Cyclic Characters of Alternating Groups

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Abstract. We determine the decomposition of cyclic characters of alternating groups into irreducible characters. As an application, we characterize pairs (w, V), where $w \in A_n$ and V is an irreducible representation of A_n such that w admits a non-zero invariant vector in V. We also establish new global conjugacy classes for alternating groups, thereby giving a new proof of a result of Heide and Zalessky on the existence of such classes.

1. Introduction

1.1. Decomposition of Cyclic Characters. Let G be a finite group. Let V be an irreducible representation of G with character χ. Suppose $g \in G$ has order m, then the eigenvalues of g in V are mth roots of unity. Let ζ_m be a primitive mth root of unity. Let $a^{\chi}_{g,i}$ denote the multiplicity of ζ_m^i as an eigenvalue of g in V. We have

$$a^{\chi}_{g,i} = \frac{1}{m}\sum_{i=0}^{m-1} \chi(g^i)\zeta_m^{-i}. \tag{1}$$

By Frobenius reciprocity, $a^{\chi}_{g,i}$ is the multiplicity of V in the representation $\mathrm{Ind}_C^G \zeta_m^i$, where C is the cyclic subgroup of G generated by g, and (by abuse of notation) ζ_m^i denotes the character of C that takes g to ζ_m^i. A representation of the form $\mathrm{Ind}_C^G \zeta_m^i$ is called a *cyclic representation* of G.

For any classical Coxeter group, Kraśkiewicz and Weyman [**5**] identified cyclic representations induced from the cyclic subgroup generated by a Coxeter element g as a sum of certain graded pieces in its co-invariant algebra. This allowed them to give a combinatorial interpretation of $a^{\chi}_{g,i}$. Stembridge [**10**] gave a combinatorial

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interpretation of all $a^\chi_{g,i}$ for all g in symmetric groups and wreath product groups. Jöllenbeck and Schocker [**4**] built on [**5**] to give a new approach to Stembridge's result for symmetric groups using Lie idempotents in the symmetric group algebra.

In this article, we determine the multiplicities $a^\chi_{g,i}$ for alternating groups. Most irreducible representations of alternating groups are restrictions of irreducible representations of symmetric groups. For these representations, the multiplicities $a^\chi_{g,i}$ are determined by Stembridge's result. The remaining irreducible representations of alternating groups are of the form $V^\pm$, where $V^+ \oplus V^-$ is the restriction of an irreducible representation V of S_n. Let χ (resp., $\chi^\pm$) denote the character of V (resp., $V^\pm$). Given $g \in A_n$ of order m, we know the multiplicity $a^\chi_{g,i}$ of ζ_m^i as an eigenvalue of g in V from Stembridge's result. It remains to determine how this eigenspace splits into eigenspaces of g in V^+ and V^-. We find that, in almost all cases, it splits equally: $a^{\chi^+}_{g,i} = a^{\chi^-}_{g,i}$. In the few remaining cases, we determine this splitting by giving a closed formula for the difference $a^{\chi^+}_{g,i} - a^{\chi^-}_{g,i}$.

1.2. Outline. In Section 2.1, we recall the character theory of the alternating group. An important step in computing the difference $a^{\chi^+}_{g,i} - a^{\chi^-}_{g,i}$ is to understand how cyclic subgroups of A_n split into conjugacy classes. The interesting case is where the cyclic subgroup is generated by a permutation whose cycle type has distinct odd parts. Theorem 2.3 (which is of interesting in its own right) describes which conjugacy classes the generators of such a cyclic subgroup fall into in terms of the Jacobi symbol. In Section 2.2, we recall Stembridge's combinatorial interpretation of $a^\chi_{g,i}$ for cyclic representations of symmetric groups.

In Section 3 we compute the difference $a^{\chi^+}_{g,i} - a^{\chi^-}_{g,i}$. In most cases, this difference is zero. In the potentially non-zero cases, the answer is given by Theorem 3.1. Using Theorem 3.1, we can exactly characterize the cases where the difference is non-zero. Theorem 3.1 also gives a simple formula for the absolute value of the difference, from which a simple upper bound is obtained in Corollary 3.4. For representations induced from trivial and primitive characters, the absolute value of the difference takes a particularly simple form (Corollary 3.5).

Section 4 gives a characterization (Theorem 4.3) of which irreducible representations of A_n occur in a representation induced from the subgroup generated by a longest cycle in A_n. This result was obtained by different methods by Yang and Staroletov [**14**]. An analogous result for S_n was obtained by Swanson [**12**]. We use Swanson's result and the upper bound in Corollary 3.4.

In Section 5, we use our results to characterize pairs (w, V), where $w \in A_n$ and V is an irreducible representation of A_n such that w admits a non-zero invariant vector in V. This result is based on a similar characterization for symmetric groups in [**7**] and Theorem 4.3. As a consequence, we show that the trivial representation of A_n is immersed in almost every irreducible representation of A_n, in the sense of Prasad and Raghunathan [**9**].

Heide and Zalessky[**3**] called a conjugacy class in a finite group G *global* if every irreducible representation of G occurs in the corresponding permutation representation. They proved the existence of global conjugacy classes of A_n for all $n > 4$. Sundaram [**11**] characterized the global conjugacy classes of S_n. In Section 6, we use our results along with those of Sundaram to establish new global conjugacy classes for alternating groups, thereby giving a new proof of the result of Heide and Zalessky.

1.3. Open Questions. We conclude this introduction by mentioning two interesting open questions. In Theorems 5.2 and 6.1 we have seen several examples of subgroup H of A_n such that the representation induced from the trivial representation of H contains every irreducible representation of A_n.

QUESTION 1.1. Find subgroups H of A_n that are maximal with respect to the property that the representation induced from the trivial representation of H contains every irreducible representation of A_n.

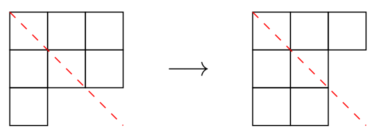
In Section 6, we have constructed a plethora of global conjugacy classes for A_n. In the spirit of Sundaram's result for S_n (Theorem 6.2), we ask the following.

QUESTION 1.2. Characterize the global conjugacy classes of A_n.

2. Preliminaries

2.1. Characters of the Alternating Group. In this section, we outline how Frobenius [**1**] expressed irreducible characters of alternating groups in terms of irreducible characters of symmetric groups. For a detailed exposition, see [**8**].

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of n. The Young diagram of λ is a left-justified array of boxes with λ_i boxes in the i-th row. The Young diagram of the conjugate partition λ' is obtained by reflecting the Young diagram of λ along the main diagonal. Thus $(3, 3, 1)' = (3, 2, 2)$, as shown below.



For every partition λ of n, let χ_λ denote the irreducible character of S_n corresponding to λ. We have

THEOREM 2.1 (Frobenius [**1**]). *Let λ be a partition of n.*

(1) *If $\lambda \neq \lambda'$ then the restriction of χ_λ to A_n is irreducible and is equal to the restriction of $\chi_{\lambda'}$ to A_n.*
(2) *If $\lambda = \lambda'$, then the restriction of χ_λ to A_n is the sum of two irreducible characters of A_n;*
$$\chi_\lambda = \chi_\lambda^+ + \chi_\lambda^-,$$
where $\chi_\lambda^-(g) = \chi_\lambda^+(wgw^{-1})$ for any $w \in S_n \setminus A_n$.

Moreover, every irreducible character of A_n arises in the above manner.

To understand the values of the characters $\chi_\lambda^\pm$, we introduce Frobenius coordinates of a partition. The Frobenius coordinates of a partition λ are $(a_1, \ldots, a_d \mid b_1, \ldots, b_d)$, where $d = \#\{i \mid \lambda_i \geq i\}$, and for each $i = 1, \ldots, d$, $a_i = \lambda_i - i$ and $b_i = \lambda'_i - i$. Thus the Frobenius coordinates of $(3, 3, 1)$ are $(2, 1|2, 0)$, while those of $(3, 2, 2)$ are $(2, 0|2, 1)$. The partition λ is self-conjugate if and only if $a_i = b_i$ for all $i = 1, \ldots, d$.

For every partition μ of n, the set $[\mu]$ of permutations with cycle type μ forms a conjugacy class in S_n. Note that $[\mu] \subset A_n$ if and only if μ has an even number of even parts. The conjugacy class $[\mu]$ of S_n remains a single conjugacy class in A_n unless its parts are all distinct and odd, in which case it is a union of two classes:
$$[\mu] = [\mu]^+ \sqcup [\mu]^-, \textit{where}$$
$$[\mu]^- = w[\mu]^+ w^{-1}, \text{ for any } w \in S_n \setminus A_n.$$

Given a partition $\mu = (2a_1+1, \dots, 2a_d+1)$ with $a_1 > \cdots > a_d$, define

$$\phi(\mu) = (a_1, \dots, a_d \mid a_1, \dots, a_d) \text{ in Frobenius coordinates.}$$

Then ϕ gives rise to a bijection from the set of partitions with distinct odd parts onto the set of self-conjugate partitions.

For each partition μ of n, choose a permutation $w_\mu \in [\mu]$. When $\mu = (2a_1 + 1, \dots, 2a_d+1)$, let $\epsilon_\mu = (-1)^{a_1+\cdots+a_d}$. Let z_μ denote the cardinality of the centralizer of w_μ in S_n. Frobenius showed the following.

Theorem 2.2. *Let λ be a self-conjugate partition of n. The values of the characters $\chi_\lambda^\pm$ are given by*

$$\chi_\lambda^\pm(w_\mu) = \begin{cases} \frac{1}{2}(\epsilon_\mu \pm \sqrt{\epsilon_\mu z_\mu}) & \text{if } \lambda = \phi(\mu) \\ \frac{1}{2}\chi_\lambda(w_\mu) & \text{otherwise.} \end{cases} \tag{2}$$

$$\chi_\lambda^\pm(w w_\mu w^{-1}) = \chi_\lambda^\mp(w_\mu), \text{ for any } w \in S_n \setminus A_n. \tag{3}$$

Recall that for an integer i and a prime p, the Legendre symbol is defined by

$$\left(\frac{i}{p}\right) = \begin{cases} 1 & \text{if } i \text{ is a non-zero quadratic residue modulo } p, \\ -1 & \text{if } i \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } p \mid i. \end{cases}$$

For an odd integer $n = p_1^{e_1} \cdots p_k^{e_k}$, the Jacobi symbol is defined by

$$\left(\frac{i}{n}\right) = \left(\frac{i}{p_1}\right)^{e_1} \cdots \left(\frac{i}{p_k}\right)^{e_k}.$$

We have the following.

Theorem 2.3. *Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of n with distinct odd parts and i be an integer. Let $M = \prod_{j=1}^k \mu_j$. Then $w_\mu^i \in [\mu]^+$ (resp., $w_\mu^i \in [\mu]^-$) if and only if $(\frac{i}{M}) = 1$ (resp., $(\frac{i}{M}) = -1$).*

Proof. Let Z_k denote the ring of integers modulo k. Consider the disjoint union $Z = Z_{\mu_1} \coprod \cdots \coprod Z_{\mu_k}$, which has cardinality n. The map $w : Z \to Z$ given by $w(a) = a+1$ for $a \in Z_{\mu_j}$ for any $1 \leq j \leq k$, is a permutation of Z with cycle type μ. The map $w^i : Z \to Z$ is given by $w^i(a) = a + i$ for any $a \in Z_{\mu_j}$. Define $\sigma_i : Z \to Z$ by $\sigma_i(a) = ia$ for any $a \in Z_{\mu_j}$. Then $\sigma_i \circ w(a) = i(a+1) = w^i \circ \sigma_i(a)$ for any $a \in Z_{\mu_j}$. Thus $\sigma_i w \sigma_i^{-1} = w^i$. It follows that w^i is conjugate to w in A_n if and only if σ_i is an even permutation of Z. Since the sign of a permutation on a disjoint union of invariant sets is the product of the signs of the permutations on the invariant sets, it suffices to show that, for any positive integer m, the sign of the permutation $\sigma_i : a \mapsto ia$ on Z_m is $(\frac{i}{m})$ for any i such that i is coprime to m.

For any odd prime p and any positive integer e, the units group $Z_{p^e}^*$ of Z_{p^e} is a cyclic group of order $p^{e-1}(p-1)$. Let i be a generator of $Z_{p^e}^*$. Then the cycles of $\sigma_i : a \mapsto ia$ on Z_{p^e} are the $Z_{p^e}^*$-orbits. These are the sets $p^j Z_{p^e} - p^{j+1} Z_{p^e}$ for $j = 0, \dots, e$. Thus the cycle type of σ_i on Z_{p^e} is $(p^{e-1}(p-1), p^{e-2}(p-1), \dots, p-1, 1)$. So σ_i has e even cycles. Since $(\frac{i}{p}) = -1$, σ_i has sign $(-1)^e = (\frac{i}{p^e})$. Since the map $i \mapsto \operatorname{sgn} \sigma_i$ and the Jacobi symbol $(\frac{i}{p^e})$ are both multiplicative in i, it follows that for any i coprime to p, the sign of σ_i on Z_{p^e} is $(\frac{i}{p^e})$.

Let m have prime factorization $p_1^{e_1}\cdots p_k^{e_r}$. Then Z_m is the direct product of the cyclic groups $Z_{p_1^{e_1}},\dots,Z_{p_k^{e_r}}$. For each $1\le j\le r$ let σ_i^j denote the map that multiplies the jth factor in the decomposition $Z_m = Z_{p_1^{e_1}}\times\cdots\times Z_{p_k^{e_r}}$ by i and fixes the other factors. Then $\sigma_i = \sigma_i^1\circ\cdots\circ\sigma_i^r$. Therefore, $\operatorname{sgn}(\sigma_i) = \prod_{j=1}^r \operatorname{sgn}(\sigma_i^j) = \prod_{j=1}^r(\frac{i}{p_j^{e_j}}) = (\frac{i}{m})$. □

2.2. Cyclic Characters of Symmetric Groups. For each partition μ, let C_μ denote the cyclic subgroup of S_n generated by the element w_μ with cycle type μ. The order $m = m_\mu$ of w_μ is the least common multiple of the parts of μ. Let ζ_m denote a primitive mth root of unity. By abuse of notation, we will use ζ_m^i to denote the character of C_μ that takes w_μ to ζ_m^i. The representations $\operatorname{Ind}_{C_\mu}^{S_n}\zeta_m^i$, for $i = 0,\dots,m-1$, are cyclic representations of S_n. Define

$$a_{\mu,i}^\lambda = \langle \chi_\lambda, \operatorname{Ind}_{C_\mu}^{S_n}\zeta_m^i\rangle.$$

Stembridge [**10**] gave a combinatorial interpretation of $a_{\mu,i}^\lambda$ in terms of a statistic on standard Young tableaux which was called the *multi major index* by Jöllenbeck and Schocker [**4**]. Recall that, for a partition λ of n, a Young tableau of shape λ is a filling of the boxes of the Young diagram of λ with the numbers $1,\dots,n$ such that the entries increase along each row and column.

Definition 2.4 (Descent). Let λ be a partition of n and let T be a standard Young tableau of shape λ. An integer i is called a descent of T if $i+1$ appears in a row of T strictly below the row in which i appears. The set of descents of T is denoted by $\operatorname{Des}(T)$.

Definition 2.5. Let $q = (q_1,\dots,q_k)$ be a sequence of non-negative integers summing to n. Let $s_i = \sum_{j=1}^i q_j$ be its sequence of partial sums. The multi major index of T with respect to q is defined as the k-tuple $\operatorname{maj}_q(T)$ whose jth term is

$$\operatorname{maj}_q(T)_j = \sum_{\{i\in\operatorname{Des}(T)\mid s_{j-1}<i<s_j\}} (i - s_{j-1}) \text{ for } 1\le j\le k.$$

Example 2.6. Let $T = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&7\\\hline 6&8\\\cline{1-2}\end{array}$. Then $\operatorname{Des}(T) = \{1,4,5,7\}$. We have $\operatorname{maj}_{(3,2,3)}(T) = (1,1,2)$.

The interpretation [**4**, p. 158] of $a_{\mu,i}^\lambda$ is

$$a_{\mu,i}^\lambda = \#\Big\{T\in\operatorname{SYT}(\lambda)\mid \sum_{j=1}^k (m/\mu_j)\operatorname{maj}_\mu(T)_j \equiv i \mod m\Big\}. \tag{4}$$

3. Computation of Multiplicities

3.1. The Easy Cases. Suppose λ is a partition of n such that $\lambda\ne\lambda'$. Let V_λ denote the irreducible representation of S_n with character χ_λ. Then V_λ is irreducible when restricted to A_n. It follows that, for every $w\in A_n$ with cycle type μ, the multiplicity of ζ_m^i as an eigenvalue of w_μ in V_λ is $a_{\mu,i}^\lambda$ from Section 2.2.

If $\lambda=\lambda'$, then the restriction of V_λ to A_n is the sum of two irreducible representations V_λ^+ and V_λ^-, with characters χ_λ^+ and χ_λ^- respectively. Let $a_{\mu,i}^{\lambda\pm}$ denote the multiplicity of ζ_m^i in $V_\lambda^\pm$. It follows from Theorem 2.2 that, unless μ has distinct odd parts and $\lambda = \phi(\mu)$, $\chi_\lambda^+(w_\mu^i) = \chi_\lambda^-(w_\mu^i)$ for all i. Hence, by (1) $a_{\mu,i}^{\lambda^+} = a_{\mu,i}^{\lambda^-} = \frac{1}{2}a_{\mu,i}^\lambda$.

3.2. The Interesting Case. Suppose that $\lambda = \lambda'$, and μ is a partition with distinct odd parts such that $\lambda = \phi(\mu)$. Let $\delta_\lambda = \chi_\lambda^+ - \chi_\lambda^-$. Let $d_{\mu,i}^\lambda = a_{\mu,i}^{\lambda^+} - a_{\mu,i}^{\lambda^-}$. Then $a_{\mu,i}^{\lambda^\pm} = \frac{1}{2}(a_{\mu,i}^\lambda \pm d_{\mu,i}^\lambda)$. The value of $d_{\mu,i}^\lambda$ is given by the following theorem.

Theorem 3.1. *Let $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ with distinct odd parts, and i be an integer. Let $M = \prod_{j=1}^k \mu_j$ and $m = \mathrm{lcm}(\mu_1, \ldots, \mu_k)$. Write $M = \prod_{j=1}^r p_j^{e_j}$, where $p_1, \ldots, p_r$ are distinct primes, $e_1, \ldots, e_s$ are odd, and $e_{s+1}, \ldots, e_r$ are even. Suppose that $m = \prod_{j=1}^r p_j^{f_j}$ and $i \equiv u_j p_j^{d_j} \mod p_j^{f_j}$ with u_j coprime to p_j and $0 \le d_j \le f_j$. Then $d_{\mu,i}^{\phi(\mu)} \neq 0$ if and only if $d_j = f_j - 1$ for $j = 1, \ldots, s$, and $d_j \in \{f_j - 1, f_j\}$ for $j = s+1, \ldots, r$. When this happens, we have*

$$d_{\mu,i}^{\phi(\mu)} = \frac{\sqrt{\epsilon_\mu M}}{m} \prod_{j=1}^s p_j^{f_j - 1} \left(\frac{u_j m / p_j^{f_j}}{p_j}\right) g(p_j) \prod_{j=s+1}^r (-p_j^{f_j-1}) \prod_{d_j = f_j} (1 - p_j),$$

where $\epsilon_\mu = (-1)^{\sum_{j=1}^k (\mu_j - 1)/2}$. In particular,

$$|d_{\mu,i}^{\phi(\mu)}| = \sqrt{\frac{M}{\prod_{j=1}^s p_j} \frac{\prod_{d_j=f_j}(p_j - 1)}{\prod_{j=s+1}^r p_j}}.$$

Example 3.2. Consider $\mu = (15, 9, 3)$. Then $\prod_{j=1}^3 \mu_j = 405 = 5^1 \times 3^4$ and $\mathrm{lcm}(15, 9, 3) = 45 = 5^1 \times 3^2$. Theorem 3.1 allows us to easily compute

$$d_{\mu,0}^{\phi(\mu)} = d_{\mu,1}^{\phi(\mu)} = d_{\mu,15}^{\phi(\mu)} = 0, |d_{\mu,3}^{\phi(\mu)}| = 3, |d_{\mu,9}^{\phi(\mu)}| = 6.$$

Proof. By Theorem 2.2, we have

$$\delta_\lambda(w_\mu^i) = \begin{cases} \sqrt{\epsilon_\mu z_\mu} & \text{if } w_\mu^i \in [\mu]^+, \\ -\sqrt{\epsilon_\mu z_\mu} & \text{if } w_\mu^i \in [\mu]^-, \\ 0 & \text{otherwise.} \end{cases}$$

Also, $w_\mu^i \in [\mu]^\pm$ if and only if $(m, i) = 1$. Therefore

$$\begin{aligned} d_{\mu,i}^\lambda &= \frac{1}{m} \sum_{l=0}^{m-1} \delta_\lambda(w_\mu^l) \zeta_m^{-il} \\ &= \frac{\sqrt{\epsilon_\mu z_\mu}}{m} \left(\sum_{w_\mu^l \in [\mu]^+} \zeta_m^{-il} - \sum_{w_\mu^l \in [\mu]^-} \zeta_m^{-il} \right) \\ &= \frac{\sqrt{\epsilon_\mu M}}{m} \sum_{l=0}^{m-1} \left(\frac{l}{M}\right) \zeta_m^{-il} \qquad \text{[by Theorem 2.3].} \end{aligned}$$

Let $n_0 = \prod_{j=1}^s p_j$. Then

$$\left(\frac{l}{M}\right) = \begin{cases} \left(\frac{l}{n_0}\right) & \text{if } (l, M) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h_j = m/p_j^{f_j}$. Since the greatest common divisor of $h_1, \ldots, h_r$ is 1, there exist integers $c_1, \ldots, c_r$ such that $\sum_{t=1}^r c_t h_t = 1$. Note that c_t is a unit modulo p_k for each $t = 1, \ldots, r$. We have

$$\left(\frac{l}{n_0}\right) = \left(\frac{\sum_{t=1}^r lc_t h_t}{n_0}\right) = \prod_{j=1}^s \left(\frac{\sum_{t=1}^r lc_t h_t}{p_j}\right) = \prod_{j=1}^s \left(\frac{lc_j h_j}{p_j}\right),$$

since $\sum_{t=1}^r lc_t h_t \equiv lc_j h_j \mod p_j$. Similarly,

$$\zeta_m^{il} = \zeta_m^{i\sum_{j=1}^r lc_j h_j} = \prod_{j=1}^r \zeta_{p_j^{f_j}}^{ilc_j}.$$

Therefore,

$$\begin{aligned}
\sum_{l=0}^{m-1} \left(\frac{l}{M}\right) \zeta_m^{-il} &= \sum_{l\in Z_m^*} \left(\frac{l}{n_0}\right) \zeta_m^{il} \\
&= \sum_{l\in Z_m^*} \prod_{j=1}^s \left(\frac{lc_j h_j}{p_j}\right) \prod_{j=1}^r \zeta_{p_j^{f_j}}^{ilc_j} \\
&= \prod_{j=1}^s \left(\frac{h_j}{p_j}\right) \sum_{l\in Z_m^*} \prod_{j=1}^s \left(\frac{lc_j}{p_j}\right) \prod_{j=1}^r \zeta_{p_j^{f_j}}^{ilc_j} \\
&= \prod_{j=1}^s \left(\frac{h_j}{p_j}\right) \sum_{(l_1,\ldots,l_r)\in Z^*_{p_1^{f_1}}\times\cdots\times Z^*_{p_r^{f_r}}} \prod_{j=1}^s \left(\frac{l_j}{p_j}\right) \zeta_{p_j^{f_j}}^{il_j} \prod_{j=s+1}^r \zeta_{p_j^{f_j}}^{il_j} \\
&= \prod_{j=1}^s \left(\frac{h_j}{p_j}\right) \prod_{j=1}^s \sum_{l_j\in Z^*_{p_j^{f_j}}} \left(\frac{l_j}{p_j}\right) \zeta_{p_j^{f_j}}^{il_j} \prod_{j=s+1}^r \sum_{l_j\in Z^*_{p_j^{f_j}}} \zeta_{p_j^{f_j}}^{il_j}. \qquad (5)
\end{aligned}$$

The sums in (5) are evaluated in the following lemma, which is easy to prove.

Lemma 3.3. *Let p be odd prime p and let $f \geq 1$ be an integer. Suppose that $i \equiv up^d \mod p^f$, where u is coprime to p, and $0 \leq d \leq f$. Let $g(p) = \sum_{l=0}^{p-1} \left(\frac{l}{p}\right) \zeta_p^l$ denote the quadratic Gauss sum. Then*

$$\sum_{l\in Z^*_{p^f}} \left(\frac{l}{p}\right) \zeta_{p^f}^{il} = \begin{cases} p^d \left(\frac{u}{p}\right) g(p) & \text{if } d = f-1, \\ 0 & \text{otherwise.} \end{cases}$$

Also,

$$\sum_{l\in Z^*_{p^f}} \zeta_{p^f}^{il} = \begin{cases} p^f - p^{f-1} & \text{if } d = f, \\ -p^{f-1} & \text{if } d = f-1, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluating $d^\lambda_{\mu,i}$ using (5) and the above lemma the formula for $d^{\phi(\mu)}_{\mu,i}$ is obtained. To get the formula for $|d^{\phi(\mu)}_{\mu,i}|$ we use the fact that $|g(p)| = \sqrt{p}$. □

Corollary 3.4. *For every partition $\mu = (\mu_1, \ldots, \mu_k)$ of an integer $n > 1$ with distinct odd parts, let $M = \prod_{j=1}^k \mu_j$. Then $|d^{\phi(\mu)}_{\mu,i}| < \sqrt{M}$ for every integer i.*

Corollary 3.5. *For every partition $\mu = (\mu_1, \dots, \mu_k)$ with distinct odd parts, let $M = \prod_{j=1}^k \mu_j$ and $m = \mathrm{lcm}(\mu_1, \dots, \mu_k)$. Suppose $M = \prod_{j=1}^k p_j^{e_j}$, where $p_1, \dots, p_k$ are distinct primes, and $e_1, \dots, e_s$ are odd, and $e_{s+1}, \dots, e_k$ are even.*

(1) *$d_{\mu,0}^{\phi(\mu)} \neq 0$ if and only if M is a square. When this happens,*
$$|d_{\mu,0}^{\phi(\mu)}| = \sqrt{M}\prod_{p|M}(1-p^{-1}),$$
the product running over primes dividing M.

(2) *$d_{\mu,1}^{\phi(\mu)} \neq 0$ if and only if m is square-free. When this happens,*
$$|d_{\mu,1}^{\phi(\mu)}| = \sqrt{\frac{M}{\prod_{j=1}^s p_j}}\cdot\frac{1}{\prod_{j=s+1}^r p_j}.$$

4. Some Cyclic Characters of Alternating Groups

For representations V and W of a group G, say that $V \geq W$ if V contains a subrepresentation isomorphic to W.

Lemma 4.1. *Suppose $n \neq 3$ is odd. Let $\mu = (n)$. Then for every integer r, $|d_{\mu,r}^{\phi(\mu)}| < a_{\mu,r}^{\phi(\mu)}$. In other words, $\mathrm{Ind}_{C_\mu}^{A_\mu} \zeta_n^r \geq V_{\phi(\mu)}^{\pm}$.*

Proof. When $\mu = (n)$, where $n = 2m+1$, $\phi(\mu) = (m+1, 1^m)$. Let f^λ denote the dimension of the representation V_λ of S_n. By [**12**, Theorem 1.9],

$$\left|\frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n}\right| < \frac{1}{n^2} \text{ for every } \lambda \vdash n \text{ such that } f^\lambda > n^5. \tag{6}$$

In our case, $f_{\phi(\mu)} = \binom{2m}{m}$, since any standard tableau of shape $\phi(\mu)$ is determined by which m out of the $2m$ numbers $2, \dots, 2m+1$ are in the first row. Note that

$$\binom{2m}{m} > \frac{4^m}{2m+1} = \frac{2^{n-1}}{n}. \tag{7}$$

For $n \geq 31$, $2^{n-1}/n > n^5$, so (6) gives us

$$a_{\mu,r}^{\phi(\mu)} > f_{\phi(\mu)}\left(\frac{1}{n} - \frac{1}{n^2}\right) > \frac{2^{n-1}(n-1)}{n^3} > \sqrt{n} > |d_{\mu,r}^{\phi(\mu)}|. \tag{8}$$

The cases $n < 31$ are easily checked using SageMath [**13**]. □

Lemma 4.2. *Suppose $n > 4$ is even. Let $\mu = (n-1, 1)$. Then for every integer r, $|d_{\mu,r}^{\phi(\mu)}| < a_{\mu,r}^{\phi(\mu)}$.*

Proof. We have

$$\begin{aligned} a_{\mu,r}^{\phi(\mu)} &= \langle \mathrm{Ind}_{C_\mu}^{A_n} \zeta_{n-1}^r, V_{\phi(\mu)}^{\pm}\rangle_{A_n} \\ &= \langle \mathrm{Ind}_{A_{n-1}}^{A_n} \mathrm{Ind}_{C_{(n-1)}}^{A_{n-1}} \zeta_{n-1}^r, V_{\phi(\mu)}^{\pm}\rangle_{A_n} \\ &= \langle \mathrm{Ind}_{C_{n-1}}^{A_{n-1}} \zeta_{n-1}^r, \mathrm{Res}_{A_{n-1}}^{A_n} V_{\phi(\mu)}^{\pm}\rangle_{A_{n-1}} \end{aligned}$$

by Frobenius reciprocity. In [**2**] it was shown that $\mathrm{Res}_{A_{n-1}}^{A_n} V_{\phi((n-1,1))}^{\pm}$ contains at least one of the irreducible representations $V_{(n-1)}^{\pm}$. Therefore,

$$a_{\mu,r}^{\phi(\mu)} \geq \langle \mathrm{Ind}_{C_{n-1}}^{A_{n-1}} \zeta_{n-1}^r, V_{\phi((n-1))}^{\pm}\rangle_{A_{n-1}} > 0$$

by Lemma 4.1. □

The following theorem is a special case of a result of Yang and Staroletov [**14**, Corollary 1.2]. Here we shall deduce it from Swanson's result [**12**, Theorem 1.5] for symmetric groups and the Lemma 4.1.

THEOREM 4.3. *For integer $n > 3$, let $\mu = (n)$ if n is odd, and let $\mu = (n-1, 1)$ if n is even. Let m be the order of w_μ. Then for every irreducible representation V of A_n, and $0 \leq r \leq m-1$, we have*

$$\mathrm{Ind}_{C_\mu}^{A_n} \zeta_m^r \geq V$$

except when V is one of the following:

(1) *$V = V_{(n-1,1)}$ for n odd and $r = 0$,*
(2) *$V = V_{(n)}$ and $r \neq 0$.*

PROOF. Let n be odd. Then by [**12**, Theorem 1.5], $a^\lambda_{(n),r} > 0$ except in the following cases:

(1) $\lambda = (n-1, 1)$ and $r = 0$
(2) $\lambda = (2, 1^{n-2})$ and $r = 0$
(3) $\lambda = (n)$ and $r \neq 0$
(4) $\lambda = (1^n)$ and $r \neq 0$.

It follows that, if λ is not self-conjugate or $\lambda \neq \phi((n))$, then $\mathrm{Ind}_{C_{(n)}}^{A_n} \zeta_n^r \geq V_\lambda$ except in the following cases:

(1) $\lambda = (n-1, 1)$ and $r = 0$
(2) $\lambda = (n)$ and $r \neq 0$.

If $\lambda = \phi(n)$, then $a^{\lambda^\pm}_{(n),r} = (a^\lambda_{(n),r} \pm d^\lambda_{(n),r})/2$, which are positive by Lemma 4.1.

Let n be even. By the Pieri rule, $a^\lambda_{(n-1,1),r} \geq 0$ if and only if there exists a partition η whose Young diagram is obtained by removing a box from the Young diagram of λ such that $a^\eta_{(n-1,1),r} \geq 0$. Applying the previous argument to S_{n-1}, we see that $a^\lambda_{(n-1,1),r} \geq 0$ except in the following cases:

(1) $\lambda = (n)$ and $r \neq 0$
(2) $\lambda = (1^n)$ and $r \neq 0$.

Hence, if λ is not self-conjugate or $\lambda \neq \phi((n-1, 1))$, then $\mathrm{Ind}_{C_{(n-1,1)}}^{A_n} \zeta_{n-1}^r \geq V_\lambda$, except when $\lambda = (n)$ and $r \neq 0$. If $\lambda = \phi((n-1, 1))$, then $a^{\lambda^\pm}_{(n-1,1),r} = (a^\lambda_{(n-1,1),r} \pm d^\lambda_{(n-1,1),r})/2$, which are positive by Lemma 4.1. □

5. Elementwise Invariant Vectors

Given partitions λ and μ of n, the following characterization of when w_μ admits a non-zero invariant vector in the representation V_λ of S_n was obtained in [**7**].

THEOREM 5.1. *The only pairs of partitions (λ, μ) of a given integer n such that w_μ does not admit a nonzero invariant vector in V_λ are the following:*

(1) *$\lambda = (1^n)$, μ is any partition of n for which w_μ is odd,*
(2) *$\lambda = (n-1, 1)$, $\mu = (n)$, $n \geq 2$,*
(3) *$\lambda = (2, 1^{n-2})$, $\mu = (n)$, $n \geq 3$ is odd,*
(4) *$\lambda = (2^2, 1^{n-4})$, $\mu = (n-2, 2)$, $n \geq 5$ is odd,*
(5) *$\lambda = (2, 2)$, $\mu = (3, 1)$,*
(6) *$\lambda = (2^3)$, $\mu = (3, 2, 1)$,*
(7) *$\lambda = (2^4)$, $\mu = (5, 3)$,*

(8) $\lambda = (4,4)$, $\mu = (5,3)$,
(9) $\lambda = (2^5)$, $\mu = (5,3,2)$.

In this section, we carry out a similar analysis for representations of A_n.

THEOREM 5.2. *For every irreducible representation V of A_n, and every $w \in A_n$, there exists a non-zero vector in V that is invariant under w unless one of the following holds:*

(1) $V = V^{\pm}_{(2,1)}$ *and* $w \in [(3)]^{\pm}$,
(2) $V = V^{\pm}_{(2,2)}$ *and* $w \in [(3,1)]^{\pm}$,
(3) $V = V_{(4,4)}$ *and* $w \in [(5,3)]^{\pm}$.
(4) $V = V_{(n-1,1)}$ *and* $w \in [(n)]^{\pm}$, *where* $n > 3$ *is odd,*

PROOF. The exceptions in Theorem 5.2 are restrictions to A_n of the exceptions in Theorem 5.1, or subrepresentations thereof. Therefore they cannot admit non-zero invariant vectors.

It only remains to check that for a partition μ with distinct odd parts such that the representation $V_{\phi(\mu)}$ of S_n admits a nonzero invariant vector for w_μ, $V^{\pm}_{\phi(mu)}$ admits a nonzero invariant vector for w_μ. The case where $\mu = (n)$, $n \neq 3$, was proved in Lemma 4.1.

Now consider the case where μ is a partition with distinct odd parts, but 3 is not a part of μ. By induction in stages,

$$\operatorname{Ind}_{C_\mu}^{A_n} 1 \geq \operatorname{Ind}^{A_n}_{\prod_{j=1}^k A_{\mu_j}} \bigotimes_{j=1}^k \operatorname{Ind}^{A_{\mu_j}}_{C_{(\mu_j)}} 1 \geq \operatorname{Ind}^{A_n}_{\prod_{j=1}^k A_{\mu_j}} \bigotimes_{j=1}^k V_{\phi((\mu_j))}. \tag{9}$$

The first inequality follows from the fact that C_μ is a subgroup of $\prod_{j=1}^k C_{(\mu_j)}$. The second inequality follows from Lemma 4.1. It remains to show that

$$\operatorname{Ind}^{A_n}_{\prod_{j=1}^k A_{\mu_j}} \bigotimes_{j=1}^k V_{\phi((\mu_j))} \geq V^{\pm}_{\phi(\mu)}. \tag{10}$$

The character of the left-hand side is invariant under conjugation by elements of S_n. Therefore, the left hand side contains $V^+_{\phi(\mu)}$ if and only if it contains $V^-_{\phi(\mu)}$ (in which case it contains $V_{\phi(\mu)}$).

Since $\operatorname{Ind}^{S_{\mu_j}}_{A_{\mu_j}} V^{\pm}_{\phi((\mu_j))} = V_{\phi((\mu_j))}$, we have

$$\operatorname{Ind}^{S_n}_{A_n} \operatorname{Ind}^{A_n}_{\prod_{j=1}^k A_{\mu_j}} \bigotimes_{j=1}^k V^{\pm}_{\phi((\mu_j))} \geq \operatorname{Ind}^{S_n}_{\prod_{j=1}^k S_{\mu_j}} \bigotimes_{j=1}^k V_{\phi((\mu_j))}. \tag{11}$$

By the Littlewood-Richardson rule [**6**, Section I.9] the multiplicity of $V_{\phi(\mu)}$ in $\operatorname{Ind}^{S_n}_{S_{\mu_1} \times S_{n-\mu_1}} V_{\phi((\mu_1))} \otimes V_{\phi((\mu_2,\dots,\mu_k))}$ is the number of semistandard Young tableaux of shape $\phi(\mu)/\phi((\mu_1))$ and weight $\phi((\mu_2,\dots,\mu_k))$ whose reverse reading word is a ballot sequence. But semistandard Young tableau of shape $\phi(\mu)/\phi((\mu_1))$ are in bijection with semistandard tableau of shape $\phi((\mu_2,\dots,\mu_k))$. Filling all the cells of the ith row of the Young diagram of $\phi((\mu_2,\dots,\mu_k))$ with i results in a reverse reading word that is a ballot sequence. Therefore,

$$\operatorname{Ind}^{S_n}_{S_{\mu_1} \times S_{n-\mu_1}} V_{\phi((\mu_1))} \otimes V_{\phi((\mu_2,\dots,\mu_k))} \geq V_{\phi(\mu)}.$$

Working recursively with respect to k, we get

$$\mathrm{Ind}_{\prod_{j=1}^k S_{\mu_j}}^{S_n} \bigotimes_{j=1}^k V_{\phi((\mu_j))} \geq V_{\phi(\mu)}.$$

Now using (11), we get

$$\mathrm{Ind}_{A_n}^{S_n} \mathrm{Ind}_{\prod_{j=1}^k A_{\mu_j}}^{A_n} \bigotimes_{j=1}^k V_{\phi((\mu_j))} \geq V_{\phi(\mu)}^{\pm}.$$

But the only irreducible representations of A_n, which upon induction to S_n contain $V_{\phi(\mu)}$, are $V_{\phi(\mu)}^{\pm}$, so (10) must hold.

If μ has 3 as a part, since the cases $\mu = (3)$ and $\mu = (3,1)$ are among the exceptions in Theorem 5.2 we may assume that μ has a part that is greater than $\mu_l = 3$ (l has to be $k-1$ or k). Suppose $\mu_{l-1} = 2m+1$. By Theorem 4.3, $\mathrm{Ind}_{C_{\mu_{l-1}}}^{A_{\mu_{l-1}}} 1 \geq V_{(m,2,1^{m-1})}$. Also $\mathrm{Ind}_{C_{\mu_l}}^{A_{\mu_l}} 1$ is the trivial representation of A_3. In place of (9), we can insert different tensor factors in the $l-1$st and lth places to get

$$\begin{aligned}\mathrm{Ind}_{C_\mu}^{A_n} 1 &\geq \mathrm{Ind}_{\prod_{j=1}^k A_{\mu_j}}^{A_n} \bigotimes_{j=1}^k \mathrm{Ind}_{C_{(\mu_j)}}^{A_{\mu_j}} 1 \\ &\geq \mathrm{Ind}_{\prod_{j=1}^k A_{\mu_j}}^{A_n} \bigotimes_{j\neq l-1,1}^k V_{\phi((\mu_j))} \otimes V_{(m,2,1^{m-1})} \otimes V_{(1^3)}.\end{aligned}$$

By Pieri's rule, $\mathrm{Ind}_{S_{\mu_{l-1}}\times S_3}^{S_{\mu_{l-1}+3}} V_{(m,2,1^{m-1})} \otimes V_{(1^3)} \geq V_{(m+1,3,2,1^{m-2})} = V_{\phi(\mu_{l-1},3)}$. Proceeding as with (11), we get $\mathrm{Ind}_{C_\mu}^{A_n} 1 \geq V_{\phi(\mu)}^{\pm}$. $\square$

Prasad and Raghunathan introduced the notion of *immersion* of automorphic representations in [**9**]. In the context of representations of a finite group, it can be formulated as follows:

Definition 5.3. Let U and V be representations of a finite group G. We say that U is immersed in V if, for every $g \in G$ and every $\lambda \in \mathbf{C}$, the multiplicity of λ as an eigenvalue of g in U does not exceed the multiplicity of λ as an eigenvalue of g in V.

Theorem 5.2 gives the following result on immersion.

Theorem 5.4. *For $n \geq 3$, the trivial representation of A_n is immersed in every irreducible representation V of A_n except when*

(1) $V = V_{(2,1)}^{\pm}$,
(2) $V_{(2,2)}^{\pm}$ *and* $V_{(4,4)}$.
(3) $V = V_{(n-1,1)}$ *for $n > 3$ odd.*

6. Global Conjugacy Classes

A group G acts on any of its conjugacy classes C by conjugation. Following Heide and Zalessky [**3**], a conjugacy class C of a finite group G is called a *global conjugacy class* if the corresponding permutation representation of G contains every irreducible representation of G as a subrepresentation. Equivalently $\mathrm{Ind}_C^G 1$ contains every irreducible representation of G as a subrepresentation.

Heide and Zalessky showed that A_n has a global conjugacy class for $n > 4$. We recover their result while establishing a larger family of global conjugacy classes (compare with the proof of Theorem 4.3 in [**3**]).

Theorem 6.1. *For any positive integer n, let μ be a partition of n with at least two parts different from $(3,1)$ and $(5,3)$ whose parts are odd and distinct. Then $[\mu]^+$ and $[\mu]^-$ are global conjugacy classes in A_n.*

The proof is based on the following theorem of Sundaram [**11**, Theorem 5.1].

Theorem 6.2. *Let $n \neq 4, 8$. The permutations with cycle type $\mu \vdash n$ form a global conjugacy class in S_n if and only if μ has at least two parts, and all its parts are odd and distinct.*

Proof of Theorem 6.1. Let μ be as in the statement of Theorem 6.1. By Theorem 6.2 (and explicit calculation for $\mu = (7,1)$), permutations with cycle type μ form a global conjugacy class in S_n. Let Z_μ denote the centralizer of w_μ in A_n. Since μ has distinct odd parts, Z_μ is also the centralizer of w_μ in S_n. Thus $\mathrm{Ind}_{Z_\mu}^{S_n} 1 \geq V_\lambda$ for every partition of n. If $\lambda \neq \lambda'$ then this implies $\mathrm{Ind}_{Z_\mu}^{A_n} 1 \geq V_\lambda$.

Now suppose $\lambda = \lambda'$. The character of $\mathrm{Ind}_{Z_\mu}^{A_n} 1$ is supported on conjugacy classes of powers of w_μ. The only such classes whose cycle types have distinct odd parts are $[\mu]^\pm$. Therefore, if $\lambda \neq \phi(\mu)$, then Schur inner product of $\mathrm{Ind}_{Z_\mu}^{A_n} 1$ and δ_λ is zero. It follows that the multiplicities of V_λ^+ and V_λ^- in $\mathrm{Ind}_{Z_\mu}^{A_n} 1$ are equal. Theorem 6.2 tells us that their sum is positive, so each of them has to be positive.

Finally, consider the case where $\lambda = \phi(\mu)$. Since the parts of μ are distinct,

$$\mathrm{Ind}_{Z_\mu}^{A_n} 1 = \mathrm{Ind}_{\prod_{j=1}^k A_{\mu_j}}^{A_n} \bigotimes_{j=1}^k \mathrm{Ind}_{C_{(\mu_j)}}^{A_{\mu_j}} 1.$$

Following the proof of Theorem 5.2 from (9) onwards establishes that $\mathrm{Ind}_{Z_\mu}^{A_n} 1 \geq V_{\phi(\mu)}^\pm$. □

Theorem 6.3. *For every odd integer $p > 3$, permutations of cycle type (p,p) form a global conjugacy class in A_{2p}.*

Proof. For $p = 5$, the result can be verified by direct calculation. Assume $p > 5$. The centralizer of $w_{(p,p)}$ in A_{2p} is isomorphic to $C_p \times C_p$. In the proof of [**7**, Lemma 9], it is shown that, for almost all $p \geq q$, and all $\lambda \vdash p+q$ different from (1^{p+q}), there exist partitions $\alpha \vdash p$ and $\beta \vdash q$ such that $\mathrm{Ind}_{C_p}^{S_p} 1 \geq V_\alpha$, $\mathrm{Ind}_{C_q}^{S_q} 1 \geq V_\beta$, and $\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} V_\alpha \otimes V_\beta \geq V_\lambda$. The only exception occurs when q is even. Also, since $C_p \times C_p \subset A_{2p}$, the sign representation $V_{(1^{p+q})}$ also occurs in $\mathrm{Ind}_{C_p \times C_p}^{S_{2p}} 1$. Thus $\mathrm{Ind}_{C_p \times C_p}^{S_{2p}} 1 \geq V_\lambda$ for all $\lambda \vdash 2p$.

It follows that $\mathrm{Ind}_{C_p \times C_p}^{A_{2p}} 1 \geq V_\lambda$ for all $\lambda \vdash 2p$ that are not self-conjugate. The support of the character of $\mathrm{Ind}_{C_p \times C_p}^{A_{2p}} 1$ only contains permutations that are conjugate to an element of $C_p \times C_p$. Hence it does not contain any permutations with cycle type having distinct odd parts. Therefore, the multiplicities of V_λ^+ and V_λ^- in $\mathrm{Ind}_{C_p \times C_p}^{A_{2p}} 1$ are equal and positive. □

For partitions λ and μ, let $\lambda \cup \mu$ denote the partition obtained by concatenating the parts of λ and μ and rearranging in weakly decreasing order.

THEOREM 6.4. *Suppose λ and μ are partitions with odd parts such that permutations with cycle type λ and μ lie in global conjugacy classes. If $\lambda\cup\mu$ is a partition where no part is repeated more than twice, then every permutation with cycle type $\lambda\cup\mu$ lies in a global conjugacy class in $A_{|\lambda\cup\mu|}$.*

PROOF. Suppose that $\lambda\vdash l$, $\mu\vdash m$ and $n=l+m$. The hypotheses on λ and μ imply that the centralizer $Z_{\lambda\cup\mu}$ of a permutation with cycle type $\lambda\cup\mu$ in A_n is $Z_\lambda\times Z_\mu\subset A_l\times A_m\subset A_n$. Therefore,

$$\mathrm{Ind}_{Z_{\lambda\cup\mu}}^{A_n}1=\mathrm{Ind}_{A_l\times A_m}^{A_n}\mathrm{Ind}_{Z_\lambda}^{A_l}1\otimes\mathrm{Ind}_{Z_\mu}^{A_m}1.$$

If V is any irreducible representation of A_n, let $U\otimes W$ be some irreducible representation of $A_l\times A_m$ that occurs in the restriction of V to $A_l\times A_m$. By Theorem 6.1, $\mathrm{Ind}_{Z_\lambda}^{A_l}1\geq U$ and $\mathrm{Ind}_{Z_\mu}^{A_m}1\geq W$, so $\mathrm{Ind}_{Z_{\lambda\cup\mu}}^{A_n}1\geq\mathrm{Ind}_{A_l\times A_m}^{A_n}U\otimes W\geq V$ (by Frobenius reciprocity). $\square$

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