General Mneimneh-type Binomial Sum involving Harmonic Numbers

Ende Pan[a,*] and Ce Xu[b,†]

a. College of Teacher Education, Quzhou University, Quzhou 324022, P.R. China

b. School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, P.R. China

Abstract. Recently, Mneimneh proved the remarkable identity

$$\sum_{k=0}^{n} H_k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{i=1}^{n} \frac{1-(1-p)^i}{i} \quad (p \in [0,1])$$

as the main result of a 2023 *Discrete Mathematics* paper, where $H_k := \sum_{i=1}^{k} 1/i$ is the classical k-th harmonic number. Thereafter, Campbell provided several other proofs of Mneimneh's formula as above in a note published in *Discrete Mathematics* in 2023. Moreover, Campbell also considered how Mneimneh's identity may be proved and generalized using the *Mathematica package Sigma*. In particular, he found the generalized Mneimneh's identity

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k = (x+y)^n \left(H_n - \sum_{i=1}^{n} \frac{y^i (x+y)^{-i}}{i} \right).$$

In this paper, we will prove a more generalization of Mneimneh's identity involving Bell numbers and some Mneimneh-type identities involving (alternating) harmonic numbers by using a few results of our previous papers.

Keywords: Mneimneh's identity; harmonic numbers; Binomial coefficients and sums; (unsigned) Stirling numbers; Bell polynomials; Bell numbers.

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1 Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In his recent paper [7], Mneimneh used the method of probabilistic analysis to establish the following binomial sum identity involving harmonic numbers

$$\sum_{k=0}^{n} H_k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{i=1}^{n} \frac{1-(1-p)^i}{i} \quad (p \in [0,1]), \tag{1.1}$$

*Email: 13052094150@163.com

†Email: cexu2020@ahnu.edu.cn

where H_k is the *classical k-th harmonic number* defined by

$$H_k := \sum_{i=1}^{k} \frac{1}{i} \quad \text{and} \quad H_0 := 0.$$

Quite recently, Campbell [1] gave two new proofs of (1.1) by using Zeilberger's algorithm and beta-type integral formula. Further, using the Sigma package for the Mathematica Computer Algebra System, Campbell gave the more general Mneimneh's identity

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k = (x+y)^n \left(H_n - \sum_{i=1}^{n} \frac{y^i (x+y)^{-i}}{i} \right). \tag{1.2}$$

Clearly, setting $(x, y) = (p, 1-p)$ in (1.2) gives (1.1). Moreover, Campbell also emphasized that this approach may also be used to derive identities for expressions such as

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k^2.$$

In a 2024 *Discrete Mathematics* article [6], Komatsu and Wang extended Mneimneh's formula to the generalized hyperharmonic numbers. Genčev [4] studied of the binomial sum

$$\sum_{k=0}^{n} \binom{n}{k} \left(\sum_{j=1}^{k} \frac{z^j}{j} \right) p^k (1-p)^k \quad (r \in \mathbb{N},\ p, z \in \mathbb{R},\ p \neq 1)$$

and established explicit formula, see [4, Thm. 2.1].

In this paper, using the method of integrals of natural logarithms, we will establish the explicit formulas of the following general Mneimneh-type binomial sums involving Bell numbers and some Mneimneh-type binomial sums involving (alternating) harmonic numbers.

Theorem 1.1. *For any reals x, y with $x/(x+y) \geq 0$ and $n, p \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} Y_p(k) = (-1)^{p-1} p! (x+y)^n \sum_{j=1}^{n} \left(1 - \left(\frac{y}{x+y} \right)^j \right) \sum_{i=0}^{p-1} (-1)^i \frac{Y_i(n)}{i!} \frac{s(j, p-i)}{j!}, \tag{1.3}$$

where $s(n, k)$ and $Y_k(n)$ stand for the (unsigned) Stirling numbers of the first kind and Bell numbers (see Section 2), respectively.

Remark 1.2. All Stirling numbers $s(n, k)$ and Bell numbers $Y_k(n)$ can be expressed in terms of a linear combination of products of harmonic numbers.

In particular, letting $p = 1, 2$ in Theorem 1.1 and noting the facts that $s(n, 1) = (n-1)!, s(n, 2) = (n-1)! H_{n-1}, Y_0(n) = 1, Y_1(n) = H_n, Y_2(n) = H_n^2 + H_n^{(2)}$, we also obtain (1.2) and

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} (H_k^2 + H_k^{(2)}) = 2(x+y)^n \sum_{j=1}^{n} \left(1 - \left(\frac{y}{x+y} \right)^j \right) \frac{H_n - H_{j-1}}{j}. \tag{1.4}$$

Theorem 1.3. *For any reals x, y and $z \in (-\infty, 1)$ with $x/(x+y) \geq 0$ and $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} \left(\sum_{j=1}^{k} \frac{z^j}{j} \right) = \sum_{j=1}^{n} \frac{(y+zx)^j - y^j}{j} (x+y)^{n-j}. \tag{1.5}$$

Obviously, letting $z \to 1$ in (1.5) gives (1.2). Setting $z = -1$ in Theorem 1.3 yields the following corollary.

Corollary 1.4. *For any reals x, y with $x/(x+y) \geq 0$ and $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} \bar{H}_k = \sum_{j=1}^{n} \frac{y^j - (y-x)^j}{j} (x+y)^{n-j}, \tag{1.6}$$

where $\bar{H}_k$ is the alternating k-th harmonic number *defined by*

$$\bar{H}_k := \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \quad \text{and} \quad \bar{H}_0 := 0. \tag{1.7}$$

Specially, setting $(x, y) = (p, 1-p)$ ($p \in [0, 1]$) in (1.6) yields the following Mneimneh-type identity

$$\sum_{k=0}^{n} \bar{H}_k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{j=1}^{n} \frac{(1-p)^j - (1-2p)^j}{j}. \tag{1.8}$$

From equation (1.4) and Theorem 1.3, we can also obtain the following corollary.

Corollary 1.5. *For any reals x, y with $x/(x+y) \geq 0$ and $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k^{(2)} = (x+y)^n \left\{ \sum_{j=1}^{n} \frac{y^j (x+y)^{-j}}{j} \sum_{i=1}^{j} \frac{y^{-i} (x+y)^i}{i} - \sum_{j=1}^{n} \frac{y^j (x+y)^{-j}}{j} H_j \right\}, \tag{1.9}$$

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k^2 = (x+y)^n \left\{ \begin{array}{c} H_n^2 + H_n^{(2)} - 2 \displaystyle\sum_{j=1}^{n} \frac{y^j (x+y)^{-j}}{j^2} - 2 H_n \sum_{j=1}^{n} \frac{y^j (x+y)^{-j}}{j} \\ +3 \displaystyle\sum_{j=1}^{n} \frac{y^j (x+y)^{-j}}{j} H_j - \sum_{j=1}^{n} \frac{y^j (x+y)^{-j}}{j} \sum_{i=1}^{j} \frac{y^{-i} (x+y)^i}{i} \end{array} \right\}. \tag{1.10}$$

We will prove Theorem 1.1 in Section 3, and Theorem 1.3 and Corollary 1.5 in Section 4.

Conjecture 1.6. *The Theorems 1.1 and 1.3 hold for any reals x, y and z.*

2 Preliminaries and lemmas

2.1 (unsigned) Stirling number of the first kind

We recall the definition of *(unsigned) Stirling number of the first kind.* Let $s(n,k)$ denote the (unsigned) Stirling number of the first kind, which is defined by [2, 3]

$$n!x\,(1+x)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right)=\sum_{k=0}^{n}s(n+1,k+1)x^{k+1},\tag{2.1}$$

with $s(n,k):=0$ if $n<k$, and $s(n,0)=s(0,k):=0,\ s(0,0):=1$, or equivalently, by the generating function:

$$\log^k(1-x)=(-1)^k k!\sum_{n=1}^{\infty}s(n,k)\frac{x^n}{n!}\quad(x\in[-1,1)).$$

The Stirling numbers $s\,(n,k)$ of the first kind satisfy a recurrence relation in the form

$$s(n,k)=s(n-1,k-1)+(n-1)\,s(n-1,k)\quad(n,k\in\mathbb{N}).$$

Obviously, $s\,(n,k)$ can be expressed in terms of a linear combinations of products of harmonic numbers (see (2.5)). In particular,

$$\begin{aligned}
&s(n,1)=(n-1)!,\\
&s\,(n,2)=(n-1)!H_{n-1},\\
&s(n,3)=\frac{(n-1)!}{2}\left[H_{n-1}^2-H_{n-1}^{(2)}\right],\\
&s(n,4)=\frac{(n-1)!}{6}\left[H_{n-1}^3-3H_{n-1}H_{n-1}^{(2)}+2H_{n-1}^{(3)}\right],\\
&s(n,5)=\frac{(n-1)!}{24}\left[H_{n-1}^4-6H_{n-1}^{(4)}-6H_{n-1}^2H_{n-1}^{(2)}+3(H_{n-1}^{(2)})^2+8H_{n-1}H_{n-1}^{(3)}\right].
\end{aligned}$$

In [9, Thm 2.5], we proved

$$s\,(n,k)=(n-1)!\zeta_{n-1}(\{1\}_{k-1})\quad(k,n\in\mathbb{N}),\tag{2.2}$$

where $\zeta_n(\{1\}_k)$ is a special multiple harmonic sum defined by ($\{l\}_r$ denotes the sequence obtained by repeating l exactly r times)

$$\zeta_n(\{1\}_r):=\sum_{n\ge n_1>n_2>\cdots>n_r>0}\frac{1}{n_1n_2\cdots n_r}.\tag{2.3}$$

For $\boldsymbol{k}=(k_1,\ldots,k_r)\in\mathbb{N}^r$ and positive integer n, the generalized *multiple harmonic sums* (MHSs) are defined by

$$\zeta_n(\boldsymbol{k})\equiv\zeta_n(k_1,\ldots,k_r):=\sum_{n\ge n_1>\cdots>n_r>0}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}.\tag{2.4}$$

We set $\zeta_n(\emptyset):=1$ if $\boldsymbol{k}=\emptyset$ and $\zeta_n(\boldsymbol{k}):=0$ if $n<r$. For $\boldsymbol{k}=(k)\in\mathbb{N}$,

$$\zeta_n(k)\equiv H_n^{(k)}=\sum_{j=1}^{n}\frac{1}{j^k}\tag{2.5}$$

is the n-th generalized *harmonic number* of order k, and furthermore, if $k = 1$ then $H_n \equiv H_n^{(1)}$ is the classical n-th harmonic number. When taking the limit $n \to \infty$ in (2.4) we get the so-called classical *multiple zeta values* (MZVs) (see [5, 12]),

$$\zeta(\boldsymbol{k}) := \lim_{n\to\infty} \zeta_n(\boldsymbol{k}),$$

defined for $\boldsymbol{k} \in \mathbb{N}^r$ and $k_1 > 1$ to ensure convergence of the series.

2.2 Bell polynomials

Define the *exponential partial Bell polynomials* $B_{n,k}$ by

$$\frac{1}{k!}\left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_n)\frac{t^n}{n!}\,, \quad k = 0, 1, 2, \dots\,,$$

and the *exponential complete Bell polynomials* Y_n by

$$Y_n(x_1, x_2, \dots, x_n) := \sum_{k=0}^{n} B_{n,k}(x_1, x_2, \dots, x_n)$$

(see [2, Section 3.3]). According to [8, Eq. (2.44)], the complete Bell polynomials Y_n satisfy the recurrence

$$Y_0 = 1\,, \quad Y_n(x_1, x_2, \dots, x_n) = \sum_{j=0}^{n-1} \binom{n-1}{j} x_{n-j} Y_j(x_1, x_2, \dots, x_j)\,, \quad n \geq 1\,,$$

from which, the first few polynomials can be obtained immediately:

$$\begin{aligned}
&Y_0 = 1\,, \quad Y_1(x_1) = x_1\,, \quad Y_2(x_1, x_2) = x_1^2 + x_2\,, \quad Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3\,,\\
&Y_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4\,.
\end{aligned}$$

Define the *Bell number* $Y_k(n)$ by

$$Y_k(n) := Y_k(H_n, 1!H_n^{(2)}, 2!H_n^{(3)}, \cdots, (k-1)!H_n^{(k)}). \tag{2.6}$$

Clearly, the Bell number $Y_k(n)$ is a rational linear combination of products of harmonic numbers. We have

$$\begin{aligned}
&Y_1(n) = H_n,\\
&Y_2(n) = H_n^2 + H_n^{(2)},\\
&Y_3(n) = H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)},\\
&Y_4(n) = H_n^4 + 8H_nH_n^{(3)} + 6H_n^2H_n^{(2)} + 3(H_n^{(2)})^2 + 6H_n^{(4)},\\
&Y_5(n) = H_n^5 + 10H_n^3H_n^{(2)} + 20H_n^2H_n^{(3)} + 15H_n(H_n^{(2)})^2 + 30H_nH_n^{(4)} + 20H_n^{(2)}H_n^{(3)} + 24H_n^{(5)}.
\end{aligned}$$

In [9, Eq. (2.9)], we showed

$$\zeta_n^\star(\{1\}_m) = \frac{1}{m!} Y_m(n) \quad (n, m \in \mathbb{N}_0), \tag{2.7}$$

where $\zeta_n^\star(\{1\}_r)$ is a special multiple harmonic star sum defined by

$$\zeta_n^\star(\{1\}_r) := \sum_{n\geq n_1\geq n_2\geq\cdots\geq n_r>0} \frac{1}{n_1 n_2\cdots n_r}.$$

For $\boldsymbol{k} = (k_1,\ldots,k_r) \in \mathbb{N}^r$ and positive integer n, the generalized *multiple harmonic star sums* (MHSSs) are defined by

$$\zeta_n^\star(\boldsymbol{k}) \equiv \zeta_n^\star(k_1,\ldots,k_r) := \sum_{n\geq n_1\geq\cdots\geq n_r>0} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}. \tag{2.8}$$

Similarly, we set $\zeta_n^\star(\emptyset) := 1$ if $\boldsymbol{k} = \emptyset$.

2.3 Some Lemmas

Next, we present some lemmas, which are useful in the development of our main theorems.

Lemma 2.1. *([10, Thm. 2.9]) For $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$, we have*

$$\int_0^1 t^{n-1}\log^p(1-t)dt = (-1)^p \frac{Y_p(n)}{n}. \tag{2.9}$$

Lemma 2.2. *([9, Thm. 2.2]) For $n,m \in \mathbb{N}$ and $x \in (-\infty,1)$, we have*

$$\begin{aligned}\int_0^x t^{n-1}\log^m(1-t)dt &= m!\frac{(-1)^m}{n}\zeta_n^\star(\{1\}_m;x)\\ &+\frac{1}{n}\sum_{j=0}^{m-1}(-1)^j j!\binom{m}{j}\log^{m-j}(1-x)\left(\zeta_n^\star(\{1\}_j;x)-\zeta_n^\star(\{1\}_j)\right),\end{aligned} \tag{2.10}$$

where $\zeta_n^\star(\{1\}_r)$ $(r \in \mathbb{N}_0)$ is defined by

$$\zeta_n^\star(\{1\}_r;x) := \sum_{n\geq n_1\geq n_2\geq\cdots\geq n_r>0} \frac{x^{n_r}}{n_1 n_2\cdots n_r} \quad \textit{and} \quad \zeta_n^\star(\emptyset;x) := x^n.$$

(Note that in [9, Thm. 2.2], the range of values for x is $x \in [-1,1)$, but in fact, the above equation also holds for $x \in (-\infty,-1]$.)

Lemma 2.3. *([11, Thm. 4.3]) For $\boldsymbol{k} = (k_1,\ldots,k_r) \in \mathbb{N}^r$ and $l \in \mathbb{N}_0, n \in \mathbb{N}$, we have*

$$\sum_{n\geq n_1\geq n_2\geq\cdots\geq n_r>0} \prod_{j=1}^r \frac{1}{(n_j+l)^{k_j}} = (-1)^r \sum_{j=0}^r (-1)^j \zeta_{n+l}^\star(\overrightarrow{\boldsymbol{k}}_j)\zeta_l(\overleftarrow{\boldsymbol{k}}_{j+1}), \tag{2.11}$$

where $\overrightarrow{\boldsymbol{k}}_j := (k_1,\ldots,k_j)$ and $\overleftarrow{\boldsymbol{k}}_j := (k_r,\ldots,k_j)$ for all $1 \leq j \leq r$.

3 Proof of Theorem 1.1

For $x/(x+y)=0$, namely $x=0$, (1.3) is obviously holds. For $x/(x+y)>0$, applying Lemma 2.1, the left hand side of (1.3) can be rewritten as

$$
\begin{aligned}
\sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k} Y_p(k) &= (-1)^p \sum_{k=1}^{n} x^k y^{n-k}\binom{n}{k} k \int_0^1 t^{k-1}\log^p(1-t)dt \\
&= (-1)^p \int_0^1 \log^p(1-t) \sum_{k=1}^{n} x^k y^{n-k}\binom{n}{k} k t^{k-1} dt \\
&= (-1)^p \int_0^1 \log^p(1-t) \sum_{k=0}^{n-1} x^{k+1} y^{n-k-1}\binom{n}{k+1} (k+1) t^{k} dt \\
&= (-1)^p \int_0^1 \log^p(1-t) \sum_{k=0}^{n-1} x^{k+1} y^{n-k-1} n\binom{n-1}{k} t^{k} dt \\
&= (-1)^p nx \int_0^1 \log^p(1-t) \sum_{k=0}^{n-1} (xt)^{k} y^{n-k-1} \binom{n-1}{k} dt \\
&= (-1)^p nx \int_0^1 \log^p(1-t)(tx+y)^{n-1} dt \qquad \text{(letting } u=tx+y) \\
&= (-1)^p n \int_y^{x+y} u^{n-1}\log^p\left(\frac{x+y-u}{x}\right)du \\
&= (-1)^p n \int_y^{x+y} u^{n-1}\left\{\log\left(\frac{x+y}{x}\right)+\log\left(1-\frac{u}{x+y}\right)\right\}^p du \\
&= (-1)^p n \sum_{l=0}^{p}\binom{p}{l}\log^{p-l}\left(\frac{x+y}{x}\right)\int_y^{x+y} u^{n-1}\log^l\left(1-\frac{u}{x+y}\right)du \\
&= (-1)^p n \sum_{l=1}^{p}\binom{p}{l}\log^{p-l}\left(\frac{x+y}{x}\right)\int_y^{x+y} u^{n-1}\log^l\left(1-\frac{u}{x+y}\right)du \\
&\quad + (-1)^p\Big((x+y)^n-y^n\Big)\log^p\left(\frac{x+y}{x}\right) \qquad \text{(letting } u=v(x+y)) \\
&= (-1)^p n(x+y)^n \sum_{l=1}^{p}\binom{p}{l}\log^{p-l}\left(\frac{x+y}{x}\right)\int_{y(x+y)^{-1}}^{1} v^{n-1}\log^l\Big(1-v\Big)dv \\
&\quad + (-1)^p\Big((x+y)^n-y^n\Big)\log^p\left(\frac{x+y}{x}\right). \qquad (3.1)
\end{aligned}
$$

(In order to utilize Lemmas 2.1 and 2.2, the integration in the second to last row of the above equation needs to satisfy $y(x+y)^{-1}<1$, namely $x/(x+y)>0$) Using Lemmas 2.1 and 2.2, by an elementary calculation, the (3.1) is equal to

$$
\begin{aligned}
\sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k} Y_p(k) &= (x+y)^n \sum_{1\le i\le l\le p} (-1)^{l-i} i! \binom{l}{i}\binom{p}{l}\log^{p-i}\left(\frac{x}{x+y}\right) \\
&\qquad \times \left(\zeta_n^\star(\{1\}_i) - \zeta_n^\star\Big(\{1\}_i; y(x+y)^{-1}\Big)\right) \\
&= (x+y)^n \sum_{1\le i\le l\le p} (-1)^{l-i} i! \binom{l}{i}\binom{p}{l}\log^{p-i}\left(\frac{x}{x+y}\right)
\end{aligned}
$$

$$\times \sum_{n\geq k_1\geq\cdots\geq k_i\geq 1} \frac{1-\left(\frac{y}{x+y}\right)^{k_i}}{k_1\cdots k_i}. \tag{3.2}$$

Noting the fact that the (3.2) can be rewritten as

$$\sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k} Y_p(k) = (x+y)^n \sum_{i=1}^{p}(-1)^i i!\log^{p-i}\left(\frac{x}{x+y}\right) \sum_{n\geq k_1\geq\cdots\geq k_i\geq 1} \frac{1-\left(\frac{y}{x+y}\right)^{k_i}}{k_1\cdots k_i}$$
$$\times \sum_{l=i}^{p}(-1)^l\binom{l}{i}\binom{p}{l} \tag{3.3}$$

and

$$\sum_{l=i}^{p}(-1)^l\binom{l}{i}\binom{p}{l} = \begin{cases} (-1)^p, & (p=i), \\ 0, & (\text{otherwise}), \end{cases}$$

we obtain

$$\sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k} Y_p(k) = p!(x+y)^n \sum_{n\geq k_1\geq\cdots\geq k_p\geq 1} \frac{1-\left(\frac{y}{x+y}\right)^{k_p}}{k_1\cdots k_p}. \tag{3.4}$$

In Lemma 2.3, replacing n by $n-j+1$ and letting $r=p-1$, $(k_1,\ldots,k_r)=(\{1\}_{p-1})$ and $l=j-1$, we easily obtain

$$\sum_{n-j+1\geq i_1\geq\cdots\geq i_{p-1}\geq 1} \frac{1}{(i_1+j-1)\cdots(i_{p-1}+j-1)}$$
$$= (-1)^{p-1}\sum_{i=0}^{p-1}(-1)^i \zeta_n^\star(\{1\}_i)\zeta_{j-1}(\{1\}_{p-1-i}). \tag{3.5}$$

Hence,

$$\begin{aligned}
\sum_{n\geq k_1\geq\cdots\geq k_p\geq 1} \frac{1-\left(\frac{y}{x+y}\right)^{k_p}}{k_1\cdots k_p} &= \sum_{j=1}^{n} \frac{1-\left(\frac{y}{x+y}\right)^{j}}{j} \sum_{n\geq k_1\geq\cdots\geq k_{p-1}\geq j} \frac{1}{k_1\cdots k_{p-1}} \\
&= \sum_{j=1}^{n} \frac{1-\left(\frac{y}{x+y}\right)^{j}}{j} \sum_{n-j+1\geq i_1\geq\cdots\geq i_{p-1}\geq 1} \frac{1}{(i_1+j-1)\cdots(i_{p-1}+j-1)} \\
&= (-1)^{p-1}\sum_{j=1}^{n} \frac{1-\left(\frac{y}{x+y}\right)^{j}}{j} \sum_{i=0}^{p-1}(-1)^i \zeta_n^\star(\{1\}_i)\zeta_{j-1}(\{1\}_{p-1-i}) \\
&= (-1)^{p-1}\sum_{j=1}^{n} \left(1-\left(\frac{y}{x+y}\right)^{j}\right) \sum_{i=0}^{p-1}(-1)^i \frac{Y_i(n)}{i!}\frac{s(j,p-i)}{j!},
\end{aligned} \tag{3.6}$$

where we used the equations (2.2) and (2.7).

Finally, substituting (3.6) into (3.4) yields the desired evaluation (1.3). Thus, this concludes the proof of Theorem 1.1.

4 Proofs of Theorem 1.3 and Corollary 1.5

For $x/(x+y)=0$, namely $x=0$, Theorem 1.3 and Corollary 1.5 are obviously hold. For $x/(x+y)>0$, in (2.10), setting $m=1$ and replacing x by z gives

$$\int_0^z t^{n-1}\log(1-t)dt = \frac{1}{n}\left\{x^n\log(1-z) - \sum_{j=1}^{n}\frac{z^j}{j} - \log(1-z)\right\}. \tag{4.1}$$

Hence,

$$\sum_{j=1}^{n}\frac{z^j}{j} = (z^n-1)\log(1-z) - n\int_0^z t^{n-1}\log(1-t)dt.$$

Hence, by a similar argument as in the proof of (3.1) gives

$$\begin{aligned}
\sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k}\left(\sum_{j=1}^{k}\frac{z^j}{j}\right) &= -nx\int_0^z \log(1-t)(xt+y)^{n-1}dt + \log(1-z)\left((xz+y)^n-(x+y)^n\right)\\
&= -n\int_y^{xz+y} u^{n-1}\log\left(\frac{x+y-u}{x}\right)du + \log(1-z)\left((xz+y)^n-(x+y)^n\right)\\
&= -n\int_y^{xz+y} u^{n-1}\left\{\log\left(\frac{x+y}{x}\right)+\log\left(1-\frac{u}{x+y}\right)\right\}du\\
&\quad + \log(1-z)\left((xz+y)^n-(x+y)^n\right)\\
&= -\log\left(\frac{x+y}{x}\right)\left((xz+y)^n-y^n\right) + \log(1-z)\left((xz+y)^n-(x+y)^n\right)\\
&\quad - n\int_y^{xz+y} u^{n-1}\log\left(1-\frac{u}{x+y}\right)du\\
&= -\log\left(\frac{x+y}{x}\right)\left((xz+y)^n-y^n\right) + \log(1-z)\left((xz+y)^n-(x+y)^n\right)\\
&\quad - n(x+y)^n\int_{y(x+y)^{-1}}^{(xz+y)(y+x)^{-1}} v^{n-1}\log(1-v)dv.
\end{aligned} \tag{4.2}$$

Therefore, applying (4.1), by a direct calculation, we deduce

$$\sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k}\left(\sum_{j=1}^{k}\frac{z^j}{j}\right) = (x+y)^n\sum_{j=1}^{n}\frac{\left(\frac{xz+y}{y+x}\right)^j - \left(\frac{y}{y+x}\right)^j}{j}. \tag{4.3}$$

This completes the proof of Theorem 1.3 (Noting that from $x/(x+y)>0$ and $z\in(-\infty,1)$ gives $(xz+y)(x+y)^{-1}<1$).

Multiplying (1.5) by $1/z$ and integrating over the interval (0,1), we have

$$\sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k}H_k^{(2)} = \sum_{j=1}^{n}\frac{(x+y)^{n-j}}{j}\int_0^1\frac{(xz+y)^j-y^j}{z}dz$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \frac{(x+y)^{n-j}}{j} y^j \sum_{i=1}^{j} \int_0^1 \left(\frac{xz}{y} + 1 \right)^{i-1} d\frac{xz}{y} \\
&= (x+y)^n \left\{ \sum_{j=1}^{n} \frac{y^j (x+y)^{-j}}{j} \sum_{i=1}^{j} \frac{y^{-i}(x+y)^i}{i} - \sum_{j=1}^{n} \frac{y^j(x+y)^{-j}}{j} H_j \right\}.
\end{aligned} \tag{4.4}
$$

Then, applying the well-known identity

$$
\sum_{j=1}^{n} \frac{H_j}{j} = \zeta_n^\star(1,1) = \frac{1}{2} Y_2(n) = \frac{H_n^2 + H_n^{(2)}}{2}
$$

and substituting (4.4) into (1.4) yields (1.10). Thus, we complete the proof of Corollary 1.5.

5 Conclusion

We presented the explicit formulas of the following Mneimneh-type binomial sum of (alternating) harmonic numbers

$$
\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} Y_p(k), \quad \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} \bar{H}_k, \quad \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k^{(2)} \quad \text{and} \quad \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k^2.
$$

It is possible that some of other Mneimneh-type binomial sums can be obtained using techniques of the present paper. For example, multiplying (1.5) by $\log^{p-1}(z)/z$ ($p \in \mathbb{N}$) and integrating over the interval (0,1), we have

$$
\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} H_k^{(p+1)} = \frac{(-1)^{p-1}}{(p-1)!} \sum_{j=1}^{n} \frac{(x+y)^{n-j}}{j} \int_0^1 \log^{p-1}(z) \frac{(xz+y)^j - y^j}{z} dz. \tag{5.1}
$$

Hence, if the evaluation of the integral on the right hand of above can be established, we can obtain the explicit formula of Mneimneh-type binomial sums on the left hand of above. We leave the detail to the interested reader. It should be emphasized that Komatsu-Wang [6, Eq. (4)] gave an evaluation of (5.1) with $x = 1-q$ and $y = q$, where q is a real number. Genčev [4, Thm. 2.1] established a generalization of Mneimneh summation formula

$$
\sum_{k=0}^{n} \binom{n}{k} \left(\sum_{j=1}^{k} \frac{z^j}{j^r} \right) p^k (1-p)^k = \sum_{n \geq n_1 \geq \cdots n_r \geq 1} \frac{(1-p)^{n_1} \left(\left(1 + \frac{zp}{1-p} \right)^{n_r} - 1 \right)}{n_1 \cdots n_r}, \tag{5.2}
$$

where $r \in \mathbb{N}, p, z \in \mathbb{R}$ and $p \neq 1$.

Declaration of competing interest. The authors declares that they has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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