continuous $d\varphi$ -invariant bundles $x \in T^1M \rightarrow E_x^{s/u}$ and continuous foliations $\{\widehat{W}^{ss/uu}(x) : x \in T^1M\}$ tangent to $E^{s/u}$ and invariant under φ such that:

- (1) E^s, E^u are orthogonal to X and $\dim(E^s) = \dim(E^u) = \dim(M) - 1$.
- (2) $x \in \text{Reg}$ if and only if $E_x^s \oplus \langle X_x \rangle \oplus E_x^u = T_x(T^1M)$.
- (3) Reg is dense in T^1M .
- (4) $\widehat{W}^{s/u}(x) := \bigcup_{t \in \mathbb{R}} \widehat{W}^{ss/uu}(\varphi^t(x))$ is a φ -invariant connected manifold of dimension $\dim(M)$ and tangent to $E^{s/u} \oplus \langle X \rangle$.
- (5) There is a universal constant $C = C(M)$ such that if d_x is the induced distance on $\widehat{W}^s(x)$ then $d_x(\varphi^t(x), \varphi^t(y)) \leq C d_x(x, y)$ for all $y \in \widehat{W}^s(x)$ and all $t \geq 0$; a similar statement holds for $\widehat{W}^u(x)$ and $t \leq 0$.
- (6) $\widehat{W}^{s/u}(x)$ is dense in T^1M for every $x \in T^1M$.

See [Ebe01] for proofs of (1)–(5), and [Bal82] for a proof of (6). For an orbit \mathcal{O} , define $\widehat{W}^{s/u}(\mathcal{O}) := \widehat{W}^{s/u}(x)$ for any $x \in \mathcal{O}$. We also recall the definition of the geometric potential.

THE GEOMETRIC POTENTIAL OF φ [BR75]: The *geometric potential* of φ is the function $\psi^u : T^1M \rightarrow \mathbb{R}$ defined as

$$\psi^u(x) = - \left. \frac{d}{dt} \right|_{t=0} \log \det(d\varphi_x^t|_{E_x^u}) = - \lim_{t \rightarrow 0} \frac{1}{t} \log \det(d\varphi_x^t|_{E_x^u}).$$

2.2. Homoclinic classes and symbolic dynamics. In this section, we combine results of [LS19, ALP20, BCS22, BCL23] on codings of hyperbolic measures. Let $\mathcal{G} = (V, E)$ be an oriented graph, where V, E are the vertex and edge sets. We denote edges by $v \rightarrow w$, and assume that V is countable.

TOPOLOGICAL MARKOV SHIFT (TMS): It is a pair (Σ, σ) where

$$\Sigma := \{\mathbb{Z}\text{-indexed paths on } \mathcal{G}\} = \{\underline{v} = \{v_n\}_{n \in \mathbb{Z}} \in V^{\mathbb{Z}} : v_n \rightarrow v_{n+1}, \forall n \in \mathbb{Z}\}$$

is the symbolic space and $\sigma : \Sigma \rightarrow \Sigma$, $[\sigma(\underline{v})]_n = v_{n+1}$, is the *left shift*. We endow Σ with the distance $d(\underline{v}, \underline{w}) := \exp[-\inf\{|n| \in \mathbb{Z} : v_n \neq w_n\}]$. The *regular set* of Σ is

$$\Sigma^\# := \left\{ \underline{v} \in \Sigma : \exists v, w \in V \text{ s.t. } \begin{array}{l} v_n = v \text{ for infinitely many } n > 0 \\ v_n = w \text{ for infinitely many } n < 0 \end{array} \right\}.$$

We will sometimes omit σ from the definition, referring to Σ as a TMS. We only consider TMS that are *locally compact*, i.e. for all $v \in V$ the number of ingoing edges $u \rightarrow v$ and outgoing edges $v \rightarrow w$ is finite. Although we use the same notation, the regular set of φ and the regular set of Σ are not related; we maintain the notation because they are classical in their contexts.

Given (Σ, σ) a TMS, let $r : \Sigma \rightarrow (0, +\infty)$ be a continuous function. For $n \geq 0$, let $r_n = r + r \circ \sigma + \dots + r \circ \sigma^{n-1}$ be the n -th *Birkhoff sum* of r , and extend this definition for $n < 0$ in the unique way such that the *cocycle identity* holds: $r_{m+n} = r_m + r_n \circ \sigma^m$, $\forall m, n \in \mathbb{Z}$.

TOPOLOGICAL MARKOV FLOW (TMF): The TMF defined by (Σ, σ) and *roof function* r is the pair (Σ_r, σ_r) where $\Sigma_r := \{(\underline{v}, t) : \underline{v} \in \Sigma, 0 \leq t < r(\underline{v})\}$ and $\sigma_r : \Sigma_r \rightarrow \Sigma_r$ is the flow on Σ_r given by $\sigma_r^t(\underline{v}, t') = (\sigma^n(\underline{v}), t' + t - r_n(\underline{v}))$, where n is the unique integer such that $r_n(\underline{v}) \leq t' + t < r_{n+1}(\underline{v})$. We endow Σ_r with a natural

metric $d_r(\cdot, \cdot)$, called the *Bowen-Walters metric*, such that σ_r is a continuous flow [BW72]. The *regular set* of (Σ_r, σ_r) is $\Sigma_r^\# = \{(\underline{v}, t) \in \Sigma_r : \underline{v} \in \Sigma^\#\}$.

Similarly, we will sometimes omit σ_r and refer to Σ_r as a TMF. The roof functions we consider will always be Hölder continuous, in which case $\exists \kappa, C > 0$ such that $d_r(\sigma_r^t(z), \sigma_r^t(z')) \leq C d_r(z, z')^\kappa$ for all $|t| \leq 1$ and $z, z' \in \Sigma_r$, see [LS19, Lemma 5.8].

IRREDUCIBLE COMPONENT: If Σ is a TMS defined by an oriented graph $\mathcal{G} = (V, E)$, its *irreducible components* are the subshifts $\Sigma' \subset \Sigma$ defined over maximal subsets $V' \subset V$ satisfying the following condition:

$$\forall v, w \in V', \exists \underline{v} \in \Sigma \text{ and } n \geq 1 \text{ such that } v_0 = v \text{ and } v_n = w.$$

An *irreducible component* Σ'_r of Σ_r is a set of the form $\Sigma'_r = \{(\underline{v}, t) \in \Sigma_r : \underline{v} \in \Sigma'\}$ where Σ' is an irreducible component of Σ .

χ -HYPERBOLIC MEASURE: A φ -invariant probability measure μ on T^1M is χ -*hyperbolic* if for μ -a.e. $x \in T^1M$ all the Lyapunov exponents are greater than χ in absolute value, except for the zero exponent in the flow direction.

Let μ be a hyperbolic φ -invariant probability measure. For μ -a.e. $x \in T^1M$,

$$W^{ss}(x) = \left\{ y \in T^1M : \limsup_{t \rightarrow +\infty} \frac{1}{t} \log d(\varphi^t(x), \varphi^t(y)) < 0 \right\}$$

denotes the *strong stable manifold* of x and

$$W^s(x) = \bigcup_{t \in \mathbb{R}} \varphi^t[W^{ss}(x)]$$

denotes the *stable manifold* of x . We define similarly $W^{uu}(x)$ and $W^u(x)$ the *strong unstable* and *unstable* manifolds of x .

Given a hyperbolic periodic orbit \mathcal{O} , we let $W^{s/u}(\mathcal{O}) = W^{s/u}(x)$ denote the stable/unstable manifold of \mathcal{O} , for any $x \in \mathcal{O}$.

HOMOCLINIC CLASS OF HYPERBOLIC PERIODIC ORBIT: The *homoclinic class* of a hyperbolic periodic orbit \mathcal{O} is the set

$$\text{HC}(\mathcal{O}) = \overline{W^u(\mathcal{O}) \cap W^s(\mathcal{O})}.$$

To simplify the notation, we will sometimes write $N := T^1M$.

HOMOCLINIC RELATION OF MEASURES [BCS22, BCL23]: We say that two ergodic hyperbolic measures μ, ν are *homoclinically related* if for μ -a.e. x and ν -a.e. y there exist transverse intersections $W^s(x) \cap W^u(y) \neq \emptyset$ and $W^u(x) \cap W^s(y) \neq \emptyset$, i.e. points $z_1 \in W^s(x) \cap W^u(y)$ and $z_2 \in W^u(x) \cap W^s(y)$ such that $T_{z_1}N = T_{z_1}W^s(x) + T_{z_1}W^u(y)$ and $T_{z_2}N = T_{z_2}W^u(x) + T_{z_2}W^s(y)$.

The homoclinic relation is an equivalence relation among ergodic hyperbolic measures, see [BCL23, Prop. 10.1]. Next, we state the coding theorem for non-uniformly hyperbolic flows that we will use to prove Theorems 1.1, 1.2, and 1.3. This theorem mixes the versions presented in [ALP20, Theorem 10.1] and [BCL23, Theorem 1.1].

Theorem 2.1. *Let X be a $C^{1+\beta}$ vector field with $X \neq 0$ everywhere, let $\varphi : N \rightarrow N$ be the flow generated by X , and let $\chi > 0$. If μ_1, μ_2 are homoclinically related χ -hyperbolic probability measures, then there is an irreducible topological Markov flow (Σ_r, σ_r) and a Hölder continuous map $\pi_r : \Sigma_r \rightarrow N$ s.t.:*

- (1) $r : \Sigma \rightarrow \mathbb{R}^+$ is Hölder continuous and bounded away from zero and infinity.
- (2) $\pi_r \circ \sigma_r^t = \varphi^t \circ \pi_r$ for all $t \in \mathbb{R}$.
- (3) $\pi_r[\Sigma_r^\#]$ has full measure with respect to μ_1 and μ_2 .
- (4) Every $x \in N$ has finitely many pre-images in $\Sigma_r^\#$.

Proof. If M has dimension two, then we can apply [BCL23, Theorem 1.1] directly. Since this latter theorem is not available in higher dimension, we have to argue differently. Let $\mathcal{O}_1, \mathcal{O}_2, \dots$ be the χ -hyperbolic periodic orbits homoclinically related to μ_1 (and hence to μ_2).

CLAIM: There is a global Poincaré section Λ such that if $f : \Lambda \rightarrow \Lambda$ is the Poincaré return map and \mathcal{S} is the *singular set* of f , then:

- (1) Λ is *adapted* for μ_1 and μ_2 , i.e.

$$\lim_{|t| \rightarrow \infty} \frac{1}{t} \log d(\varphi^t(x), \mathcal{S}) = 0 \quad (2.1)$$

for μ_i -a.e. $x \in \Lambda$, $i = 1, 2$.¹

- (2) For every n , there exists a compact, φ -invariant, transitive, locally maximal, χ -hyperbolic set K_n that contains $\mathcal{O}_1, \dots, \mathcal{O}_n$ and such that $K_n \cap \mathcal{S} = \emptyset$.

The singular set is $\mathcal{S} = \{x \in \Lambda : \{x, f(x), f^{-1}(x)\} \cap \partial\Lambda \neq \emptyset\}$.

Proof of the claim. Fix $x_i \in \mathcal{O}_i$ and $z_{ij} \in W^u(x_i) \pitchfork W^s(x_j)$, and let $\tau_{ij} \in \mathbb{R}$ such that $z_{ij} \in W^{uu}(x_i) \cap W^{ss}(\varphi^{\tau_{ij}}(x_j))$. We construct a one-parameter family of global Poincaré sections Λ_r , where r varies in an interval $[a, b]$ and each Λ_r is the union of finitely many codimension 1 disjoint balls $D_1(r), \dots, D_k(r) \subset N$ centered at points $y_1, \dots, y_k \in N$, each of them with radius r and almost orthogonal to X . The details of the construction can be found in [LS19, Section 2] for three-dimensional flows and in [ALP20, Section 10] for any dimension. We can assume that the roof functions of the Poincaré return maps to Λ_r are all larger than some $\varepsilon_0 > 0$. As the radius r varies, the boundary of Λ_r varies as well. Applying a double counting argument and the Borel-Cantelli lemma, for Lebesgue almost every choice $r \in [a, b]$ the section Λ_r satisfies (1) above, see [LS19, Theorem 2.8] for details. Hence we focus on showing that (2) holds in a set of parameters $r \in [a, b]$ of positive measure. We will prove this using parameter selection. Observe that if K is φ -invariant, then $K \cap \mathcal{S} = \emptyset$ if and only if $K \cap \partial\Lambda_r = \emptyset$.

Write $z_{ii} = x_i$, and let \mathcal{O}_{ij} denote the orbit of z_{ij} . Consider the countable union of orbits $\bigcup_{i,j} \mathcal{O}_{ij}$. Clearly, the set of parameters $r \in [a, b]$ such that $(\bigcup_{i,j} \mathcal{O}_{ij}) \cap \partial\Lambda_r \neq \emptyset$ is countable. Below, we consider r in the complement of this set. We will use the following notation: given two functions $g(\delta), h(\delta)$, write $g = O_n(h)$ when there is a constant $C > 0$ that depends on n and $\delta_0 > 0$ such that $|g(\delta)| \leq C|h(\delta)|$ for all $|\delta| < \delta_0$.

Fix n and $\delta > 0$. In the sequel, we construct a δ -neighborhood of $\bigcup_{1 \leq i, j \leq n} \mathcal{O}_{ij}$. Let $C = C(n) > 0$ such that

$$\begin{aligned} d(\varphi^{-t}(x_i), \varphi^{-t}(z_{ij})) &\leq Ce^{-\chi t} \\ d(\varphi^{t+\tau_{ij}}(x_j), \varphi^t(z_{ij})) &\leq Ce^{-\chi t} \end{aligned} \quad , \quad \forall 1 \leq i, j \leq n, \forall t \geq 0.$$

¹This is equivalent to the projection of μ_i to Λ being adapted in the notation of [LS19].

Let $t_{ij} = O_n(|\log \delta|)$ positive such that $\varphi^{-t_{ij}}(z_{ij}) \in B_\delta(\mathcal{O}_i)$ and $\varphi^{t_{ij}}(z_{ij}) \in B_\delta(\mathcal{O}_j)$. The set $Z_n = \bigcup_i \mathcal{O}_i \cup \bigcup \{\varphi^t(z_{ij}) : 1 \leq i, j \leq n, |t| \leq t_{ij}\}$ is the union of finitely many pieces of orbits and $\bigcup_{1 \leq i, j \leq n} \mathcal{O}_{ij} \subset B_\delta(Z_n)$. Arguing as in [BCS22, Lemma 3.11], we can construct inside $B_\delta(Z_n)$ a compact, φ -invariant, transitive, locally maximal, χ -hyperbolic set that contains $\mathcal{O}_1, \dots, \mathcal{O}_n$. Thus, we estimate the measure of the set of $r \in [a, b]$ such that $B_\delta(Z_n) \cap \partial\Lambda_r \neq \emptyset$.

Denoting by $|\mathcal{O}_i|$ the length of \mathcal{O}_i , the total length of Z_n is

$$\sum_{1 \leq i \leq n} |\mathcal{O}_i| + \sum_{1 \leq i, j \leq n} 2t_{ij} = O_n(|\log \delta|).$$

Recalling that ε_0 is a lower bound for the roof function of Λ_r , the set $Z_n \cap \Lambda_r$ has at most $\frac{1}{\varepsilon_0} O_n(|\log \delta|) = O_n(|\log \delta|)$ elements. Then the intersection $B_\delta(Z_n) \cap \Lambda_b$ is contained in $O_n(|\log \delta|)$ balls of radius 2δ . The set of parameters $\text{Bad}(n) = \{r \in [a, b] : B_\delta(Z_n) \cap \partial\Lambda_r \neq \emptyset\}$ is thus contained in the union of $O_n(|\log \delta|)$ intervals of length 4δ , and so has Lebesgue measure $O_n(\delta |\log \delta|)$. Since $\lim_{\delta \rightarrow 0} \delta \log \delta = 0$, we can take δ_n such that $\text{Leb}[\text{Bad}(n)] < \varepsilon/n^2$ for small $\varepsilon > 0$ and so the complement $[a, b] \setminus \bigcup_{n \geq 1} \text{Bad}(n)$ has positive Lebesgue measure.

In summary, we can choose $r \in [a, b]$ such that:

- Condition (2.1) is satisfied: the space of $r \in [a, b]$ satisfying it has full measure.
- $(\bigcup_{i,j} \mathcal{O}_{ij}) \cap \partial\Lambda_r = \emptyset$: the space of $r \in [a, b]$ satisfying it is the complement of a countable set, hence has full measure.
- $B_{\delta_n}(Z_n) \cap \partial\Lambda_r = \emptyset$ for all $n \geq 1$: the space of $r \in [a, b]$ satisfying it has positive measure.

This concludes the proof of the claim. \square

Once the section $\Lambda = \Lambda_r$ is chosen, apply [ALP20, Theorem 10.1] to construct a TMF (Σ_r, σ_r) and a Hölder continuous map $\pi_r : \Sigma_r \rightarrow N$ satisfying (1)–(4) in Theorem 2.1, with the exception that Σ_r might not be irreducible. Since $K_n \cap \mathcal{S} = \emptyset$ and K_n is compact and χ -hyperbolic, [ALP20] implies that $K_n \subset \pi_r[\Sigma_r]$. Since K_n is uniformly hyperbolic, we actually have $K_n \subset \pi_r[\Sigma_r^\#]$.

The final step is to find an irreducible component Σ'_r of Σ_r that lifts both μ_1 and μ_2 . For that, we proceed as in [BCL23, Lemma 10.4]:

- For each $n \geq 1$, there is an invariant, compact, transitive set $X_n \subset \Sigma_r^\#$ that lifts K_n .
- Since \mathcal{O}_1 has finitely many lifts in $\Sigma_r^\#$, the sequence X_1, X_2, \dots has a subsequence X_{n_1}, X_{n_2}, \dots containing a same lift (v, t) of \mathcal{O}_1 . Take Σ'_r to be the irreducible component of (v, t) . Then \mathcal{O}_1, \dots all lift to periodic orbits in Σ'_r .

Now proceed as in [BCL23, Theorem 1.1] to lift generic points for μ_1 and μ_2 to Σ'_r . This completes the proof of the theorem. \square

3. PROOFS

Proof of Main Theorem. Let \mathcal{O} be a hyperbolic periodic orbit.

CLAIM: $W^{s/u}(\mathcal{O}) = \widehat{W}^{s/u}(\mathcal{O})$.

Proof of the claim. We prove the claim for $W^s(\mathcal{O})$ (the other proof is analogous). We first show that $W^s(\mathcal{O}) \subset \widehat{W}^s(\mathcal{O})$. Fix $\varepsilon > 0$ small enough so that

$$W_\varepsilon^s(\mathcal{O}) = \{y \in T^1M : d(\varphi^t(y), \mathcal{O}) < \varepsilon, \forall t \geq 0\} \subset W^s(\mathcal{O}).$$

Let $\widehat{W}_\delta^s(\mathcal{O}) \subset \widehat{W}^s(\mathcal{O})$ be the δ -neighborhood of \mathcal{O} in the d_x metric, for (any) $x \in \mathcal{O}$. If $y \in \widehat{W}_{\varepsilon/C}^s(\mathcal{O})$ with $d_x(y, x) < \varepsilon/C$ where $x \in \mathcal{O}$, then by property (5) of Section 2.1 we have

$$d(\varphi^t(y), \mathcal{O}) \leq d_x(\varphi^t(y), \varphi^t(x)) \leq C d_x(y, x) < \varepsilon$$

for all $t \geq 0$, hence $\widehat{W}_{\varepsilon/C}^s(\mathcal{O}) \subset W_\varepsilon^s(\mathcal{O})$. Since $\widehat{W}_{\varepsilon/C}^s(\mathcal{O})$ and $W_\varepsilon^s(\mathcal{O})$ both have dimension $\dim(M)$, we conclude that $W_\delta^s(\mathcal{O}) \subset \widehat{W}_{\varepsilon/C}^s(\mathcal{O})$ for small $\delta > 0$. Therefore $W_\delta^s(\mathcal{O}) \subset \widehat{W}^s(\mathcal{O})$ and by the invariance of \widehat{W}^s the same occurs for $W^s(\mathcal{O}) = \bigcup_{t \geq 0} \varphi^{-t}[W_\delta^s(\mathcal{O})]$.

Now we prove that $W^s(\mathcal{O}) = \widehat{W}^s(\mathcal{O})$. Since $\widehat{W}^s(\mathcal{O})$ is connected (see e.g. [BP07, Prop. 12.2.4]), it is enough to show that $W^s(\mathcal{O})$ is open and dense in the induced topology of $\widehat{W}^s(\mathcal{O})$. Recall that $\varepsilon > 0$ is small such that $W^s(\mathcal{O}) = \bigcup_{t \geq 0} \varphi^{-t}[W_\varepsilon^s(\mathcal{O})]$, where $W_\varepsilon^s(\mathcal{O}) = \{y \in T^1M : d(\varphi^t(y), \mathcal{O}) < \varepsilon, \forall t \geq 0\}$. Since $W_\varepsilon^s(\mathcal{O})$ is open in the induced topology of $\widehat{W}^s(\mathcal{O})$, the same holds for $W^s(\mathcal{O})$. To prove that $W^s(\mathcal{O})$ is closed in the induced topology of $\widehat{W}^s(\mathcal{O})$, let $z \in \widehat{W}^s(\mathcal{O})$ and $z_k \in W^s(\mathcal{O})$ such that $d_x(z_k, z) \rightarrow 0$, for (any) $x \in \mathcal{O}$. We wish to show that $z \in W^s(\mathcal{O})$. By the triangle inequality and property (5) in Section 2.1, we have

$$\begin{aligned} d(\varphi^t z, \mathcal{O}) &\leq d(\varphi^t z, \varphi^t z_k) + d(\varphi^t z_k, \mathcal{O}) \\ &\leq d_x(\varphi^t z, \varphi^t z_k) + d(\varphi^t z_k, \mathcal{O}) \leq C d_x(z, z_k) + d(\varphi^t z_k, \mathcal{O}). \end{aligned}$$

Let k such that $d_x(z_k, z) < \varepsilon/2C$, and then let $T_0 > 0$ such that $d(\varphi^t z_k, \mathcal{O}) < \varepsilon/2$ for all $t \geq T_0$. Hence $d(\varphi^t z, \mathcal{O}) < \varepsilon$ for all $t \geq T_0$, and so $\varphi^{T_0} z \in W_\varepsilon^s(\mathcal{O})$. This proves that $z \in W^s(\mathcal{O})$, as wished. \square

By the claim and property (6) of Section 2.1, we conclude that $W^{s/u}(\mathcal{O}) = \widehat{W}^{s/u}(\mathcal{O})$ is dense in T^1M . Fix $y \in \text{Reg}$, and let $U \subset \text{Reg}$ be an open set containing y . Let $\varepsilon, \delta > 0$ small such that:

- For every $z \in B_\varepsilon(y)$, $\widehat{W}^{s/u}(z)$ has internal radius at z larger than δ .
- If N^s, N^u are submanifolds tangent to $E^s \oplus \langle X \rangle, E^u \oplus \langle X \rangle$ respectively and if there is $z_{s/u} \in N^{s/u} \cap B_\varepsilon(y)$ such that $N^{s/u}$ has internal radius at $z_{s/u}$ larger than δ , then N^s, N^u have a transverse intersection at some $z \in U$.

The parameter $\varepsilon > 0$ exists because $E^{s/u}$ varies continuously and $E_y^s \oplus \langle X_y \rangle \oplus E_y^u = T_y(T^1M)$. Since $W^{s/u}(\mathcal{O})$ is dense, then $W^s(\mathcal{O}) \cap B_\varepsilon(y) \neq \emptyset$ and $W^u(\mathcal{O}) \cap B_\varepsilon(y) \neq \emptyset$, hence $W^s(\mathcal{O}) \pitchfork W^u(\mathcal{O})$ at some point $z \in U$. This implies that $y \in \text{HC}(\mathcal{O})$. Since $y \in \text{Reg}$ is arbitrary, $\text{HC}(\mathcal{O}) \supset \text{Reg}$. Finally, since Reg is dense in T^1M , we conclude that $\text{HC}(\mathcal{O}) = T^1M$. \square

Proof of Theorem 1.1. Let ψ be Hölder continuous or of the form $\psi = q\psi^u$ with $q \in \mathbb{R}$. In particular, ψ is continuous, hence the existence of an equilibrium state is guaranteed by the C^∞ regularity of φ [New89]. For the uniqueness, let μ_1, μ_2 be two ergodic equilibrium states. Since $P(\text{Sing}, \psi) < P(\psi)$, we have $\mu_1(\text{Reg}) = \mu_2(\text{Reg}) = 1$ and so μ_1, μ_2 are hyperbolic. Since μ_1, μ_2 are ergodic, we can take $\chi > 0$ small so that μ_1, μ_2 are χ -hyperbolic. The measures μ_1 and μ_2 are homoclinically related.

Indeed, let \mathcal{O} be a hyperbolic periodic orbit and $x \in \mathcal{O}$. Let $y_i \in \text{Reg}$ be a generic point for μ_i , $i = 1, 2$. Since $W^{s/u}(\mathcal{O}) = \widehat{W}^{s/u}(\mathcal{O})$ is dense in T^1M , we have $W_{\text{loc}}^s(y_1) \cap W^u(x)$ and $W_{\text{loc}}^u(y_2) \cap W^s(x)$. By the inclination lemma, we conclude that $W^s(y_1) \cap W^u(y_2)$ near x . Interchanging the roles of y_1, y_2 , we also have $W^s(y_2) \cap W^u(y_1)$.

Applying Theorem 2.1 to μ_1 and μ_2 , we get an irreducible TMF (Σ_r, σ_r) and a Hölder continuous map $\pi_r : \Sigma_r \rightarrow \mathbb{R}$ such that $\mu_i[\pi_r(\Sigma_r^\#)] = 1$ for $i = 1, 2$. Therefore, μ_1, μ_2 lift to ergodic measures $\widehat{\mu}_1, \widehat{\mu}_2$ on Σ_r . These measures are equilibrium states of the potential $\widehat{\psi} = \psi \circ \pi_r$.

CLAIM: $\widehat{\psi}$ is Hölder continuous.

Proof of the claim. When ψ is Hölder continuous, $\widehat{\psi} = \psi \circ \pi_r$ is the composition of two Hölder maps, hence Hölder continuous.

When $\psi = q\psi^u$ for some $q \in \mathbb{R}$, we show that $(\underline{v}, t) \in \Sigma_r \mapsto E_{\pi_r(\underline{v}, t)}^u$ is Hölder continuous, which obviously implies that $\widehat{\psi}$ is Hölder continuous. It is enough to show that $\underline{v} \in \Sigma \mapsto E_{\pi_r(\underline{v}, 0)}^u$ is Hölder continuous, since by $d\varphi$ -invariance this implies that $(\underline{v}, t) \in \Sigma_r \mapsto E_{\pi_r(\underline{v}, t)}^u$ is Hölder continuous.

As in the proof of Theorem 2.1, let $f : \Lambda \rightarrow \Lambda$ be the Poincaré return map. The map $\mathcal{F} : \underline{v} \in \Sigma \rightarrow F_x^u \subset T_x\Lambda$, where F_x^u is the unstable direction for f at $x = \pi_r(\underline{v}, 0)$, is Hölder continuous. As a matter of fact, employing the notation of [ALP20], F_x^u is the tangent direction of the u -admissible graph $V^u[\underline{v}] = \lim_{n \rightarrow \infty} (\mathcal{F}_{-1} \circ \dots \circ \mathcal{F}_{-n})(V_{-n})$, where \mathcal{F}_{-n} is the unstable graph transform defined by the edge $v_n \rightarrow v_{n+1}$ and V_{-n} is any u -admissible graph at v_{-n} , see [ALP20, Section 4.5].² Therefore, if two sequences $\underline{v}, \underline{w}$ are close then $V^u[\underline{v}], V^u[\underline{w}]$ are C^1 -close. For the details, see [BO18, Proposition. 3.12(5)], whose proof applies verbatim in the context of [ALP20].

Let $\mathbf{p}_x : T_x\Lambda \rightarrow X_x^\perp$ be the orthogonal projection. Such map exists and is an isomorphism because both $X_x^\perp, T_x\Lambda$ have dimension $\dim(T^1M) - 1$ and $T_x\Lambda$ is almost orthogonal to X_x . We have $E_x^u = \mathbf{p}_x(F_x^u)$. Since $x \in T^1M \mapsto X_x^\perp$ and $x \in \Lambda \mapsto T_x\Lambda$ are C^∞ , the map $\mathcal{P} : x \in \Lambda \mapsto \mathbf{p}_x$ is C^∞ . Therefore $\underline{v} \in \Sigma \mapsto E_{\pi_r(\underline{v}, 0)}^u$, being the composition $\mathcal{P} \circ \mathcal{F}$, is Hölder continuous. \square

The measures $\widehat{\mu}_1, \widehat{\mu}_2$ project to ergodic σ -invariant probability measures $\widehat{\nu}_1, \widehat{\nu}_2$ on the irreducible component Σ which are equilibrium states of the Hölder continuous potential $\widehat{\psi}_r - P_{\text{top}}(\psi)r$ where $\widehat{\psi}_r(\underline{v}) = \int_0^{r(\underline{v})} \widehat{\psi}(\underline{v}, t) dt$, see e.g. [PP90, Proposition 6.1]. By [BS03, Theorem 1.1], we conclude that $\widehat{\nu}_1 = \widehat{\nu}_2$, and so $\mu_1 = \mu_2$.

Finally, we show that the unique equilibrium state μ is fully supported. The proof is the same of [BCS22, Corollary 3.3]. Using the same notation of the previous paragraphs, $\widehat{\nu}$ has full support in Σ by [BS03] and so $\widehat{\mu}$ has full support in Σ_r . This implies that $\text{supp}(\mu) = \overline{\pi_r(\Sigma_r)}$, so it is enough to show that $\pi_r(\Sigma_r)$ is dense in T^1M . Let \mathcal{O} be a hyperbolic periodic orbit homoclinically related to μ .³ We thus have $\text{supp}(\mu) \subset \text{HC}(\mathcal{O}) = T^1M$, hence it is enough to show that $\pi_r(\Sigma_r)$ is dense in $\{W^u(\mathcal{O}) \cap W^s(\mathcal{O})\}$. The proof of this fact in [BCS22, Corollary 3.3]

²Actually, the coding Σ is obtained from a refinement of a preliminary coding. The sequences on Σ lift to sequences on this preliminary coding, whose vertices are double Pesin charts and whose edges define stable/unstable graph transforms.

³The existence of this orbit is consequence of Katok's horseshoe theorem for flows; another way to obtain this is by the symbolic coding of [ALP20].

uses [BCS22, Proposition 3.7], which works equally well in our context. Hence we conclude that $\text{supp}(\mu) = \overline{\pi_r(\Sigma_r)} = \text{HC}(\mathcal{O}) = T^1M$. \square

Remark 3.1. When M is a surface and $\psi = q\psi^u$, [LLS16] proves uniqueness without using the Hölder continuity of $\hat{\psi}$. For that, the authors show that $\hat{\psi}_r$ above is cohomologous to the Hölder continuous function $\underline{v} \in \Sigma \mapsto -\log \det(df|_{F_{\pi_r(\underline{v},0)}^u})$, see [LLS16, Lemmas 8.1 and 8.2]. We note that the same proof applies here.

Proof of Theorem 1.2. The proof is the same of [BCFT18, Theorem C]. We include it for completeness. Let M be a surface and $\psi = q\psi^u$ for $q < 1$. Inside Sing all Lyapunov exponents are zero. Let μ be a φ -invariant measure supported on Sing. By the Ruelle inequality, $h(\mu) = 0$. Also, since $\psi^u = 0$ on Sing, it follows that

$$h(\mu) + q \int \psi^u d\mu = 0.$$

Therefore $P(\text{Sing}, \psi) = 0$. On the other hand, if λ is the Lebesgue measure, then by Pesin equality

$$h(\lambda) + q \int \psi^u d\mu = (1 - q)h(\lambda) > 0.$$

Therefore $P(\psi) > 0 = P(\text{Sing}, \psi)$. Now apply Theorem 1.1. \square

Proof of Theorem 1.3. When M is a surface we have, as in the previous proof, that $P(\text{Sing}, 0) = 0$. Since $P(0) = h_{\text{top}}(\varphi) > 0$, we get the pressure gap $P(\text{Sing}, 0) < P(0)$. Applying Theorem 1.1, we conclude the uniqueness of the measure of maximal entropy. In higher dimension, we invoke [BCFT18, Theorem B] to get the pressure gap $P(\text{Sing}, 0) < P(0)$. Again, applying Theorem 1.1, we obtain the uniqueness of the measure of maximal entropy. \square

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