Representations of not-finitely graded Lie algebras related to Virasoro algebra

Chunguang Xia*,†,‡, Tianyu Ma†, Xiao Dong†, Mingjing Zhang†

†School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China

‡School of Mathematical Sciences, Xiameng University, Xiameng 361005, China

E-mails: chgxia@cumt.edu.cn, tyma@cumt.edu.cn, xdong@cumt.edu.cn, zhangmj@nju.edu.cn

Abstract. In this paper, we study representations of not-finitely graded Lie algebras $\mathcal{W}(\epsilon)$ related to Virasoro algebra, where $\epsilon = \pm 1$. Precisely speaking, we completely classify the free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(\epsilon)$, and find that these module structures are rather different from those of other graded Lie algebras. We also determine the simplicity and isomorphism classes of these modules.

Key words: Virasoro algebra; not-finitely graded Lie algebras; free $\mathcal{U}(\mathfrak{h})$-module; simplicity

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1 Introduction

It is well-known that the Virasoro algebra, denoted $\widehat{\mathrm{Vir}}$, is an infinite dimensional Lie algebra with basis $\{L_i,\, c \,|\, i \in \mathbb{Z}\}$ satisfying

$$[L_i, L_j] = (j-i)L_{i+j} + \delta_{i+j,0}\frac{i^3-i}{12}c, \quad [L_i, c] = 0.$$

Here c is the center of $\widehat{\mathrm{Vir}}$. We denote by Vir the centerless Virasoro algebra. Clearly, Vir is a $\mathbb{Z}$-graded Lie algebra with Cartan subalgebra $\mathfrak{h}_{\mathrm{Vir}} = \mathbb{C}L_0$. The representation theories of $\widehat{\mathrm{Vir}}$ and Vir have been extensively and deeply studied due to its importance in mathematics and physics, see, e.g., the survey in [9]. One of the most important work in weigh module theory is the well-known Mathieu's theorem [13] on classification of Harish-Chandra modules, which was conjectured earlier by Kac [10].

Recently, an interesting non-weight module problem for a Lie algebra $\mathfrak{g}$, defined by an "opposite condition" relative to weight modules, was proposed by Nilsson [14, 15]. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Denote by $\mathcal{U}(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}$. This kind of non-weight modules, referred to as *free $\mathcal{U}(\mathfrak{h})$-modules* since the action of $\mathfrak{h}$ is required to be free, was constructed first for $\mathfrak{sl}_n$ by Nilsson [14] and independently by Tan and Zhao [22]. In another paper, Tan and Zhao [21] proved that any free $\mathcal{U}(\mathfrak{h}_{\mathrm{Vir}})$-module of rank one over Vir is isomorphic to $\Omega_{\mathrm{Vir}}(\lambda, \alpha)$ (cf. (2.1)) for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$. Following [14, 15, 21, 22], free $\mathcal{U}(\mathfrak{h})$-modules, especially for finitely graded Lie algebras containing Virasoro subalgebra, have been extensively

*Corresponding author: chgxia@cumt.edu.cn (C. Xia).

studied in recent years, see, e.g., [1–3, 7, 8, 24] and the references therein. It has been realized that interesting free $\mathcal{U}(\mathfrak{h})$-module results will appear under the assumption that $\mathfrak{h}$ is not a Cartan subalgebra (e.g., [1, 7, 16]). The latest work in this direction is Nilsson's work [16], where he chose $\mathfrak{h}$ to be the nilradical of a maximal parabolic subalgebra of $\mathfrak{sl}_n$.

Let $\epsilon = \pm 1$. In this paper, we focus on a class of not-finitely graded Lie algebras $\mathcal{W}(\epsilon)$ with basis $\{L_{i,m} \mid i \in \mathbb{Z}, m \in \mathbb{Z}_+\}$ and relations

$$[L_{i,m}, L_{j,n}] = (j-i)L_{i+j,m+n} + \epsilon(m-n)L_{i+j,m+n-\epsilon}. \tag{1.1}$$

Note that the subspace spanned by $\{L_{i,0} \in \mathcal{W}(\epsilon) \mid i \in \mathbb{Z}\}$ is isomorphic to the centerless Virasoro algebra Vir. For simplicity, we refer to $\mathcal{W}(\epsilon)$ as *not-finitely graded Virasoro algebras*. The case for $\epsilon = 1$ was first constructed in [20, 25] as the simple Lie algebra $W(0,1,0;\mathbb{Z})$ of Witt type. The case for $\epsilon = -1$ was naturally realized in [4] as the non-simple Lie algebra $W(\mathbb{Z})$ consisting of smooth function $L_{i,m} := -(1+t)^{-m}e^{-it} \in C^{\infty}_{[0,+\infty)}$ under bracket $[f,g] = fg' - f'g$ for $f, g \in W(\mathbb{Z})$.

For not-finitely graded Lie algebras with Virasoro subalgebra, motivated by Mathieu's work [13], it is natural to consider the classification of the so-called quasifinite modules, such as the W-infinity algebra $\mathcal{W}_{1+\infty}$ [11], Lie algebras of Weyl type [17], Lie algebras of Block type [18, 19] and so on. However, for the Lie algebra $\mathcal{W}(\epsilon)$, although it admits a natural principal $\mathbb{Z}$-gradation $\mathcal{W}(\epsilon) = \oplus_{i\in\mathbb{Z}}\mathcal{W}(\epsilon)_i$ with $\mathcal{W}(\epsilon)_i = \mathrm{span}\{L_{i,m} \mid m \in \mathbb{Z}_+\}$, the zero-graded part $\mathcal{W}(\epsilon)_0$ is not a commutative subalgebra and more worse it does not contain Cartan subalgebra. This leads to that one cannot cope with the quasifinite modules over $\mathcal{W}(\epsilon)$. Fortunately, in this paper, we find that one can develop the representation theory of $\mathcal{W}(\epsilon)$ by studying free $\mathcal{U}(\mathfrak{h})$-modules with $\mathfrak{h} = \mathbb{C}L_{0,0}$ (although it is not a Cartan subalgebra). To best of our knowledge, up to now, there are few work on free $\mathcal{U}(\mathfrak{h})$-modules over not-finitely graded Lie algebras, see [5, 6, 23] for some attempt on loop Virasoro algebra and some Lie algebras of Block type.

We surprisingly find that the module structures of $\mathcal{W}(\epsilon)$ are much more complicated than that of the Virasoro algebra [21], and rather different from those of other not-finitely graded Lie algebras [5, 6, 23]. Our first main result for $\mathcal{W}(1)$ is Theorem 3.2, in the proof of which the following combinatoric formula: for $m < n$,

$$\binom{n-1}{m} + \binom{n-1}{m-1} = \binom{n}{m} \tag{1.2}$$

will be used frequently. Our second main result for $\mathcal{W}(-1)$ is Theorem 4.2, for which we need more technical analysis. We also would like to point out that our techniques used here may be applied to analogous problems of not-finitely graded Lie algebras which are closely related to $\mathcal{W}(\epsilon)$. This is also our motivation for writing this paper.

This paper is organized as follows. In Section 2, we recall the classification of free $\mathcal{U}(\mathfrak{h}_{\mathrm{Vir}})$-modules of rank one over Vir, and present an elementary result on a number sequence. In Sections 3 and 4, we classify the free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(1)$ and $\mathcal{W}(-1)$, respectively. Along the way, we also determine the simplicity and isomorphism classes of these modules.

2 Preliminaries

Throughout this paper, we use $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{C}$ and $\mathbb{C}^*$ to denote the sets of integers, nonnegative integers, complex numbers and nonzero complex numbers, respectively. We work over the complex field $\mathbb{C}$. In this section, we recall the classification of the free $\mathcal{U}(\mathfrak{h}_{\mathrm{Vir}})$-modules of rank one over the centerless Virasoro algebra Vir. We also present an elementary result on a number sequence, which will be used in Section 4.

2.1 Free $\mathcal{U}(\mathfrak{h}_{\mathrm{Vir}})$-modules of rank one over Vir

Recall that $\mathfrak{h}_{\mathrm{Vir}} = \mathbb{C}L_0$ is the Cartan subalgebra of Vir. Let $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$. If we define the action of Vir on the vector space of polynomials in one variable

$$\Omega_{\mathrm{Vir}}(\lambda, \alpha) := \mathbb{C}[t]$$

by

$$L_i \cdot f(t) = \lambda^i (t - i\alpha) f(t - i), \tag{2.1}$$

where $i \in \mathbb{Z}, f(t) \in \mathbb{C}[t]$, then $\Omega_{\mathrm{Vir}}(\lambda, \alpha)$ becomes a Vir-module. Equivalently, we can rewrite (2.1) in a more simple form by restricting the action of L_i on monomials:

$$L_i \cdot t^k = \lambda^i (t - i\alpha)(t - i)^k. \tag{2.2}$$

These modules firstly appeared in [12] as quotient modules of fraction Virasoro modules. From (2.2), we see that $\Omega_{\mathrm{Vir}}(\lambda, \alpha)$ is free of rank one when restricted to $\mathfrak{h}_{\mathrm{Vir}}$. In fact, we have the following classification result [21].

Lemma 2.1 *Any free $\mathcal{U}(\mathfrak{h}_{\mathrm{Vir}})$-module of rank one over* Vir *is isomorphic to $\Omega_{\mathrm{Vir}}(\lambda, \alpha)$ defined by* (2.2) *for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$.*

The simplicity and isomorphism classification of $\Omega_{\mathrm{Vir}}(\lambda, \alpha)$ are as follows [21].

Lemma 2.2 (1) *$\Omega_{\mathrm{Vir}}(\lambda, \alpha)$ is simple if and only if $\alpha \neq 0$.*

(2) *$\Omega_{\mathrm{Vir}}(\lambda, 0)$ has a unique proper submodule $t\Omega_{\mathrm{Vir}}(\lambda, 0) \cong \Omega_{\mathrm{Vir}}(\lambda, 1)$, and $\Omega_{\mathrm{Vir}}(\lambda, 0)/t\Omega_{\mathrm{Vir}}(\lambda, 0)$ is a one-dimensional trivial* Vir*-module.*

(3) *$\Omega_{\mathrm{Vir}}(\lambda_1, \alpha_1) \cong \Omega_{\mathrm{Vir}}(\lambda_2, \alpha_2)$ if and only if $\lambda_1 = \lambda_2$ and $\alpha_1 = \alpha_2$.*

2.2 An elementary result

Lemma 2.3 *Let $\{\beta_m \,|\, m \in \mathbb{Z}_+\}$ be a sequence of complex numbers satisfying $\beta_0 = 1$ and*

$$\beta_{m+n} + (n - m)\beta_{m+n+1} = \beta_n \beta_m + n\beta_m \beta_{n+1} - m\beta_n \beta_{m+1}. \tag{2.3}$$

Then $\beta_m = \beta^m$, where $\beta = \beta_1$.

Proof. We prove this lemma by induction on m. First, the cases for $m = 0, 1$ are clear. Taking $(m, n) = (1, 1)$ in (2.3), we immediately see that $\beta_2 = \beta_1^2$. Then, taking $(m, n) = (1, 2)$ and $(2, 1)$ respectively in (2.3), we have

$$\begin{aligned} \beta_3 + \beta_4 &= \beta_1^3 + 2\beta_1\beta_3 - \beta_2^2, \\ \beta_3 - \beta_4 &= \beta_1^3 + \beta_2^2 - 2\beta_1\beta_3. \end{aligned}$$

Adding the above two equations, we see that $\beta_3 = \beta_1^3$.

Let $m \geq 4$. Assume that the lemma holds for $k \leq m - 1$. Namely, we assume $\beta_k = \beta_1^k$ for $k \leq m - 1$. Next, we consider the case for m. Taking $(m, n) \to (m - 2, 1)$ in (2.3), we have

$$\beta_{m-1} + (3 - m)\beta_m = \beta_1\beta_{m-2} + \beta_{m-2}\beta_2 - (m - 2)\beta_1\beta_{m-1}.$$

A direct computation shows that $\beta_m = \beta_1^m$. This completes the proof. □

3 Free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(1)$

Note that the Lie algebra $\mathcal{W}(1)$ is $\mathbb{Z}\times\mathbb{Z}_+$-graded, and recall that $\mathfrak{h} = \mathbb{C}L_{0,0}$ is the $(0, 0)$-graded part. In this section, we completely classify the free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(1)$.

3.1 Construction of free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(1)$

Let $\lambda \in \mathbb{C}^*$ and $\alpha, \beta \in \mathbb{C}$. Define the action of $\mathcal{W}(1)$ on the vector space of polynomials in one variable

$$\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta) := \mathbb{C}[t]$$

by

$$L_{i,m} \cdot t^k = \sum_{s=0}^{\min\{m,k\}} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s-1}((m - s)\alpha - i\alpha\beta + \beta t)(t - i)^{k-s}, \tag{3.1}$$

where $i \in \mathbb{Z}, m \in \mathbb{Z}_+$ and $t^k \in \mathbb{C}[t]$. It is important to note here that we allow $\beta = 0$: if $\beta = 0$ and $s = m$ in the sum of (3.1), although $\beta^{m-s-1} = \beta^{-1}$ formally appears, we view

$$\beta^{m-s-1}((m - s)\alpha - i\alpha\beta + \beta t) = t - i\alpha$$

as a whole, and thus (3.1) still make sense. Note further that if we take $i = m = 0$ in (3.1), then we have $L_{0,0} \cdot t^k = t^{k+1}$. Hence, $\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$ is a $\mathcal{U}(\mathfrak{h})$-module, which is free of rank one. Furthermore, we find that $\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$ is in fact a $\mathcal{W}(1)$-module.

Proposition 3.1 *The space $\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$ is a $\mathcal{W}(1)$-module under the action* (3.1).

Proof. Taking $k = 0$ in (3.1), we have

$$L_{i,m} \cdot 1 = \lambda^i \beta^{m-1}(m\alpha - i\alpha\beta + \beta t). \tag{3.2}$$

By (3.2), we can rewrite (3.1) as

$$L_{i,m} \cdot t^k = \sum_{s=0}^{\min\{m,k\}} s! \binom{m}{s} \binom{k}{s} (t-i)^{k-s} L_{i,m-s} \cdot 1. \tag{3.3}$$

This in particular implies that the action of $L_{i,m}$ on the base element t^k of $\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$ is a linear combination of the actions of $L_{i,j}$ ($m - \min\{m,k\} \le j \le m$) on 1. Hence, to prove this proposition, we only need to check the action of $L_{i,m}$ on 1. In fact, on the one hand, by (1.1) and (3.2), we have

$$\begin{aligned}
\frac{1}{\lambda^{i+j}}[L_{i,m}, L_{j,n}] \cdot 1 \;&=\; \frac{1}{\lambda^{i+j}}(j-i)L_{i+j,m+n} \cdot 1 + \frac{1}{\lambda^{i+j}}(m-n)L_{i+j,m+n-1} \cdot 1 \\
&=\; (i^2 - j^2)\alpha\beta^{m+n} + (2nj - 2mi)\alpha\beta^{m+n-1} + (m^2 - m - n^2 + n)\alpha\beta^{m+n-2} \\
&\quad +(j-i)\beta^{m+n}t + (m-n)\beta^{m+n-1}t.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{\lambda^{i+j}} L_{i,m} \cdot L_{j,n} \cdot 1 \;&=\; \frac{1}{\lambda^i} L_{i,m} \cdot (n\alpha\beta^{n-1} - j\alpha\beta^n + \beta^n t) \\
&=\; ij\alpha^2\beta^{m+n} - (mj + ni)\alpha^2\beta^{m+n-1} + mn\alpha^2\beta^{m+n-2} + i^2\alpha\beta^{m+n} \\
&\quad -2mi\alpha\beta^{m+n-1} + m(m-1)\alpha\beta^{m+n-2} - (i+j)\alpha\beta^{m+n}t \\
&\quad +(n+m)\alpha\beta^{m+n-1}t - i\beta^{m+n}t + m\beta^{m+n-1}t + \beta^{m+n}t^2,
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{1}{\lambda^{i+j}}(L_{i,m} \cdot L_{j,n} \cdot 1 - L_{j,n} \cdot L_{i,m} \cdot 1) \;&=\; (i^2 - j^2)\alpha\beta^{m+n} + (2nj - 2mi)\alpha\beta^{m+n-1} \\
&\quad +(m^2 - m - n^2 + n)\alpha\beta^{m+n-2} + (j-i)\beta^{m+n}t \\
&\quad +(m-n)\beta^{m+n-1}t.
\end{aligned}$$

Hence we have $[L_{i,m}, L_{j,n}] \cdot 1 = L_{i,m} \cdot L_{j,n} \cdot 1 - L_{j,n} \cdot L_{i,m} \cdot 1$. This completes the proof. □

3.2 Classification of free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(1)$

Now we show that $\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$ constructed in (3.1) in fact exhaust all free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(1)$. This is our first main result.

Theorem 3.2 *Any free $\mathcal{U}(\mathfrak{h})$-module of rank one over $\mathcal{W}(1)$ is isomorphic to $\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$ defined by* (3.1) *for some $\lambda \in \mathbb{C}^*$ and $\alpha, \beta \in \mathbb{C}$.*

Proof. Let M be a free $\mathcal{U}(\mathfrak{h})$-module of rank one over $\mathcal{W}(1)$. By viewing M as a Vir-module, from Lemma 2.1, we may assume that $M = \mathbb{C}[t]$ and there exist some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$ such that the action of Vir $\subseteq \mathcal{W}(1)$ on M is as follows

$$L_{i,0} \cdot t^k = \lambda^i (t - i\alpha)(t-i)^k. \tag{3.4}$$

We reduce the remaining proof to the following Lemma 3.3 and Lemma 3.4. $\square$

Lemma 3.3 *The action of $L_{i,m}$ on t^k is a linear combination of the actions of $L_{i,j}$ ($m - \min\{m,k\} \le j \le m$) on 1, and more precisely we have (cf. (3.3))*

$$L_{i,m} \cdot t^k = \sum_{s=0}^{\min\{m,k\}} s! \binom{m}{s} \binom{k}{s} (t-i)^{k-s} L_{i,m-s} \cdot 1. \tag{3.5}$$

Proof. We shall prove (3.5) by induction on k. First, by (1.1), we have

$$L_{i,m} \cdot t = L_{i,m} \cdot L_{0,0} \cdot 1 = (t-i) L_{i,m} \cdot 1 + m L_{i,m-1} \cdot 1,$$

which implies that (3.5) holds for $k = 1$. Let $k \ge 2$. Assume that (3.5) holds for $k-1$, namely we have

$$L_{i,m} \cdot t^{k-1} = \sum_{s=0}^{\min\{m,k-1\}} s! \binom{m}{s} \binom{k-1}{s} (t-i)^{k-s-1} L_{i,m-s} \cdot 1. \tag{3.6}$$

Next, we prove that (3.5) holds for k in two cases.

Case 1: $m < k$. In this case, by (3.6), we have

$$\begin{aligned}
L_{i,m} \cdot t^k &= L_{i,m} \cdot L_{0,0} \cdot t^{k-1} = (t-i) L_{i,m} \cdot t^{k-1} + m L_{i,m-1} \cdot t^{k-1} \\
&= (t-i) \sum_{s=0}^{\min\{m,k-1\}} s! \binom{m}{s} \binom{k-1}{s} (t-i)^{k-s-1} L_{i,m-s} \cdot 1 \\
&\quad + m \sum_{s=0}^{\min\{m-1,k-1\}} s! \binom{m-1}{s} \binom{k-1}{s} (t-i)^{k-s-1} L_{i,m-s-1} \cdot 1 \\
&= (t-i)^k L_{i,m} \cdot 1 + \sum_{s=1}^{m} s! \binom{m}{s} \binom{k-1}{s} (t-i)^{k-s} L_{i,m-s} \cdot 1 \\
&\quad + m \sum_{s=0}^{m-1} s! \binom{m-1}{s} \binom{k-1}{s} (t-i)^{k-s-1} L_{i,m-s-1} \cdot 1 \\
&= (t-i)^k L_{i,m} \cdot 1 + \sum_{s=1}^{m} s! \binom{m}{s} \binom{k-1}{s} (t-i)^{k-s} L_{i,m-s} \cdot 1 \\
&\quad + \sum_{s=1}^{m} s! \binom{m}{s} \binom{k-1}{s-1} (t-i)^{k-s} L_{i,m-s} \cdot 1 \\
&= \sum_{s=0}^{m} s! \binom{m}{s} \binom{k}{s} (t-i)^{k-s} L_{i,m-s} \cdot 1.
\end{aligned}$$

Here we have used the combinatoric formula (cf. (1.2))

$$\binom{k-1}{s}+\binom{k-1}{s-1}=\binom{k}{s}$$

in the last equality.

Case 2: $m \geq k$. Similarly, in this case, we have

$$\begin{aligned}
L_{i,m}\cdot t^k &= L_{i,m}\cdot L_{0,0}\cdot t^{k-1}=(t-i)L_{i,m}\cdot t^{k-1}+mL_{i,m-1}\cdot t^{k-1}\\
&= (t-i)\sum_{s=0}^{\min\{m,k-1\}} s!\binom{m}{s}\binom{k-1}{s}(t-i)^{k-s-1}L_{i,m-s}\cdot 1\\
&\quad +m\sum_{s=0}^{\min\{m-1,k-1\}} s!\binom{m-1}{s}\binom{k-1}{s}(t-i)^{k-s-1}L_{i,m-s-1}\cdot 1\\
&= (t-i)^kL_{i,m}\cdot 1+\sum_{s=1}^{k-1}s!\binom{m}{s}\binom{k-1}{s}(t-i)^{k-s}L_{i,m-s}\cdot 1\\
&\quad +m\sum_{s=0}^{k-2}s!\binom{m-1}{s}\binom{k-1}{s}(t-i)^{k-s-1}L_{i,m-s-1}\cdot 1+m(k-1)!\binom{m-1}{k-1}L_{i,m-k}\cdot 1\\
&= (t-i)^kL_{i,m}\cdot 1+\sum_{s=1}^{k-1}s!\binom{m}{s}\binom{k-1}{s}(t-i)^{k-s}L_{i,m-s}\cdot 1\\
&\quad +\sum_{s=1}^{k-1}s!\binom{m}{s}\binom{k-1}{s-1}(t-i)^{k-s}L_{i,m-s}\cdot 1+m(k-1)!\binom{m-1}{k-1}L_{i,m-k}\cdot 1\\
&= \sum_{s=0}^{k}s!\binom{m}{s}\binom{k}{s}(t-i)^{k-s}L_{i,m-s}\cdot 1.
\end{aligned}$$

This completes the proof. □

Lemma 3.4 *There exists some $\beta\in\mathbb{C}$ such that $L_{i,m}\cdot 1=\lambda^i\beta^{m-1}(m\alpha-i\alpha\beta+\beta t)$.*

Proof. Denote $F_{i,m}(t)=L_{i,m}\cdot 1$. We determine $F_{i,m}(t)$ by induction on m.

The case for $m=0$ has been given by (3.4) with $k=0$ (note that, as before, if $m=\beta=0$ in this lemma, we have viewed $\beta^{-1}(m\alpha-i\alpha\beta+\beta t)=t-i\alpha$ as a whole).

Let us first determine $F_{i,1}(t)$. Applying

$$[L_{0,1},L_{i,0}]=iL_{i,1}+L_{i,0},\quad [L_{-i,0},L_{i,1}]=2iL_{0,1}-L_{0,0}$$

respectively on 1, by (3.4), we obtain

$$iF_{i,1}(t)=\lambda^i(t-i\alpha)F_{0,1}(t)-\lambda^i(t-i\alpha)F_{0,1}(t-i)+\lambda^i i\alpha, \tag{3.7}$$

$$2iF_{0,1}(t)=\lambda^{-i}(t+i\alpha)F_{i,1}(t+i)-\lambda^{-i}(t+i\alpha-i)F_{i,1}(t)+i\alpha. \tag{3.8}$$

Multiplying (3.8) by i, and then using the relation (3.7), we can derive that

$$2i^2\alpha = 2(t^2 - i^2\alpha^2 + i^2\alpha + i^2)F_{0,1}(t) - (t^2 + it - i^2\alpha^2 + i^2\alpha)F_{0,1}(t+i) \\ -(t^2 - it - i^2\alpha^2 + i^2\alpha)F_{0,1}(t-i). \tag{3.9}$$

If $F_{0,1}(t) = 0$, then by (3.7) and (3.9), we see that $F_{i,1}(t) = 0$ for all $i \in \mathbb{Z}$. This proves this lemma for the case $(\alpha, \beta) = (0,0)$. If $F_{0,1}(t) \neq 0$, we let $\deg F_{0,1}(t) = K$, and assume that

$$F_{0,1}(t) = \sum_{r=0}^{K} a_r t^r, \quad \text{where } a_r \in \mathbb{C} \text{ and } a_K \neq 0. \tag{3.10}$$

If $K = 0$, then $F_{0,1}(t) = a_0$. By (3.9) with $i = 1$, we have $a_0 = \alpha$, and thus $F_{0,1}(t) = \alpha$ $(\neq 0)$. If $K = 1$, then $F_{0,1}(t) = a_0 + a_1 t$. By (3.9), one can also derive that $a_0 = \alpha$. Redenote a_1 by β, we have $F_{0,1}(t) = \alpha + \beta t$. If $K \geq 2$, substituting (3.10) into (3.9) with $i = 1$ and comparing the coefficients of t^K on both sides, we obtain $(K^2 + K - 2)a_K = 0$, a contradiction. Hence, in general, we have $F_{0,1}(t) = \alpha + \beta t$. Then, by (3.7), we obtain $F_{i,1}(t) = \lambda^i(\alpha - i\alpha\beta + \beta t)$. This proves this lemma for the case $(\alpha, \beta) \neq (0,0)$.

Let $m \geq 1$. Assume that this lemma holds for $0 \leq k \leq m-1$, namely,

$$F_{i,k}(t) = \lambda^i \beta^{k-1}(k\alpha - i\alpha\beta + \beta t),\ 0 \leq k \leq m-1. \tag{3.11}$$

Next, we determine $F_{i,m}(t)$. Using the same arguments as for the case $F_{i,1}(t)$, by relations

$$[L_{0,m}, L_{i,0}] = iL_{i,m} + mL_{i,m-1}, \quad [L_{-i,0}, L_{i,m}] = 2iL_{0,m} - mL_{0,m-1}$$

and (3.11), one can derive that $F_{i,m}(t) = \lambda^i(m\alpha\beta^{m-1} - i\alpha b_m + b_m t)$ for some $b_m \in \mathbb{C}$. Then, applying

$$[L_{1,1}, L_{0,m-1}] = -L_{1,m} - (m-2)L_{1,m-1}$$

on 1, we obtain $b_m = \beta^m$, and thus $F_{i,m}(t) = \lambda^i \beta^{m-1}(m\alpha - i\alpha\beta + \beta t)$. This completes the proof. □

3.3 Simplicity and isomorphism classification of $\mathcal{W}(1)$-modules

Next, we determine the simplicity of $\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$. One will see that although the results are similar to Lemmas 2.2(1) and (2), the proofs, especially for Theorem 3.5(2), are rather non-trivial.

Theorem 3.5 (1) *$\Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$ is simple if and only if $\alpha \neq 0$.*

(2) *$\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta)$ has a unique proper submodule $t\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta) \cong \Omega_{\mathcal{W}(1)}(\lambda, 1, \beta)$, and the quotient $\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta)/t\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta)$ is a one-dimensional trivial $\mathcal{W}(1)$-module.*

Proof. (1) Let $M = \Omega_{\mathcal{W}(1)}(\lambda, \alpha, \beta)$. If $\alpha \neq 0$, by viewing M as a Vir-module, from Lemma 2.2(1), we see that M is simple. If $\alpha = 0$, one can easily see that tM is a submodule of M.

(2) From definition, one can easily see that $t\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta)$ is the unique proper submodule of $\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta)$. Next, we prove $t\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta) \cong \Omega_{\mathcal{W}(1)}(\lambda, 1, \beta)$ by comparing the actions of $\mathcal{W}(1)$ on these two modules.

First, we consider the action of $\mathcal{W}(1)$ on $t\Omega_{\mathcal{W}(1)}(\lambda, 0, \beta)$. For $k \geq 0$, by (3.1), we have

$$L_{i,m} \cdot (t \cdot t^k) = t \left(\sum_{s=0}^{\min\{m, k+1\}} s! \binom{m}{s} \binom{k+1}{s} \lambda^i \beta^{m-s} (t-i)^{k-s+1} \right). \tag{3.12}$$

Next, we consider the action of $\mathcal{W}(1)$ on $\Omega_{\mathcal{W}(1)}(\lambda, 1, \beta)$ in two cases. Let $k \geq 0$.

Case 1: $m \leq k$. In this case, by (3.1), we have

$$\begin{aligned}
L_{i,m} \cdot t^k &= \sum_{s=0}^{m} s! \binom{m}{s} \binom{k}{s} \lambda^i \beta^{m-s-1} (\beta(t-i) + (m-s))(t-i)^{k-s} \\
&= \sum_{s=0}^{m} s! \binom{m}{s} \binom{k}{s} \lambda^i \beta^{m-s} (t-i)^{k-s+1} + \sum_{s=0}^{m} s! \binom{m}{s} \binom{k}{s} \lambda^i \beta^{m-s-1} (m-s)(t-i)^{k-s} \\
&= \lambda^i \beta^m (t-i)^{k+1} + \sum_{s=1}^{m} s! \binom{m}{s} \binom{k}{s} \lambda^i \beta^{m-s} (t-i)^{k-s+1} \\
&\quad + \sum_{s=0}^{m-1} s! \binom{m}{s} \binom{k}{s} \lambda^i \beta^{m-s-1} (m-s)(t-i)^{k-s} \\
&= \lambda^i \beta^m (t-i)^{k+1} + \sum_{s=1}^{m} s! \binom{m}{s} \binom{k}{s} \lambda^i \beta^{m-s} (t-i)^{k-s+1} \\
&\quad + \sum_{s=1}^{m} (s-1)! \binom{m}{s-1} \binom{k}{s-1} \lambda^i \beta^{m-s} (m-s+1)(t-i)^{k-s+1} \\
&= \lambda^i \beta^m (t-i)^{k+1} + \sum_{s=1}^{m} s! \binom{m}{s} \binom{k}{s} \lambda^i \beta^{m-s} (t-i)^{k-s+1} \\
&\quad + \sum_{s=1}^{m} s! \binom{m}{s} \binom{k}{s-1} \lambda^i \beta^{m-s} (t-i)^{k-s+1} \\
&= \sum_{s=0}^{m} s! \binom{m}{s} \binom{k+1}{s} \lambda^i \beta^{m-s} (t-i)^{k-s+1}.
\end{aligned}$$

Here, we have again used the combinatoric formula (1.2) in the last equality.

Case 2: $m > k$. Similarly, in this case, we have

$$\begin{aligned}
L_{i,m} \cdot t^k &= \sum_{s=0}^{k} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s-1}(\beta(t-i)+(m-s))(t-i)^{k-s} \\
&= \sum_{s=0}^{k} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s}(t-i)^{k-s+1} + \sum_{s=0}^{k} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s-1}(m-s)(t-i)^{k-s} \\
&= \lambda^i\beta^m(t-i)^{k+1} + \sum_{s=1}^{k} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s}(t-i)^{k-s+1} \\
&\quad + \sum_{s=0}^{k-1} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s-1}(m-s)(t-i)^{k-s} + k!\binom{m}{k}\lambda^i\beta^{m-k-1}(m-k) \\
&= \lambda^i\beta^m(t-i)^{k+1} + \sum_{s=1}^{k} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s}(t-i)^{k-s+1} \\
&\quad + \sum_{s=1}^{k}(s-1)!\binom{m}{s-1}\binom{k}{s-1}\lambda^i\beta^{m-s}(m-s+1)(t-i)^{k-s+1} \\
&\quad + (k+1)!\binom{m}{k+1}\lambda^i\beta^{m-k-1} \\
&= \lambda^i\beta^m(t-i)^{k+1} + \sum_{s=1}^{k} s!\binom{m}{s}\binom{k}{s}\lambda^i\beta^{m-s}(t-i)^{k-s+1} \\
&\quad + \sum_{s=1}^{k} s!\binom{m}{s}\binom{k}{s-1}\lambda^i\beta^{m-s}(t-i)^{k-s+1} + (k+1)!\binom{m}{k+1}\lambda^i\beta^{m-k-1} \\
&= \sum_{s=0}^{k+1} s!\binom{m}{s}\binom{k+1}{s}\lambda^i\beta^{m-s}(t-i)^{k-s+1}.
\end{aligned}$$

Summarizing the above two cases, we have

$$L_{i,m} \cdot t^k = \sum_{s=0}^{\min\{m,\,k+1\}} s!\binom{m}{s}\binom{k+1}{s}\lambda^i\beta^{m-s}(t-i)^{k-s+1}. \tag{3.13}$$

Comparing (3.12) with (3.13), we see that $t\Omega_{\mathcal{W}(1)}(\lambda,0,\beta) \cong \Omega_{\mathcal{W}(1)}(\lambda,1,\beta)$.

At last, it is clear that the quotient $\Omega_{\mathcal{W}(1)}(\lambda,0,\beta)/t\Omega_{\mathcal{W}(1)}(\lambda,0,\beta)$ is a one-dimensional trivial $\mathcal{W}(1)$-module. This completes the proof. $\square$

The isomorphism classification of $\Omega_{\mathcal{W}(1)}(\lambda,\alpha,\beta)$ is as follows.

Theorem 3.6 $\Omega_{\mathcal{W}(1)}(\lambda_1,\alpha_1,\beta_1) \cong \Omega_{\mathcal{W}(1)}(\lambda_2,\alpha_2,\beta_2)$ *if and only if* $\lambda_1 = \lambda_2$, $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$.

Proof. Suppose that φ is a module isomorphism from $\Omega_{\mathcal{W}(1)}(\lambda_1, \alpha_1, \beta_1)$ to $\Omega_{\mathcal{W}(1)}(\lambda_2, \alpha_2, \beta_2)$. Denote by φ^{-1} the inverse of φ. Let $f(t) = \varphi^{-1}(1)$. Since $L_{0,0} \cdot t^k = t^{k+1}$, we have

$$1 = \varphi(f(t)) = \varphi(f(L_{0,0}) \cdot 1) = f(L_{0,0}) \cdot \varphi(1) = f(t)\varphi(1).$$

Hence, $\varphi(1) \in \mathbb{C}^*$. Computing $\varphi(L_{i,0} \cdot 1)$, we have

$$\varphi(L_{i,0} \cdot 1) = \varphi(\lambda_1^i(t - i\alpha_1)) = \varphi(\lambda_1^i(L_{0,0} - i\alpha_1) \cdot 1) = \lambda_1^i(L_{0,0} - i\alpha_1) \cdot \varphi(1) = \lambda_1^i(t - i\alpha_1)\varphi(1).$$

On the other hand, we have (note that $\varphi(1) \in \mathbb{C}^*$)

$$\varphi(L_{i,0} \cdot 1) = L_{i,0} \cdot \varphi(1) = \lambda_2^i(t - i\alpha_2)\varphi(1).$$

Comparing the above two formulas, we must have $\lambda_1 = \lambda_2$ and $\alpha_1 = \alpha_2$. Similarly, computing $\varphi(L_{i,1} \cdot 1)$, one can derive that

$$\lambda_1^i(\alpha_1 - i\alpha_1\beta_1 + \beta_1 t)\varphi(1) = \lambda_2^i(\alpha_2 - i\alpha_2\beta_2 + \beta_2 t)\varphi(1),$$

which implies that $\beta_1 = \beta_2$. This completes the proof. □

4 Free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(-1)$

Recall that $\mathfrak{h} = \mathbb{C}L_{0,0}$. In this section, we completely classify the free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(-1)$. Comparing with Section 3, one will see that the problem becomes more difficult, especially when determining the actions on 1 (one can compare the following Lemma 4.4 with Lemma 3.4). Thus we need more technical analysis.

4.1 Construction of free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(-1)$

Let $\lambda \in \mathbb{C}^*$ and $\alpha, \beta \in \mathbb{C}$. We define the action of $\mathcal{W}(-1)$ on the vector space of polynomials in one variable

$$\Omega_{\mathcal{W}(-1)}(\lambda, \alpha, \beta) := \mathbb{C}[t]$$

by

$$L_{i,m} \cdot t^k = \sum_{s=0}^{k} (-1)^s s! \binom{m+s-1}{s} \binom{k}{s} \lambda^i \beta^{m+s} (t - i\alpha - (m+s)\alpha\beta)(t-i)^{k-s}, \tag{4.1}$$

where $i \in \mathbb{Z}$, $m \in \mathbb{Z}_+$ and $t^k \in \mathbb{C}[t]$. It should note here that $\binom{-1}{0} = 1$ and $\binom{n-1}{n} = 0$ if $n > 0$. This guarantees that $\Omega_{\mathcal{W}(-1)}(\lambda, \alpha, \beta)$ is a $\mathcal{U}(\mathfrak{h})$-module, which is free of rank one. Similar to Proposition 3.1, one can further prove that $\Omega_{\mathcal{W}(-1)}(\lambda, \alpha, \beta)$ is a $\mathcal{W}(-1)$-module; the details are omitted.

Proposition 4.1 *The space $\Omega_{\mathcal{W}(-1)}(\lambda, \alpha, \beta)$ is a $\mathcal{W}(-1)$-module under the action* (4.1).

4.2 Classification of free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(-1)$

The following is our second main result, which states that $\Omega_{\mathcal{W}(-1)}(\lambda,\alpha,\beta)$ exhausts all free $\mathcal{U}(\mathfrak{h})$-modules of rank one over $\mathcal{W}(-1)$.

Theorem 4.2 *Any free $\mathcal{U}(\mathfrak{h})$-module of rank one over $\mathcal{W}(-1)$ is isomorphic to $\Omega_{\mathcal{W}(-1)}(\lambda,\alpha,\beta)$ defined by* (4.1) *for some $\lambda\in\mathbb{C}^*$ and $\alpha,\beta\in\mathbb{C}$.*

Proof. Let M be a free $\mathcal{U}(\mathfrak{h})$-module of rank one over $\mathcal{W}(-1)$. By viewing M as an Vir-module, from Lemma 2.1, we may assume that $M=\mathbb{C}[t]$ and there exist some $\lambda\in\mathbb{C}^*$ and $\alpha\in\mathbb{C}$ such that the action of Vir $\subseteq\mathcal{W}(-1)$ on M is as follows

$$L_{i,0}\cdot t^k=\lambda^i(t-i\alpha)(t-i)^k. \tag{4.2}$$

We reduce the remaining proof to the following Lemma 4.3 and Lemma 4.4. $\square$

Lemma 4.3 *The action of $L_{i,m}$ on t^k is a linear combination of the actions of $L_{i,j}$ $(m\le j\le m+k)$ on* 1*, and more precisely we have*

$$L_{i,m}\cdot t^k=\sum_{s=0}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}(t-i)^{k-s}L_{i,m+s}\cdot 1. \tag{4.3}$$

Proof. The proof is similar to that of Lemma 3.3 and is omitted. $\square$

Lemma 4.4 *There exists some $\beta\in\mathbb{C}$ such that $L_{i,m}\cdot 1=\lambda^i\beta^m(t-i\alpha-m\alpha\beta)$.*

Proof. Denote $G_{i,m}(t)=L_{i,m}\cdot 1$. Since the Lie structure of $\mathcal{W}(-1)$ is essentially different from that of $\mathcal{W}(1)$, one cannot prove this lemma by induction on m as in Lemma 3.4. In fact, the situation becomes more difficult, and thus we need more technical analysis. To simplify the proof, we shall use the shifted notation $Y_{i,m}(t)=\lambda^{-i}G_{i,m}(t)$.

Let $N_i=\deg Y_{i,1}(t)$, $i\in\mathbb{Z}$. Assume that

$$Y_{i,1}(t)=\sum_{r=0}^{N_i}a_r^{(i)}t^r,\quad \text{where}\ \ a_r^{(i)}\in\mathbb{C}\ \text{and}\ a_{N_i}^{(i)}\neq 0. \tag{4.4}$$

Let $f(x_1,x_2,\ldots,x_n)\in\mathbb{C}[x_1,x_2,\ldots,x_n]$ be a polynomial in n variables. In the following, we shall use the notation $\mathrm{Coeff}_{f(x_1,x_2,\ldots,x_n)}x_r^i$ to denote the coefficient of x_r^i in $f(x_1,x_2,\ldots,x_n)\in\mathbb{C}[x_r]$. In particular, if $i=0$, we use it to denote the constant term.

Claim 1 *We have $N_i=N_0$ and $a_{N_i}^{(i)}=a_{N_0}^{(0)}$ for all $i\in\mathbb{Z}$. We shall simply denote N_0 by N, and $a_{N_0}^{(0)}$ by a_N.*

Applying

$$[L_{j,1}, L_{i-j,0}] = (i-2j)L_{i,1} - L_{i,2}, \quad [L_{-j,0}, L_{j,1}] = 2jL_{0,1} + L_{0,2} \tag{4.5}$$

respectively on 1, by (4.2), we obtain

$$(t-j-(i-j)\alpha)Y_{j,1}(t)-(t-(i-j)\alpha)Y_{j,1}(t-i+j)-Y_{j,2}(t) = (i-2j)Y_{i,1}(t) - Y_{i,2}(t), \tag{4.6}$$

$$(t+j\alpha)Y_{j,1}(t+j) - (t-j+j\alpha)Y_{j,1}(t) + Y_{j,2}(t) = 2jY_{0,1}(t) + Y_{0,2}(t). \tag{4.7}$$

Taking $j = 0$ in (4.6), we have

$$(t-i\alpha)Y_{0,1}(t) - (t-i\alpha)Y_{0,1}(t-i) - Y_{0,2}(t) = iY_{i,1}(t) - Y_{i,2}(t). \tag{4.8}$$

Computing (4.6) + (4.7) − (4.8), we obtain

$$\begin{aligned}&(t+j\alpha)Y_{j,1}(t+j) - i\alpha Y_{j,1}(t) - (t-(i-j)\alpha)Y_{j,1}(t-i+j) + 2jY_{i,1}(t)\\ =\ &(t-i\alpha+2j)Y_{0,1}(t) - (t-i\alpha)Y_{0,1}(t-i).\end{aligned} \tag{4.9}$$

Taking $j = i$ in (4.9), we have

$$(t+i\alpha)Y_{i,1}(t+i) - (t+i\alpha-2i)Y_{i,1}(t) = (t-i\alpha+2i)Y_{0,1}(t) - (t-i\alpha)Y_{0,1}(t-i). \tag{4.10}$$

Let $\mathbf{L}_1(t)$ and $\mathbf{R}_1(t)$ denote respectively the left- and right-hand sides of (4.10). Considering the coefficients of the highest degree terms, we have

$$\mathrm{Coeff}_{\mathbf{L}_1(t)} t^{N_i} = (N_i+2)ia^{(i)}_{N_i}, \quad \mathrm{Coeff}_{\mathbf{R}_1(t)} t^{N_0} = (N_0+2)ia^{(0)}_{N_0}.$$

By (4.10) with $i \neq 0$, we must have $N_i = N_0$ and $a^{(i)}_{N_i} = a^{(0)}_{N_0}$. Namely, Claim 1 holds.

Claim 2 *The situation $N \geq 3$ is impossible.*

Let $N \geq 3$. Taking $j = 1$ in (4.9), we have

$$\begin{aligned}Y_{i,1}(t) = \ &\frac{1}{2}\big((t-i\alpha+2)Y_{0,1}(t) - (t-i\alpha)Y_{0,1}(t-i) - (t+\alpha)Y_{1,1}(t+1)\\ &+i\alpha Y_{1,1}(t) + (t-(i-1)\alpha)Y_{1,1}(t-i+1)\big).\end{aligned} \tag{4.11}$$

Note that if we substitute (4.11) into (4.9), then we obtain an equation on functions $Y_{0,1}$ and $Y_{1,1}$, and both sides can be viewed as functions on i, j and t. Let $\mathbf{L}_2(i,j,t)$ and $\mathbf{R}_2(i,j,t)$ denote respectively the left- and right-hand sides of (4.9). By Claim 1, a direct computation shows that there always exist $K_{N-2}(i,j), K_0(i,j) \in \mathbb{C}[i,j]$ such that

$$\begin{aligned}\mathrm{Coeff}_{\mathbf{L}_2(i,j,t)-\mathbf{R}_2(i,j,t)} t^{N-2} &= \frac{1}{2}ijK_{N-2}(i,j),\\ \mathrm{Coeff}_{\mathbf{L}_2(i,j,t)-\mathbf{R}_2(i,j,t)} t^{0} &= \frac{1}{2}ij\alpha K_0(i,j).\end{aligned}$$

By (4.9), we have $K_{N-2}(i,j)=0$ if $ij\neq 0$, and $K_0(i,j)=0$ if $ij\alpha\neq 0$. Through a more detailed analysis, we observe that $\deg K_{N-2}(i,j)\leq 1$ and $\deg K_0(i,j)\leq N-1$. Since one can first fix $i\neq 0$ (resp., $j\neq 0$) and then let $j\to\infty$ (resp., $i\to\infty$), the above two observations imply that (see Example 4.5 for concrete equations in the case of $N=3$)

Case 1: $\alpha\neq 0$.

$$\begin{cases} \mathrm{Coeff}_{K_{N-2}(i,j)} i = 0, \\ \mathrm{Coeff}_{K_{N-2}(i,j)} j = 0, \\ \mathrm{Coeff}_{K_0(i,j)} i^{N-1} = 0, \\ \mathrm{Coeff}_{K_0(i,j)} j^{N-1} = 0; \end{cases}$$

Case 2: $\alpha = 0$.

$$\begin{cases} \mathrm{Coeff}_{K_{N-2}(i,j)} i\big|_{\alpha=0} = 0, \\ \mathrm{Coeff}_{K_{N-2}(i,j)} j\big|_{\alpha=0} = 0. \end{cases}$$

In both cases, one can derive that $a_N=0$, which contradicts to our previous assumption (4.4). Hence, Claim 2 holds.

Claim 3 *The situation $N=2$ is impossible.*

Let $N=2$. Recall Claim 1. In this case, we have

$$Y_{i,1}(t) = a_2t^2 + a_1^{(i)}t + a_0^{(i)},$$

which is essentially derived from relations (4.5). Similarly, start from relations

$$[L_{j,2}, L_{i-j,0}] = (i-2j)L_{i,2} - 2L_{i,3}, \quad [L_{-j,0}, L_{j,2}] = 2jL_{0,2} + 2L_{0,3}, \tag{4.12}$$

one can derive that

$$Y_{i,2}(t) = b_2t^2 + b_1^{(i)}t + b_0^{(i)}.$$

By comparing the coefficients of t^2 on both sides of (4.8) with $i=1$, we can derive that $a_2=0$, a contradiction. Hence, Claim 3 holds.

Claim 4 *If $N\leq 1$, then $Y_{i,m}(t)=\beta_m(t-i\alpha)+\gamma_m$, where $\beta_m,\gamma_m\in\mathbb{C}$ and $\beta_0=1,\gamma_0=0$.*

Let $N\leq 1$. Recall Claim 1. In this case, we have

$$Y_{i,1}(t) = a_1t + a_0^{(i)}.$$

By (4.10), we can derive that $a_0^{(i)} = a_0^{(0)} - i\alpha a_1$. Redenote a_1 by β_1, and $a_0^{(0)}$ by γ_1. The above formula becomes

$$Y_{i,1}(t) = \beta_1(t-i\alpha) + \gamma_1.$$

Generally, start from relations (cf. (4.5) for the case $m = 1$, and (4.12) for the case $m = 2$)

$$[L_{j,m}, L_{i-j,0}] = (i - 2j)L_{i,m} - mL_{i,m+1}, \quad [L_{-j,0}, L_{j,m}] = 2jL_{0,m} + mL_{0,m+1},$$

one can derive that $Y_{i,m}(t) = \beta_m(t - i\alpha) + \gamma_m$ for some $\beta_m, \gamma_m \in \mathbb{C}$. Finally, from (4.2) we see that $\beta_0 = 1$ and $\gamma_0 = 0$. This completes the proof of Claim 4.

Claim 5 *We have $\beta_m = \beta^m$ and $\gamma_m = -m\alpha\beta^{m+1}$, where $\beta \in \mathbb{C}$.*

First, let us determine β_m. By Claim 4, applying the relation

$$[L_{0,m}, L_{1,n}] = L_{1,m+n} + (n - m)L_{1,m+n+1}$$

on 1, one can obtain an equation on β_m and γ_m. By comparing the coefficients of t on both sides of this equation, we obtain

$$\beta_{m+n} + (n - m)\beta_{m+n+1} = \beta_m\beta_n + n\beta_m\beta_{n+1} - m\beta_n\beta_{m+1}.$$

Recall that $\beta_0 = 1$ (by Claim 4). By Lemma 2.3, we have $\beta_m = \beta^m$, where $\beta = \beta_1$.

Next, we determine γ_m. By Claim 4, applying the relation $(i \neq 0)$

$$[L_{0,m}, L_{i,0}] = iL_{i,m} - mL_{i,m+1}$$

on 1, one can easily obtain $\gamma_m = -m\alpha\beta^{m+1}$. This completes the proof of Claim 5.

Finally, from Claims 1–5 we see that this lemma holds. □

Example 4.5 For the convenience of the reader we explicitly write the systems of equations in Cases 1 and 2 in Claim 2 under the assumption $N = 3$. If $\alpha \neq 0$, the system of equations is

$$\begin{cases} (1+2\alpha)\Big(2(a_2^{(0)} - a_2^{(1)}) - 3(1+\alpha)a_3^{(0)}\Big) = 0, \\ (1+2\alpha)(a_2^{(0)} - a_2^{(1)}) - (1+3\alpha+8\alpha^2)a_3^{(0)} = 0, \\ 2(a_2^{(0)} - a_2^{(1)}) - 3(1+\alpha)a_3^{(0)} = 0, \\ (1-\alpha)(a_2^{(0)} - a_2^{(1)}) - a_3^{(0)} = 0; \end{cases}$$

and if $\alpha = 0$, the system of equations is

$$\begin{cases} 2(a_2^{(0)} - a_2^{(1)}) - 3a_3^{(0)} = 0, \\ (a_2^{(0)} - a_2^{(1)}) - a_3^{(0)} = 0. \end{cases}$$

It is straightforward to check that both cases yield $a_3 = a_3^{(0)} = 0$.

4.3 Simplicity and isomorphism classification of $\mathcal{W}(-1)$-modules

Similar to Theorem 3.5, we have the following result on the simplicity of $\Omega_{\mathcal{W}(-1)}(\lambda,\alpha,\beta)$.

Theorem 4.6 (1) *$\Omega_{\mathcal{W}(-1)}(\lambda,\alpha,\beta)$ is simple if and only if $\alpha\neq 0$.*

(2) *$\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)$ has a unique proper submodule $t\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)\cong\Omega_{\mathcal{W}(-1)}(\lambda,1,\beta)$, and the quotient $\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)/t\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)$ is a one-dimensional trivial $\mathcal{W}(-1)$-module.*

Proof. (1) Let $M=\Omega_{\mathcal{W}(-1)}(\lambda,\alpha,\beta)$. If $\alpha\neq 0$, by viewing M as a Vir-module, from Lemma 2.2(1), we see that M is simple. If $\alpha=0$, one can easily see that tM is a submodule of M.

(2) From definition, one can easily see that $t\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)$ is the unique proper submodule of $\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)$. Next, we prove $t\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)\cong\Omega_{\mathcal{W}(-1)}(\lambda,1,\beta)$ by comparing the actions of $\mathcal{W}(-1)$ on these two modules.

First, we consider the action of $\mathcal{W}(-1)$ on $t\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)$. For $k\geq 0$, by (4.1), we have

$$L_{i,m}\cdot(t\cdot t^k)=t\left(\sum_{s=0}^{k+1}(-1)^s s!\binom{m+s-1}{s}\binom{k+1}{s}\lambda^i\beta^{m+s}(t-i)^{k-s+1}\right). \tag{4.13}$$

Next, we consider the action of $\mathcal{W}(-1)$ on $\Omega_{\mathcal{W}(-1)}(\lambda,1,\beta)$. Let $k\geq 0$. By (4.1), we have

$$\begin{aligned}
L_{i,m}\cdot t^k &= \sum_{s=0}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}\lambda^i\beta^{m+s}(t-i-(m+s)\beta)(t-i)^{k-s}\\
&= \sum_{s=0}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}\lambda^i\beta^{m+s}(t-i)^{k-s+1}\\
&\quad -\sum_{s=0}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}\lambda^i\beta^{m+s+1}(m+s)(t-i)^{k-s}\\
&= \lambda^i\beta^m(t-i)^{k+1}+\sum_{s=1}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}\lambda^i\beta^{m+s}(t-i)^{k-s+1}\\
&\quad -\sum_{s=0}^{k-1}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}\lambda^i\beta^{m+s+1}(m+s)(t-i)^{k-s}\\
&\quad -(-1)^k k!\binom{m+k-1}{k}\lambda^i\beta^{m+k+1}(m+k)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^i\beta^m(t-i)^{k+1}+\sum_{s=1}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}\lambda^i\beta^{m+s}(t-i)^{k-s+1}\\
&\quad+\sum_{s=1}^{k}(-1)^s(s-1)!\binom{m+s-2}{s-1}\binom{k}{s-1}\lambda^i\beta^{m+s}(m+s-1)(t-i)^{k-s+1}\\
&\quad+(-1)^{k+1}(k+1)!\binom{m+k}{k+1}\lambda^i\beta^{m+k+1}\\
&= \lambda^i\beta^m(t-i)^{k+1}+\sum_{s=1}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s}\lambda^i\beta^{m+s}(t-i)^{k-s+1}\\
&\quad+\sum_{s=1}^{k}(-1)^s s!\binom{m+s-1}{s}\binom{k}{s-1}\lambda^i\beta^{m+s}(t-i)^{k-s+1}\\
&\quad+(-1)^{k+1}(k+1)!\binom{m+k}{k+1}\lambda^i\beta^{m+k+1}\\
&= \sum_{s=0}^{k+1}(-1)^s s!\binom{m+s-1}{s}\binom{k+1}{s}\lambda^i\beta^{m+s}(t-i)^{k-s+1}. \qquad (4.14)
\end{aligned}
$$

Comparing (4.13) with (4.14), we see that $t\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)\cong\Omega_{\mathcal{W}(-1)}(\lambda,1,\beta)$.

At last, it is clear that the quotient $\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)/t\Omega_{\mathcal{W}(-1)}(\lambda,0,\beta)$ is a one-dimensional trivial $\mathcal{W}(-1)$-module. This completes the proof. $\square$

Similar to Theorem 3.6, one can also prove the following result on the isomorphism classification of $\Omega_{\mathcal{W}(-1)}(\lambda,\alpha,\beta)$; the details are omitted.

Theorem 4.7 $\Omega_{\mathcal{W}(-1)}(\lambda_1,\alpha_1,\beta_1)\cong\Omega_{\mathcal{W}(-1)}(\lambda_2,\alpha_2,\beta_2)$ *if and only if* $\lambda_1=\lambda_2$, $\alpha_1=\alpha_2$ *and* $\beta_1=\beta_2$.

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