ON A SERIES OF SIMPLE AFFINE VOAS AT NON-ADMISSIBLE LEVEL ARISING FROM RANK ONE 4D SCFTS

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ABSTRACT. We study the representations of the simple affine vertex algebras at non-admissible level arising from rank one 4D SCFTs. In particular, we classify the irreducible highest weight modules of $L_{-2}(G_2)$ and $L_{-2}(B_3)$. It is known by the works of Adamović and Perše that these vertex algebras can be conformally embedded into $L_{-2}(D_4)$. We also compute the associated variety of $L_{-2}(G_2)$, and show that it is the orbifold of the associated variety of $L_{-2}(D_4)$ by the symmetric group of degree 3 which is the Dynkin diagram automorphism group of D_4. This provides a new interesting example of associated variety satisfying a number of conjectures in the context of orbifold vertex algebras.

1. INTRODUCTION

Throughout this article, all Lie algebras are defined over $\mathbb{C}$ and all topological terms refer to the Zariski topology.

It is well known that the automorphisms of the Dynkin diagrams of the classical simple Lie algebras $B_3 = \mathfrak{so}_7(\mathbb{C})$ and $D_4 = \mathfrak{so}_8(\mathbb{C})$ induce embeddings from the exceptional simple Lie algebra G_2 into B_3, and from B_3 into D_4:

$$G_2 \overset{\iota_2}{\hookrightarrow} B_3 \overset{\iota_3}{\hookrightarrow} D_4. \tag{1}$$

For an arbitrary simple Lie algebra $\mathfrak{g}$, consider the corresponding extended affine Kac–Moody Lie algebra

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

with usual Lie bracket, see Section 2. Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with $\mathfrak{g}$ at level k, and $L_k(\mathfrak{g})$ its simple quotient. It will be always assumed in the article that $k \neq -h_{\mathfrak{g}}^\vee$ is not critical, where $h_{\mathfrak{g}}^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Thus $V^k(\mathfrak{g})$ is conformal with conformal grading given by the semisimple element $L_0 = -D$.

The embeddings (1) induce embeddings for the corresponding universal affine vertex algebras at any level:

$$V^k(G_2) \hookrightarrow V^k(B_3) \hookrightarrow V^k(D_4),$$

but this does not hold in general for the simple quotients. However, remarkably, the following conformal embeddings at negative integer level $k = -2$ were established by Adamović and Perše in [AP]:

$$L_{-2}(G_2) \hookrightarrow L_{-2}(B_3) \hookrightarrow L_{-2}(D_4). \tag{2}$$

Here a conformal vertex algebra U is said to be *conformally embedded* into a conformal vertex algebra V if U can be realized as a vertex subalgebra of V with the same conformal vector. Moreover, Adamović and Perše proved that $L_{-2}(D_4)$ is a finite extension of $L_{-2}(B_3)$ and that $L_{-2}(B_3)$ is a finite extension of $L_{-2}(G_2)$.

1.1. **Main results.** In this article, we are interested in the representations in the category $\mathcal{O}$ of the simple affine vertex algebras $L_{-2}(G_2)$ and $L_{-2}(B_3)$; those of $L_{-2}(D_4)$ were previously studied in [AM1]. We also compute the associated variety of $L_{-2}(G_2)$; that of $L_{-2}(B_3)$ and of $L_{-2}(D_4)$ were described in [AM1, AM3].

Recall that to an arbitrary vertex algebra V one attaches, in a functorial manner, a certain affine Poisson variety X_V referred to as the *associated variety* [Ar1], see Section 2.3. To describe our result about the associated variety, note that the embedding $\iota_2 \colon G_2 \hookrightarrow D_4$ induces a projection map $D_4^* \twoheadrightarrow G_2^*$. Hence we get a linear map

$$\pi_2 \colon D_4 \longrightarrow\!\!\!\!\!\rightarrow G_2,$$

identifying D_4 and G_2 with their duals through their respective Killing forms. More concretely, given $x \in D_4$, then $\pi_2(x)$ is the unique element of G_2 defined by $\kappa_{G_2}(\pi_2(x), y) = \kappa_{D_4}(x, y)$ for all y in G_2, where $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. Denoting by $\mathbb{O}_{\text{sreg}}$ the subregular nilpotent orbit in G_2, by $\mathbb{O}_{\min}$ the minimal nilpotent orbit in D_4 and by $\overline{\mathbb{O}}_{\text{sreg}}$, $\overline{\mathbb{O}}_{\min}$ their Zariski closures, we have by [LS],

$$\overline{\mathbb{O}}_{\text{sreg}} = \pi_2(\overline{\mathbb{O}}_{\min}). \tag{3}$$

Note that $\mathbb{O}_{\text{sreg}}$ and $\mathbb{O}_{\min}$ have both dimension 10. Furthermore, by [AM1], $\overline{\mathbb{O}}_{\min}$ is precisely the associated variety of the vertex algebra $L_{-2}(D_4)$.

The following result was conjectured in [F2, Conjecture 4.5], and agrees with the physical expectation [LXY, Table 4], see also Section 1.2 below.

Theorem A. *The associated variety of $L_{-2}(G_2)$ is $\overline{\mathbb{O}}_{\text{sreg}}$.*

Likewise, the embedding $\iota_3 \colon B_3 \hookrightarrow D_4$ induces a linear projection map

$$\pi_3 \colon D_4 \longrightarrow\!\!\!\!\!\rightarrow B_3.$$

The associated variety of $L_{-2}(B_3)$ was obtained in [AM3]:

$$X_{L_{-2}(B_3)} = \overline{\mathbb{O}}_{\text{short}} = \pi_3(\overline{\mathbb{O}}_{\min}), \tag{4}$$

where $\mathbb{O}_{\text{short}}$ is the unique short nilpotent orbit in B_3 and $\overline{\mathbb{O}}_{\text{short}}$ is its Zariski closure. Here, a nilpotent element f of a simple Lie algebra $\mathfrak{g}$ is called *short* if for (e, f, h) an $\mathfrak{sl}_2$-triple,

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_j = \{x \in \mathfrak{g} \colon [h, x] = 2jx\}$. In $B_3 = \mathfrak{so}_7(\mathbb{C})$, the partition corresponding to $\mathbb{O}_{\text{short}}$ is $(3, 1^4)$ and $\mathbb{O}_{\text{short}}$ has dimension 10, too. Similarly to the computations of $X_{L_{-2}(D_4)}$ and $X_{L_{-2}(B_3)}$, the proof of Theorem A is based on the analysis of singular vectors and the theory of $\mathcal{W}$-algebras. Here, obtaining a singular vector is much harder, and the novelty is the use of the explicit OPE's between the generators of the subregular $\mathcal{W}$-algebra in G_2 computed in [F2]. It was observed in [F2, Corollary 4.2] that $\mathcal{W}_{-2}(G_2, f_{\text{sreg}}) \cong \mathbb{C}$ which prompted to conjecture $X_{L_{-2}(G_2)} = \overline{\mathbb{O}}_{\text{sreg}}$.

Our next results give a complete classification of the simple highest-weight $L_{-2}(\mathfrak{g})$-modules and the simple ordinary $L_{-2}(\mathfrak{g})$-modules for $\mathfrak{g} = G_2$ and $\mathfrak{g} = B_3$. Here, a module is called *ordinary* if L_0 acts semisimply on it with finite-dimensional graded components and a grading bounded from below. Let us denote by $L_{\mathfrak{g}}(k, \mu)$ the irreducible highest-weight modules of $\widetilde{\mathfrak{g}}$ at level k with highest-weight $\mu + k\Lambda_0$, where μ is in the dual of the Cartan subalgebra of $\mathfrak{g}$ and Λ_0 is the dual of the central element K in the dual of the Cartan of $\widetilde{\mathfrak{g}}$.

Theorem B. *The set* $\{L_{G_2}(-2,\mu_i)\colon i = 1,\dots,20\}$, *where the* μ_i*'s are given by Proposition 3.4, provides the complete list of irreducible* $L_{-2}(G_2)$*-modules from the category* $\mathcal{O}$. *Among them,* $L_{G_2}(-2,0)$, $L_{G_2}(-2,\varpi_1)$ *and* $L_{G_2}(-2,\varpi_2)$ *are precisely the irreducible ordinary modules of* $L_{-2}(G_2)$.

Exploiting our singular vector in $V^{-2}(G_2)$ and the notion of subsingular vectors (see Definition 2.1) in $V^{-2}(B_3)$, we succeed to describe the maximal ideal of $V^{-2}(B_3)$. This leads us to the following classification result.

Theorem C. *The set* $\{L_{B_3}(-2,\mu_i)\colon i = 1,\dots,13\}$, *where the* μ_i*'s are given by Proposition 5.7, provides the complete list of irreducible* $L_{-2}(B_3)$*-modules from the category* $\mathcal{O}$. *Among them,* $L_{B_3}(-2,0)$ *and* $L_{B_3}(-2,\varpi_1)$ *are precisely the irreducible ordinary modules for* $L_{-2}(B_3)$.

We also establish the following result.

Theorem D. *We have the following decomposition*

$$L_{D_4}(-2,-2\varpi_1) = L_{B_3}(-2,-2\varpi_1) \oplus L_{B_3}(-2,-3\varpi_1)$$

as $L_{-2}(B_3)$*-modules.*

Note that $L_{D_4}(-2,-2\varpi_1)$ is not an ordinary module since its L_0-eigenspaces are not finite-dimensional. Since $L_{-2}(B_3) \hookrightarrow L_{-2}(D_4)$ is a conformal embedding and since both $L_{B_3}(-2,-2\varpi_1)$ and $L_{B_3}(-2,-3\varpi_1)$ have the same conformal dimension -1 (see the formula (7)), Theorem D in particular implies the non-trivial decomposition

$$L_{D_4}(-2\varpi_1) = L_{B_3}(-2\varpi_1) \oplus L_{B_3}(-3\varpi_1)$$

of an infinite-dimensional representation of the finite-dimensional Lie algebra B_3, where $L_{\mathfrak{g}}(\mu)$ denotes the irreducible highest-weight modules of $\mathfrak{g}$ with highest-weight μ. The authors do not know whether this has been known in the literature.

1.2. **Motivations from physics.** The vertex algebras $L_{-2}(G_2)$, $L_{-2}(B_3)$ and $L_{-2}(D_4)$ are neither rational nor lisse. Therefore, they are not related in any sense with rational conformal field theories in two dimensions. However, they are remarkably related with superconformal field theories in *four* dimensions, via the 4D/2D correspondence discovered in [BLL+].

In more details, for any four-dimensional $\mathcal{N} = 2$ superconformal field theory (SCFT), there is a subsector which can be described by a two-dimensional vertex operator algebra (VOA). The normalized character of the corresponding VOA reproduces the special limit of the superconformal index, called the *Schur index*. On the one hand, four-dimensional SCFTs lead to some interesting conjectures for large classes of VOAs. For example, it is expected [BR] that the Higgs branch of such a 4D theory is the associated variety X_V of the corresponding VOA V. On the other hand, the representation theory of the VOA produces new physical observables of the 4D SCFT, such as the ordinary Schur index and the Schur index in the presence of boundary conditions, line defects and surface defects.

One of the major advancement in the last ten years is that one can engineer a large class of new 4D SCFTs by geometric methods.

The classification of $\mathcal{N} = 2$ rank one SCFTs have been studied based on the analysis of their Coulomb branch geometries and all possible deformations of planar special Kähler singularities, labeled by their Kodaira type which are consistent with the low energy Dirac quantization condition [AL+1, AL+2]. One particularly interesting class of theories in these frameworks is the class of Argyres–Douglas theories [DG, DX] which cannot be studied like usual quantum field theory, since they are strongly coupled interacting 4D SCFT which have no known Lagrangian description in general.

In [LXY], the authors found a universal formula for the rank of the theory so that a complete search is possible. They listed all rank one, rank two, rank three Argyres–Douglas theories based on this formula and found the corresponding VOA and the associated Higgs branch for these theories. This classification gives some very interesting rank one SCFTs such that the Higgs branches are not given by one-instanton moduli spaces on $\mathbb{R}^4$ for a flavor symmetry group G. All these rank one Argyres–Douglas theories coincide with [AL+2] but arise from entirely different constructions. For exemple, the simple affine vertex algebras $L_{-2}(G_2)$ and $L_{-2}(B_3)$ appear as the vertex operator algebras corresponding to rank one Argyres–Douglas theories in four dimension with flavour symmetry G_2 and B_3.

Consequently, one expects [SXY] that the representations of these vertex algebras are closely connected with the Coulomb branch of the circle compactified corresponding 4D theory; for $L_{-2}(D_4)$, the corresponding Coulomb branch is related to [GMN] the moduli space of the SL_2-Higgs bundles on the sphere with four punctures. However, for the other two theories it seems there are no precise description of the corresponding Coulomb branches at the moment.

In this context, studying the representation theory of these simple affine vertex algebras becomes very important. For example, the decomposition in Theorem D suggests the decomposability of generalized Schur index.

1.3. **Connections with mathematical conjectures.** First of all, as a negative integer, -2 is a not an admissible level for G_2. Therefore, Theorem A provides a new example of a vertex algebra whose associated variety has a finite number of symplectic leaves outside the admissible levels. Indeed, in the setting of affine vertex algebras associated with $\mathfrak{g}$, this condition is equivalent to that of being contained in the nilpotent cone of $\mathfrak{g}$ (see for instance [AM4, Proposition 12.1]), and the symplectic leaves are nothing but the coadjoint orbits of $\mathfrak{g}^*$, identified with the adjoint orbits of $\mathfrak{g}$ through the Killing form.

Vertex algebras whose associated variety has a finite number of symplectic leaves are referred to as *quasi-lisse vertex algebras* [AK], the lisse ones corresponding to the case where the associated variety has dimension zero. The following was conjectured in [AM3].

Conjecture A. *If V is a simple quasi-lisse conformal vertex algebra, then X_V is irreducible.*

Theorem A thus gives a new example where Conjecture A holds.

Our result is also interesting in the context of orbifold vertex algebras. By (3), $\overline{\mathbb{O}}_{\text{sreg}}$ is the orbifold of $\overline{\mathbb{O}}_{\text{min}}$ by the symmetric group $\mathfrak{S}_3$ of degree 3 which is the group of automorphisms of the Dynkin diagram of D_4. The group $\mathfrak{S}_3$ naturally acts on $V^k(D_4)$ at any level. According to [AP], this action passes through the quotient $L_{-2}(D_4)$ at the level -2 and, remarkably, we have $L_{-2}(G_2) = L_{-2}(D_4)^{\mathfrak{S}_3}$, where for V a vertex algebra and G a finite subgroup of the automorphism group of V, V^G denotes the fixed point vertex subalgebra. Hence, Theorem A can be reformulated as follows:

$$X_{V^{\mathfrak{S}_3}} = X_V/\mathfrak{S}_3,$$

for $V := L_{-2}(D_4)$. In general, it is not true for an arbitrary vertex algebra V acted by a finite group G that $X_{V^G} = X_V/G$. The following was proved by Miyamoto in [M] though:

if V is a lisse simple conformal vertex algebra and G is a finite solvable subgroup of the automorphism group of V, then V^G is also lisse. It is believed that an analogue conjecture holds for quasi-lisse vertex algebras[1].

Conjecture B. *Let $V = \bigoplus_{n\geqslant 0} V_n$ be a simple positively graded quasi-lisse vertex algebra such that $V_0 \cong \mathbb{C}$ and G a finite solvable automorphism group of V, then V^G is also quasi-lisse.*

Theorem A supports Conjecture B, and also the equalities (4). Indeed, by [AP], we also have $L_{-2}(B_3) = L_{-2}(D_4)^{\mathbb{Z}/2\mathbb{Z}}$ where $\mathbb{Z}/2\mathbb{Z}$ is the group of automorphisms of the Dynkin diagram of B_3.

Then, our result gives new evidences for the following conjecture stated in [AEM].

Conjecture C. *If W is a finite extension of the vertex algebra V then the corresponding morphism of Poisson algebraic varieties $\pi\colon X_W \to X_V$ is a dominant morphism.*

As mentioned above, by [AP], $L_{-2}(D_4)$ is a finite extension of both $L_{-2}(G_2)$ and $L_{-2}(B_3)$ and the restriction of π_2 (resp. π_3) to $\overline{\mathbb{O}}_{\text{min}}$ is precisely the corresponding morphism between $X_{L_{-2}(D_4)}$ and $X_{L_{-2}(G_2)}$ (resp. $X_{L_{-2}(B_3)}$). Since

$$\dim \overline{\mathbb{O}}_{\text{min}} = \dim \overline{\mathbb{O}}_{\text{sreg}} = \dim \overline{\mathbb{O}}_{\text{short}} = 10,$$

Theorem A and the equalities (4) furnish new examples where Conjecture C holds. Most examples so far occurred between simple affine vertex algebras and $\mathcal{W}$-algebras at admissible levels [AEM].

Finally, in the course of the proof of Theorem A, it will be proved that

$$H^0_{DS,f_{\text{sreg}}}(L_{-2}(G_2)) = \mathcal{W}_{-2}(G_2, f_{\text{sreg}}) \cong \mathbb{C}, \tag{5}$$

where $H^0_{DS,f}(-)$ denotes the Drinfeld–Sokolov reduction with respect to the nilpotent element f of $\mathfrak{g}$, $\mathcal{W}_k(\mathfrak{g}, f)$ is the simple quotient of the universal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g}))$ associated with $\mathfrak{g}$ and f, and f_{sreg} is an element of the subregular nilpotent orbit of G_2, see §2.4 and Section 4. The next conjecture ([KRW, KW]) was proved for many cases, but mainly for k an admissible level.

Conjecture D. *$H^0_{DS,f}(L_k(\mathfrak{g}))$ is either zero or isomorphic to $\mathcal{W}_k(\mathfrak{g}, f)$.*

The identities (5) give a new case where Conjecture D holds for a non-admissible level.

1.4. **Organization of the paper.** The rest of the article is organized as follows. Section 2 regroups a few preliminary results on Zhu's algebra and Zhu's correspondence, associated varieties and $\mathcal{W}$-algebras. We fix in this section the main notation of the article. In Section 3, we study the representations in the category $\mathcal{O}$ of the simple affine $L_{-2}(G_2)$. This is based on the obtaining of a singular vector. The computation of the associated variety of $L_{-2}(G_2)$ is achieved in Section 4. Section 5 is about the representations of $L_{-2}(B_3)$. We first study the representations in the category $\mathcal{O}$ exploiting the results about G_2. Furthermore, we study non-ordinary modules using spectral flows from ordinary modules of $L_{-2}(B_3)$. There are two appendices: Appendix A gives the explicit formulas of a singular vector in $V^{-2}(G_2)$ and of its image in the Zhu's algebra. Appendix B describes useful polynomials in the symmetric algebra of B_3 related to subsingular vectors in $V^{-2}(B_3)$.

[1]This was suggested to one of the authors by Dražen Adamović in a private communication.

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2. PRELIMINARIES

Let $\mathfrak{g}$ be a simple Lie algebra with Killing form $\kappa_{\mathfrak{g}}$ as in the introduction, and let $\widetilde{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the extended affine Kac-Moody Lie algebra associated with $\mathfrak{g}$ and the inner product

$$(-|-) = \frac{1}{2h_{\mathfrak{g}}^{\vee}} \times \kappa_{\mathfrak{g}},$$

with the commutation relations

$$[x(m), y(n)] = [x,y](m+n) + m(x|y)\delta_{m+n,0}K, \quad [D, x(m)] = mx(m), \quad [K, \widetilde{\mathfrak{g}}] = 0,$$

for $m, n \in \mathbb{Z}$ and $x, y \in \mathfrak{g}$, where $x(m) = x \otimes t^m$.

Let $\widehat{\mathfrak{g}} = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ so that

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+ \quad \text{and} \quad \widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$$

are triangular decompositions for $\widetilde{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}$, respectively, with $\widehat{\mathfrak{n}}_- = \mathfrak{n}_- + t^{-1}\mathfrak{g}[t^{-1}]$, $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ + t\mathfrak{g}[t]$, $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ and $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$. The Cartan subalgebra $\widetilde{\mathfrak{h}}$ is equipped with a bilinear form extending that on $\mathfrak{h}$ given by

$$(K|D) = 1, \quad (\mathfrak{h}|\mathbb{C}K \oplus \mathbb{C}D) = (K|K) = (D|D) = 0.$$

We write Λ_0 and δ for the elements of $\widetilde{\mathfrak{h}}^*$ orthogonal to $\mathfrak{h}^*$ and dual to K and D, respectively. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$ with basis $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$, and denote by θ the highest positive root. We write $\varpi_1, \ldots, \varpi_\ell$ for the fundamental weights of $\mathfrak{g}$ with respect to $\alpha_1, \ldots, \alpha_\ell$, and $\Lambda_0, \Lambda_1, \ldots, \Lambda_\ell$ for those of $\widetilde{\mathfrak{g}}$.

For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}) = U(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$ on which $\mathfrak{g}[t] \oplus \mathbb{C}D$ acts trivially and K acts as multiplication by k. The space $V^k(\mathfrak{g})$ is naturally a vertex algebra, called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at level* k. By the PBW theorem, we have $V^k(\mathfrak{g}) \cong U(\mathfrak{g}[t^{-1}]t^{-1})$ as $\mathbb{C}$-vector spaces.

The vertex algebra $V^k(\mathfrak{g})$ is graded by D:

$$V^k(\mathfrak{g}) = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_d, \quad V^k(\mathfrak{g})_d = \{a \in V^k(\mathfrak{g}) \colon Da = -da\}.$$

This grading gives a conformal structure provided that k is not critical, that is, $k \neq -h_{\mathfrak{g}}^{\vee}$. A $V^k(\mathfrak{g})$-module is the same as a smooth $\widetilde{\mathfrak{g}}$-module of level k, where a $\widetilde{\mathfrak{g}}$-module M is called smooth if $x(n)m = 0$ for n sufficiently large for all $x \in \mathfrak{g}$, $m \in M$.

2.1. **Singular vectors and highest-weight modules.** For each $\alpha \in \Delta$, fix a nonzero root vector e_α. Recall that a vector $v \in V^k(\mathfrak{g})$ is called *singular* if $e_\alpha(0)v = 0$ for all $\alpha \in \Pi$ and $e_{-\theta}(1)v = 0$. In other words, v is a singular vector if v is singular for $\widetilde{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{n}}_+$. If v is singular for $V^k(\mathfrak{g})$, denote by $\langle v \rangle$ the ideal in $V^k(\mathfrak{g})$ generated by v, that is, $\langle v \rangle = U(\widetilde{\mathfrak{g}})v$. We set

$$\widetilde{V}_k(\mathfrak{g}) = V^k(\mathfrak{g})/\langle v \rangle, \tag{6}$$

the associated quotient vertex algebra.

Let $L_k(\mathfrak{g})$ be the unique simple graded quotient of $V^k(\mathfrak{g})$. As a $\widetilde{\mathfrak{g}}$-module, $L_k(\mathfrak{g})$ is isomorphic to the irreducible highest-weight representation of $\widetilde{\mathfrak{g}}$ with highest-weight $k\Lambda_0$. If N_k denotes the unique maximal ideal of $V^k(\mathfrak{g})$, then

$$L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k,$$

and $L_k(\mathfrak{g})$ is a quotient of $\widetilde{V}_k(\mathfrak{g})$. We will also make use of the notion of *subsingular vector*.

Definition 2.1. *A vector $v_{\text{sub}} \in N_k$ is* subsingular *if there exists a proper submodule N_k' of N_k such that the following conditions hold:*

$$v_{\text{sub}} \notin N_k', \quad e_\alpha(0)v_{\text{sub}} \in N_k' \quad \textit{for all } \alpha \in \Pi, \quad e_{-\theta}(1)v_{\text{sub}} \in N_k'.$$

Note that the image of a subsingular vector in $V^k(\mathfrak{g})/N_k'$ is a singular vector of $V^k(\mathfrak{g})/N_k'$.

For $\lambda \in \mathfrak{h}^*$, we denote by $L_{\mathfrak{g}}(\lambda)$ the irreducible highest-weight representation of $\mathfrak{g}$ with highest-weight λ. Similarly, for $\tilde{\lambda} \in \widetilde{\mathfrak{h}}^*$ we denote by $L_{\widetilde{\mathfrak{g}}}(\tilde{\lambda})$ the irreducible highest-weight representation of $\widetilde{\mathfrak{g}}$. In the case where $\tilde{\lambda} = \lambda + k\Lambda_0$, we shall sometimes write $L_{\mathfrak{g}}(k,\lambda)$ instead of $L_{\widetilde{\mathfrak{g}}}(\tilde{\lambda})$. In this way, we have

$$L_k(\mathfrak{g}) = L_{\widetilde{\mathfrak{g}}}(k\Lambda_0) = L_{\mathfrak{g}}(k,0).$$

A finitely generated module M over a conformal vertex algebra V is called *ordinary* if L_0 acts semisimply, M_d being finite-dimensional for all d, where

$$M_d = \{m \in M \colon L_0 m = dm\},$$

and the conformal weights of M are bounded from below, i.e. there exists d_0 so that $M_d = 0$ for $d \leqslant d_0$. Call the *conformal dimension* of a simple ordinary V-module M the minimum conformal weight of M. More generally, a V-module M is said to be of *positive energy* if it is $\mathbb{Z}_{\geqslant 0}$-graded, $M = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} M_{d_0+d}$, with $M_{d_0} \neq 0$, such that $a(n)M_k \subset M_{k-n}$, where for $a \in V$ of conformal weight Δ we write $a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-\Delta}$.

The highest-weight $\widetilde{\mathfrak{g}}$-module $L_{\mathfrak{g}}(k,\lambda)$, regarded as a $V^k(\mathfrak{g})$-module, has conformal dimension

$$h_{L(\lambda)} = \frac{(\lambda|\lambda+2\rho)}{2(k+h_{\mathfrak{g}}^{\vee})}, \tag{7}$$

where ρ is the half–sum of positive roots.

2.2. **Zhu's algebra and the characteristic variety.** For a positively $\mathbb{Z}$-graded vertex algebra $V = \bigoplus_d V_d$, let $A(V)$ be the Zhu's algebra of V,

$$A(V) = V/V \circ V,$$

where $V \circ V$ is the $\mathbb{C}$-span of the vectors

$$a \circ b := \sum_{i \geqslant 0} \binom{\Delta}{i} a_{(i-2)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$, and $V \to (\mathrm{End} V)[\![z, z^{-1}]\!]$, $a \mapsto \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, denotes the state-field correspondence. The space $A(V)$ is a unital associative algebra with respect to the multiplication defined by

$$a * b := \sum_{i \geqslant 0} \binom{\Delta}{i} a_{(i-1)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$. Denote by $[a]$ the image of $a \in V$ in $A(V)$.

Let $M = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} M_{d_0+d}$, with $M_{d_0} \neq 0$, be a positive energy representation of V. Then $A(V)$ naturally acts on its top weight space $M_{\text{top}} := M_{d_0}$, and the correspondence $M \mapsto M_{top}$ defines a bijection between isomorphism classes of simple positive energy representations of V and simple $A(V)$-modules [Z].

The Zhu's algebra $A(V^k(\mathfrak{g}))$ is naturally isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ [FZ], where the isomorphism $F\colon A(V^k(\mathfrak{g})) \to U(\mathfrak{g})$ is given by

$$F([a_1(-n_1-1) \dots a_m(-n_m-1)\mathbf{1}]) = (-1)^{n_1+\dots+n_m} a_m \dots a_1, \tag{8}$$

for $a_1, \dots, a_m \in \mathfrak{g}$ and $n_1, \dots, n_m \in \mathbb{Z}_{\geqslant 0}$.

We have an exact sequence

$$A(N_k) \to U(\mathfrak{g}) \to A(L_k(\mathfrak{g})) \to 0$$

since the functor $A(-)$ is right exact, and thus $A(L_k(\mathfrak{g}))$ is the quotient of $U(\mathfrak{g})$ by the image J_k of the maximal ideal N_k in $A(V^k(\mathfrak{g})) = U(\mathfrak{g})$:

$$A(L_k(\mathfrak{g})) = U(\mathfrak{g})/J_k.$$

In particular, if v is a singular vector,

$$A(\widetilde{V}_k(\mathfrak{g})) \cong U(\mathfrak{g})/\langle v' \rangle,$$

where $\langle v' \rangle$ is the two-sided ideal in $U(\mathfrak{g})$ generated by the vector

$$v' := F([v]).$$

The top degree component of $L_{\widetilde{\mathfrak{g}}}(\lambda)$ is $L_{\mathfrak{g}}(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of λ to $\mathfrak{h}$. Hence, by Zhu's correspondence, a level k representation $L_{\widetilde{\mathfrak{g}}}(\lambda)$, that is $\lambda(K) = k$, is a $L_k(\mathfrak{g})$-module if and only if $J_k L_{\mathfrak{g}}(\bar{\lambda}) = 0$.

Set $U(\mathfrak{g})^{\mathfrak{h}} := \{u \in U(\mathfrak{g}) \colon [h, u] = 0 \text{ for all } h \in \mathfrak{h}\}$ and let

$$\Upsilon\colon U(\mathfrak{g})^{\mathfrak{h}} \to U(\mathfrak{h}) \tag{9}$$

be the *Harish–Chandra projection map* which is the restriction of the projection map $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) \to U(\mathfrak{h})$ to $U(\mathfrak{g})^{\mathfrak{h}}$. It is known that Υ is an algebra homomorphism. For a two-sided ideal I of $U(\mathfrak{g})$, the *characteristic variety of* I is defined as [J]:

$$\mathscr{X}(I) = \{\lambda \in \mathfrak{h}^* \colon p(\lambda) = 0 \text{ for all } p \in \Upsilon(I^{\mathfrak{h}})\},$$

where $I^{\mathfrak{h}} = I \cap U(\mathfrak{g})^{\mathfrak{h}}$. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through $(-|-)$, and thus $\mathfrak{h}^*$ with $\mathfrak{h}$, we view $\mathscr{X}(I)$ as a subset of $\mathfrak{h}$.

The following result was established in [Ar3, Lemma 2.1].

Lemma 2.2. *For $\lambda \in \mathfrak{h}^*$, $\lambda \in \mathscr{X}(I)$ if and only if $IL_{\mathfrak{g}}(\lambda) = 0$.*

In other words, the characteristic variety $\mathscr{X}(I)$ classifies the simple $U(\mathfrak{g})/I$-modules in category $\mathcal{O}^{\mathfrak{g}}$, where $\mathcal{O}^{\mathfrak{g}}$ is the BGG category $\mathcal{O}$ of $\mathfrak{g}$.

According to [Ad, AM, Ar3], we have the following result.

Proposition 2.3. *Let $v \in V^k(\mathfrak{g})$ be a singular vector, $\widetilde{V}_k(\mathfrak{g}) = V/\langle v\rangle$ as in (6), $v' := F([v])$ the corresponding image in $U(\mathfrak{g})$ and R the $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector v'. The following statements are equivalent:*

(i) *$L_{\mathfrak{g}}(\mu)$ is an $A(\widetilde{V}_k(\mathfrak{g}))$-module,*
(ii) *$RL_{\mathfrak{g}}(\mu) = 0$,*
(iii) *$R^{\mathfrak{h}} v_\mu = 0$, where $R^{\mathfrak{h}} := R \cap U(\mathfrak{g})^{\mathfrak{h}}$,*

where v_μ is a highest-weight vector of $L_{\mathfrak{g}}(\mu)$.

In the notation of the Proposition 2.3, given $r \in R^{\mathfrak{h}}$, there exists a unique polynomial $p_r \in \Upsilon(R^{\mathfrak{h}})$ such that $rv_\mu = p_r(\mu)v_\mu$. Define the polynomial set of $\mathfrak{h}$ by

$$\mathscr{P}_v = \{p_r \colon r \in R^{\mathfrak{h}}\}. \tag{10}$$

If v is a subregular vector, one can define similarly $\mathscr{P}_v$ using the $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector $v' := F([v])$.

As a consequence of Proposition 2.3, we obtain:

Corollary 2.4. *Let $v \in V^k(\mathfrak{g})$ be a singular vector and $\widetilde{V}_k(\mathfrak{g}) = V/\langle v\rangle$. There is a one-to-one correspondence between the irreducible $A(\widetilde{V}_k(\mathfrak{g}))$-modules in the category $\mathcal{O}^{\mathfrak{g}}$ and the weights $\mu \in \mathfrak{h}^*$ such that $p(\mu) = 0$ for all $p \in \mathscr{P}_v$.*

Define the left-adjoint action on $U(\mathfrak{g})$ by

$$x_L f = [x, f] \text{ for } x \in \mathfrak{g} \text{ and } f \in U(\mathfrak{g}). \tag{11}$$

This action extends to $U(\mathfrak{g})$ and we still denote it by $x_L f$ for $x \in U(\mathfrak{g})$ and $f \in U(\mathfrak{g})$.

2.3. **Associated variety.** As in the introduction, let X_V be the *associated variety* [Ar1] of a vertex algebra V, that is the reduced scheme associated with the *Zhu's C_2-algebra* of V

$$R_V := V/C_2(V),$$

with $C_2(V) = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}b \colon a, b \in V\}$. In the case that V is a quotient of $V^k(\mathfrak{g})$, $V/C_2(V) = V/\mathfrak{g}[t^{-1}]t^{-2}V$ and we have a surjective Poisson algebra homomorphism

$$\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \twoheadrightarrow R_V = V/\mathfrak{g}[t^{-1}]t^{-2}V, \quad x \mapsto \overline{x(-1)} + \mathfrak{g}[t^{-1}]t^{-2}V, \tag{12}$$

where $\overline{x(-1)}$ denotes the image of $x(-1)$ in the quotient R_V. Then X_V is just the zero locus of the kernel of the above map in $\mathfrak{g}^*$. It is a G-invariant and conic subvariety of $\mathfrak{g}^*$, with G the adjoint group of $\mathfrak{g}$. Similarly to the characteristic variety, identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through $(-|-)$, we view it as a subset of $\mathfrak{g}$.

For $V = V^k(\mathfrak{g})$, we get

$$R_{V^k(\mathfrak{g})} \cong S(\mathfrak{g})$$

under the algebra isomorphism (12). For $v \in V^k(\mathfrak{g})$, denote by v'' the image of $\overline{v}$ in $S(\mathfrak{g})$ by the above isomorphism. If v is a singular vector of $V^k(\mathfrak{g})$, then

$$R_{\widetilde{V}_k(\mathfrak{g})} \cong S(\mathfrak{g})/I_M,$$

where M is the $\mathfrak{g}$-module generated by v'' under the adjoint action, and I_M is the ideal of $S(\mathfrak{g})$ generated by M.

It will be also useful to consider the *Chevalley projection map*

$$\Psi\colon S(\mathfrak{g})^{\mathfrak{h}} \to S(\mathfrak{h}), \tag{13}$$

where $S(\mathfrak{g})^{\mathfrak{h}} = \{x \in S(\mathfrak{g})\colon [h,x] = 0 \text{ for all } h \in \mathfrak{h}\}$.

2.4. **Affine $\mathcal{W}$-algebras.** For a nilpotent element f of $\mathfrak{g}$, let $\mathcal{W}^k(\mathfrak{g}, f)$ be the universal $\mathcal{W}$-algebra associated with $(\mathfrak{g}, f)$ at level k, defined by the generalized quantized Drinfeld–Sokolov reduction [FF, KRW]:

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0_{DS,f}(V^k(\mathfrak{g})),$$

where $H^0_{DS,f}(M)$ is the corresponding BRST cohomology with coefficient in a $\widetilde{\mathfrak{g}}$-module M. We have a natural Poisson algebra isomorphism $R_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathbb{C}[\mathscr{S}_f]$, where $\mathscr{S}_f = f + \mathfrak{g}^e$, with $\mathfrak{g}^e = \{x \in \mathfrak{g}\colon [x,e] = 0\}$, is the Slodowy slice associated with an $\mathfrak{sl}_2$-triple (e,h,f) [DK, Ar2]. It follows that

$$X_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathscr{S}_f.$$

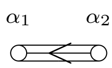
Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. Then $X_{\mathcal{W}_k(\mathfrak{g},f)}$ is a $\mathbb{C}^*$-invariant closed Poisson subvariety of $\mathscr{S}_f$. Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widetilde{\mathfrak{g}}$ at level k. We have a functor

$$\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_{DS,f}(M),$$

where $\mathcal{W}^k(\mathfrak{g}, f)$-Mod denotes the category of $\mathcal{W}^k(\mathfrak{g}, f)$-modules. According to [Ar2], for any quotient V of $V^k(\mathfrak{g})$, $X_{H^0_{DS,f}(V)}$ is isomorphic, as a Poisson variety, to the intersection $X_V \cap \mathscr{S}_f$. In particular, $H^0_{DS,f}(V) \neq 0$ if and only if $\overline{G.f} \subset X_V$ and $H^0_{DS,f}(V)$ is lisse if $X_V = \overline{G.f}$.

3. On the representations of $L_{-2}(G_2)$

In this section, $\mathfrak{g}$ is the simple exceptional Lie algebra of type G_2 with simple roots α_1, α_2 and Dynkin diagram



In particular, α_1 is the simple short root. One can fix the root vectors so that

$$[e_{\alpha_1}, e_{\alpha_2}] = e_{\alpha_1+\alpha_2}, \quad [e_{\alpha_1}, e_{\alpha_1+\alpha_2}] = 2e_{2\alpha_1+\alpha_2},$$
$$[e_{\alpha_1}, e_{2\alpha_1+\alpha_2}] = 3e_{3\alpha_1+\alpha_2}, \quad [e_{\alpha_2}, e_{3\alpha_1+\alpha_2}] = e_{3\alpha_1+2\alpha_2}.$$

All other commutation relations can be obtained by using the Jacobi identity. It will be convenient to number the other positive roots as follows:

$$\alpha_3 = \alpha_1 + \alpha_2, \quad \alpha_4 = 2\alpha_1 + \alpha_2, \quad \alpha_5 = 3\alpha_1 + \alpha_2, \quad \alpha_6 = 3\alpha_1 + 2\alpha_2 = \theta.$$

Denote by ϖ_1, ϖ_2 the fundamental weights of G_2 with respect to α_1, α_2, and by $\{h_1 = \alpha_1^\vee, h_2 = \alpha_2^\vee\}$ a basis of the Cartan subalgebra.

Theorem 3.1. *There is a singular vector v_{sing} of $V^{-2}(G_2)$ of weight $-2\Lambda_0 + 4\varpi_1 - 6\delta$. In particular, v_{sing} has conformal weight six and they is no singular vector of conformal weight strictly smaller than six.*

Due to the complexity of the specific form, we place an explicit form of such a singular vector in the Appendix A.

Proof. The affine space $\{\lambda + k\Lambda_0 \colon \lambda \in \mathfrak{h}^*\}$ is identified with an affine subvariety of $\widetilde{\mathfrak{h}}^*$ by the correspondence

$$\lambda + k\Lambda_0 \longmapsto \lambda + k\Lambda_0 - h_{L(\lambda)}\delta,$$

where $h_{L(\lambda)}$ is the conformal dimension given by (7) and $h^\vee_{G_2} = 4$.

The strategy in order to find a singular vector is the following. We search for a G_2-weight of a potential singular vector v that makes the conformal dimension an integer. Then we test the conditions of a singular vector,

$$e_{\alpha_1}(0)v = 0, \quad e_{\alpha_2}(0)v = 0, \quad e_{-\theta}(1)v = 0,$$

with $\theta = 3\alpha_1 + 2\alpha_2$ from smaller to larger conformal dimensions. For any $\lambda = a_1\varpi_1 + a_2\varpi_2$, we have

$$h_{L(\lambda)} = \frac{1}{12}\left(2(a_1+2) + 3(a_2+2)\right) + a_2\left((a_1+2) + 2(a_2+2)\right).$$

We list the integer solutions for the conformal dimension from 2 to 6 in Table 1. We observe

conformal dimension	weight
0	0
1	ϖ_1
2	ϖ_2
3	no solution
4	$3\varpi_1$
5	$2\varpi_2$
6	$4\varpi_1$

TABLE 1. The integer solutions for the conformal dimension

that there is no corresponding solution for a singular vector with conformal weight $1, 2, 4, 5$. However, there is indeed a singular vector with conformal weight 6. Our candidate, that we denote by v_{sing}, is described in Appendix A and has G_2-weight $4\varpi_1$. Then it is straightforward to check that

$$e_{\alpha_1}(0)v_{\text{sing}} = 0, \quad e_{\alpha_2}(0)v_{\text{sing}}, \quad e_{-\theta}(1)v_{\text{sing}} = 0.$$

Alternatively, we can compare the first few terms of the character formulas of $V^{-2}(G_2)$ and $L_{-2}(G_2)$ by using Kazhdan–Lusztig polynomials to determine the existence of a singular vector with conformal weight 6.[2] $\square$

Keep the notation relative to v_{sing} as in Section 2: $[v_{\text{sing}}]$ denotes its image in the Zhu's algebra of $V^{-2}(G_2)$, v'_{sing} the corresponding element of $U(G_2)$ via the isomorphism (8), $\overline{v}_{\text{sing}}$ the image of v_{sing} in the Zhu's C_2-algebra and v''_{sing} the corresponding element of $S(G_2)$ through the isomorphism (12).

Let also $\langle v_{\text{sing}}\rangle$ be the submodule of $V^{-2}(G_2)$ generated by v_{sing}, and $\widetilde{V}_{-2}(G_2) = V^{-2}(G_2)/\langle v_{\text{sing}}\rangle$ the associated quotient vertex algebra. Then the Zhu's algebra $A(\widetilde{V}_{-2}(G_2))$ is isomorphic to $U(G_2)/\langle v'_{\text{sing}}\rangle$, where $\langle v'_{\text{sing}}\rangle$ is the two-sided ideal in $U(G_2)$ generated by the vector v'_{sing}. The explicit form of v'_{sing} can be found in Appendix A as well.

Lemma 3.2. *The zero-weight subspace $L_{G_2}(4\varpi_1)^{\mathfrak{h}}$ has dimension 8, spanned by the linearly independent vectors below*

$$v_1 = (e^4_{-\alpha_4})_L v'_{\text{sing}}, \quad v_2 = (e_{-\alpha_3}e^2_{-\alpha_4}e_{-\alpha_5})_L v'_{\text{sing}}, \quad v_3 = (e^2_{-\alpha_3}e^2_{-\alpha_5})_L v'_{\text{sing}}$$
$$v_4 = (e_{-\alpha_2}e_{-\alpha_4}e^2_{-\alpha_5})_L v'_{\text{sing}}, \quad v_5 = (e_{-\alpha_1}e^2_{-\alpha_4}e_{-\theta})_L v'_{\text{sing}}$$
$$v_6 = (e_{-\alpha_1}e_{-\alpha_3}e_{-\alpha_5}e_{-\theta})_L v'_{\text{sing}}, \quad v_7 = (e^2_{-\alpha_1}e^2_{-\theta})_L v'_{\text{sing}}, \quad v_8 = (e_{-\alpha_1}e_{-\alpha_3}e^3_{-\alpha_4})_L v'_{\text{sing}}$$

Lemma 3.3. *Let $p_i \in U(\mathfrak{h})$ be the image of v_i for $i = 1, \dots, 8$ by the Harish–Chandra projection (9). Then, the polynomial set $\{p_1, \dots, p_7\}$ forms a basis for $\mathscr{P}_{v_{\text{sing}}}$, where*

$$\begin{aligned}
p_1(h) =& -24\,(2h_1+3h_2+3)\,(8h_1^5+60h_2h_1^4+36h_1^4+178h_2^2h_1^3+212h_2h_1^3+24h_1^3+261h_2^3h_1^2\\
&+438h_2^2h_1^2+48h_2h_1^2-44h_1^2+189h_2^4h_1+369h_2^3h_1-28h_2^2h_1-152h_2h_1-24h_1+54h_2^5\\
&+108h_2^4-42h_2^3-96h_2^2-24h_2)\\
p_2(h) =& 2(32h_1^6+294h_2h_1^5+192h_1^5+1081h_2^2h_1^4+1398h_2h_1^4+312h_1^4+2070h_2^3h_1^3+3928h_2^2h_1^3\\
&+1542h_2h_1^3-32h_1^3+2223h_2^4h_1^2+5346h_2^3h_1^2+2557h_2^2h_1^2-630h_2h_1^2-360h_1^2+1296h_2^5h_1\\
&+3582h_2^4h_1+1656h_2^3h_1-1554h_2^2h_1-1164h_2h_1-144h_1+324h_2^6+972h_2^5+396h_2^4\\
&-828h_2^3-720h_2^2-144h_2)\\
p_3(h) =& -4(16h_1^6+161h_2h_1^5+96h_1^5+634h_2^2h_1^4+763h_2h_1^4+156h_1^4+1254h_2^3h_1^3+2260h_2^2h_1^3\\
&+865h_2h_1^3-16h_1^3+1332h_2^4h_1^2+3123h_2^3h_1^2+1558h_2^2h_1^2-271h_2h_1^2-180h_1^2+729h_2^5h_1\\
&+2016h_2^4h_1+1041h_2^3h_1-708h_2^2h_1-582h_2h_1-72h_1+162h_2^6+486h_2^5+198h_2^4\\
&-414h_2^3-360h_2^2-72h_2)\\
p_4(h) =& 6h_1h_2\,(h_1+h_2+1)\,(h_1+2h_2+2)\,(h_1+3h_2+3)\,(2h_1+3h_2)\\
p_5(h) =& 2h_1\,(h_1+2h_2+2)\,(32h_1^4+218h_2h_1^3+128h_1^3+555h_2^2h_1^2+614h_2h_1^2+56h_1^2+612h_2^3h_1\\
&+882h_2^2h_1-10h_2h_1-144h_1+243h_2^4+360h_2^3-249h_2^2-390h_2-72)\\
p_6(h) =& -2h_1\,(h_1+2h_2+2)\,(16h_1^4+109h_2h_1^3+64h_1^3+264h_2^2h_1^2+289h_2h_1^2+28h_1^2+279h_2^3h_1\\
&+396h_2^2h_1-17h_2h_1-72h_1+108h_2^4+153h_2^3-132h_2^2-189h_2-36)\\
p_7(h) =& -4(h_1-1)h_1(h_1+2h_2+2)(h_1+2h_2+3)(16h_1^2+63h_2h_1+32h_1+63h_2^2+63h_2+12)\\
p_8(h) =& 6(64h_1^6+534h_2h_1^5+384h_1^5+1829h_2^2h_1^4+2556h_2h_1^4+624h_1^4+3168h_2^3h_1^3+6210h_2^2h_1^3\\
&+2594h_2h_1^3-64h_1^3+2727h_2^4h_1^2+6426h_2^3h_1^2+2863h_2^2h_1^2-1412h_2h_1^2-720h_1^2+918h_2^5h_1\\
&+2322h_2^4h_1+402h_2^3h_1-2538h_2^2h_1-1824h_2h_1-288h_1)
\end{aligned}$$

[2] This was suggested to one of the authors by Yiwen Pan in a private communication.

Proof. According to Lemma 3.2, we have $\dim L_{G_2}(4\varpi_1)^{\mathfrak{h}} = 8$. Furthermore, one obtains by direct calculations that $v_i \equiv p_i(h) \mod \mathfrak{n}_- U(G_2) + U(G_2)\mathfrak{n}_+$ for $i = 1, \ldots, 8$. It is easily checked that $\{p_1, \ldots, p_8\}$ is linearly dependent, whereas $\{p_1, \ldots, p_7\}$ form a linearly independent set. □

Corollary 2.4 implies that the highest-weights λ of irreducible $A(\widetilde{V}_{-2}(G_2))$-modules from the category $\mathcal{O}$ are given by the solutions of the polynomial equations:

$$\lambda(p_i(h)) = 0, \quad i = 1, 2, \ldots, 7.$$

Proposition 3.4. *The complete list of irreducible $A(\widetilde{V}_{-2}(G_2))$-modules in the category $\mathcal{O}$ is given by the set $\{L_{G_2}(\mu_i) \colon i = 1, 2, \ldots, 20\}$, where the μ_i's are given by Table 2.*

μ_1	0	μ_{11}	$-\frac{1}{3}\varpi_2$
μ_2	ϖ_1	μ_{12}	$-\frac{2}{3}\varpi_2$
μ_3	ϖ_2	μ_{13}	$-\frac{2}{3}\varpi_1 + \frac{1}{2}\varpi_2$
μ_4	$-2\varpi_1$	μ_{14}	$-\frac{1}{2}\varpi_1 - \frac{1}{2}\varpi_2$
μ_5	$-3\varpi_1$	μ_{15}	$\varpi_1 - \frac{3}{2}\varpi_2$
μ_6	$-\varpi_2$	μ_{16}	$\varpi_1 - \frac{4}{3}\varpi_2$
μ_7	$-2\varpi_2$	μ_{17}	$\varpi_1 - \frac{2}{3}\varpi_2$
μ_8	$\varpi_1 - 2\varpi_2$	μ_{18}	$2\varpi_1 - \frac{5}{3}\varpi_2$
μ_9	$-\frac{1}{2}\varpi_1$	μ_{19}	$2\varpi_1 - \frac{4}{3}\varpi_2$
μ_{10}	$-\frac{3}{2}\varpi_1$	μ_{20}	$3\varpi_1 - \frac{5}{2}\varpi_2$

TABLE 2. The weights μ_i for G_2

From Zhu's correspondence, we deduce the following result.

Theorem 3.5. *The set $\{L_{G_2}(-2, \mu_i) \colon i = 1, \ldots, 20\}$ provides the complete list of irreducible $\widetilde{V}_{-2}(G_2)$-modules from the category $\mathcal{O}$, and the set $\{L_{G_2}(-2, 0),\ L_{G_2}(-2, \varpi_1),\ L_{G_2}(-2, \varpi_2)\}$ provides the complete list of irreducible ordinary modules for $\widetilde{V}_{-2}(G_2)$.*

Proof. The first part of the Theorem follows directly from the Proposition 3.4 and Zhu's correspondence. We look for the irreducible ordinary $\widetilde{V}_{-2}(G_2)$-modules among the list of modules in the category $\mathcal{O}$. An ordinary $\widetilde{V}_{-2}(G_2)$-module M has finite-dimensional graded spaces. In particular, the space corresponding to the graded space with the minimal conformal weight is a finite-dimensional G_2-module. Hence, the irreducible ordinary $\widetilde{V}_{-2}(G_2)$-modules correspond exactly to the dominant integral weights in the Table 2. □

Theorem 3.6. *The vertex algebra $\widetilde{V}_{-2}(G_2)$ is simple, and hence $\widetilde{V}_{-2}(G_2) = L_{-2}(G_2)$.*

Proof. According to Theorem 3.5, the only possible G_2-weights for a subsingular vector (see Definition 2.1) in $V^{-2}(G_2)$ with respect to $\langle v_{\text{sing}} \rangle$ besides 0 are ϖ_1 and ϖ_2. The conformal weights of those potential subsingular vectors are respectively:

$$\frac{(\varpi_1, \varpi_1 + 2\rho)}{2(k + h^\vee_{G_2})} = 1, \quad \frac{(\varpi_2, \varpi_2 + 2\rho)}{2(k + h^\vee_{G_2})} = 2, \tag{14}$$

with $h^\vee_{G_2} = 4$ and $k = -2$. It is clear that there are no subsingular vectors in the subspace $V^{-2}(G_2)_1$ of elements of conformal weight one. So it suffices to show that there is no subsingular vector of weight ϖ_2 with conformal weight two. Notice that we have the following decomposition as a G_2-module of the symmetric algebra (see [Ga, Chapter IV, Proposition 2]):

$$S^2(G_2) \cong L_{G_2}(2\theta) \oplus L_{G_2}(\theta+\alpha) \oplus L_{G_2}(0).$$

Therefore, we obtain that, as G_2-modules,

$$\begin{aligned} V^{-2}(G_2)_2 &\cong S^2(G_2) \oplus G_2 \\ &\cong L_{G_2}(2\theta) \oplus L_{G_2}(\theta+\alpha) \oplus L_{G_2}(0) \oplus L_{G_2}(\theta). \end{aligned}$$

Since $\theta = \varpi_2$ and $\theta + \alpha = \varpi_1$, it suffices to show that $e_\theta(-2)\mathbf{1}$ is not a singular vector with respect to $\widetilde{V}_{-2}(G_2)$. This is obvious due to the fact that

$$e_{-\alpha_2}(1)e_\theta(-2)\mathbf{1} = e_{3\alpha_1+\alpha_2}(-1)\mathbf{1},$$

and $e_{-\alpha_2}(1) \in \widehat{\mathfrak{n}}_+$ while $e_{3\alpha_1+\alpha_2}(-1)\mathbf{1} \notin \langle v_{\text{sing}}\rangle$. □

Combining Theorem 3.5 and Theorem 3.6, we obtain the classification of the irreducible modules of $L_{-2}(G_2)$ in the category $\mathcal{O}$. This completes the proof of Theorem B.

4. Associated variety of $L_{-2}(G_2)$

We continue to assume that $\mathfrak{g}$ is the simple Lie algebra G_2, and we keep the related notations of the previous sections, in particular about the numbering of positive roots in G_2. First, using the computations of the previous section, we establish the following result.

Proposition 4.1. *The associated variety of $L_{-2}(G_2)$ is contained in the nilpotent cone of G_2. In other words, the simple vertex algebra $L_{-2}(G_2)$ is quasi-lisse.*

Proof. We follow the strategy adopted in [AM2]. Let M be the G_2-module generated by v''_{sing} under the adjoint action, and I_M the ideal of $S(G_2)$ generated by M. Then $R_{L_{-2}(G_2)} \cong S(G_2)/I_M$. Set

$$I_M^{\mathfrak{h}} := \Psi(I_M \cap S(G_2)^{\mathfrak{h}}),$$

where Ψ is the Chevalley projection map (13). We extract seven linearly independent polynomials of $\mathfrak{h}$ in $I_M^{\mathfrak{h}}$:

$$[e_{-\theta},[e_{-\alpha_5},[e_{-\alpha_4},v''_{\text{sing}}]]] = -3h_1h_2\,(h_1+h_2)\,(h_1+2h_2)\,(h_1+3h_2)\,(2h_1+3h_2)\,,$$

$$[e_{-\alpha_4},[e_{-\alpha_4},[e_{-\alpha_4},[e_{-\alpha_4},v''_{\text{sing}}]]]] = -24\,(h_1+h_2)\,(h_1+2h_2)\,(2h_1+3h_2)^4\,,$$

$$[e_{-\alpha_5},[e_{-\alpha_4},[e_{-\alpha_4},[e_{-\alpha_3},v''_{\text{sing}}]]]] = 4\,(h_1+h_2)\,(h_1+2h_2)\,(h_1+3h_2)\,(2h_1+3h_2)^2\,(4h_1+3h_2)\,,$$

$$[e_{-\alpha_5},[e_{-\alpha_5},[e_{-\alpha_3},[e_{-\alpha_3},v''_{\text{sing}}]]]] = -4\,(h_1+h_2)^2\,(h_1+3h_2)^2\,\left(16h_1^2+33h_2h_1+18h_2^2\right),$$

$$[e_{-\theta},[e_{-\alpha_4},[e_{-\alpha_4},[e_{-\alpha},v''_{\text{sing}}]]]] = 4h_1\,(h_1+h_2)\,(h_1+2h_2)\,(2h_1+3h_2)^2\,(4h_1+9h_2)\,,$$

$$[e_{-\theta},[e_{-\alpha_5},[e_{-\alpha_3},[e_{-\alpha_1},v''_{\text{sing}}]]]] = -2h_1\,(h_1+h_2)\,(h_1+2h_2)\,(h_1+3h_2)\,\left(16h_1^2+51h_2h_1+45h_2^2\right),$$

$$[e_{-\theta},[e_{-\theta},[e_{-\alpha_1},[e_{-\alpha_1},v''_{\text{sing}}]]]] = -4h_1^2\,(h_1+2h_2)^2\,\left(16h_1^2+63h_2h_1+63h_2^2\right),$$

where the equalities are modulo $\mathfrak{n}_-S(G_2)+S(G_2)\mathfrak{n}_+$. The only semisimple element vanishing these seven polynomial equations is 0. Because the associated variety is invariant under the adjoint group, this implies that the associated variety of $L_{-2}(G_2)$ has no (non-zero) semisimple element, and so is contained in the nilpotent cone of G_2. The proportion follows. □

Set

$$f = f_{\text{sreg}} = e_{-\alpha_2} + e_{-\alpha_4}, \quad x = \frac{h}{2} = h_1 + 2h_2$$

so that (e, h, f) is a subregular $\mathfrak{sl}_2$-triple in G_2. It defines a grading with respect to f_{sreg}. The nilpotent element f_{sreg} is even and we have

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\begin{aligned}
&\mathfrak{g}_2 = \mathbb{C}e_\theta, && \mathfrak{g}_{-2} = \mathbb{C}e_{-\theta},\\
&\mathfrak{g}_1 = \mathbb{C}e_{\alpha_2} \oplus \mathbb{C}e_{\alpha_4} \oplus \mathbb{C}e_{\alpha_3} \oplus \mathbb{C}e_{\alpha_5}, && \mathfrak{g}_{-1} = \mathbb{C}e_{-\alpha_2} \oplus \mathbb{C}e_{-\alpha_4} \oplus \mathbb{C}e_{-\alpha_3} \oplus \mathbb{C}e_{-\alpha_5},\\
&\mathfrak{g}_0 = \mathbb{C}h_1 \oplus \mathbb{C}h_2 \oplus \mathbb{C}e_{\alpha_1} \oplus \mathbb{C}e_{-\alpha_1}.
\end{aligned}$$

The centralizer of f in G_2 is a four-dimensional vector space $\mathfrak{g}^f = \mathfrak{g}^f_{-2} \oplus \mathfrak{g}^f_{-1}$, with:

$$\mathfrak{g}^f_{-1} = \mathbb{C}(e_{-\alpha_2} + e_{-\alpha_4}) \oplus \mathbb{C}e_{-\alpha_2} \oplus \mathbb{C}(e_{-\alpha_3} - 3e_{-\alpha_5}), \quad \mathfrak{g}^f_{-2} = \mathbb{C}e_{-\theta}.$$

Consider the $\mathcal{W}$-algebra $\mathcal{W}^k(G_2, f_{\text{sreg}})$ associated with G_2 and f_{sreg} (see Section 2.4). Since $\dim \mathfrak{g}^f = 4$, we know that the $\mathcal{W}$-algebra $\mathcal{W}^k(G_2, f_{\text{sreg}})$ is strongly generated by the fields $J^{\{f_{\text{sreg}}\}}$, $J^{\{e_{-\alpha_2}\}}$, $J^{\{e_{-\alpha_3}-3e_{-\alpha_5}\}}$ and $J^{\{e_{-\theta}\}}$ defined bellow. The OPEs between the generators have been computed in [F2]:

$$\begin{aligned}
J^{\{f_{\text{sreg}}\}} &= J^{f_{\text{sreg}}} - \frac{1}{3} :J^{h_1}J^{h_1}: - :J^{h_1}J^{h_2}: - :J^{h_2}J^{h_2}: - \frac{1}{3} :J^{e_{-\alpha_1}}J^{e_{-\alpha_1}}: - \left(\frac{7}{3}+k\right)\partial J^{h_1} - (5+2k)\,\partial J^{h_2}\\
J^{\{e_{-\alpha_2}\}} &= J^{e_{-\alpha_2}} - \frac{1}{4} :J^{h_2}J^{h_2}: - \frac{1}{12} :J^{e_{\alpha_1}}J^{e_{\alpha_1}}: - \frac{1}{4}(5+2k)\,\partial J^{h_2}\\
J^{\{e_{-\alpha_3}-3e_{-\alpha_5}\}} &= J^{e_{-\alpha_3}-3e_{-\alpha_5}} - \frac{2}{3} :J^{h_1}J^{e_{\alpha_1}}: + :J^{h_1}J^{e_{-\alpha_1}}: - :J^{h_2}J^{e_{\alpha_1}}:\\
&\quad + :J^{h_2}J^{e_{-\alpha_1}}: - \left(\frac{7}{3}+k\right)\partial J^{e_{\alpha_1}} + (3+k)\,\partial J^{e_{-\alpha_1}}\\
6J^{\{e_{-\theta}\}} &= 6J^{e_{-\theta}} + \partial J^{e_{-\alpha_3}} + 3\partial J^{e_{-\alpha_5}} + :J^{e_{-\alpha_2}}J^{e_{\alpha_1}}: - 2 :J^{e_{-\alpha_2}}J^{e_{-\alpha_2}}: + 2 :J^{e_{-\alpha_2}}J^{h_1}:\\
&\quad + 3 :J^{e_{-\alpha_3}}J^{h_2}: - :J^{e_{-\alpha_4}}J^{e_{\alpha_1}}: + 3 :J^{e_{-\alpha_5}}J^{h_2}: - \frac{1}{3}\partial^2 J^{e_{\alpha_1}} - \frac{1}{3} :J^{h_1}\partial J^{e_{\alpha_1}}:\\
&\quad - :J^{h_2}\partial J^{e_{\alpha_1}}: + \frac{1}{3} :\partial J^{h_1}J^{e_{\alpha_1}}: - :\partial J^{e_{-\alpha_1}}J^{h_1}: - :\partial J^{e_{-\alpha_1}}J^{h_2}: - \frac{1}{3} :J^{h_1}J^{h_1}J^{e_{\alpha_1}}:\\
&\quad - :J^{h_1}J^{h_2}J^{e_{\alpha_1}}: - :J^{h_2}J^{h_2}J^{e_{-\alpha_1}}: - \frac{1}{9} :J^{e_{\alpha_1}}J^{e_{\alpha_1}}J^{e_{\alpha_1}}: - :J^{e_{-\alpha_1}}J^{h_1}J^{h_2}:\\
&\quad - :J^{e_{-\alpha_1}}J^{h_2}J^{h_2}: + \frac{1}{3} :J^{e_{-\alpha_1}}J^{e_{\alpha_1}}J^{e_{\alpha_1}}:
\end{aligned}$$

For $k \neq -4$, we can redefine the generators as follows:

$$\begin{aligned}
&\text{conformal weight } 2: L = -\frac{J^{\{f_{\text{sreg}}\}}}{4+k}\\
&\text{conformal weight } 2: G^+ = -J^{\{f_{\text{sreg}}\}} + 4J^{\{e_{-\alpha_2}\}}\\
&\text{conformal weight } 2: G^- = -J^{\{e_{-\alpha_3}-3e_{-\alpha_5}\}}\\
&\text{conformal weight } 3: F = 6J^{\{e_{-\theta}\}}.
\end{aligned}$$

Let $(-|-)$ be an invariant inner product of G_2. Define $\chi \in \mathfrak{g}^*_{>0}$ by $\chi(x) = -(f_{\text{sreg}}|x)$ for $x \in \mathfrak{g}_{>0}$. Set

$$\mathfrak{m} := \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad J_\chi := \sum_{x \in \mathfrak{m}} \mathbb{C}[\mathfrak{g}^*](x - \chi(x)).$$

We obtain

$$(f_{\text{sreg}}|e_{\alpha_2}) = 1, \quad (f_{\text{sreg}}|e_{\alpha_3}) = 0, \quad (f_{\text{sreg}}|e_{\alpha_4}) = 3, \quad (f_{\text{sreg}}|e_{\alpha_5}) = 0, \quad (f_{\text{sreg}}|e_{\theta}) = 0,$$

and

$$v'' = 9(e_{-\alpha_2} + e_{-\alpha_4}) - 12e_{-\alpha_2} + e_{\alpha_1}^2 - 3e_{\alpha_1}e_{-\alpha_1} - 3h_1^2 - 9h_1h_2 - 6h_2^2 \mod J_\chi$$

Moreover,

$$\begin{aligned}9J^{\{f_{\text{sreg}}\}} - 12J^{\{e_{-\alpha_2}\}} &= 9J^{e_{-\alpha_2}+e_{-\alpha_4}} - 12J^{e_{-\alpha_2}} + :J^{e_{\alpha_1}}J^{e_{\alpha_1}}: -3:J^{e_{\alpha_1}}J^{e_{-\alpha_1}}:\\ &\quad -3:J^{h_1}J^{h_1}: -9:J^{h_1}J^{h_2}: -6:J^{h_2}J^{h_2}: -(21+9k)\partial J^{h_1} - 6(5+2k)\partial J^{h_2}.\end{aligned}$$

Next, consider the term $e_{-\alpha_1}(0)v_{\text{sing}}$ which preserves the conformal weight. It is easy to select the possible nonzero terms inside v_{sing} that contribute for the evaluation of $e_{-\alpha_1}(0)v_{\text{sing}}$, namely

$$\begin{aligned}v^{\text{nonzero}} &= e_{\alpha_4}(-1)^3e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2\mathbf{1} - e_{\alpha_4}(-1)^3e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1} - e_{\alpha_4}(-1)^4e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\ &\quad + e_{\alpha_5}(-1)^4e_{\alpha_2}(-1)e_{-\alpha_2}(-1)\mathbf{1} - e_{\alpha_5}(-1)^4e_{\alpha_3}(-1)e_{-\alpha_3}(-1)\mathbf{1} - e_{\alpha_4}(-1)^5e_{-\alpha_4}(-1)\mathbf{1}\\ &\quad - e_{\alpha_4}(-1)^4h_1(-1)^2\mathbf{1} - 3e_{\alpha_4}(-1)^4h_1(-1)h_2(-1)\mathbf{1} - 2e_{\alpha_4}(-1)^4h_2(-1)^2\mathbf{1}\\ &\quad + 10e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)h_1(-1)\mathbf{1} + 15e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}\\ &\quad + 8e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3e_{\alpha_2}(-1)e_{-\alpha_3}(-1)\mathbf{1} - 7e_{\alpha_5}(-1)e_{\alpha_4}(-1)^4e_{-\alpha_5}(-1)\mathbf{1}\\ &\quad - 4e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3h_1(-1)e_{-\alpha_1}(-1)\mathbf{1} - 2e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3h_2(-1)e_{-\alpha_1}(-1)\mathbf{1}.\end{aligned}$$

By calculation, we have

$$\begin{aligned}\chi(e_{-\alpha_1}(0)v^{\text{nonzero}}) &= -3^3h_1e_{\alpha_1} - 3^3e_{\alpha_1}h_1 - 3^4e_{\alpha_1}h_2 + 3^4h_1e_{-\alpha_1} - 3^4e_{-\alpha_3}\\ &= -3^5e_{-\alpha_3} + 3^6e_{-\alpha_5} - 2\cdot 3^4\left(e_{-\alpha_1}h_1 - h_1e_{-\alpha_1}\right) - 2\cdot 3^5e_{-\alpha_1}h_2 + 3^5h_1e_{-\alpha_1}\\ &\quad + 2\cdot 3^4\left(e_{-\alpha_1}h_2 + h_2e_{-\alpha_1}\right) + 10\cdot 3^3e_{\alpha_1}h_1 + 15\cdot 3^3e_{\alpha_1}h_2 + 8\cdot 3^4e_{-\alpha_3}\\ &\quad - 7\cdot 3^5e_{-\alpha_5} - 4\cdot 3^4h_1e_{-\alpha_1} - 2\cdot 3^4h_2e_{-\alpha_1}.\end{aligned}$$

Hence

$$(e_{-\alpha_1}(0)v_{\text{sing}})'' = 3^3\cdot 12\left(-(e_{-\alpha_3} - 3e_{-\alpha_5}) + \frac{2}{3}h_1e_{\alpha_1} - h_1e_{-\alpha_1} + h_2e_{\alpha_1} - h_2e_{-\alpha_1}\right) \mathrm{mod}\, J_\chi.$$

The following result is known, see for instance [Ar2, DK].

Lemma 4.2. *Denoting by M the connected nilpotent subgroup with Lie algebra $\mathfrak{m}$ of the adjoint group of G_2, we have:*

$$R_{\mathcal{W}^k(\mathfrak{g},f)} \cong (S(\mathfrak{g})/J_\chi)^M.$$

Theorem 4.3. *We have*

$$H^0_{DS,f_{\text{sreg}}}(L_{-2}(G_2)) \cong \mathcal{W}_{-2}(G_2, f_{\text{sreg}})$$

Proof. Set for simplicity $f := f_{\text{sreg}}$. Writing $I_{G_2} = \langle v_{\text{sing}}\rangle$, we get the short exact sequence

$$0 \longrightarrow I_{G_2} \longrightarrow V^{-2}(G_2) \longrightarrow L_{-2}(G_2) \longrightarrow 0. \tag{15}$$

Applying the quantum Drinfeld–Sokolov reduction $H^0_{DS,f}(-)$ to the above sequence, we obtain the short exact sequence

$$0 \longrightarrow H^0_{DS,f}(I_{G_2}) \longrightarrow \mathcal{W}^{-2}(G_2, f) \longrightarrow H^0_{DS,f}(L_{-2}(G_2)) \longrightarrow 0$$

due to the exactness of the quantum Drinfeld–Sokolov reduction functor.

Suppose that v_{sing} maps to $\tilde{v}$ in $\mathcal{W}^{-2}(G_2, f)$. One can easily verify that the conformal weight of $\tilde{v}$ equals 2. By Lemma 4.2, its image in $R_{\mathcal{W}^{-2}(G_2,f)}$ is the image of the vector

$(e_{-\alpha_1}(0)v_{\text{sing}})''$ in $(S(\mathfrak{g})/J_\chi)^M$. It is clear that v_{sing} maps to $-12L - 3G^+$ in $\mathcal{W}^{-2}(G_2, f)$, and similarly $e_{-\alpha_1}(0)v_{\text{sing}}$ maps to $324G^-$ in $\mathcal{W}^{-2}(G_2, f)$. From the OPEs of the four strong generators $L, G^\pm, F$ (see [F2]), we see that $-12L - 3G^+$ does not generate the maximal ideal in $\mathcal{W}^{-2}(G_2, f)$, but the element G^- does. Thus, $H^0_{DS,f}(I_{G_2})$ is the maximal ideal in $\mathcal{W}^{-2}(G_2, f)$, and hence $H^0_{DS,f}(L_{-2}(G_2)) \cong \mathbb{C}$. □

Remark 4.4. *From the above proof, we recover that $\mathcal{W}_{-2}(G_2, f) \cong \mathbb{C}$ from* [F2, Corollary 4.2].

We are now in a position to prove Theorem A.

Proof of Theorem A. As in the previous proof, set for simplicity $f := f_{\text{sreg}}$. Write $\mathfrak{g}$ for the simple exceptional Lie algebra G_2, and G for its adjoint group. Let $\mathbb{O}_f = G.f$ be the adjoint orbit of f. We have to show that $X_{L_{-2}(G_2)} = \overline{\mathbb{O}_f}$.

On the one hand, by [Ar2], the associated variety of $H^0_{DS,f}(L_{-2}(G_2))$ is the intersection of $X_{L_{-2}(G_2)}$ with the Slodowy slice $\mathscr{S}_f := f + \mathfrak{g}^e$, whence

$$\{f\} = X_{H^0_{DS,f}(L_{-2}(G_2))} = X_{L_{-2}(G_2)} \cap \mathscr{S}_f$$

using $H^0_{DS,f}(L_{-2}(G_2)) \cong \mathcal{W}_{-2}(G_2, f) \cong \mathbb{C}$ from Theorem 4.3. As a consequence, the associated variety $X_{L_{-2}(G_2)}$ contains f and so $\overline{\mathbb{O}_f}$, because $X_{L_{-2}(G_2)}$ is closed and G-invariant.

On the other hand, the associated variety $X_{L_{-2}(G_2)}$ is included in the nilpotent cone $\mathcal{N}$ of G_2 by Proposition 4.1. We conclude that

$$\overline{\mathbb{O}_f} \subset X_{L_{-2}(G_2)} \subset \mathcal{N},$$

and so $X_{L_{-2}(G_2)}$ is the closure of the regular or the subregular nilpotent orbit of G_2. The intersection between the nilpotent cone and the Slodowy slice $\mathscr{S}_f$ is two-dimensional whereas $X_{L_{-2}(G_2)} \cap \mathscr{S}_f = \{f\}$, the only possibility is $X_{L_{-2}(G_2)} = \overline{\mathbb{O}_f}$.

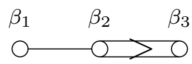
Theorem 4.3 also allows to prove that $X_{L_{-2}(G_2)} \subset \mathcal{N}$ without having recourse to Proposition 4.1. The argument first appears in [F1, Proposition 6.3.1]. We reproduce it for completeness of the paper. Suppose that there exists a non nilpotent element $x \in X_{L_{-2}(G_2)}$. Denote by $x = x_n + x_s$ its Jordan decomposition with x_n nilpotent and x_s a nonzero semisimple element. The G-invariant closed cone $C(x) := G.\mathbb{C}^* x$ generated by x is included in the associated variety. But according to [CM, Theorem 2.9], $C(x)$ contains the induced nilpotent orbit $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{x_s}}(\mathbb{O}_{x_n})$ from the adjoint orbit of x_n in $\mathfrak{g}^{x_s}$. The only induced nilpotent orbits in G_2 are the regular and the subregular orbits, so $C(x)$ strictly contains the subregular nilpotent orbit. The variety $C(x)$ is G-invariant, reduced and irreducible. Thus by [Gi, Corollary 1.3.8],

$$0 = \dim(X_{L_{-2}(G_2)} \cap \mathscr{S}_f) \geqslant \dim(C(x) \cap \mathscr{S}_f) = \dim C(x) - \dim \mathbb{O}_f > 0,$$

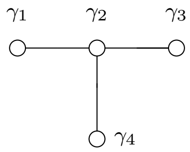
whence a contradiction. □

5. The representation theory of $L_{-2}(B_3)$

In this section, we study the representations of the simple affine vertex algebra $L_{-2}(B_3)$. Let us consider the simple exceptional Lie algebra of type B_3 with simple roots $\beta_1, \beta_2, \beta_3$ and Dynkin diagram



and the simple Lie algebra D_4 with simple roots $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and Dynkin diagram



We describe below the explicit embeddings $\iota_2 \colon G_2 \hookrightarrow B_3$ and $\iota_3 \colon B_3 \hookrightarrow D_4$ induced by the automorphisms of the Dynkin diagrams. First, one can express a Chevalley basis of G_2 in terms of that for B_3, which gives the embedding $\iota_2 \colon G_2 \hookrightarrow B_3$:

$$
\begin{aligned}
& e_{\alpha_1} = e_{\beta_1} + e_{\beta_3}, && e_{-\alpha_1} = e_{-\beta_1} + e_{-\beta_3}, \\
& e_{\alpha_2} = e_{\beta_2}, && e_{-\alpha_2} = e_{-\beta_2} \\
& e_{\alpha_3} = e_{\beta_1+\beta_2} - e_{\beta_2+\beta_3}, && e_{-\alpha_3} = e_{-(\beta_1+\beta_2)} - e_{-(\beta_2+\beta_3)}, \\
& e_{\alpha_4} = -e_{\beta_2+2\beta_3} - e_{\beta_1+\beta_2+\beta_3}, && e_{-\alpha_4} = -e_{-(\beta_2+2\beta_3)} - e_{-(\beta_1+\beta_2+\beta_3)}, \\
& e_{\alpha_5} = -e_{\beta_1+\beta_2+2\beta_3}, && e_{-\alpha_5} = -e_{-(\beta_1+\beta_2+2\beta_3)}; \\
& e_{\alpha_6} = -e_{\beta_1+2\beta_2+2\beta_3}, && e_{-\alpha_6} = -e_{-(\beta_1+2\beta_2+2\beta_3)}, \\
& h_1 = h_{\alpha_1} = h_{\beta_1} + h_{\beta_3}, && h_2 = h_{\alpha_2} = h_{\beta_2}.
\end{aligned}
$$

Similarly, let us describe the Chevalley basis of B_3 in terms of that for D_4 in order to get the embedding $\iota_3 \colon B_3 \hookrightarrow D_4$:

$$
\begin{aligned}
& e_{\beta_1} = e_{\gamma_1}, && e_{-\beta_1} = e_{-\gamma_1}, \\
& e_{\beta_2} = e_{\gamma_2}, && e_{-\beta_2} = e_{-\gamma_2}, \\
& e_{\beta_3} = e_{\gamma_3} + e_{\gamma_4}, && e_{-\beta_3} = e_{-\gamma_3} + e_{-\gamma_4} \\
& e_{\beta_1+\beta_2} = e_{\gamma_1+\gamma_2}, && e_{-(\beta_1+\beta_2)} = e_{-(\gamma_1+\gamma_2)}, \\
& e_{\beta_2+\beta_3} = e_{\gamma_2+\gamma_3} + e_{\gamma_2+\gamma_4}, && e_{-(\beta_2+\beta_3)} = e_{-(\gamma_2+\gamma_3)} + e_{-(\gamma_2+\gamma_4)}, \\
& e_{\beta_1+\beta_2+\beta_3} = e_{\gamma_1+\gamma_2+\gamma_3} + e_{\gamma_1+\gamma_2+\gamma_4}, && e_{-(\beta_1+\beta_2+\beta_3)} = e_{-(\gamma_1+\gamma_2+\gamma_3)} + e_{-(\gamma_1+\gamma_2+\gamma_4)}, \\
& e_{\beta_2+2\beta_3} = e_{\gamma_2+\gamma_3+\gamma_4}, && e_{-(\beta_2+2\beta_3)} = e_{-(\gamma_2+\gamma_3+\gamma_4)}, \\
& e_{\beta_1+\beta_2+2\beta_3} = e_{\gamma_1+\gamma_2+\gamma_3+\gamma_4}, && e_{-(\beta_1+\beta_2+2\beta_3)} = e_{-(\gamma_1+\gamma_2+\gamma_3+\gamma_4)}, \\
& e_{\beta_1+2\beta_2+2\beta_3} = e_{\gamma_1+2\gamma_2+\gamma_3+\gamma_4}, && e_{-(\beta_1+2\beta_2+2\beta_3)} = e_{-(\gamma_1+2\gamma_2+\gamma_3+\gamma_4)}.
\end{aligned}
$$

$$
h_{\beta_1} = h_{\gamma_1}, \quad h_{\beta_2} = h_{\gamma_2}, \quad h_{\beta_3} = h_{\gamma_3} + h_{\gamma_4}.
$$

We can compose these linear maps so that

$$
G_2 \overset{\iota_2}{\hookrightarrow} B_3 \overset{\iota_3}{\hookrightarrow} D_4 \hookrightarrow V^{-2}(D_4) \twoheadrightarrow L_{-2}(D_4). \tag{16}
$$

In [AP], Adamović and Perše proved that the vertex subalgebra generated by G_2 (resp. B_3) in $L_{-2}(D_4)$ is isomorphic to the irreducible affine vertex algebra $L_{-2}(G_2)$ (resp. $L_{-2}(B_3)$). Consider the vertex algebra homomorphism $\hat{\iota}_2 \colon V^{-2}(G_2) \to V^{-2}(B_3)$ induced from ι_2. A direct consequence of [AP] is that the vertex algebra homomorphism $\bar{\iota}_2 \colon L_{-2}(G_2) \to L_{-2}(B_3)$

is well defined and satisfies the following commutative diagram

$$\begin{tikzcd} V^{-2}(G_2) \arrow[r, "\hat{\iota}_2"] \arrow[d, "\pi_{G_2}"'] & V^{-2}(B_3) \arrow[d, "\pi_{B_3}"] \\ L_{-2}(G_2) \arrow[r, "\bar{\iota}_2"'] & L_{-2}(B_3) \end{tikzcd}$$

where π_{G_2} and π_{B_3} are the natural projection maps.

Let $N_{-2}^{B_3}$ be the maximal ideal in $V^{-2}(B_3)$ and let

$$v_{\text{sing}}^{G_2} := v_{\text{sing}}$$

be the singular vector of $V^{-2}(G_2)$ as in Theorem 3.1. It is clear from the commutative diagram that $\hat{\iota}_2(\langle v_{\text{sing}}^{G_2}\rangle) \subset N_{-2}^{B_3}$, because $\bar{\iota}_2 \circ \pi_{G_2}(\langle v_{\text{sing}}^{G_2}\rangle) = 0$. Hence the vector

$$w := \hat{\iota}_2(v_{\text{sing}}^{G_2})$$

is contained in $N_{-2}^{B_3}$ with conformal weight 6. Fix $\mathfrak{h}_{B_3} = \text{span}_{\mathbb{C}}\{h_{\beta_1}, h_{\beta_2}, h_{\beta_3}\}$ a Cartan subalgebra of B_3. Since the embedding $\hat{\iota}_2$ does not preserve the $\mathfrak{h}_{B_3}$-weight, we decompose w into a sum of $\mathfrak{h}_{B_3}$-weight vectors

$$w = \sum_{\mu \in \mathfrak{h}_{B_3}^*} w_\mu \tag{17}$$

where $h.w_\mu = \mu(h)w_\mu$ for all $h \in \mathfrak{h}_{B_3}$. It is known by [AM2] that there is a singular vector $v_{\text{sing}}^{B_3}$ of conformal weight two in $V^{-2}(B_3)$ given by:

$$v_{\text{sing}}^{B_3} = \\ e_{\beta_1+2\beta_2+2\beta_3}(-1)e_{\beta_3}(-1)\mathbf{1} - e_{\beta_1+\beta_2+2\beta_3}(-1)e_{\eta_2}(-1)\mathbf{1} + e_{\beta_1+\beta_2+\beta_3}(-1)e_{\beta_2+2\beta_3}(-1)\mathbf{1}.$$

We denote by I^{B_3} the left-ideal generated by $v_{\text{sing}}^{B_3}$ in $V^{-2}(B_3)$, and consider the quotient vertex algebra

$$\mathcal{V}_{-2}(B_3) = V^{-2}(B_3)/I^{B_3}.$$

The $U(B_3)$-submodule R^{B_3} generated by the vector ${v'}_{\text{sing}}^{B_3}$ under the adjoint action is isomorphic to $L_{B_3}(2\varpi_3)$. By using the same method as for G_2, we can determine a basis of the space of polynomials $\mathscr{P}_{v_{\text{sing}}^{B_3}}$, defined as in (10) with respect to R^{B_3}.

Lemma 5.1. *We have* $\mathscr{P}_{v_{\text{sing}}^{B_3}} = \text{span}_{\mathbb{C}}\{p_1^{B_3}, p_2^{B_3}, p_3^{B_3}\}$, *where*

$$\begin{aligned} p_1^{B_1}(h) &= (2h_2 + h_3)(h_1 + h_2 + h_3) + 2(h_2 + h_3), \\ p_2^{B_3}(h) &= (h_2 + h_3)(2h_1 + 2h_2 + h_3 + 2), \\ p_3^{B_3}(h) &= h_3(h_1 + 2h_2 + h_3 + 2). \end{aligned}$$

One gets the complete list of irreducible $A(\mathcal{V}_{-2}(B_3))$-modules in the category $\mathcal{O}$ by solving the polynomial equations

$$p_1^{B_3}(h) = p_2^{B_3}(h) = p_3^{B_3}(h) = 0.$$

Using Zhu's correspondence, we obtain the following results.

Theorem 5.2. *The complete list of irreducible $\mathcal{V}_{-2}(B_3)$-modules in the category $\mathcal{O}$ is given by the following set:*

$$\{L_{B_3}(-2, \mu_i(t))\colon i = 1, 2, 3,\ t \in \mathbb{C}\}$$

where,

$$\mu_1(t) = t\varpi_1, \quad \mu_2(t) = (-1-t)\varpi_1 + t\varpi_2, \quad \mu_3(t) = t\varpi_2 - 2(1+t)\varpi_3.$$

Corollary 5.3. *The complete list of irreducible ordinary modules for $\mathcal{V}_{-2}(B_3)$ is given by the following set:*

$$\{L_{B_3}(-2, k\varpi_1)\colon k \in \mathbb{Z}_{\geqslant 0}\}.$$

The vertex algebra $\mathcal{V}_{-2}(B_3)$ is not simple and the following lemma gives a description of the structure of the quotient vertex algebra, see [AKM+, Corollary 7.6][3].

Lemma 5.4. *The vertex algebra $\mathcal{V}_{-2}(B_3)$ contains a unique ideal $I \cong L_{B_3}(-2, -6\Lambda_0 + 4\Lambda_1)$, where $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3$ are the fundamental weights of $\hat{B}_3$.*

The key observation is that the unique singular vector in $\mathcal{V}_{-2}(B_3)$ comes from a subsingular vector in $V^{-2}(B_3)$.

Lemma 5.5. *In the decomposition (17), assume that $w_{4\varpi_1} \neq 0$ and $w_{4\varpi_1} \notin I^{B_3}$, then the maximal ideal $N_{-2}^{B_3}$ is generated by $w_{4\varpi_1}$ and $v_{\text{sing}}^{B_3}$.*

Proof. Since $N_{-2}^{B_3}$ is homogeneous with respect to $\mathfrak{h}_{B_3}^*$, we deduce that $w_{4\varpi_1} \in N_{-2}^{B_3}$. Let v_{sub} be a homogeneous subsingular vector in $V^{-2}(B_3)$ which maps through the quotient map $V^{-2}(B_3) \twoheadrightarrow \mathcal{V}_{-2}(B_3)$ to the unique singular vector in $\mathcal{V}_{-2}(B_3)$. So v_{sub} has $\mathfrak{h}_{B_3}$-weight $4\varpi_1$ and conformal weight 6. By Lemma 5.4, we have $N_{-2}^{B_3} = \langle v_{\text{sub}}, v_{\text{sing}}^{B_3}\rangle$. For any ideal J, we denote by J_6 the conformal weight 6 subspace of J. It is clear that for any $u = \sum_{\mu \in \mathfrak{h}_{B_3}^*} u_\mu$ in $\langle v_{\text{sub}}\rangle_6 \backslash I^{B_3}$, $u_\mu \neq 0$ implies that $\mu \leqslant 4\varpi_1$, where the equality holds if and only if $u_\mu = u_{4\varpi_1} = c v_{\text{sub}} \mod I^{B_3}$ for some constant $c \neq 0$. Since $w_{4\varpi_1} \in N_{-2}^{B_3} \backslash I^{B_3}$, $w_{4\varpi_1} = c v_{\text{sub}} \mod I^{B_3}$ for $c \neq 0$. Therefore $v_{\text{sub}} \in \langle w_{4\varpi_1}, I^{B_3}\rangle$. $\square$

Under the adjoint action of B_3, the submodule of $U(B_3)$ generated by vector $w_{4\varpi_1}$ is isomorphic to $L_{B_3}(4\varpi_1)$, the zero-weight space of $L_{B_3}(4\varpi_1)$ has dimension six. Let

$$\tilde{L}_{-2}(B_3) := V^{-2}(B_3)/\langle w_{4\varpi_1}, I^{B_3}\rangle.$$

Hence the irreducible highest-weight modules of $A(\tilde{L}_{-2}(B_3))$ are determined by polynomials in Lemma 5.1 and 5.6 below.

Lemma 5.6. *Let $\mathscr{P}_{w_{4\varpi_1}}$ be the polynomial set (10) relatively to $w_{4\varpi_1}$ defined by the decomposition (17). Then $\mathscr{P}_{w_{4\varpi_1}} = \operatorname{span}_{\mathbb{C}}\{p_4^{B_3}, p_5^{B_3}, \cdots, p_9^{B_3}\}$.*

Proof. By direct calculation we show that the following six polynomials are linearly independent, modulo $\mathfrak{n}_- U(B_3) + U(B_3)\mathfrak{n}_+$:

$$p_4^{B_3} = (e_{-\beta_1-\beta_2-\beta_3} e_{-\beta_1-\beta_2-\beta_3} e_{-\beta_1-\beta_2-\beta_3} e_{-\beta_1-\beta_2-\beta_3})_L (w_{4\varpi_1}),$$
$$p_5^{B_3} = (e_{-\beta_1-\beta_2} e_{-\beta_1-\beta_2-\beta_3} e_{-\beta_1-\beta_2-\beta_3} e_{-\beta_1-\beta_2-2\beta_3})_L (w_{4\varpi_1}),$$

[3] There is a typo in [AKM+, Corollary 7.6]: $L_{B_3}(-2, -4\Lambda_0 + 2\Lambda_1)$ should be $L_{B_3}(-2, -6\Lambda_0 + 4\Lambda_1)$.

$$p_6^{B_3} = (e_{-\beta_1-\beta_2}e_{-\beta_1-\beta_2}e_{-\beta_1-\beta_2-2\beta_3}e_{-\beta_1-\beta_2-2\beta_3})_L\,(w_{4\varpi_1})\,,$$
$$p_7^{B_3} = (e_{-\beta_1}e_{-\beta_1-\beta_2-\beta_3}e_{-\beta_1-\beta_2-\beta_3}e_{-\beta_1-2\beta_2-2\beta_3})_L\,(w_{4\varpi_1})\,,$$
$$p_8^{B_3} = (e_{-\beta_1}e_{-\beta_1-\beta_2}e_{-\beta_1-\beta_2-2\beta_3}e_{-\beta_1-2\beta_2-2\beta_3})_L\,(w_{4\varpi_1})\,,$$
$$p_9^{B_3} = (e_{-\beta_1}e_{-\beta_1}e_{-\beta_1-2\beta_2-2\beta_3}e_{-\beta_1-2\beta_2-2\beta_3})_L\,(w_{4\varpi_1})\,.$$

The explicit form of these polynomials can be found in Appendix B. □

Proposition 5.7. *The complete list of irreducible $A(\tilde{L}_{-2}(B_3))$-modules in the category $\mathcal{O}$ is given by the set $\{L_{B_3}(\mu_i)\colon i = 1, 2, \ldots, 13\}$, where the μ_i's are given by Table 3.*

μ_1	0	μ_8	$-\frac{5}{2}\varpi_2 + 3\varpi_3$
μ_2	ϖ_1	μ_9	$-\frac{3}{2}\varpi_2 + \varpi_3$
μ_3	$-2\varpi_1$	μ_{10}	$-\frac{1}{2}\varpi_1 - \frac{1}{2}\varpi_2$
μ_4	$-3\varpi_1$	μ_{11}	$-\frac{3}{2}\varpi_1$
μ_5	$-\varpi_2$	μ_{12}	$-\frac{1}{2}\varpi_1$
μ_6	$-2\varpi_3$	μ_{13}	$-\frac{3}{2}\varpi_1 + \frac{1}{2}\varpi_2$
μ_7	$\varpi_1 - 2\varpi_2$		

TABLE 3. The weights μ_i for B_3

Proof. The assertion can be established through a straightforward computation involving the polynomials in Lemma 5.1 and 5.6. □

Theorem 5.8. *We have $\tilde{L}_{-2}(B_3) \cong L_{-2}(B_3)$.*

Proof. According to Proposition 5.7, the set of the solutions of the equations in $\mathscr{P}_{v_{\text{sing}}^{B_3}} \cup \mathscr{P}_{w_{4\varpi_1}}$ is distinct from the solution set of $\mathscr{P}_{v_{\text{sing}}^{B_3}}$. Hence $w_{4\varpi_1}$ is nonzero and not contained in I^{B_3}. We complete the proof due to Lemma 5.5. □

Using the Zhu's correspondence, we have achieved the proof of Theorem C.

5.1. **Decomposition of non-ordinary modules.** In this paragraph, we obtain a nontrivial decomposition theorem of non-ordinary modules coming from the spectral flows of the ordinary modules. These modules are still graded by L_0, however, they are not necessarily bounded below.

According to [AP], we have

$$L_{D_4}(-2, 0) = L_{B_3}(-2, 0) \oplus L_{B_3}(-2, \varpi_1). \tag{18}$$

Identifying $\mathfrak{h}_{B_3}$ with its dual using $(-|-)$, the fundamental weights of B_3 are $\varpi_1 = \beta_1 + \beta_2 + \beta_3$, $\varpi_2 = \beta_1 + 2\beta_2 + 2\beta_3$, $\varpi_3 = \frac{1}{2}\beta_1 + \beta_2 + \frac{3}{2}\beta_3$. Consider the standard representation $L := L_{B_3}(\varpi_1) \cong \mathbb{C}^7$ of B_3 with canonical basis ϵ_i, $i \in \{1, \ldots, 7\}$. We schematize the representation by the following graph.

$$L_{\beta_1+\beta_2+\beta_3} \ni \epsilon_2 \xrightarrow{f_1} \epsilon_3 \xrightarrow{f_2} \epsilon_4 \xrightarrow{f_3} \epsilon_1 \xrightarrow{f_3} \epsilon_7 \xrightarrow{f_2} \epsilon_6 \xrightarrow{f_1} \epsilon_5 \in L_{-\beta_1-\beta_2-\beta_3},$$

where $f_j := e_{-\beta_j}$ is the negative β_j-root vector.

Assume for awhile that $\mathfrak{g}$ is an arbitrary simple Lie algebra as in Section 2. For an arbitrary $\widetilde{\mathfrak{g}}$-module M, one obtains a new $\widetilde{\mathfrak{g}}$-module structure on M by twisting the action by a certain automorphism σ of $\widetilde{\mathfrak{g}}$ as follows:

$$x(n)\sigma^*(v) = \sigma^*(\sigma^{-1}(x(n))v), \quad \text{for any } x \in \mathfrak{g}, n \in \mathbb{Z} \text{ and } v \in M.$$

To distinguish the two module structures, we will denote the new module by $\sigma^*(M)$. Among to the automorphisms of $\widetilde{\mathfrak{g}}$, the spectral flows are of particular interest. We refer [L] (or to [R, Appendix A] and references therein) for precise definitions and motivations.

In the following, we consider spectral flow automorphisms which correspond to translations of the extended Weyl group of $\widetilde{\mathfrak{g}}$. More concretely, each simple coroots $\alpha_i^\vee$ of $\mathfrak{g}$ defines a transformation τ_i that acts on the generators of $\widetilde{\mathfrak{g}}$ as follows:

$$\tau_i(e_\alpha(n)) = e_\alpha(n - \langle\alpha, \alpha_i^\vee\rangle), \qquad \tau_i(h_j(n)) = h_j(n) - (\alpha_j^\vee|\alpha_i^\vee)\delta_{n,0}K,$$
$$\tau_i(K) = K, \qquad \tau_i(L_0) = L_0 - h_i(0) + \frac{(\alpha_i^\vee|\alpha_i^\vee)}{2}K,$$

with $n \in \mathbb{Z}$, $\alpha \in \Delta$ and $L_0 = -D$. The powers of τ_i acts as follows:

$$\tau_i^s(e_\alpha(n)) = e_\alpha(n - s\langle\alpha, \alpha_i^\vee\rangle), \qquad \tau_i^s(h_j(n)) = h_j(n) - s(\alpha_j^\vee|\alpha_i^\vee)\delta_{n,0}K,$$
$$\tau_i^s(K) = K, \qquad \tau_i^s(L_0) = L_0 - s\,h_i(0) + \frac{s^2}{2}(\alpha_i^\vee|\alpha_i^\vee)K.$$

Return to the case of $\mathfrak{g} = D_4$ and consider the spectral flow automorphism along the direction Λ_1, which is defined by: $\sigma^{-1} := \tau_1^1\tau_2^1\tau_3^{1/2}\tau_4^{1/2}$. It is direct to check that σ is determined by the following maps

$$e_{\gamma_1}(n) \longmapsto e_{\gamma_1}(n+1), \quad e_{-\gamma_1}(n) \longmapsto e_{-\gamma_1}(n-1), \quad e_{\pm\gamma_i}(n) \longmapsto e_{\pm\gamma_i}(n),$$
$$K \longmapsto K, \quad h_1^{D_4}(0) \longmapsto h_1^{D_4}(0) + K, \quad h_i^{D_4}(0) \longmapsto h_i^{D_4}(0), \text{ for } i = 2, 3, 4.$$

In particular $\sigma^{-1}(e_{-\theta}(n)) = e_{-\theta}(n-1)$.

Applying the spectral flow to both sides of (18), one obtain the decomposition of Theorem D.

Proof of Theorem D. It is clear that the spectral flow σ preserves $\hat{B}_3$, and hence the highest-weight module structures for $L_{-2}(D_4)$ and for $L_{-2}(B_3)$. It suffices to show that $\sigma^* L_{B_3}(-2, \varpi_1) = L_{B_3}(-2, -3\varpi_1)$, which follows from the following lemma.

Lemma 5.9. *Let $\mathbf{1}_{\varpi_1}$ be the highest-weight vector of $L_{B_3}(-2, \varpi_1)$. Then $\sigma^*(e_{-\beta_1}(0)e_{-\theta}(0)\mathbf{1}_{\varpi_1})$ is the highest-weight vector of $\sigma^* L_{B_3}(-2, \varpi_1) = L_{B_3}(-2, -3\varpi_1)$.*

Proof. From the realization of the standard representation $L_{B_3}(\varpi_1)$, we deduce that $v = e_{-\beta_1}(0)e_{-\theta}(0)\mathbf{1}_{\varpi_1} \neq 0$ is the lowest weight vector in $L_{B_3}(\varpi_1)$ of weight $-\varpi_1$. By calculating the conformal weight, we have

$$\sigma^{-1}(e_{\beta_1}(0))v = e_{\beta_1}(1)e_{-\beta_1}(0)e_{-\theta}(0)\mathbf{1}_{\varpi_1} = 0.$$

For $i = 2, 3$, we have

$$\sigma^{-1}(e_{\beta_i}(0))v = e_{\beta_i}(0)e_{-\beta_1}(0)e_{-\theta}(0)\mathbf{1}_{\varpi_1} = 0,$$

where the last equality is due to the fact that $-\varpi_1 + \beta_i = \beta_1 - \beta_i$ is not a weight in $L_{B_3}(\varpi_1)$. Similarly, we have

$$\sigma^{-1}(e_{-\theta}(1))v = e_{-\theta}(0)v = 0,$$

as $-\varpi_1 - \theta$ is not a weight in $L_{B_3}(\varpi_1)$. Therefore v is a singular vector in $\sigma^*(e_{-\beta_1}(0)e_{-\theta}(0)\mathbf{1}_{\varpi_1})$ of highest-weight $-2\Lambda_0 - 3\varpi_1$. $\square$

Lemma 5.9 concludes the proof of Theorem D. $\square$

Appendix A. Singular vector for $V^{-2}(G_2)$

We give in this appendix an explicit description of a singular vector v_{sing} in the affine vertex algebra $V^{-2}(G_2)$ with weight $-2\Lambda_0 + 4\varpi_1 - 6\delta$ in $V^{-2}(G_2)$ (385 terms) as obtained in Theorem 3.1. We also obtain from this the image $v'_{\text{sing}} := F([v_{\text{sing}}])$ of v_{sing} is the Zhu's algebra, where F is the isomorphism (8).

$$
\begin{aligned}
v_{\text{sing}} = {}& -60e_\theta(-3)e_{\alpha_5}(-2)e_{\alpha_4}(-1)\mathbf{1} + 12e_\theta(-3)e_{\alpha_5}(-1)e_{\alpha_4}(-2)\mathbf{1} + 60e_\theta(-2)e_{\alpha_5}(-3)e_{\alpha_4}(-1)\mathbf{1} - 120e_\theta(-2)e_{\alpha_5}(-2)e_{\alpha_4}(-2)\mathbf{1} \\
& + 36e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-3)\mathbf{1} + 84e_\theta(-3)e_{\alpha_5}(-1)e_{\alpha_4}(-2)\mathbf{1} - 84e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-3)\mathbf{1} + 8e_{\alpha_4}(-3)e_{\alpha_4}(-1)^3\mathbf{1} \\
& - 12e_{\alpha_4}(-2)^2e_{\alpha_4}(-1)^2\mathbf{1} - 12e_{\alpha_5}(-3)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)\mathbf{1} + 48e_{\alpha_5}(-3)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2\mathbf{1} - 36e_{\alpha_5}(-3)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)\mathbf{1} \\
& + 60e_{\alpha_5}(-2)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_3}(-1)\mathbf{1} - 24e_{\alpha_5}(-2)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)\mathbf{1} - 60e_{\alpha_5}(-2)^2e_{\alpha_3}(-1)^2\mathbf{1} + 60e_{\alpha_5}(-2)^2e_{\alpha_4}(-1)e_{\alpha_2}(-1)\mathbf{1} \\
& - 24e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-2)e_{\alpha_3}(-1)\mathbf{1} - 60e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_2}(-1)\mathbf{1} + 48e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-2)\mathbf{1} \\
& - 28e_{\alpha_5}(-1)e_{\alpha_4}(-3)e_{\alpha_4}(-1)e_{\alpha_3}(-1)\mathbf{1} - 12e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-2)e_{\alpha_3}(-1)\mathbf{1} + 48e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_3}(-2)\mathbf{1} \\
& - 8e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-3)\mathbf{1} + 24e_{\alpha_5}(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-3)e_{\alpha_3}(-1)\mathbf{1} - 24e_{\alpha_5}(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-2)e_{\alpha_3}(-2)\mathbf{1} \\
& + 36e_{\alpha_5}(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-3)e_{\alpha_2}(-1)\mathbf{1} - 48e_{\alpha_5}(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_2}(-2)\mathbf{1} - 12e_\theta(-3)e_{\alpha_4}(-1)e_{\alpha_4}(-1)e_{\alpha_1}(-1)\mathbf{1} \\
& + 48e_\theta(-3)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)\mathbf{1} + 48e_\theta(-3)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-1)\mathbf{1} + 48e_\theta(-3)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_2(-1)\mathbf{1} \\
& - 36e_\theta(-3)e_{\alpha_5}(-1)^2e_{-\alpha_1}(-1)\mathbf{1} + 48e_\theta(-3)e_\theta(-1)e_{\alpha_1}(-1)^2\mathbf{1} + 24e_\theta(-3)e_\theta(-1)e_{\alpha_4}(-1)e_{-\alpha_2}(-1)\mathbf{1} \\
& + 48e_\theta(-3)e_\theta(-1)e_{\alpha_5}(-1)e_{-\alpha_3}(-1)\mathbf{1} + 60e_\theta(-2)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_1}(-1)\mathbf{1} - 12e_\theta(-2)e_{\alpha_4}(-1)^2e_{\alpha_1}(-2)\mathbf{1} \\
& - 120e_\theta(-2)e_{\alpha_5}(-2)e_{\alpha_3}(-1)e_{\alpha_1}(-1)\mathbf{1} + 60e_\theta(-2)e_{\alpha_5}(-2)e_{\alpha_4}(-1)h_2(-1)\mathbf{1} + 72e_\theta(-2)e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{-\alpha_1}(-1)\mathbf{1} \\
& - 36e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-2)e_{\alpha_1}(-1)\mathbf{1} + 48e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-2)\mathbf{1} + 72e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-2)h_1(-1)\mathbf{1} \\
& + 36e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-2)h_2(-1)\mathbf{1} - 16e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-2)\mathbf{1} - 36e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_2(-2)\mathbf{1} \\
& - 48e_\theta(-2)e_{\alpha_5}(-1)^2e_{-\alpha_1}(-2)\mathbf{1} - 60e_\theta(-2)^2e_{\alpha_1}(-1)^2\mathbf{1} - 60e_\theta(-2)^2e_{\alpha_4}(-1)e_{-\alpha_2}(-1)\mathbf{1} - 72e_\theta(-2)^2e_{\alpha_5}(-1)e_{-\alpha_3}(-1)\mathbf{1} \\
& - 72e_\theta(-2)e_\theta(-1)e_{\alpha_1}(-2)e_{\alpha_1}(-1)\mathbf{1} + 156e_\theta(-2)e_\theta(-1)e_{\alpha_4}(-2)e_{-\alpha_2}(-1)\mathbf{1} - 48e_\theta(-2)e_\theta(-1)e_{\alpha_4}(-1)e_{-\alpha_2}(-2)\mathbf{1} \\
& + 72e_\theta(-2)e_\theta(-1)e_{\alpha_5}(-2)e_{-\alpha_3}(-1)\mathbf{1} + 48e_\theta(-2)e_\theta(-1)e_{\alpha_5}(-1)e_{-\alpha_3}(-2)\mathbf{1} - 36e_\theta(-1)e_{\alpha_4}(-3)e_{\alpha_4}(-1)e_{\alpha_1}(-1)\mathbf{1} \\
& - 12e_\theta(-1)e_{\alpha_4}(-2)^2e_{\alpha_1}(-1)\mathbf{1} + 60e_\theta(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_1}(-2)\mathbf{1} - 12e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-3)\mathbf{1} \\
& + 48e_\theta(-1)e_{\alpha_5}(-3)e_{\alpha_3}(-1)e_{\alpha_1}(-1)\mathbf{1} - 48e_\theta(-1)e_{\alpha_5}(-3)e_{\alpha_4}(-1)h_1(-1)\mathbf{1} - 84e_\theta(-1)e_{\alpha_5}(-3)e_{\alpha_4}(-1)h_2(-1)\mathbf{1} \\
& + 36e_\theta(-1)e_{\alpha_5}(-3)e_{\alpha_5}(-1)e_{-\alpha_1}(-1)\mathbf{1} + 60e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_3}(-2)e_{\alpha_1}(-1)\mathbf{1} - 120e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_3}(-1)e_{\alpha_1}(-2)\mathbf{1} \\
& + 24e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-2)h_1(-1)\mathbf{1} + 16e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)h_1(-2)\mathbf{1} + 84e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)h_2(-2)\mathbf{1} \\
& - 72e_\theta(-1)e_{\alpha_5}(-2)^2e_{-\alpha_1}(-1)\mathbf{1} + 48e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{-\alpha_1}(-2)\mathbf{1} + 48e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-3)e_{\alpha_1}(-1)\mathbf{1} \\
& - 96e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-2)e_{\alpha_1}(-2)\mathbf{1} + 48e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-3)\mathbf{1} + 36e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-3)h_1(-1)\mathbf{1} \\
& + 84e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-3)h_2(-1)\mathbf{1} - 32e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)h_1(-2)\mathbf{1} - 84e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)h_2(-2)\mathbf{1} \\
& + 48e_\theta(-1)^2e_{\alpha_1}(-3)e_{\alpha_1}(-1)\mathbf{1} - 60e_\theta(-1)^2e_{\alpha_1}(-2)^2\mathbf{1} - 42e_\theta(-1)^2e_{\alpha_4}(-3)e_{-\alpha_2}(-1)\mathbf{1} \\
& + 42e_\theta(-1)^2e_{\alpha_4}(-2)e_{-\alpha_2}(-2)\mathbf{1} - 42e_\theta(-1)^2e_{\alpha_5}(-3)e_{-\alpha_3}(-1)\mathbf{1} - 48e_\theta(-1)^2e_{\alpha_5}(-2)e_{-\alpha_3}(-2)\mathbf{1} \\
& + 6e_{\alpha_4}(-2)e_{\alpha_4}(-1)^3h_1(-1)\mathbf{1} + 8e_{\alpha_4}(-2)e_{\alpha_4}(-1)^3h_2(-1)\mathbf{1} + 2e_{\alpha_4}(-1)^3e_{\alpha_3}(-2)e_{\alpha_1}(-1)\mathbf{1} - e_{\alpha_4}(-1)^4h_1(-2)\mathbf{1} \\
& - 2e_{\alpha_4}(-1)^4h_2(-2)\mathbf{1} - 12e_{\alpha_5}(-2)e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)\mathbf{1} - 12e_{\alpha_5}(-2)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_1(-1)\mathbf{1} \\
& - 12e_{\alpha_5}(-2)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_2(-1)\mathbf{1} + 6e_{\alpha_5}(-2)e_{\alpha_4}(-1)^3e_{-\alpha_1}(-1)\mathbf{1} + 48e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)\mathbf{1} \\
& + 48e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2h_1(-1)\mathbf{1} + 48e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2h_2(-1)\mathbf{1} - 24e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)h_1(-1)\mathbf{1} \\
& - 36e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)h_2(-1)\mathbf{1} - 24e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_1}(-1)\mathbf{1} \\
& + 18e_{\alpha_5}(-2)e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1} - 12e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)\mathbf{1} \\
& - 24e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_3}(-1)h_1(-1)\mathbf{1} - 28e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_3}(-1)h_2(-1)\mathbf{1} \\
& - 2e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)^2e_{-\alpha_1}(-1)\mathbf{1} - 4e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-2)e_{\alpha_3}(-1)e_{\alpha_1}(-1)\mathbf{1} - 6e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_2}(-2)e_{\alpha_1}(-1)\mathbf{1} \\
& + 10e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-2)\mathbf{1} - 2e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-2)h_1(-1)\mathbf{1} - 8e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-2)h_2(-1)\mathbf{1} \\
& + 8e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_1(-2)\mathbf{1} + 12e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_2(-2)\mathbf{1} - 4e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3e_{-\alpha_1}(-2)\mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&+18e_{\alpha_5}(-1)^2e_{\alpha_3}(-2)e_{\alpha_2}(-1)e_{\alpha_1}(-1)\mathbf{1}+18e_{\alpha_5}(-1)^2e_{\alpha_3}(-2)e_{\alpha_3}(-1)h_1(-1)\mathbf{1}+24e_{\alpha_5}(-1)^2e_{\alpha_3}(-2)e_{\alpha_3}(-1)h_2(-1)\mathbf{1}\\
&+6e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{\alpha_2}(-2)e_{\alpha_1}(-1)\mathbf{1}-30e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-2)\mathbf{1}-16e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2h_1(-2)\mathbf{1}\\
&-18e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2h_2(-2)\mathbf{1}+18e_{\alpha_5}(-1)^2e_{\alpha_4}(-2)e_{\alpha_2}(-1)h_1(-1)\mathbf{1}+36e_{\alpha_5}(-1)^2e_{\alpha_4}(-2)e_{\alpha_2}(-1)h_2(-1)\mathbf{1}\\
&+18e_{\alpha_5}(-1)^2e_{\alpha_4}(-2)e_{\alpha_3}(-1)e_{-\alpha_1}(-1)\mathbf{1}-12e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_2}(-2)h_1(-1)\mathbf{1}+3e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_2}(-1)h_1(-2)\mathbf{1}\\
&-12e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_3}(-2)e_{-\alpha_1}(-1)\mathbf{1}+13e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_1}(-2)\mathbf{1}+36e_{\alpha_5}(-1)^3e_{\alpha_2}(-2)e_{-\alpha_1}(-1)\mathbf{1}\\
&-9e_{\alpha_5}(-1)^3e_{\alpha_2}(-1)e_{-\alpha_1}(-2)\mathbf{1}-12e_\theta(-2)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)h_1(-1)\mathbf{1}-24e_\theta(-2)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)h_2(-1)\mathbf{1}\\
&-12e_\theta(-2)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)e_{-\alpha_2}(-1)\mathbf{1}-6e_\theta(-2)e_{\alpha_4}(-1)^3e_{-\alpha_3}(-1)\mathbf{1}+42e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2\mathbf{1}\\
&+48e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_1(-1)\mathbf{1}+54e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}\\
&+6e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2e_{-\alpha_2}(-1)\mathbf{1}-28e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1}+72e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&-4e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_3}(-1)\mathbf{1}-40e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{-\alpha_4}(-1)\mathbf{1}-16e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-1)^2\mathbf{1}\\
&-60e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-1)h_2(-1)\mathbf{1}-36e_\theta(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_2(-1)^2\mathbf{1}+90e_\theta(-2)e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+48e_\theta(-2)e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{-\alpha_4}(-1)\mathbf{1}-126e_\theta(-2)e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{-\alpha_5}(-1)\mathbf{1}-48e_\theta(-2)e_{\alpha_5}(-1)^2h_1(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&-18e_\theta(-2)e_{\alpha_5}(-1)^2h_2(-1)e_{-\alpha_1}(-1)\mathbf{1}+48e_\theta(-2)e_\theta(-1)e_{\alpha_1}(-1)^2h_1(-1)\mathbf{1}+96e_\theta(-2)e_\theta(-1)e_{\alpha_1}(-1)^2.h_2(-1)\mathbf{1}\\
&+48e_\theta(-2)e_\theta(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)e_{-\alpha_2}(-1)\mathbf{1}+24e_\theta(-2)e_\theta(-1)e_{\alpha_4}(-1)e_{\alpha_1}(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+24e_\theta(-2)e_\theta(-1)e_{\alpha_4}(-1)h_1(-1)e_{-\alpha_2}(-1)\mathbf{1}+36e_\theta(-2)e_\theta(-1)e_{\alpha_4}(-1)h_2(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&+48e_\theta(-2)e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_1}(-1)e_{-\alpha_4}(-1)\mathbf{1}-108e_\theta(-2)e_\theta(-1)e_{\alpha_5}(-1)e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&-126e_\theta(-2)e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{-\theta}(-1)\mathbf{1}+48e_\theta(-2)e_\theta(-1)e_{\alpha_5}(-1)h_1(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+108e_\theta(-2)e_\theta(-1)e_{\alpha_5}(-1)h_2(-1)e_{-\alpha_3}(-1)\mathbf{1}+18e_\theta(-2)e_\theta(-1)^2e_{-\alpha_3}(-1)e_{-\alpha_2}(-1)\mathbf{1}-24e_\theta(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_1}(-1)h_1(-1)\mathbf{1}\\
&-36e_\theta(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}-4e_\theta(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_2}(-1)\mathbf{1}+8e_\theta(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)^2e_{-\alpha_3}(-1)\mathbf{1}\\
&-6e_\theta(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-2)e_{\alpha_1}(-1)^2\mathbf{1}-12e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-2)h_1(-1)\mathbf{1}-13e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-2)h_2(-1)\mathbf{1}\\
&+8e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)h_1(-2)\mathbf{1}+6e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)h_2(-2)\mathbf{1}+16e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-2)e_{-\alpha_2}(-1)\mathbf{1}\\
&-6e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)e_{-\alpha_2}(-2)\mathbf{1}+4e_\theta(-1)e_{\alpha_4}(-1)^3e_{-\alpha_3}(-2)\mathbf{1}+6e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2\mathbf{1}\\
&+48e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_1(-1)\mathbf{1}+90e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}+42e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_3}(-1)^2e_{-\alpha_2}(-1)\mathbf{1}\\
&+4e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1}-72e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&+28e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_3}(-1)\mathbf{1}+40e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)^2e_{-\alpha_4}(-1)\mathbf{1}+16e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)h_1(-1)^2\mathbf{1}\\
&+36e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_4}(-1)h_1(-1)h_2(-1)\mathbf{1}-108e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_2}(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&-48e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{-\alpha_4}(-1)\mathbf{1}+126e_\theta(-1)e_{\alpha_5}(-2)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{-\alpha_5}(-1)\mathbf{1}\\
&+48e_\theta(-1)e_{\alpha_5}(-2)e_\theta(-1)h_1(-1)e_{-\alpha_1}(-1)\mathbf{1}+36e_\theta(-1)e_{\alpha_5}(-2)e_\theta(-1)h_2(-1)e_{-\alpha_1}(-1)\mathbf{1}-12e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_2}(-2)e_{\alpha_1}(-1)^2\mathbf{1}\\
&+24e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-2)e_{\alpha_1}(-1)\mathbf{1}-12e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-2)e_{\alpha_1}(-1)h_1(-1)\mathbf{1}\\
&+6e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-2)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}-12e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-2)e_{\alpha_3}(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&+48e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-2)h_1(-1)\mathbf{1}+30e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-2)h_2(-1)\mathbf{1}\\
&-32e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_1(-2)\mathbf{1}-30e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_2(-2)\mathbf{1}+6e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2e_{-\alpha_2}(-2)\mathbf{1}\\
&+16e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1}-36e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_2}(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&-24e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_3}(-1)e_{-\alpha_3}(-1)\mathbf{1}-6e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{\alpha_4}(-1)e_{-\alpha_4}(-1)\mathbf{1}-32e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)h_1(-1)^2\mathbf{1}\\
&-48e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-2)h_1(-1)h_2(-1)\mathbf{1}-15e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_1}(-2)e_{-\alpha_1}(-1)\mathbf{1}\\
&+13e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_1}(-1)e_{-\alpha_1}(-2)\mathbf{1}-90e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-2)e_{-\alpha_2}(-1)\mathbf{1}\\
&+36e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{-\alpha_2}(-2)\mathbf{1}-6e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-2)e_{-\alpha_3}(-1)\mathbf{1}\\
&-13e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_3}(-2)\mathbf{1}+10e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{-\alpha_4}(-2)\mathbf{1}\\
&+3e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-2)h_2(-1)\mathbf{1}-12e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-1)h_2(-2)\mathbf{1}-90e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_2}(-2)e_{-\alpha_3}(-1)\mathbf{1}\\
&+9e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)e_{-\alpha_3}(-2)\mathbf{1}-54e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_3}(-2)e_{-\alpha_4}(-1)\mathbf{1}-30e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{-\alpha_4}(-2)\mathbf{1}\\
&-126e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_4}(-2)e_{-\alpha_5}(-1)\mathbf{1}+36e_\theta(-1)e_{\alpha_5}(-1)^2h_2(-2)e_{-\alpha_1}(-1)\mathbf{1}-9e_\theta(-1)e_{\alpha_5}(-1)^2h_2(-1)e_{-\alpha_1}(-2)\mathbf{1}\\
&+48e_\theta(-1)^2e_{\alpha_1}(-2)e_{\alpha_1}(-1)h_1(-1)\mathbf{1}+84e_\theta(-1)^2e_{\alpha_1}(-2)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}-16e_\theta(-1)^2e_{\alpha_1}(-1)^2h_1(-2)\mathbf{1}\\
&-21e_\theta(-1)^2e_{\alpha_1}(-1)^2h_2(-2)\mathbf{1}-48e_\theta(-1)^2e_{\alpha_3}(-2)e_{\alpha_1}(-1)e_{-\alpha_2}(-1)\mathbf{1}+54e_\theta(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-2)e_{-\alpha_2}(-1)\mathbf{1}\\
&-12e_\theta(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-1)e_{-\alpha_2}(-2)\mathbf{1}-22e_\theta(-1)^2e_{\alpha_4}(-2)e_{\alpha_1}(-1)e_{-\alpha_3}(-1)\mathbf{1}-78e_\theta(-1)^2e_{\alpha_4}(-2)h_1(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&-126e_\theta(-1)^2e_{\alpha_4}(-2)h_2(-1)e_{-\alpha_2}(-1)\mathbf{1}+15e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_1}(-2)e_{-\alpha_3}(-1)\mathbf{1}-13e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_1}(-1)e_{-\alpha_3}(-2)\mathbf{1}\\
&-3e_\theta(-1)^2e_{\alpha_4}(-1)h_1(-2)e_{-\alpha_2}(-1)\mathbf{1}+12e_\theta(-1)^2e_{\alpha_4}(-1)h_1(-1)e_{-\alpha_2}(-2)\mathbf{1}-63e_\theta(-1)^2e_{\alpha_4}(-1)h_2(-2)e_{-\alpha_2}(-1)\mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&+ 63e_\theta(-1)^2e_{\alpha_4}(-1)h_2(-1)e_{-\alpha_2}(-2)\mathbf{1} - 48e_\theta(-1)^2e_{\alpha_5}(-2)e_{\alpha_1}(-1)e_{-\alpha_4}(-1)\mathbf{1} + 90e_\theta(-1)^2e_{\alpha_5}(-2)e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&+ 126e_\theta(-1)^2e_{\alpha_5}(-2)e_{\alpha_4}(-1)e_{-\theta}(-1)\mathbf{1} - 48e_\theta(-1)^2e_\theta(-2)h_1(-1)e_{-\alpha_3}(-1)\mathbf{1} - 126e_\theta(-1)^2e_\theta(-2)h_2(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&- 30e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_1}(-1)e_{-\alpha_4}(-2)\mathbf{1} - 9e_\theta(-1)^2e_{\alpha_5}(-1)e_{-\alpha_2}(-2)e_{-\alpha_1}(-1)\mathbf{1} + 9e_\theta(-1)^2e_{\alpha_5}(-1)e_{-\alpha_2}(-1)e_{-\alpha_1}(-2)\mathbf{1}\\
&- 126e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_4}(-2)e_{-\theta}(-1)\mathbf{1} - 63e_\theta(-1)^2e_\theta(-1)h_2(-2)e_{-\alpha_3}(-1)\mathbf{1} + 9e_\theta(-1)e_\theta(-1)e_\theta(-1)h_2(-1)e_{-\alpha_3}(-2)\mathbf{1}\\
&- 9e_\theta(-1)^3e_{-\alpha_3}(-2)e_{-\alpha_2}(-1)\mathbf{1} + 63e_\theta(-1)^3e_{-\alpha_3}(-1)e_{-\alpha_2}(-2)\mathbf{1} + e_{\alpha_4}(-1)^3e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2\mathbf{1} - e_{\alpha_4}(-1)^3e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}\\
&- e_{\alpha_4}(-1)^3e_{\alpha_3}(-1)^2e_{-\alpha_2}(-1)\mathbf{1} - e_{\alpha_4}(-1)^4e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1} + e_{\alpha_4}(-1)^4e_{\alpha_2}(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&- e_{\alpha_4}(-1)^4e_{\alpha_3}(-1)e_{-\alpha_3}(-1)\mathbf{1} - e_{\alpha_4}(-1)^5e_{-\alpha_4}(-1)\mathbf{1} - e_{\alpha_4}(-1)^4h_1(-1)^2\mathbf{1} - 3e_{\alpha_4}(-1)^4h_1(-1)h_2(-1)\mathbf{1} - 2e_{\alpha_4}(-1)^4h_2(-1)^2\mathbf{1}\\
&- 2e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2\mathbf{1} + 2e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)^2e_{\alpha_1}(-1)h_2(-1)\mathbf{1} + 2e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)^3e_{-\alpha_2}(-1)\mathbf{1}\\
&+ 10e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)h_1(-1)\mathbf{1} + 15e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}\\
&+ 3e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1} + 6e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&+ 3e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)^2e_{-\alpha_3}(-1)\mathbf{1} + 8e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_1(-1)^2\mathbf{1}\\
&+ 19e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_1(-1)h_2(-1)\mathbf{1} + 12e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_2(-1)^2\mathbf{1}\\
&+ 8e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3e_{\alpha_2}(-1)e_{-\alpha_3}(-1)\mathbf{1} + 4e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3e_{\alpha_3}(-1)e_{-\alpha_4}(-1)\mathbf{1} - 7e_{\alpha_5}(-1)e_{\alpha_4}(-1)^4e_{-\alpha_5}(-1)\mathbf{1}\\
&- 4e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3h_1(-1)e_{-\alpha_1}(-1)\mathbf{1} - 2e_{\alpha_5}(-1)e_{\alpha_4}(-1)^3h_2(-1)e_{-\alpha_1}(-1)\mathbf{1} - 9e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)^2e_{\alpha_1}(-1)^2\mathbf{1}\\
&- 30e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)h_1(-1)\mathbf{1} - 36e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)h_2(-1)\mathbf{1}\\
&- e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1} - 9e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2e_{\alpha_2}(-1)e_{-\alpha_2}(-1)\mathbf{1} - e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^3e_{-\alpha_3}(-1)\mathbf{1}\\
&- 16e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2h_1(-1)^2\mathbf{1} - 33e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2h_1(-1)h_2(-1)\mathbf{1} - 18e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2h_2(-1)^2\mathbf{1}\\
&+ 3e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)e_{-\alpha_1}(-1)\mathbf{1} - 27e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_2}(-1)^2e_{-\alpha_2}(-1)\mathbf{1}\\
&+ 3e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_2}(-1)h_1(-1)^2\mathbf{1} + 9e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_2}(-1)h_1(-1)h_2(-1)\mathbf{1}\\
&- 18e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{-\alpha_3}(-1)\mathbf{1} - 3e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_3}(-1)^2e_{-\alpha_4}(-1)\mathbf{1}\\
&+ 13e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_3}(-1)h_1(-1)e_{-\alpha_1}(-1)\mathbf{1} + 9e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)e_{\alpha_3}(-1)h_2(-1)e_{-\alpha_1}(-1)\mathbf{1} + 5e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)^2e_{-\alpha_1}(-1)^2\mathbf{1}\\
&+ 18e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{-\alpha_4}(-1)\mathbf{1} + 42e_{\alpha_5}(-1)^2e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)e_{-\alpha_5}(-1)\mathbf{1} - 27e_{\alpha_5}(-1)^3e_{\alpha_2}(-1)^2e_{-\alpha_3}(-1)\mathbf{1}\\
&- 9e_{\alpha_5}(-1)^3e_{\alpha_2}(-1)h_1(-1)e_{-\alpha_1}(-1)\mathbf{1} - 27e_{\alpha_5}(-1)^3e_{\alpha_2}(-1)h_2(-1)e_{-\alpha_1}(-1)\mathbf{1} - 15e_{\alpha_5}(-1)^3e_{\alpha_3}(-1)e_{-\alpha_1}(-1)^2\mathbf{1}\\
&- 54e_{\alpha_5}(-1)^3e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{-\alpha_4}(-1)\mathbf{1} - 63e_{\alpha_5}(-1)^3e_{\alpha_3}(-1)^2e_{-\alpha_5}(-1)\mathbf{1} - 2e_\theta(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)^3\mathbf{1}\\
&+ 2e_\theta(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)^2h_2(-1)\mathbf{1} + 2e_\theta(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)^2e_{\alpha_1}(-1)e_{-\alpha_2}(-1)\mathbf{1} + 3e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)^2e_{-\alpha_1}(-1)\mathbf{1}\\
&+ 8e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)h_1(-1)^2\mathbf{1} + 29e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)h_1(-1)h_2(-1)\mathbf{1} + 27e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)h_2(-1)^2\mathbf{1}\\
&+ 6e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)e_{-\alpha_2}(-1)\mathbf{1} + 3e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+ 10e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_1(-1)e_{-\alpha_2}(-1)\mathbf{1} + 15e_\theta(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)h_2(-1)e_{-\alpha_2}(-1)\mathbf{1} + 4e_\theta(-1)e_{\alpha_4}(-1)^3e_{\alpha_1}(-1)e_{-\alpha_4}(-1)\mathbf{1}\\
&- 8e_\theta(-1)e_{\alpha_4}(-1)^3e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1} - 7e_\theta(-1)e_{\alpha_4}(-1)^4e_{-\theta}(-1)\mathbf{1} + 4e_\theta(-1)e_{\alpha_4}(-1)^3h_1(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+ 10e_\theta(-1)e_{\alpha_4}(-1)^3h_2(-1)e_{-\alpha_3}(-1)\mathbf{1} - 30e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2h_1(-1)\mathbf{1} - 54e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2h_2(-1)\mathbf{1}\\
&- 2e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)^2e_{-\alpha_1}(-1)\mathbf{1} - 32e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_1(-1)^2\mathbf{1}\\
&- 96e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_1(-1)h_2(-1)\mathbf{1} - 72e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_2(-1)^2\mathbf{1}\\
&- 36e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)e_{-\alpha_2}(-1)\mathbf{1} - 2e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2e_{\alpha_1}(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&- 30e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2h_1(-1)e_{-\alpha_2}(-1)\mathbf{1} - 36e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_3}(-1)^2h_2(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&+ 13e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_1}(-1)h_1(-1)e_{-\alpha_1}(-1)\mathbf{1} + 12e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_1}(-1)h_2(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&- 21e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{\alpha_1}(-1)e_{-\alpha_3}(-1)\mathbf{1} - 27e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_2}(-1)h_2(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&- 6e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)e_{-\alpha_4}(-1)\mathbf{1} + 21e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&- 13e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)h_1(-1)e_{-\alpha_3}(-1)\mathbf{1} - 27e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)e_{\alpha_3}(-1)h_2(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+ 42e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)e_{-\alpha_5}(-1)\mathbf{1} - 10e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{-\alpha_3}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&+ 42e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2e_{\alpha_3}(-1)e_{-\theta}(-1)\mathbf{1} + 18e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)^2h_2(-1)e_{-\alpha_4}(-1)\mathbf{1}\\
&+ 3e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-1)^2h_2(-1)\mathbf{1} + 9e_\theta(-1)e_{\alpha_5}(-1)e_{\alpha_4}(-1)h_1(-1)h_2(-1)^2\mathbf{1} - 15e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_1}(-1)e_{-\alpha_1}(-1)^2\mathbf{1}\\
&- 54e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)e_{-\alpha_4}(-1)\mathbf{1} + 27e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&+ 9e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)h_1(-1)e_{-\alpha_3}(-1)\mathbf{1} - 27e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_2}(-1)h_2(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&- 126e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-1)e_{-\alpha_5}(-1)\mathbf{1} + 30e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)e_{-\alpha_3}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&- 63e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)^2e_{-\theta}(-1)\mathbf{1} - 54e_\theta(-1)e_{\alpha_5}(-1)^2e_{\alpha_3}(-1)h_2(-1)e_{-\alpha_4}(-1)\mathbf{1}
\end{aligned}
$$

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\begin{aligned}
&- 9e_\theta(-1)e_{\alpha_5}(-1)^2h_1(-1)h_2(-1)e_{-\alpha_1}(-1)\mathbf{1} - 27e_\theta(-1)e_{\alpha_5}(-1)^2h_2(-1)^2e_{-\alpha_1}(-1)\mathbf{1}\\
&- e_\theta(-1)^2e_{\alpha_1}(-1)^3e_{-\alpha_1}(-1)\mathbf{1} - 16e_\theta(-1)^2e_{\alpha_1}(-1)^2h_1(-1)^2\mathbf{1} - 63e_\theta(-1)^2e_{\alpha_1}(-1)^2h_1(-1)h_2(-1)\mathbf{1}\\
&- 63e_\theta(-1)^2e_{\alpha_1}(-1)^2h_2(-1)^2\mathbf{1} - 9e_\theta(-1)^2e_{\alpha_2}(-1)e_{\alpha_1}(-1)^2e_{-\alpha_2}(-1)\mathbf{1} - e_\theta(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-1)^2e_{-\alpha_3}(-1)\mathbf{1}\\
&- 30e_\theta(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_1(-1)e_{-\alpha_2}(-1)\mathbf{1} - 54e_\theta(-1)^2e_{\alpha_3}(-1)e_{\alpha_1}(-1)h_2(-1)e_{-\alpha_2}(-1)\mathbf{1} - 9e_\theta(-1)^2e_{\alpha_3}(-1)^2e_{-\alpha_2}(-1)^2\mathbf{1}\\
&- 3e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_1}(-1)^2e_{-\alpha_4}(-1)\mathbf{1} + 18e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_1}(-1)e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1}\\
&- 13e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_1}(-1)h_1(-1)e_{-\alpha_3}(-1)\mathbf{1} - 30e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_1}(-1)h_2(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+ 27e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_2}(-1)e_{-\alpha_2}(-1)^2\mathbf{1} - 3e_\theta(-1)^2e_{\alpha_4}(-1)e_{\alpha_3}(-1)e_{-\alpha_3}(-1)e_{-\alpha_2}(-1)\mathbf{1} + 42e_\theta(-1)^2e_{\alpha_4}(-1)^2e_{\alpha_1}(-1)e_{-\theta}(-1)\mathbf{1}\\
&- 18e_\theta(-1)^2e_{\alpha_4}(-1)^2e_{-\alpha_4}(-1)e_{-\alpha_2}(-1)\mathbf{1} + 5e_\theta(-1)^2e_{\alpha_4}(-1)^2e_{-\alpha_3}(-1)^2\mathbf{1} - 3e_\theta(-1)^2e_{\alpha_4}(-1)h_1(-1)^2e_{-\alpha_2}(-1)\mathbf{1}\\
&- 9e_\theta(-1)^2e_{\alpha_4}(-1)h_1(-1)h_2(-1)e_{-\alpha_2}(-1)\mathbf{1} - 63e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_1}(-1)^2e_{-\alpha_5}(-1)\\
&+ 30e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_1}(-1)e_{-\alpha_3}(-1)e_{-\alpha_1}(-1)\mathbf{1} - 54e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_1}(-1)h_2(-1)e_{-\alpha_4}(-1)\mathbf{1}\\
&+ 27e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_2}(-1)e_{-\alpha_3}(-1)e_{-\alpha_2}(-1)\mathbf{1} - 126e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{\alpha_1}(-1)e_{-\theta}(-1)\mathbf{1}\\
&+ 54e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{-\alpha_4}(-1)e_{-\alpha_2}(-1)\mathbf{1} - 15e_\theta(-1)^2e_{\alpha_5}(-1)e_{\alpha_3}(-1)e_{-\alpha_3}(-1)^2\mathbf{1}\\
&+ 9e_\theta(-1)^2e_\theta(-1)h_1(-1)e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1} + 9e_\theta(-1)^2e_\theta(-1)h_1(-1)h_2(-1)e_{-\alpha_3}(-1)\mathbf{1}\\
&+ 54e_\theta(-1)^2e_\theta(-1)h_2(-1)e_{-\alpha_2}(-1)e_{-\alpha_1}(-1)\mathbf{1} - 63e_\theta(-1)^3e_{\alpha_1}(-1)^2e_{-\theta}(-1)\mathbf{1} + 54e_\theta(-1)^3e_{\alpha_1}(-1)e_{-\alpha_4}(-1)e_{-\alpha_2}(-1)\mathbf{1}\\
&- 15e_\theta(-1)^3e_{\alpha_1}(-1)e_{-\alpha_3}(-1)^2\mathbf{1} - 27e_\theta(-1)^3e_{-\alpha_2}(-1)^2e_{-\alpha_1}(-1)\mathbf{1} - 9e_\theta(-1)^3h_1(-1)e_{-\alpha_3}(-1)e_{-\alpha_2}(-1)\mathbf{1}
\end{aligned}
$$

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\begin{aligned}
v'_{\text{sing}} = {}& 72e_{\alpha_4}e_{\alpha_5}e_\theta - 96e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta - 72e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta + 36e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta + 6e_{-\alpha_3}e_{\alpha_5}e_\theta e_\theta + 72e_{-\alpha_2}e_{\alpha_4}e_\theta e_\theta - 36e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}\\
&+ 24e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - 4e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} + 100h_1e_{\alpha_4}e_{\alpha_5}e_\theta + 10e_{-\alpha_1}e_{\alpha_1}e_{\alpha_4}e_{\alpha_5}e_\theta + 18e_{\alpha_1}e_{-\alpha_2}e_{\alpha_5}e_\theta e_\theta - 45e_{-\alpha_1}e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_{\alpha_5}\\
&+ 5e_{-\alpha_1}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - 60e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_5}e_\theta + 6e_{\alpha_1}e_{\alpha_1}e_{\alpha_3}e_{\alpha_4}e_\theta - 42e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 20e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}\\
&+ 108h_2e_{\alpha_4}e_{\alpha_5}e_\theta - 45e_{-\alpha_1}h_2e_{\alpha_5}e_{\alpha_5}e_\theta + 4e_{\alpha_1}e_{\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5} - 2e_{\alpha_1}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} + 126e_{-\theta}e_{\alpha_4}e_{\alpha_5}e_\theta e_\theta\\
&+ 126e_{-\alpha_5}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5}e_\theta + 30e_{-\alpha_4}e_{\alpha_1}e_{\alpha_5}e_\theta e_\theta + 84e_{-\alpha_4}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}e_\theta - 4e_{-\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_\theta - 4e_{-\alpha_3}e_{\alpha_1}e_{\alpha_4}e_\theta e_\theta + 99e_{-\alpha_3}e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_\theta\\
&+ 19e_{-\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_\theta - 6e_{-\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_\theta - 42e_{-\alpha_2}e_{\alpha_1}e_{\alpha_3}e_\theta e_\theta - 72e_{-\alpha_2}.e_{-\alpha_3}e_\theta e_\theta e_\theta + 90e_{-\alpha_2}e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_\theta - 42e_{-\alpha_2}e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_\theta\\
&+ 6e_{-\alpha_2}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_\theta + 72e_{-\alpha_3}h_2e_{\alpha_5}e_\theta e_\theta + 45e_{-\alpha_2}h_1e_{\alpha_4}e_\theta e_\theta + 90e_{-\alpha_2}h_2e_{\alpha_4}e_\theta e_\theta - 80h_1e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta - 15e_{-\alpha_1}e_{-\alpha_1}e_{\alpha_1}e_{\alpha_5}e_{\alpha_5}e_\theta\\
&- 15e_{-\alpha_1}e_{-\alpha_1}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}e_{\alpha_5} + 5e_{-\alpha_1}e_{-\alpha_1}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - e_{-\alpha_1}e_{\alpha_1}e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta - 2e_{\alpha_1}e_{\alpha_1}e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta - 100h_1e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta\\
&+ 40h_1e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta + 15h_1e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - 50h_1e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 30h_1e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - 5h_1e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} + 32h_1h_1e_{\alpha_4}e_{\alpha_5}e_\theta\\
&- 159h_2e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta - 150h_2e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta + 67h_2e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta - 54h_2e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 36h_2e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - 6h_2e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}\\
&+ 81h_2h_1e_{\alpha_4}e_{\alpha_5}e_\theta + 36h_2h_2e_{\alpha_4}e_{\alpha_5}e_\theta + 3e_{-\alpha_1}.e_{\alpha_1}e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta + 3e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - e_{-\alpha_1}e_{\alpha_1}e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}\\
&+ 3e_{-\alpha_1}e_{\alpha_1}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - e_{-\alpha_1}e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} + 30e_{-\alpha_1}e_{-\alpha_3}e_{\alpha_1}e_{\alpha_5}e_\theta e_\theta + 30e_{-\alpha_1}e_{-\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}e_\theta - 10e_{\alpha_1}e_{-\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_\theta\\
&+ 18e_{-\alpha_1}e_{-\alpha_2}e_{\alpha_1}e_{\alpha_4}e_\theta e_\theta - 27e_{-\alpha_1}e_{-\alpha_2}e_{-\alpha_2}e_\theta e_\theta e_\theta + 27e_{-\alpha_1}e_{-\alpha_2}e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_\theta + 21e_{-\alpha_1}e_{-\alpha_2}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_\theta - 8e_{-\alpha_1}e_{-\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_\theta\\
&+ 9e_{-\alpha_1}e_{-\alpha_2}h_1e_{\alpha_5}e_\theta e_\theta + 54e_{\alpha_1}e_{-\alpha_2}h_2e_{\alpha_5}e_\theta e_\theta + 13e_{\alpha_1}h_1e_{\alpha_1}e_{\alpha_4}e_{\alpha_5}e_\theta - 9e_{-\alpha_1}h_1e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_{\alpha_5} + 13e_{-\alpha_1}h_1e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5}\\
&- 4e_{-\alpha_1}h_1e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} + 12e_{-\alpha_1}h_2e_{\alpha_1}e_{\alpha_4}e_{\alpha_5}e_\theta - 27e_{-\alpha_1}h_2.e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_{\alpha_5} + 9e_{\alpha_1}h_2e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - 2e_{-\alpha_1}h_2e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}\\
&- 9e_{-\alpha_1}h_2h_1e_{\alpha_5}e_{\alpha_5}e_\theta - 27e_{-\alpha_1}h_2h_2e_{\alpha_5}e_{\alpha_5}e_\theta - 2e_{\alpha_1}e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_\theta - 9e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_2}e_{\alpha_5}e_{\alpha_5} - 2e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}\\
&+ e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} - 63e_{-\theta}e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta e_\theta - 126e_{-\theta}e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta e_\theta + 42e_{-\theta}e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta e_\theta - 63e_{-\theta}e_{\alpha_3}.\alpha_3e_{\alpha_5}e_{\alpha_5}e_\theta\\
&+ 42e_{-\theta}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_\theta - 7e_{-\theta}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_\theta - 63e_{-\alpha_5}e_{\alpha_1}e_{\alpha_1}e_{\alpha_5}e_\theta e_\theta - 126e_{-\alpha_5}e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}e_\theta + 42e_{-\alpha_5}e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_\theta\\
&- 63e_{-\alpha_5}e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}e_{\alpha_5} + 42e_{-\alpha_5}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - 7e_{-\alpha_5}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - 3e_{-\alpha_4}e_{\alpha_1}e_{\alpha_1}e_{\alpha_4}e_\theta e_\theta - 54e_{-\alpha_4}e_{\alpha_1}e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_\theta\\
&- 6e_{-\alpha_4}e_{\alpha_1}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_\theta + 4e_{-\alpha_4}e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_\theta - 54e_{-\alpha_4}e_{\alpha_2}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}e_{\alpha_5} + 18e_{-\alpha_4}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - 3e_{-\alpha_4}e_{\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5}\\
&+ 4e_{-\alpha_4}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - e_{-\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} - 54e_{-\alpha_4}h_2.e_{\alpha_1}e_{\alpha_5}e_\theta e_\theta - 54e_{-\alpha_4}h_2e_{\alpha_3}e_{\alpha_5}e_{\alpha_5}e_\theta + 18e_{-\alpha_4}h_2e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}e_\theta\\
&- e_{-\alpha_3}e_{\alpha_1}e_{\alpha_1}e_{\alpha_3}e_\theta e_\theta - 21e_{-\alpha_3}e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_\theta - 2e_{-\alpha_3}e_{\alpha_1}e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_\theta + 3e_{-\alpha_3}e_{\alpha_1}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_\theta - 15e_{-\alpha_3}e_{-\alpha_3}e_{\alpha_1}e_\theta e_\theta e_\theta\\
&- 15e_{-\alpha_3}e_{-\alpha_3}e_{\alpha_3}e_{\alpha_5}e_\theta e_\theta + 5e_{-\alpha_3}e_{-\alpha_3}e_{\alpha_4}e_{\alpha_4}e_\theta e_\theta - 27e_{-\alpha_3}e_{\alpha_2}e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_{\alpha_5} - 18e_{-\alpha_3}e_{\alpha_2}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} + 8e_{-\alpha_3}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5}\\
&- e_{-\alpha_3}e_{\alpha_3}e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 3e_{-\alpha_3}e_{\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - e_{-\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} - 13e_{-\alpha_3}h_1e_{\alpha_1}e_{\alpha_4}e_\theta e_\theta + 9e_{-\alpha_3}h_1e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_\theta\\
&- 13e_{-\alpha_3}h_1.e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_\theta + 4e_{-\alpha_3}h_1e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_\theta - 30e_{-\alpha_3}h_2e_{\alpha_1}e_{\alpha_4}e_\theta e_\theta - 27e_{-\alpha_3}h_2e_{\alpha_2}e_{\alpha_5}e_{\alpha_5}e_\theta - 27e_{-\alpha_3}h_2e_{\alpha_3}e_{\alpha_4}e_{\alpha_5}e_\theta\\
&+ 10e_{-\alpha_3}h_2e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_\theta + 9e_{-\alpha_3}h_2h_1e_{\alpha_5}e_\theta e_\theta - 9e_{-\alpha_2}e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_\theta e_\theta - 36e_{-\alpha_2}e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}e_{\alpha_5}e_\theta + 6e_{-\alpha_2}e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_\theta\\
&+ 2e_{-\alpha_2}e_{\alpha_1}e_{\alpha_3}e_{\alpha_3}e_{\alpha_4}e_\theta + 54e_{-\alpha_2}e_{-\alpha_4}e_{\alpha_1}e_\theta e_\theta e_\theta + 54e_{-\alpha_2}e_{-\alpha_4}e_{\alpha_3}e_{\alpha_5}e_\theta e_\theta - 18e_{-\alpha_2}e_{-\alpha_4}e_{\alpha_4}e_{\alpha_4}e_\theta e_\theta + 27e_{-\alpha_2}e_{-\alpha_3}e_{\alpha_2}e_{\alpha_5}e_\theta e_\theta\\
&- 3e_{-\alpha_2}e_{-\alpha_3}e_{\alpha_3}e_{\alpha_4}e_\theta e_\theta - 9e_{-\alpha_2}e_{-\alpha_3}h_1e_\theta e_\theta e_\theta + 27e_{-\alpha_2}e_{-\alpha_2}e_{\alpha_2}e_{\alpha_4}e_\theta e_\theta - 9e_{-\alpha_2}e_{-\alpha_2}e_{\alpha_3}e_{\alpha_3}e_\theta e_\theta - 27e_{-\alpha_2}e_{\alpha_2}e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5}\\
&- 9e_{-\alpha_2}e_{\alpha_2}e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 6e_{-\alpha_2}e_{\alpha_2}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} + e_{-\alpha_2}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} + 2e_{-\alpha_2}e_{\alpha_3}e_{\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5} - e_{-\alpha_2}e_{\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}\\
&- 30e_{-\alpha_2}h_1e_{\alpha_1}e_{\alpha_3}e_\theta e_\theta - 30e_{-\alpha_2}h_1e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_\theta + 10e_{-\alpha_2}h_1e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_\theta - 3e_{-\alpha_2}h_1h_1e_{\alpha_4}e_\theta e_\theta - 54e_{-\alpha_2}h_2e_{\alpha_1}e_{\alpha_3}e_\theta e_\theta\\
&- 27e_{-\alpha_2}h_2e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_\theta - 36e_{-\alpha_2}h_2e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_\theta + 15e_{-\alpha_2}h_2e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_\theta - 9e_{-\alpha_2}h_2h_1e_{\alpha_4}e_\theta e_\theta - 30h_1e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_5}e_\theta\\
&- 30h_1e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 10h_1e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - 16h_1h_1e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta - 32h_1h_1e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta + 8h_1h_1e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta
\end{aligned}
$$

$$
\begin{aligned}
&+ 3h_1h_1e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - 16h_1h_1e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 8h_1h_1e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - h_1h_1e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} - 54h_2e_{\alpha_1}e_{\alpha_1}e_{\alpha_2}e_{\alpha_5}e_\theta \\
&+ 2h_2e_{\alpha_1}e_{\alpha_1}e_{\alpha_3}e_{\alpha_4}e_\theta - 36h_2e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 15h_2e_{\alpha_1}e_{\alpha_2}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} + 2h_2e_{\alpha_1}e_{\alpha_3}e_{\alpha_3}e_{\alpha_4}e_{\alpha_5} - h_2e_{\alpha_1}e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} \\
&- 63h_2h_1e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta - 96h_2h_1e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta + 29h_2h_1e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta + 9h_2h_1e_{\alpha_2}e_{\alpha_4}e_{\alpha_5}e_{\alpha_5} - 33h_2h_1e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} \\
&+ 19h_2h_1e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - 3h_2h_1e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} + 3h_2h_1h_1e_{\alpha_4}e_{\alpha_5}e_\theta - 63h_2h_2e_{\alpha_1}e_{\alpha_1}e_\theta e_\theta - 72h_2h_2e_{\alpha_1}e_{\alpha_3}e_{\alpha_5}e_\theta \\
&+ 27h_2h_2e_{\alpha_1}e_{\alpha_4}e_{\alpha_4}e_\theta - 18h_2h_2e_{\alpha_3}e_{\alpha_3}e_{\alpha_5}e_{\alpha_5} + 12h_2h_2e_{\alpha_3}e_{\alpha_4}e_{\alpha_4}e_{\alpha_5} - 2h_2h_2e_{\alpha_4}e_{\alpha_4}e_{\alpha_4}e_{\alpha_4} + 9h_2h_2h_1e_{\alpha_4}e_{\alpha_5}e_\theta
\end{aligned}
$$

Appendix B. Polynomials for subsingular vector of $V^{-2}(B_3)$.

We give in this appendix the explicit form of the polynomials in the symmetric algebra of the Cartan of B_3 appearing in Lemma 5.6.

$$
\begin{aligned}
p_4^{B_3} &= -24(16h_1^6 + 112h_2h_1^5 + 64h_3h_1^5 + 96h_1^5 + 320h_2^2h_1^4 + 104h_3^2h_1^4 + 496h_2h_1^4 + 368h_2h_3h_1^4 + 264h_3h_1^4 + 156h_1^4 + 480h_2^3h_1^3 \\
&+ 88h_3^3h_1^3 + 984h_2^2h_1^3 + 472h_2h_3^2h_1^3 + 260h_3^2h_1^3 + 444h_2h_1^3 + 832h_2^2h_3h_1^3 + 1032h_2h_3h_1^3 + 156h_3h_1^3 - 16h_1^3 + 400h_2^4h_1^2 \\
&+ 41h_3^4h_1^2 + 912h_2^3h_1^2 + 296h_2h_3^3h_1^2 + 106h_3^3h_1^2 + 246h_2^2h_1^2 + 792h_2^2h_3^2h_1^2 + 696h_2h_3^2h_1^2 \\
&- 77h_3^2h_1^2 - 464h_2h_1^2 + 928h_2^3h_3h_1^2 + 1412h_2^2h_3h_1^2 + 12h_2h_3h_1^2 - 358h_3h_1^2 - 180h_1^2 + 176h_2^5h_1 + 10h_3^5h_1 \\
&+ 376h_2^4h_1 + 91h_2h_3^4h_1 + 13h_3^4h_1 - 150h_2^3h_1 + 328h_2^2h_3^3h_1 + 154h_2h_3^3h_1 - 96h_3^3h_1 - 706h_2^2h_1 + 584h_2^3h_3^2h_1 \\
&+ 542h_2^2h_3^2h_1 - 415h_2h_3^2h_1 - 269h_3^2h_1 - 476h_2h_1 + 512h_2^4h_3h_1 + 760h_2^3h_3h_1 - 488h_2^2h_3h_1 - 886h_2h_3h_1 \\
&- 306h_3h_1 - 72h_1 + 32h_2^6 + h_3^6 + 48h_2^5 + 11h_2h_3^5 - h_3^5 - 76h_2^4 + 50h_2^2h_3^4 + h_2h_3^4 - 13h_3^4 - 72h_2^3 + 120h_2^3h_3^3 \\
&+ 33h_2^2h_3^3 - 111h_2h_3^3 + 13h_3^3 + 44h_2^2 + 160h_2^4h_3^2 + 100h_2^3h_3^2 - 265h_2^2h_3^2 - 27h_2h_3^2 + 36h_3^2 + 24h_2 + 112h_2^5h_3 \\
&+ 116h_2^4h_3 - 244h_2^3h_3 - 110h_2^2h_3 + 54h_2h_3 - 36h_3) \\
p_5^{B_3} &= -2(32h_1^6 + 204h_2h_1^5 + 128h_3h_1^5 + 192h_1^5 + 544h_2^2h_1^4 + 200h_3^2h_1^4 + 986h_2h_1^4 + 676h_2h_3h_1^4 + 536h_3h_1^4 + 312h_1^4 \\
&+ 776h_2^3h_1^3 + 152h_3^3h_1^3 + 1934h_2^2h_1^3 + 839h_2h_3^2h_1^3 + 500h_3^2h_1^3 + 888h_2h_1^3 + 1432h_2^2h_3h_1^3 + 2077h_2h_3h_1^3 + 356h_3h_1^3 \\
&- 32h_1^3 + 624h_2^4h_1^2 + 56h_3^4h_1^2 + 1774h_2^3h_1^2 + 474h_2h_3^3h_1^2 + 160h_3^3h_1^2 + 592h_2^2h_1^2 + 1322h_2^2h_3^2h_1^2 + 1301h_2h_3^2h_1^2 \\
&- 156h_3^2h_1^2 - 814h_2h_1^2 + 1520h_2^3h_3h_1^2 + 2808h_2^2h_3h_1^2 + 209h_2h_3h_1^2 - 636h_3h_1^2 - 360h_1^2 + 268h_2^5h_1 + 8h_3^5h_1 \\
&+ 730h_2^4h_1 + 115h_2h_3^4h_1 + 4h_3^4h_1 - 120h_2^3h_1 + 493h_2^2h_3^3h_1 + 196h_2h_3^3h_1 - 200h_3^3h_1 - 1262h_2^2h_1 + 927h_2^3h_3^2h_1 \\
&+ 974h_2^2h_3^2h_1 - 697h_2h_3^2h_1 - 580h_3^2h_1 - 824h_2h_1 + 808h_2^4h_3h_1 + 1497h_2^3h_3h_1 - 637h_2^2h_3h_1 - 1718h_2h_3h_1 \\
&- 528h_3h_1 - 144h_1 + 48h_2^6 + 96h_2^5 + 8h_2h_3^5 - 72h_2^4 + 59h_2^2h_3^4 - 14h_2h_3^4 - 144h_2^3 + 171h_2^3h_3^3 + 18h_2^2h_3^3 \\
&- 120h_2h_3^3 + 24h_2^2 + 244h_2^4h_3^2 + 167h_2^3h_3^2 - 319h_2^2h_3^2 - 118h_2h_3^2 + 48h_2 + 172h_2^5h_3 + 230h_2^4h_3 - 272h_2^3h_3 \\
&- 254h_2^2h_3 + 28h_2h_3) \\
p_6^{B_3} &= -4(16h_1^6 + 97h_2h_1^5 + 64h_3h_1^5 + 96h_1^5 + 246h_2^2h_1^4 + 96h_3^2h_1^4 + 462h_2h_1^4 + 323h_2h_3h_1^4 + 272h_3h_1^4 + 156h_1^4 \\
&+ 334h_2^3h_1^3 + 64h_3^3h_1^3 + 860h_2^2h_1^3 + 387h_2h_3^2h_1^3 + 240h_3^2h_1^3 + 477h_2h_1^3 + 654h_2^2h_3h_1^3 + 989h_2h_3h_1^3 + 200h_3h_1^3 \\
&- 16h_1^3 + 256h_2^4h_1^2 + 16h_3^4h_1^2 + 760h_2^3h_1^2 + 193h_2h_3^3h_1^2 + 48h_3^3h_1^2 + 402h_2^2h_1^2 + 586h_2^2h_3^2h_1^2 + 576h_2h_3^2h_1^2 \\
&- 68h_3^2h_1^2 - 312h_2h_1^2 + 664h_2^3h_3h_1^2 + 1268h_2^2h_3h_1^2 + 192h_2h_3h_1^2 - 284h_3h_1^2 - 180h_1^2 + 105h_2^5h_1 + 308h_2^4h_1 \\
&+ 32h_2h_3^4h_1 - 16h_3^4h_1 + 43h_2^3h_1 + 194h_2^2h_3^3h_1 + 33h_2h_3^3h_1 - 112h_3^3h_1 - 488h_2^2h_1 + 395h_2^3h_3^2h_1 + 393h_2^2h_3^2h_1 \\
&- 335h_2h_3^2h_1 - 268h_3^2h_1 - 400h_2h_1 + 338h_2^4h_3h_1 + 649h_2^3h_3h_1 - 203h_2^2h_3h_1 - 734h_2h_3h_1 - 252h_3h_1 - 72h_1 \\
&+ 18h_2^6 + 42h_2^5 - 6h_2^4 + 16h_2^2h_3^4 - 16h_2h_3^4 - 42h_2^3 + 65h_2^3h_3^3 - 15h_2^2h_3^3 - 50h_2h_3^3 - 12h_2^2 + 100h_2^4h_3^2 + 57h_2^3h_3^2 \\
&- 119h_2^2h_3^2 - 38h_2h_3^2 + 69h_2^5h_3 + 98h_2^4h_3 - 77h_2^3h_3 - 86h_2^2h_3 - 4h_2h_3) \\
p_7^{B_3} &= -2h_1\left(h_1 + 2h_2 + h_3 + 2\right)\left(32h_1^4 + 180h_2h_1^3 + 96h_3h_1^3 + 128h_1^3 + 372h_2^2h_1^2 + 104h_3^2h_1^2 + 422h_2h_1^2\right. \\
&+ 392h_2h_3h_1^2 + 216h_3h_1^2 + 56h_1^2 + 332h_2^3h_1 + 48h_3^3h_1 + 332h_2^2h_1 + 273h_2h_3^2h_1 + 76h_3^2h_1 - 176h_2h_1 \\
&+ 520h_2^2h_3h_1 + 341h_2h_3h_1 - 132h_3h_1 - 144h_1 + 108h_2^4 + 8h_3^4 - 26h_2^3 + 61h_2h_3^3 - 12h_3^3 - 512h_2^2 \\
&\left.+ 175h_2^2h_3^2 - 34h_2h_3^2 - 176h_3^2 - 402h_2 + 224h_2^3h_3 - 37h_2^2h_3 - 613h_2h_3 - 228h_3 - 72\right) \\
p_8^{B_3} &= -2h_1\left(h_1 + 2h_2 + h_3 + 2\right)\left(16h_1^4 + 80h_2h_1^3 + 48h_3h_1^3 + 64h_1^3 + 148h_2^2h_1^2 + 48h_3^2h_1^2 + 217h_2h_1^2\right. \\
&+ 176h_2h_3h_1^2 + 112h_3h_1^2 + 28h_1^2 + 120h_2^3h_1 + 16h_3^3h_1 + 188h_2^2h_1 + 112h_2h_3^2h_1 + 32h_3^2h_1 - 66h_2h_1 \\
&+ 212h_2^2h_3h_1 + 171h_2h_3h_1 - 52h_3h_1 - 72h_1 + 36h_2^4 + 3h_2^3 + 16h_2h_3^3 - 16h_3^3 - 216h_2^2 + 64h_2^2h_3^2 \\
&\left.- 46h_2h_3^2 - 80h_3^2 - 195h_2 + 84h_2^3h_3 - 31h_2^2h_3 - 261h_2h_3 - 108h_3 - 36\right) \\
p_9^{B_3} &= -4\left(h_1 - 1\right)h_1\left(h_1 + 2h_2 + h_3 + 2\right)\left(h_1 + 2h_2 + h_3 + 3\right)\left(16h_1^2 + 63h_2h_1 + 32h_3h_1 + 32h_1 + 63h_2^2\right.
\end{aligned}
$$

$+ 16h_3^2 + 63h_2 + 63h_2h_3 + 32h_3 + 12)$

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