LEPOWSKY'S AND WAKIMOTO'S PRODUCT FORMULAS FOR THE AFFINE LIE ALGEBRAS $C_l^{(1)}$

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Abstract. In this paper, we recall Lepowsky's and Wakimoto's product character formulas formulated in a new way by using arrays of specialized weighted crystals of negative roots for affine Lie algebras of type $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$. Lepowsky–Wakimoto's infinite periodic products appear as one side of (conjectured) Rogers–Ramanujan-type combinatorial identities for affine Lie algebras of type $C_l^{(1)}$.

Contents

1. Introduction

In the last several decades, numerous applications of Rogers–Ramanujan-type identities have been extensively studied. For example, in the early eighties, they emerged in the areas of statistical mechanics [ABF, B] and orthogonal polynomials [AsI, Br3]. On the other hand, in the last decade, in addition to their well-known role in combinatorics and number theory (see, e.g., [DK, GOW, W]), their close connections with modular forms [BCFK], algebraic geometry [BMS, GOR] and double affine Hecke algebras [CF] were investigated. In this paper, we are interested in a line of research going back to Lepowsky and Milne [LM], which connected the product sides of Gordon–Andrews–Bressoud's generalization [Go, A1, A2, Br1, Br2] of the Rogers–Ramanujan identities with principally specialized characters of integrable highest weight modules for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Their seminal paper [LM] motivated a research of Lepowsky and Wilson [LW1]–[LW4], which led to discovery of vertex operators in the principal picture of integrable highest weight $\widehat{\mathfrak{sl}}_2$-modules and bases of vacuum spaces for the principal Heisenberg subalgebra which are parametrized by partitions satisfying certain difference conditions.

J. Lepowsky proved that the principally specialized characters of standard modules for affine Lie algebras can be written as infinite periodic products. M. Wakimoto extended Lepowsky's argument for some other specializations of characters. The aim of this paper is to write explicitly all Lepowsky–Wakimoto's product formulas for affine Lie algebras of types $C_l^{(1)}$, $l \geq 2$, by using arrays of specialized weighted crystals of negative roots for affine Lie algebras of type $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$. These weighted crystals parametrize root vectors for negative roots in terms of crystal bases of adjoint representations of C_l and B_l, and B_l-modules $L_{B_l}(\omega_1)$ and $L_{B_l}(2\omega_1)$. Although inspired by the crystal bases theory, we use only the combinatorial notion of weighted crystals for Cartan matrices, and the realization of affine Kac–Moody Lie algebras.

As an illustration of the above described approach, we remark that, by using Wakimoto's product formula for $\mathrm{ch}^{(2,1,1)} L_{C_2^{(1)}}(1,0,0)$, and the basis for the basic module $L_{C_2^{(1)}}(1,0,0)$, we get an analogue of Capparelli's identity:

Theorem. *The number of colored partitions satisfying level* 1 *difference conditions on the array*

$$\begin{array}{cccccccccc}
 & & & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & \\
 & & & \mathit{5} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} \\
 & & \mathit{4} & & \mathit{6} & & \mathbf{9} & & \mathbf{11} & \\
 & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathbf{10} & & \mathit{13} \\
\mathit{2} & & \mathit{4} & & \mathit{6} & & \mathit{8} & & \mathit{12} &
\end{array}$$

equals the number of partitions of n into parts

$$j \not\equiv 0, \pm 1, \pm 6, \pm 7, 8 \mod 16.$$

This paper, and the above Theorem in particular, was motivated by the results on Poincaré–Birkhoff–Witt-type bases for standard modules over $C_l^{(1)}$ in [CMPP, PŠ1, PŠ2]. It is organized as follows. Sections 2–4 serve as an introduction. In Section 2, we provide some preliminary definitions and results on the affine Kac–Moody Lie algebras and their representation theory; see [Kac] for more details. Next, in Section 3, we recall the specialized character formulas of Lepowsky [L] and Wakimoto [W1]. Finally, in Section 4, we consider certain weighted crystals which arise as connected components of tensor squares of vector representation crystals associated with the complex simple Lie algebras of types B_l and C_l. An introduction to the theory of crystal bases, which, in particular, contains all notions and results used in this manuscript, can be found in [HK].

In Section 5, we use the weighted crystals from Section 4 to explicitly describe the arrays of negative root vectors for the affine Lie algebras of type $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$. By assigning degrees to the negative Chevalley generators with respect to certain specializations, in Section 6, we turn these arrays into tables of integers, which we refer to as specialized weighted arrays. Moreover, we express their generating functions in terms of product formulas. Finally, in Section 7, we employ the product formulas to establish a connection with the aforementioned specialized character formulas of Lepowsky and Wakimoto.

In Section 8, we use the generating functions from Section 6 to generalize Borcea's correspondence [Bor] between specialized characters of certain standard modules for the affine Lie algebras $C_l^{(1)}$ and $A_{2l}^{(2)}$ to an arbitrary positive integer level. At the end, in Section 9, we introduce the notion of *level k-difference conditions* for colored partitions on arrays connected with a basis of negative root vectors. Using this we recall a conjecture by [CMPP] on combinatorial identities, that served as one of the motivations for the present paper.

2. Affine Lie algebras of type $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$

2.1. **Realization of affine Lie algebras.** Let $\mathfrak{g}$ be a complex simple Lie algebra of type X_l and let μ be a diagram automorphism of $\mathfrak{g}$ of order r $(= 1, 2, \text{or } 3)$, see [Kac, Section 7.9]. Let $\langle \cdot, \cdot \rangle$ be an invariant symmetric bilinear form on $\mathfrak{g}$, normalized so that the square length of a long root is 2. The untwisted affine Lie algebra of type $X_l^{(1)}$ can then be realized as

$$\mathfrak{g}(X_l^{(1)}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with the following commutation relations: denote by $x(n) := x \otimes t^n$ for $x \in \mathfrak{g}, n \in \mathbb{Z}$, then define

$$\begin{aligned}
&[x(m), y(n)] = [x, y](m + n) + m\delta_{m+n,0}\langle x, y\rangle c,\\
&c \in \operatorname{center}(\mathfrak{g}(X_l^{(1)})),\\
&[d, x(n)] = nx(n),
\end{aligned}$$

for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$.

Following [Kac] we shall realize the (twisted) affine Lie algebra of type $X_l^{(r)}$ as a subalgebra of the affine Lie algebra $\mathfrak{g}(X_l^{(1)})$. For $j \in \mathbb{Z}$ we set $\bar{j} = j + r\mathbb{Z} \in \mathbb{Z}/r\mathbb{Z}$ and let $\mathfrak{g}_{\bar{j}}$ denote the eigenspace of μ on $\mathfrak{g}$ with eigenvalue $e^{2\pi i j/r}$. Define

$$\mathcal{L}(\mathfrak{g}, \mu) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{\bar{j}} \otimes t^j \oplus \mathbb{C}c.$$

Then $\mathcal{L}(\mathfrak{g}, \mu)$ is a realization of the (twisted) affine Lie algebra of type $X_l^{(r)}$, see [Kac, Theorem 8.3].

2.2. **Affine Lie algebras of type $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$.** We denote the Kac–Moody Lie algebra with generalized Cartan matrix A of type $X_l^{(r)}$ as $\mathfrak{g}(X_l^{(r)})$. In this paper we are interested in three types of affine Lie algebras with Dynkin diagrams in Figure 1.

In the case $X_l = C_l$ we have

$$\mathfrak{g}(C_l^{(1)}) = \mathbb{C}c \oplus \mathbb{C}d \oplus \coprod_{j\in\mathbb{Z}} \mathfrak{g}(C_l) \otimes t^j. \tag{2.1}$$

In the case $X_l = D_{l+1}$, the fixed point subalgebra is $\mathfrak{g}(B_l)$ [Kac, Proposition 7.9(b)], and the -1-eigenspace is the $\mathfrak{g}(B_l)$-module $L(\omega_1)$ of highest weight ω_1 (a fundamental

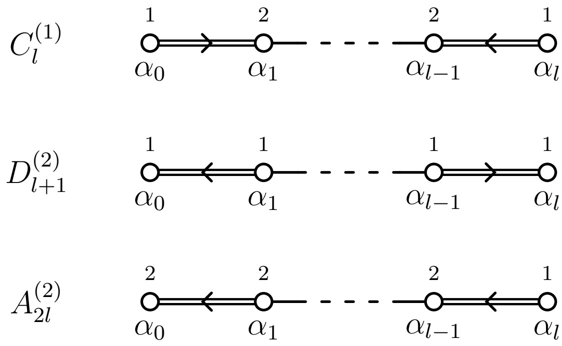

FIGURE 1. Dynkin diagrams of types $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$

weight) [Kac, Proposition 7.9(g)]. Hence

$$\mathfrak{g}(D_{l+1}^{(2)}) = \mathbb{C}c \oplus \mathbb{C}d \oplus \coprod_{j\in\mathbb{Z}} \mathfrak{g}(B_l) \otimes t^{2j} \oplus \coprod_{j\in\mathbb{Z}} L_{B_l}(\omega_1) \otimes t^{2j+1}. \tag{2.2}$$

In the case $X_l = A_{2l}$, the fixed point subalgebra is $\mathfrak{g}(B_l)$ [Kac, Proposition 7.10(b)], and the -1-eigenspace is the $\mathfrak{g}(B_l)$-module $L(2\omega_1)$ of highest weight $2\omega_1$ [Kac, Proposition 7.10(g)]. Hence

$$\mathfrak{g}(A_{2l}^{(2)}) = \mathbb{C}c \oplus \mathbb{C}d \oplus \coprod_{j\in\mathbb{Z}} \mathfrak{g}(B_l) \otimes t^{2j} \oplus \coprod_{j\in\mathbb{Z}} L_{B_l}(2\omega_1) \otimes t^{2j+1}. \tag{2.3}$$

Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_{\bar{0}}$, with $\mathfrak{g}_{\bar{0}}$ the fixed point subalgebra of μ on $\mathfrak{g}(X_l)$ as above. For a root vector $b = x \otimes t^i \in \mathfrak{g}(X_l^{(r)})$ we define the $\mathrm{wt}_{\mathfrak{h}}(b)$ as the weight of x with respect to $\mathfrak{h}$ i.e. $\mathrm{wt}_{\mathfrak{h}}(b) = \nu$ if

$$[h, x] = \nu(h)x$$

for all $h \in \mathfrak{h}$. By identifying $\mathfrak{h}^*$ with the subspace of $(\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$ of weights that take the value 0 on c and d, we then have

$$\mathrm{wt}(b) = \mathrm{wt}_{\mathfrak{h}}(b) + i\delta.$$

For affine Lie algebras of the types $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$, $l \geq 2$, we have the following relations for the imaginary root δ and the canonical central element c of the affine Lie algebras (see [Kac]):

$$\begin{aligned}
C_l^{(1)}: &\qquad \delta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l, \quad c = h_0 + h_1 + \cdots + h_{l-1} + h_l,\\
D_{l+1}^{(2)}: &\qquad \delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{l-1} + \alpha_l, \quad c = h_0 + 2h_1 + \cdots + 2h_{l-1} + h_l,\\
A_{2l}^{(2)}: &\qquad \delta = 2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l, \quad c = h_0 + 2h_1 + \cdots + 2h_{l-1} + 2h_l.
\end{aligned}$$

2.3. **Standard modules and the Weyl–Kac character formula.** Let A be a symmetrizable generalized Cartan matrix and let $\mathfrak{g}(A)$ be the associated Kac–Moody Lie algebra, see [Kac, Section 1.3]. Let h_i, e_i, f_i be the usual Kac–Moody generators and $\{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots. The root system Δ then decomposes

$$\Delta = \Delta^+ \cup -\Delta^+$$

where $\Delta^+ = \Delta \cap \{\sum_i \mathbb{Z}_{\geq 0}\alpha_i\}$. With

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- = \bigoplus_{\alpha\in-\Delta^+} \mathfrak{g}_\alpha$$

one then has the triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where $\mathfrak{h} = \bigoplus_i \mathbb{C}h_i$ is the Cartan subalgebra.

Let $\Lambda \in \mathfrak{h}^*$ be a dominant integral weight i.e. $\Lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all i. The *standard module* $L(\Lambda)$ is then the unique up to isomorphism irreducible highest weight $\mathfrak{g}(A)$-module with highest weight Λ i.e. $L(\Lambda)$ contains a nonzero vector v_Λ of weight Λ, which is annihilated by $\mathfrak{n}_+$, and generates $L(\Lambda)$ as a $\mathfrak{g}(A)$-module, cf. [Kac, Section 9.3].

The Weyl group W of $\mathfrak{g}(A)$ is the group generated by the simple reflections r_{α_i}. The length $\ell(w)$ of an element $w \in W$ is the minimal ℓ such that w has an expression $w = r_{\alpha_{i_1}} \cdots r_{\alpha_{i_\ell}}$ as a product of simple reflections. Fix an element $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all i.

To formulate the Weyl–Kac character formula we also need a certain ring of formal series. Consider formal series of the form $s = \sum_{\mu \in \mathfrak{h}^*} c_\mu e^\mu$, $c_\mu \in \mathbb{C}$. The *support* of s is then $\{\mu; c_\mu \neq 0\}$. The set of series with support contained in a finite union of sets $D(\lambda)$, where $D(\lambda) = \lambda + \sum_i \mathbb{Z}_{\geq 0}(-\alpha_i)$, forms a ring under the multiplication determined by $e^\lambda e^\mu = e^{\lambda+\mu}$. Formula (2.4) below takes place in this ring.

Define the *character* of $L(\Lambda)$ by

$$\operatorname{ch} L(\Lambda) = \sum_{\mu \in \mathfrak{h}^*} \dim(L(\Lambda)_\mu) e^\mu,$$

which is a series of the above form.

Theorem 2.1 (Weyl–Kac character formula [Kac, Theorem 10.4])**.** *The character of the standard module $L(\Lambda)$ satisfies*

$$\operatorname{ch} L(\Lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim(\mathfrak{g}_{-\alpha})}}. \tag{2.4}$$

3. Lepowsky's and Wakimoto's theorems on specializations for $C_l^{(1)}$

Let $s = (s_0, \ldots, s_l)$ be a sequence of positive integers. Following [W1], denote by F_s the homomorphism

$$F_s : \mathbb{C}[[e^{-\alpha_0}, \ldots, e^{-\alpha_l}]] \to \mathbb{C}[[q]]$$

determined by $F_s(e^{-\alpha_j}) = q^{s_j}$, called the *$s$-specialization.*

Let A be an affine GCM, and let

$$\mathfrak{g}(A) = \coprod_{\alpha \in \Delta} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

be the root space decomposition of $\mathfrak{g}(A)$. For $j \in \mathbb{Z}$ set

$$\Delta_{j,s} = \{\alpha = \sum k_i \alpha_i \in \Delta \mid \sum k_i s_i = j\},$$

and using these define

$$\mathfrak{g}_j(s; A) = \coprod_{\alpha \in \Delta_{j,s}} \mathfrak{g}_\alpha.$$

Then

$$\mathfrak{g}(A) = \coprod_{j \in \mathbb{Z}} \mathfrak{g}_j(s; A)$$

is a $\mathbb{Z}$-gradation of $\mathfrak{g}(A)$, where $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. We call this the s-gradation of $\mathfrak{g}(A)$.

Following [W1], define

$$Q(s;A) = F_s\left(\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\dim\mathfrak{g}_{-\alpha}}\right) = \prod_{j=1}^{\infty}(1-q^j)^{\dim\mathfrak{g}_j(s;A)},$$

the s-specialization of the denominator in the Weyl–Kac character formula.

For $L(\Lambda)$ an integrable $\mathfrak{g}(A)$-module, we denote by

$$\mathrm{ch}^{(s;A)} L(\Lambda) = F_s(e^{-\Lambda}\,\mathrm{ch}\, L(\Lambda))$$

the s-specialized character. For $\Lambda = \sum k_i\Lambda_i$, set $s_\Lambda = (k_0,\dots,k_l)$.

Theorem 3.1 (Lepowsky's product formula [L, Theorem 2.6.]). *Let Λ be a dominant integral weight, and set $\mathbf{1} = (1,\dots,1)$. Then*

$$\mathrm{ch}^{(1,\dots,1;A)} L(\Lambda) = \frac{Q(s_\Lambda+\mathbf{1};A^T)}{Q(\mathbf{1};A)}.$$

Remark 3.2. *In [L] this is proved more generally in the setting where A is a symmetrizable GCM.*

The case $A = C_l^{(1)}$ gives

Theorem 3.3. *For a dominant integral weight Λ,*

$$\mathrm{ch}^{(1,\dots,1;C_l^{(1)})} L(\Lambda) = \frac{Q(s_\Lambda+\mathbf{1};D_{l+1}^{(2)})}{Q(\mathbf{1};C_l^{(1)})} \tag{3.1}$$

(cf. Theorem 7.4).

Let $(k_0,\dots,k_l)$ be the coordinates of a dominant integral weight $\Lambda = \sum k_i\Lambda_i$. Set

$$\phi(q) = \prod_{j=1}^{\infty}(1-q^j).$$

Theorem 3.4 ([W1, Section 4]). *The following specializations of characters of $\mathfrak{g}(C_l^{(1)})$-modules hold:*

$$\mathrm{ch}^{(2,1,\dots,1;C_l^{(1)})} L(\Lambda) = \phi(q)^{-l}Q(k_l+1,\dots,k_1+1,2(k_0+1);A_{2l}^{(2)}), \tag{3.2}$$

(cf. Theorem 7.14),

$$\mathrm{ch}^{(1,\dots,1,2;C_l^{(1)})} L(\Lambda) = \phi(q)^{-l}Q(k_0+1,\dots,k_{l-1}+1,2(k_l+1);A_{2l}^{(2)}), \tag{3.3}$$

(cf. Theorem 7.11),

$$\begin{aligned}&\mathrm{ch}^{(2,1,\dots,1,2;C_l^{(1)})} L(\Lambda)\\ &= \frac{Q(2(k_0+1),k_1+1,\dots,k_{l-1}+1,2(k_l+1);C_l^{(1)})}{Q(2,1,\dots,1,2;C_l^{(1)})},\end{aligned} \tag{3.4}$$

(cf. Theorem 7.7),

$$\begin{aligned}&\mathrm{ch}^{(s;C_l^{(1)})} L\Big(\sum_{i=0}^{l-1}(2n-1)\Lambda_i+(n-1)\Lambda_l\Big)\\ &= Q(ns_l,2ns_{l-1},\dots,2ns_0;A_{2l}^{(2)})/Q(s;C_l^{(1)}),\end{aligned} \tag{3.5}$$

(cf. Theorem 7.17),

$$\begin{aligned} &\operatorname{ch}^{(s;C_l^{(1)})} L((n-1)\Lambda_0 + \sum_{i=1}^{l}(2n-1)\Lambda_i) \\ &= Q(ns_0, 2ns_1, \dots, 2ns_l; A_{2l}^{(2)})/Q(s; C_l^{(1)}), \end{aligned} \tag{3.6}$$

(cf. Theorem 7.19),

$$\begin{aligned} &\operatorname{ch}^{(s;C_l^{(1)})} L((n-1)\Lambda_0 + \sum_{i=1}^{l-1}(2n-1)\Lambda_i + (n-1)\Lambda_l) \\ &= Q(ns_0, 2ns_1, \dots, 2ns_{l-1}, ns_l; D_{l+1}^{(2)})/Q(s; C_l^{(1)}), \end{aligned} \tag{3.7}$$

(cf. Theorem 7.21).

4. Weighted crystals for C_l and B_l

In this section, we introduce some weighted crystals for Cartan matrices of types C_l and B_l and their tensor products—see [HK, K1, K2, KKMMNN] for the definitions and results we use. Here we shall not distinguish properly between $U_q(\mathfrak{g})$-modules $L(\lambda)$ for quantum groups and $\mathfrak{g}$-modules $L(\lambda)$ for simple Lie algebras—what we really want is a parametrization of the basis $\mathcal{B}(\lambda)$ of the $\mathfrak{g}$-module $L(\lambda)$ "suggested" by the crystal bases theory.

4.1. **Weighted crystals $\mathcal{B}_{C_l}(\omega_1)$ and $\mathcal{B}_{C_l}(\theta)$ for C_l, $l \geq 2$.** We start with $l = 2$ (see Figure 2). Kashiwara's [K1, K2] tensor product of crystals $\mathcal{B}_{C_2}(\omega_1) \otimes \mathcal{B}_{C_2}(\omega_1)$ is the union $\mathcal{B}_{C_2}(\theta) \cup \mathcal{B}_{C_2}(\omega_2) \cup \mathcal{B}_{C_2}(0)$ of C_2-crystals (see Figure 3).

$$1 \xrightarrow{1} 2 \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}\ .$$

Figure 2. The crystal of the vector representation for C_2

$$\begin{tikzcd}
1 \arrow[r, "1"] & 2 \arrow[r, "2"] & \bar{2} \arrow[r, "1"] & \bar{1}
\end{tikzcd}$$

$$\begin{tikzcd}
11 \arrow[r, "1"] & 21 \arrow[r, "2"] \arrow[d, "1"] & \bar{2}1 \arrow[r, "1"] & \bar{1}1 \arrow[d, "1"] & 1 \arrow[d, "1"] \\
12 \arrow[d, "2"] & 22 \arrow[r, "2"] & \bar{2}2 \arrow[d, "2"] & \bar{1}2 \arrow[d, "2"] & 2 \arrow[d, "2"] \\
1\bar{2} \arrow[r, "1"] & 2\bar{2} \arrow[d, "1"] & \bar{2}\bar{2} \arrow[r, "1"] & \bar{1}\bar{2} \arrow[d, "1"] & \bar{2} \arrow[d, "1"] \\
1\bar{1} & 2\bar{1} \arrow[r, "2"] & \bar{2}\bar{1} & \bar{1}\bar{1} & \bar{1}
\end{tikzcd}$$

Figure 3. The crystal of the tensor square of vector representation for C_2

Note that the crystal $\mathcal{B}_{C_2}(\theta)$ parametrizes a weight basis of the adjoint 10-dimensional representation $L_{C_2}(\theta)$ of the simple Lie algebra $\mathfrak{g} = \mathfrak{g}(C_2)$ of type C_2 with the highest weight vector 11; the crystal $\mathcal{B}_{C_2}(\omega_2)$ parametrizes a weight basis of the 5-dimensional representation $L_{C_2}(\omega_2)$ with highest weight vector 12; the crystal $\mathcal{B}_{C_2}(0) = \{1\bar{1}\}$ parametrizes a weight basis of the 1-dimensional trivial representation.

We can parametrize the weights of $\mathcal{B}_{C_2}(\omega_1) = \{1, 2, \bar{2}, \bar{1}\}$ and the root system Δ of $\mathfrak{g}$ as

$$\{\epsilon_1, \epsilon_2, -\epsilon_2, -\epsilon_1\} \quad \text{and} \quad \Delta = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i \le j \le 2\} \backslash \{0\},$$

where, as usual, $\{\epsilon_1, \epsilon_2\}$ is the canonical basis of $\mathbb{R}^2$. Then 11 is the root vector for the maximal root $\theta = 2\epsilon_1$, and the root vectors $\bar{1}2$ and $\bar{2}\bar{2}$ for the negative simple roots $-\alpha_1 = -\epsilon_1 + \epsilon_2$ and $-\alpha_2 = -2\epsilon_2$ are proportional to Chevalley generators f_1 and f_2. The elements $\bar{i}i$ are proportional to simple coroots $h_i = \alpha_i^\vee$, $i = 1, 2$ in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (cf. [Bou1, Bou2, Car, Hum]). Moreover, the i-arrow $\xrightarrow{i}$, $i = 1, 2$, denotes the action of the Kashiwara operator $\tilde{f}_i$ (which is, in this case, proportional to the action of the Chevalley generator f_i, $i = 1, 2$).

In Figure 3, the crystal for the adjoint representation has the shape of a right-angled triangle with vertices 11, $\bar{1}1$ and $\bar{1}\bar{1}$. The hypotenuse $\{11, 22, \bar{2}\bar{2}, \bar{1}\bar{1}\}$ consists of root vectors corresponding to the long roots $\{2\epsilon_1, 2\epsilon_2, -2\epsilon_2, -2\epsilon_1\}$, and the arrows on both catheti are congruent to the arrows of the crystal for the vector representation.

For $l = 3$, the crystal $\mathcal{B}_{C_3}(\omega_1)$ for the vector representation is shown in Figure 4. The

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \bar{3} \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

FIGURE 4. The crystal of the vector representation for C_3

tensor product of crystals $\mathcal{B}_{C_3}(\omega_1) \otimes \mathcal{B}_{C_3}(\omega_1)$ is the union $\mathcal{B}_{C_3}(\theta) \cup \mathcal{B}_{C_3}(\omega_2) \cup \mathcal{B}_{C_3}(0)$ of C_3-crystals (see Figure 5). Except for more complicated crystals, we see again that the

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \bar{3} \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

$$\begin{tikzcd}
11 \arrow[r, "1"] & 21 \arrow[r, "2"] \arrow[d, "1"] & 31 \arrow[r, "3"] \arrow[d, "1"] & \bar{3}1 \arrow[r, "2"] \arrow[d, "1"] & \bar{2}1 \arrow[r, "1"] & \bar{1}1 \arrow[d, "1"] & 1 \arrow[d, "1"] \\
12 \arrow[d, "2"] & 22 \arrow[r, "2"] & 32 \arrow[r, "3"] \arrow[d, "2"] & \bar{3}2 \arrow[r, "2"] & \bar{2}2 \arrow[d, "2"] & \bar{1}2 \arrow[d, "2"] & 2 \arrow[d, "2"] \\
13 \arrow[r, "1"] \arrow[d, "3"] & 23 \arrow[d, "3"] & 33 \arrow[r, "3"] & \bar{3}3 \arrow[d, "3"] & \bar{2}3 \arrow[r, "1"] \arrow[d, "3"] & \bar{1}3 \arrow[d, "3"] & 3 \arrow[d, "3"] \\
1\bar{3} \arrow[r, "1"] \arrow[d, "2"] & 2\bar{3} \arrow[r, "2"] & 3\bar{3} \arrow[d, "2"] & \bar{3}\bar{3} \arrow[r, "2"] & \bar{2}\bar{3} \arrow[r, "1"] \arrow[d, "2"] & \bar{1}\bar{3} \arrow[d, "2"] & \bar{3} \arrow[d, "2"] \\
1\bar{2} \arrow[r, "1"] & 2\bar{2} \arrow[d, "1"] & 3\bar{2} \arrow[r, "3"] \arrow[d, "1"] & \bar{3}\bar{2} \arrow[d, "1"] & \bar{2}\bar{2} \arrow[r, "1"] & \bar{1}\bar{2} \arrow[d, "1"] & \bar{2} \arrow[d, "1"] \\
1\bar{1} & 2\bar{1} \arrow[r, "2"] & 3\bar{1} \arrow[r, "3"] & \bar{3}\bar{1} \arrow[r, "2"] & \bar{2}\bar{1} & \bar{1}\bar{1} & \bar{1}
\end{tikzcd}$$

FIGURE 5. The crystal of the tensor square of vector representation for C_3

crystal $\mathcal{B}_{C_3}(\theta)$ for the adjoint representation has the shape of a right-angled triangle with vertices 11, $\bar{1}1$ and $\bar{1}\bar{1}$ and that the sequences of arrows on both catheti are congruent to the sequence of arrows in the crystal $\mathcal{B}_{C_3}(\omega_1)$. In general, for $l \geq 3$, we have the C_l-crystal $\mathcal{B}_{C_l}(\omega_1)$ for the vector representation (see Figure 6) and the tensor product of crystals $\mathcal{B}_{C_l}(\omega_1) \otimes \mathcal{B}_{C_l}(\omega_1)$ is the union $\mathcal{B}_{C_l}(\theta) \cup \mathcal{B}_{C_l}(\omega_2) \cup \mathcal{B}_{C_l}(0)$ of C_l-crystals. The crystal $\mathcal{B}_{C_l}(\theta)$ for the adjoint representation of $\mathfrak{g} = \mathfrak{g}(C_l)$ of the type C_l has the shape of a right-angled triangle with vertices 11, $\bar{1}1$ and $\bar{1}\bar{1}$ and the sequence of arrows on both catheti are congruent to the sequence of arrows in the crystal $\mathcal{B}_{C_l}(\omega_1)$.

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \qquad \cdots \xrightarrow{l-1} l \xrightarrow{l} \bar{l} \xrightarrow{l-1} \cdots \qquad \cdots \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

FIGURE 6. The crystal of the vector representation for C_l

4.2. **Weighted crystals $\mathcal{B}_{B_l}(\omega_1)$ and $\mathcal{B}_{B_l}(\theta)$ for B_l, $l \geq 2$.** We start with $l = 2$ and the crystal $\mathcal{B}_{B_2}(\omega_1)$ for the Cartan matrix B_2—it is the crystal of the 5-dimensional vector representation shown in Figure 7. The tensor product of crystals $\mathcal{B}_{B_2}(\omega_1) \otimes \mathcal{B}_{B_2}(\omega_1)$ is

$$1 \xrightarrow{1} 2 \xrightarrow{2} 0 \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}.$$

FIGURE 7. The crystal of the vector representation for B_2

the union $\mathcal{B}_{B_2}(2\omega_1) \cup \mathcal{B}_{B_2}(\theta) \cup \mathcal{B}_{B_2}(0)$ of B_2-crystals (see Figure 8). The crystal $\mathcal{B}_{B_2}(2\omega_1)$ parametrizes a weight basis of the 14-dimensional representation $L_{B_2}(2\omega_1)$ with highest weight vector 11; the crystal $\mathcal{B}_{B_2}(0) = \{1\bar{1}\}$ parametrizes a weight basis of the 1-dimensional trivial representation. We analyze $\mathcal{B}_{B_2}(2\omega_1)$ in the next subsection.

Note that the crystal $\mathcal{B}_{B_2}(\theta)$ parametrizes a weight basis of the adjoint 10-dimensional representation $L_{B_2}(\theta)$ of the simple Lie algebra $\mathfrak{g} = \mathfrak{g}(B_2)$ of type B_2 with highest weight vector 12. By translating the diagonal points 00 and $2\bar{2}$ down along the secondary diagonal $\{\bar{1}1, \bar{2}2, 00, 2\bar{2}, 1\bar{1}\}$, we rectify this crystal to become a triangle (see Figure 9). Note that we did not change the sequence of arrows going down or going to the right. We

$$1 \xrightarrow{1} 2 \xrightarrow{2} 0 \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

$$\begin{tikzcd}
11 \arrow[r, "1"] & 21 \arrow[r, "2"] \arrow[d, "1"] & 01 \arrow[r, "2"] \arrow[d, "1"] & \bar{2}1 \arrow[r, "1"] & \bar{1}1 \arrow[d, "1"] & 1 \arrow[d, "1"] \\
12 \arrow[d, "2"] & 22 \arrow[r, "2"] & 02 \arrow[r, "2"] & \bar{2}2 \arrow[d, "2"] & \bar{1}2 \arrow[d, "2"] & 2 \arrow[d, "2"] \\
10 \arrow[r, "1"] \arrow[d, "2"] & 20 \arrow[r, "2"] & 00 \arrow[d, "2"] & \bar{2}0 \arrow[r, "1"] \arrow[d, "2"] & \bar{1}0 \arrow[d, "2"] & 0 \arrow[d, "2"] \\
1\bar{2} \arrow[r, "1"] & 2\bar{2} \arrow[d, "1"] & 0\bar{2} \arrow[d, "1"] & \bar{2}\bar{2} \arrow[r, "1"] & \bar{1}\bar{2} \arrow[d, "1"] & \bar{2} \arrow[d, "1"] \\
1\bar{1} & 2\bar{1} \arrow[r, "2"] & 0\bar{1} \arrow[r, "2"] & \bar{2}\bar{1} & \bar{1}\bar{1} & \bar{1}
\end{tikzcd}$$

FIGURE 8. The crystal of the tensor square of vector representation for B_2

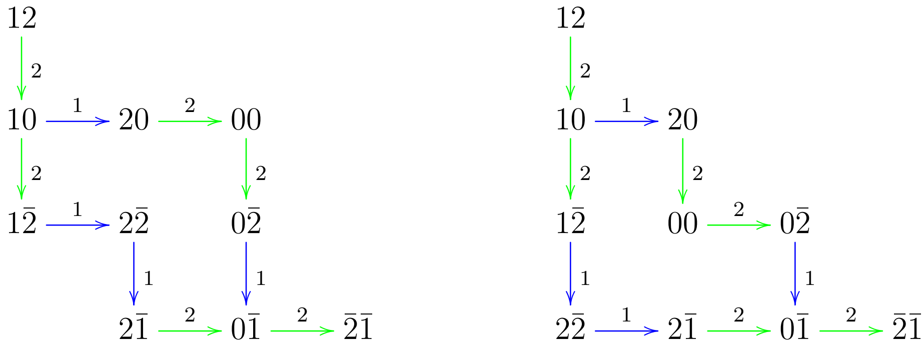

FIGURE 9. Rectifying the crystal for the adjoint B_2-module

parametrize the weights of $\mathcal{B}_{B_2}(\omega_1) = \{1, 2, 0, \bar{2}, \bar{1}\}$ and the root system Δ of $\mathfrak{g}$ as

$$\{\epsilon_1, \epsilon_2, 0, -\epsilon_2, -\epsilon_1\} \quad \text{and} \quad \Delta = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i < j \le 2\} \cup \{\pm\epsilon_i \mid i = 1, 2\}.$$

Then $12 \in \mathcal{B}_{B_2}(\theta)$ is the root vector for the maximal root $\theta = \epsilon_1 + \epsilon_2$, and the root vectors $2\bar{1}$ and $0\bar{2}$ for the negative simple roots $-\alpha_1 = -\epsilon_1 + \epsilon_2$ and $-\alpha_2 = -\epsilon_2$ are proportional to the Chevalley generators f_1 and f_2. The elements $2\bar{2}$ and 00 are proportional to the simple coroots $h_i = \alpha_i^\vee$, $i = 1, 2$ in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (cf. [Bou1, Bou2, Car, Hum]). Moreover, the i-arrow, $i = 1, 2$, denotes the action of the Kashiwara operator $\tilde{f}_i$ (which is, in this case, proportional to f_i). In Figure 9, the (rectified) crystal for the adjoint representation has the shape of a right-angled triangle with vertices 12, $2\bar{2}$ and $\bar{2}\bar{1}$. The hypotenuse $\{12, 20, 0\bar{2}, \bar{2}\bar{1}\}$ consists of root vectors corresponding to the roots $\{\epsilon_1 + \epsilon_2, \epsilon_2, -\epsilon_2, -\epsilon_1 - \epsilon_2\}$. Unlike the C_2-case, the arrows on both catheti are not mutually congruent, nor congruent to the arrows of the crystal for the vector representation. Instead, the arrows on catheti $[12, 2\bar{2}]$ are congruent to the arrows on the segment $[2, \bar{1}]$ of $\mathcal{B}_{B_2}(\omega_1)$, and the arrows on catheti $[2\bar{2}, \bar{2}\bar{1}]$ are congruent to the arrows on the segment $[1, \bar{2}]$ of $\mathcal{B}_{B_2}(\omega_1)$.

For $l = 3$, the tensor product $\mathcal{B}_{B_3}(\omega_1) \otimes \mathcal{B}_{B_3}(\omega_1)$ is the union $\mathcal{B}_{B_3}(2\omega_1) \cup \mathcal{B}_{B_3}(\theta) \cup \mathcal{B}_{B_3}(0)$ of B_3-crystals (see Figure 10). By translating the diagonal points 00, $3\bar{3}$ and $2\bar{2}$ down along the secondary diagonal $\{\bar{1}1, \bar{2}2, \bar{3}3, 00, 3\bar{3}, 2\bar{2}, 1\bar{1}\}$, we rectify crystal $\mathcal{B}_{B_3}(\theta)$ to become a right-angled triangle with vertices 12, $2\bar{2}$ and $\bar{2}\bar{1}$. Again the arrows on catheti $[12, 2\bar{2}]$ are congruent to the arrows on the segment $[2, \bar{1}]$ of $\mathcal{B}_{B_3}(\omega_1)$, and the arrows on catheti $[2\bar{2}, \bar{2}\bar{1}]$ are congruent to the arrows on the segment $[1, \bar{2}]$ of $\mathcal{B}_{B_3}(\omega_1)$. In general, for $l \ge 3$, we have the B_l crystal $\mathcal{B}_{B_l}(\omega_1)$ for the vector representation (see Figure 11) and the tensor product of crystals $\mathcal{B}_{B_l}(\omega_1) \otimes \mathcal{B}_{B_l}(\omega_1)$ is the union $\mathcal{B}_{B_l}(2\omega_1) \cup \mathcal{B}_{B_l}(\theta) \cup \mathcal{B}_{B_l}(0)$ of B_l-crystals. By translating the diagonal points diagonally down and to the left, we rectify crystal $\mathcal{B}_{B_l}(\theta)$ to become a right-angled triangle with vertices 12, $2\bar{2}$ and $\bar{2}\bar{1}$. The arrows on catheti $[12, 2\bar{2}]$ are congruent to the arrows on the segment $[2, \bar{1}]$ of $\mathcal{B}_{B_l}(\omega_1)$, and the arrows on catheti $[2\bar{2}, \bar{2}\bar{1}]$ are congruent to the arrows on the segment $[1, \bar{2}]$ of $\mathcal{B}_{B_l}(\omega_1)$.

4.3. **Weighted crystals $\mathcal{B}_{B_l}(2\omega_1)$ and $\mathcal{B}_{B_l}(\theta)$ for B_l, $l \ge 2$.** The crystal $\mathcal{B}_{B_l}(2\omega_1)$ parametrizes a weight basis of the irreducible representation $L_{B_l}(2\omega_1)$ of the simple Lie algebra $\mathfrak{g} = \mathfrak{g}(B_l)$ of type B_l with the highest weight vector $11 \in \mathcal{B}_{B_l}(\omega_1) \otimes \mathcal{B}_{B_l}(\omega_1)$ (see Figures 8 and 10 for $l = 2$ and $l = 3$). This crystal is a right-angled triangle with vertices 11, $\bar{1}1$ and $\bar{1}\bar{1}$. Note that the midpoint on hypotenuse is missing (i.e. $00 \in \mathcal{B}_{B_l}(\theta)$). The arrows on both catheti are congruent to the arrows on $\mathcal{B}_{B_l}(\omega_1)$.

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} 0 \xrightarrow{3} \bar{3} \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

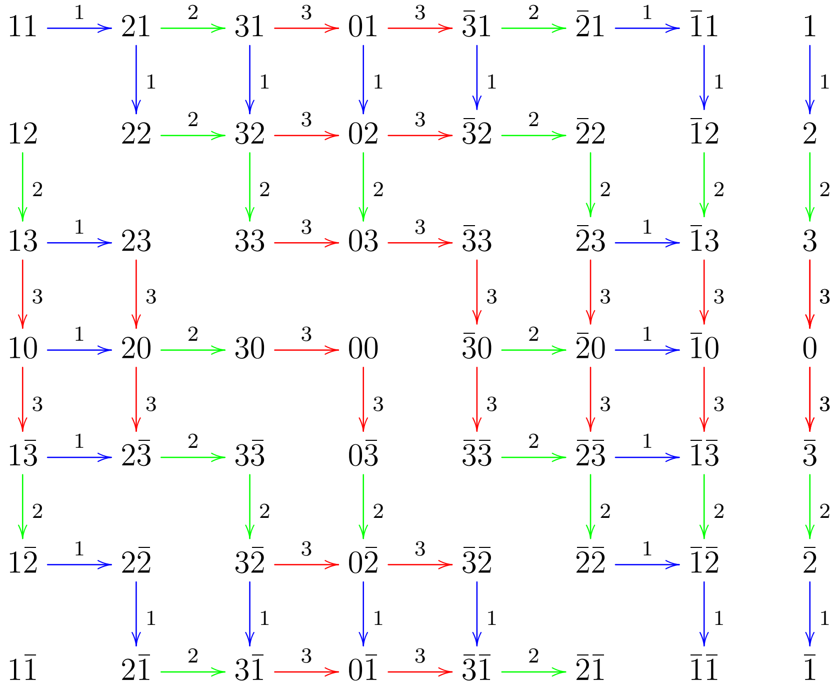

FIGURE 10. The crystal of the tensor square of vector representation for B_3

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \qquad \cdots \xrightarrow{l-1} l \xrightarrow{l} 0 \xrightarrow{l} \bar{l} \xrightarrow{l-1} \cdots \qquad \cdots \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

FIGURE 11. The crystal of the vector representation for B_l

5. Arrays of negative root vectors for $C_l^{(1)}$, $D_{l+1}^{(2)}$ and $A_{2l}^{(2)}$

5.1. **The array of negative root vectors for $C_l^{(1)}$.** Since the crystal $\mathcal{B}_{C_l}(\lambda)$ parametrizes a weight basis of the $\mathfrak{g}(C_l)$-module $L_{C_l}(\lambda)$, from (2.1) we get (a parametrization of) a weight basis of $\mathfrak{g}(C_l^{(1)})$

$$\{c, d\} \cup \mathcal{B}_{C_l^{(1)}}, \quad \text{where} \quad \mathcal{B}_{C_l^{(1)}} = \bigcup_{j \in \mathbb{Z}} \mathcal{B}_{C_l}(\theta) \otimes t^j. \tag{5.1}$$

For $b \in \mathcal{B}_{C_l}(\theta)$ and $i \in \mathbb{Z}$, set $b_i = b \otimes t^i$. Note that this is a weight vector in the affine Lie algebra $\mathfrak{g}(C_l^{(1)})$ of weight

$$\mathrm{wt}(b \otimes t^i) = \mathrm{wt}_{\mathfrak{h}}(b) + i\delta.$$

In each triangle $\mathcal{B}_{C_l}(\theta) \subset \mathcal{B}_{C_l}(\omega_1) \otimes \mathcal{B}_{C_l}(\omega_1)$ (see Figures 3 and 5 for $l = 2$ and $l = 3$), the $\mathfrak{h}$-weight changes by a negative simple root $-\alpha_i$, $i \in \{1, \ldots, l\}$, when passing from one column to the next, or from one row to the next, according to the sequence of arrows in the crystal of the vector representation in Figure 6. We can "glue" the triangles $\mathcal{B}_{C_l}(\theta) \otimes t^j$ in $\mathcal{B}_{C_l^{(1)}}$ in such a way that this rule also holds for the negative simple root

$$-\alpha_0 = \theta - \delta = 2\epsilon_1 - \delta,$$

where "gluing" means that we place together the catheti of the triangles $\mathcal{B}_{C_2}(\theta) \otimes t^{i+1}$ and $\mathcal{B}_{C_2}(\theta) \otimes t^i$ along the congruent arrows (see Figures 12 and 13 for $l = 2$ and Figure 14 in general). In this way, $\mathcal{B}_{C_l^{(1)}}$ obtains the structure of a crystal graph (i.e. weighted oriented graph), where dashed 0-arrows represent the action of the Kashiwara operator $\tilde{f}_0$ (or Chevalley's generator f_0).

Remark 5.1. *In the theory of crystal bases, the previous construction is called the affinization of $\mathcal{B}_{C_l}$ with the notation $\mathcal{B}_{C_l^{(1)}} = \bar{\mathcal{B}}_{C_l} = \mathcal{B}_{C_l}^{\mathit{aff}}$ (cf. [HK]). Since the notion of a crystal has many layers, from a simple combinatorial notion of colored directed graph to the notion of crystal basis for a quantum group $U_q(\mathfrak{g})$, for $\mathcal{B}_{C_l}$ we prefer the term* the array of root vectors for $C_l^{(1)}$ *or* the arrangement of root vectors for $C_l^{(1)}$. *For us, $\mathcal{B}_{C_l^{(1)}}$ is a part of the basis $\{c, d\} \cup \mathcal{B}_{C_l^{(1)}}$ of $\mathfrak{g}(C_l^{(1)})$ and the i-arrows, $i = 0, 1, \ldots, l$, indicate how the action of Chevalley's generators f_i changes the weight of a basis element, i.e.*

$$\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i, \qquad i = 0, 1, \ldots, l, \quad b \in \mathcal{B}_{C_l^{(1)}}.$$

$$\begin{tikzcd}
\bar{1}1_{i+1} \arrow[r, "1"] \arrow[d, dashed, "0"] & \bar{1}2_{i+1} \arrow[r, "2"] \arrow[d, dashed, "0"] & \bar{1}\bar{2}_{i+1} \arrow[r, "1"] \arrow[d, dashed, "0"] & \bar{1}\bar{1}_{i+1} \arrow[d, dashed, "0"] \\
11_i \arrow[r, "1"] & 21_i \arrow[r, "2"] & \bar{2}1_i \arrow[r, "1"] & \bar{1}1_i
\end{tikzcd}$$

FIGURE 12. Gluing the catheti of triangles in $\mathcal{B}_{C_2^{(1)}}$

Remark 5.2. *We shall usually write the array $\mathcal{B}_{C_l^{(1)}}$ rotated by $\pi/4$. We shall mainly write only negative root vectors $\mathcal{B}^-_{C_l^{(1)}}$, starting from the left column with $f_0 = 11 \otimes t^{-1}$, $f_1 = \bar{1}2 \otimes t^0$, ..., $f_l = \overline{l}l \otimes t^0$ (for $l = 2$ see Figure 15), with f_0 in the bottom row. Finally, for our purposes, it is not necessary to specify precisely the elements $b_i \in \mathcal{B}_{C_l^{(1)}}$, as all the information needed is encoded in the "colored" arrows and "positions of triangles" $\mathcal{B}_{C_l}(\theta) \otimes t^i$ (for $l = 2$ see Figure 16 where the positions of the triangles are denoted by "circles" and "bullets").*

Proposition 5.3. *The* array of negative root vectors for $C_l^{(1)}$, *denoted by $\mathcal{B}^-_{C_l^{(1)}}$, is a colored directed graph. Its nodes, which represent the basis vectors of $\mathfrak{g}(C_l^{(1)})^-$, are organized into $2l + 1$ rows and two sequences of diagonals with $2l + 1$ (or fewer) nodes. Its arrows indicate the action of Chevalley's generators $f_0, f_1, \ldots, f_l$ on the negative root subspaces and they are colored by $0, 1, \ldots, l$, respectively. Removing the arrows of color 0, which correspond to the action of $f_0 = 11 \otimes t^{-1}$, the graph decomposes into an infinite union of connected subgraphs. We shall refer to these subgraphs as* triangles. *The first triangle, positioned in the upper left corner, corresponds to the root vector basis of l^2-dimensional Lie subalgebra $\mathfrak{n}_- \otimes t^0 \cong \mathfrak{n}_-$ of the simple Lie algebra $\mathfrak{g}(C_l)$ of type C_l. The remaining triangles are crystal graphs $\mathcal{B}_{C_l}(\theta)$ of the adjoint representation of $\mathfrak{g}(C_l)$ given over $L_{C_l}(\theta) \otimes t^i$ with $i < 0$. The weights of nodes in $\mathcal{B}^-_{C_l^{(1)}}$ are periodic with period δ. We place the Chevalley generator f_0 in the bottom row.*

The main property of $\mathcal{B}^-_{C_l^{(1)}}$ is that the weights of the corresponding points on two adjacent diagonals differ by $-\alpha_i$ if there is an i-arrow between these two diagonals. The sequence of arrows between diagonals is determined by the sequence of arrows in $\mathcal{B}_{C_l}(\omega_1)$.

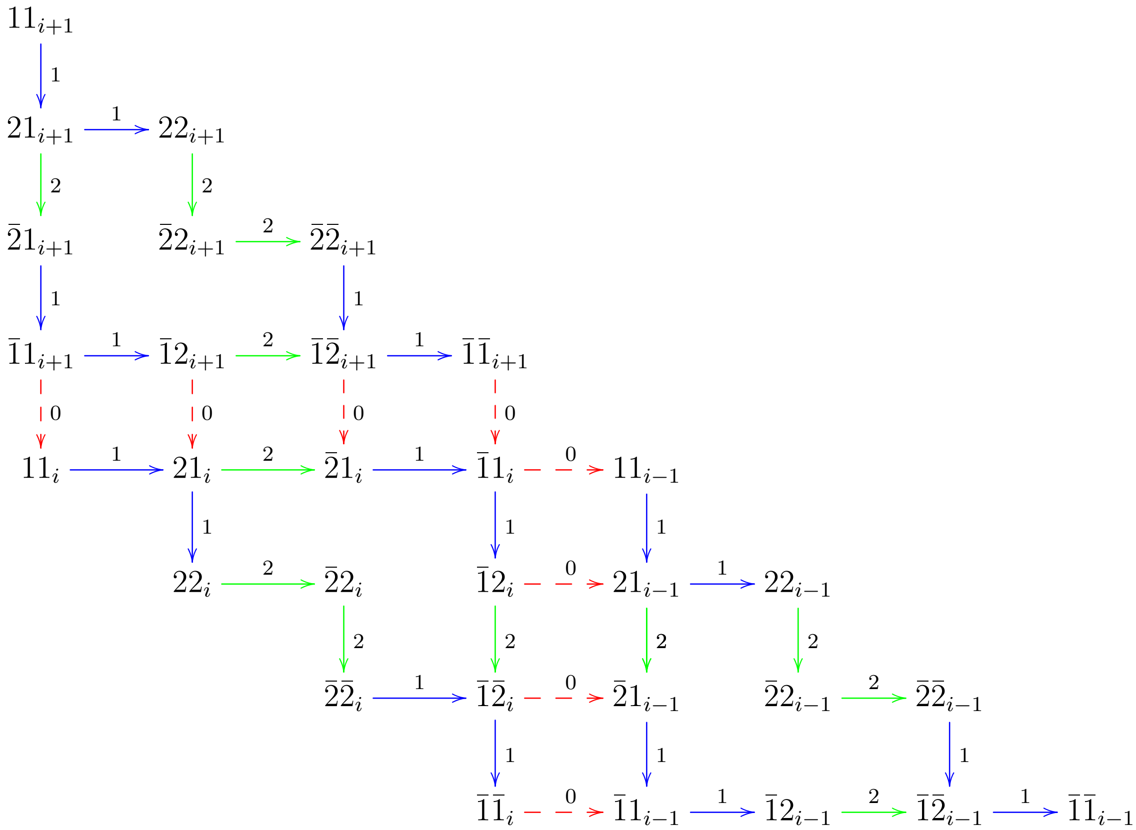

FIGURE 13. Gluing triangles in $\mathcal{B}_{C_2^{(1)}}$

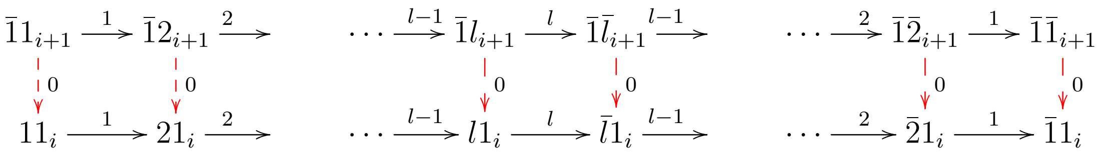

FIGURE 14. Gluing the catheti of triangles $\mathcal{B}_{C_l}(\theta) \otimes t^{i+1}$ and $\mathcal{B}_{C_l}(\theta) \otimes t^i$

Example 5.4. *The array $\mathcal{B}^-_{C_2^{(1)}}$ is given in Figure 15. The triangle with vertices $\bar{2}\bar{2}_0$, $\bar{1}\bar{2}_0$ and $\bar{1}\bar{1}_0$ corresponds to the basis of the 4-dimensional Lie subalgebra $\mathfrak{n}_- \otimes t^0$ isomorphic to the nilpotent subalgebra $\mathfrak{n}_-$ of the Lie algebra of type C_2, while the remaining triangles $\mathcal{B}_{C_l}(\theta) \otimes t^i$, $i < 0$, possess catheti consisting of two 1-arrows and one 2-arrow, which represent the action of f_1 and f_2, respectively. Finally, the triangles are connected by the action of f_0 which is indicated by the dashed 0-arrows. The same array $\mathcal{B}^-_{C_2^{(1)}}$ without specified basis elements $b_i = b \otimes t^i$ is given in Figure 16.*

5.2. **The array of negative root vectors for $D_{l+1}^{(2)}$.** Since the crystal $\mathcal{B}_{B_l}(\lambda)$ parametrizes a weight basis of $\mathfrak{g}(B_l)$-module $L_{B_l}(\lambda)$, from (2.2) we get (a parametrization of) a weight basis of $\mathfrak{g}(D_{l+1}^{(2)})$:

$$\{c, d\} \cup \mathcal{B}_{D_{l+1}^{(2)}}, \quad \text{where} \quad \mathcal{B}_{D_{l+1}^{(2)}} = \bigcup_{j\in\mathbb{Z}} \mathcal{B}_{B_l}(\theta) \otimes t^{2j} \ \cup \ \bigcup_{j\in\mathbb{Z}} \mathcal{B}_{B_l}(\omega_1) \otimes t^{2j+1}. \tag{5.2}$$

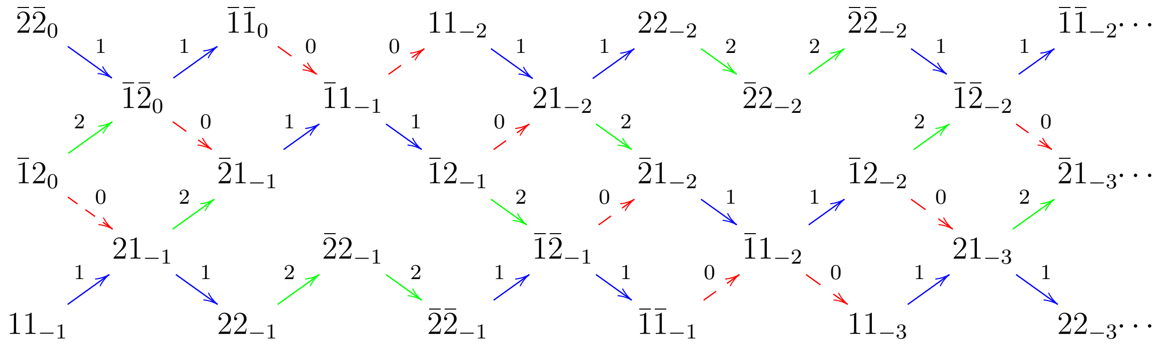

FIGURE 15. The arrangement of negative root vectors with elements b_i for $C_2^{(1)}$

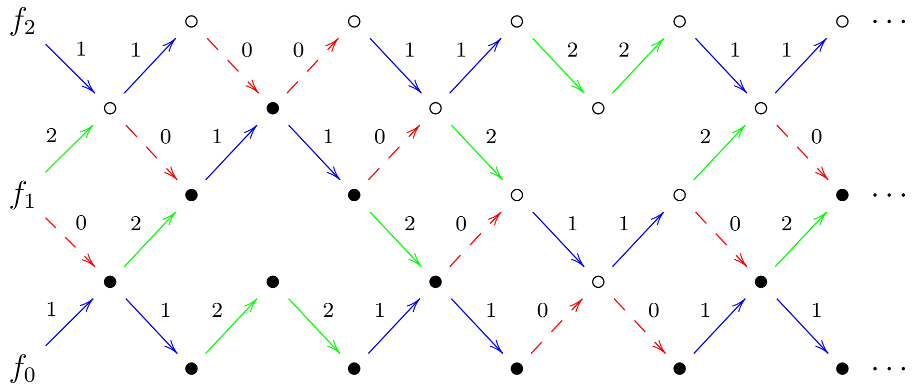

FIGURE 16. The arrangement of negative root vectors for $C_2^{(1)}$

For $i \in \mathbb{Z}$, set $b_{2i} = b \otimes t^{2i}$ for $b \in \mathcal{B}_{B_l}(\theta)$ and $b_{2i+1} = b \otimes t^{2i+1}$ for $b \in \mathcal{B}_{B_l}(\omega_1)$. Note that these are weight vectors in the affine Lie algebra $\mathfrak{g}(D_{l+1}^{(2)})$ of weight

$$\mathrm{wt}(b \otimes t^j) = \mathrm{wt}_{\mathfrak{h}}(b) + j\delta.$$

We can glue the catheti of the triangles $\mathcal{B}_{B_l}(\theta) \otimes t^{2i}$ to the lines $\mathcal{B}_{B_l}(\omega_1) \otimes t^{2i\pm1}$ in $\mathcal{B}_{D_{l+1}^{(2)}}$ along the congruent arrows; for $l = 2$, this is shown in Figure 17, and, in general, in Figure 18. The dashed 0-arrows represent the action of the Chevalley generator f_0 on the array of negative root vectors $\mathcal{B}_{D_{l+1}^{(2)}}$, which changes weights of the corresponding vectors by

$$-\alpha_0 = \omega_1 - \delta = \epsilon_1 - \delta.$$

We shall write the array of negative root vectors $\mathcal{B}^-_{D_{l+1}^{(2)}}$ rotated by $\pi/4$, starting with $f_0 = 1 \otimes t^{-1}$, $f_1 = 2\bar{1} \otimes t^0$, $\ldots$, $f_l = 0\bar{l} \otimes t^0$.

Proposition 5.5. *The* array of negative root vectors for $D_{l+1}^{(2)}$, *denoted by* $\mathcal{B}^-_{D_{l+1}^{(2)}}$, *is a colored directed graph. Its nodes, which represent the basis vectors of* $\mathfrak{g}(D_{l+1}^{(2)})^-$, *are organized into* $2l+1$ *rows and two sequences of diagonals with* $2l+1$ *(or fewer) nodes. Its arrows indicate the action of the Chevalley generators* $f_0, f_1, \ldots, f_l$ *on the negative root subspaces and they are colored by* $0, 1, \ldots, l$, *respectively. Removing the arrows of color* 0, *which correspond to the action of* $f_0 = 1 \otimes t^{-1}$, *the graph decomposes into an infinite union of connected subgraphs. We shall refer to these subgraphs as* triangles *and* diagonals. *The first triangle, positioned in the upper left corner, corresponds to the root*

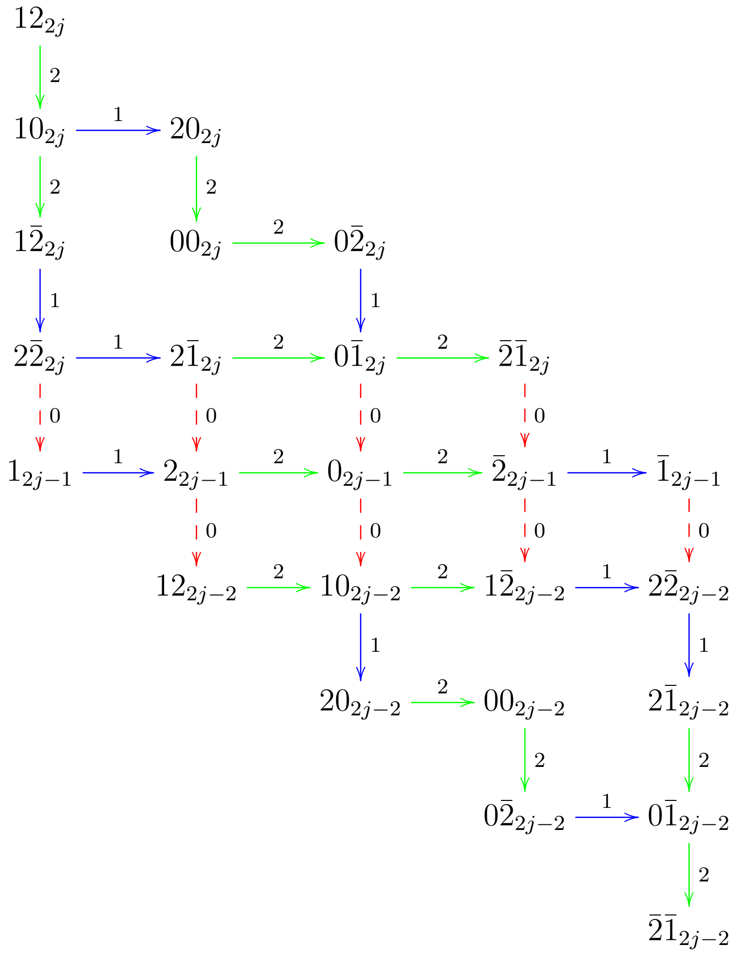

FIGURE 17. Gluing triangles to lines in $\mathcal{B}_{D_3^{(2)}}$

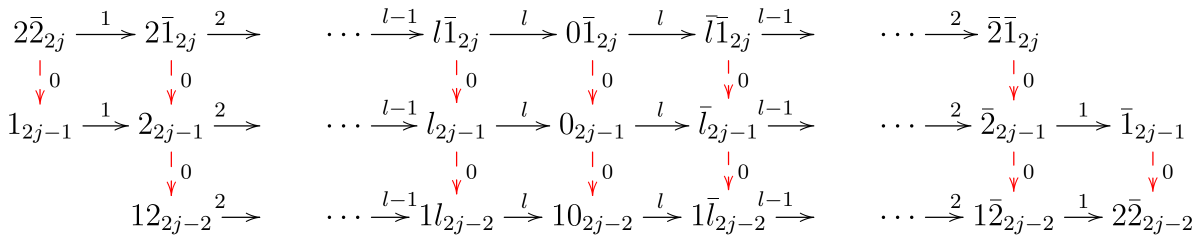

FIGURE 18. Gluing catheti of triangles to lines in $\mathcal{B}_{D_{l+1}^{(2)}}$

vector basis of the l^2-dimensional Lie subalgebra $\mathfrak{n}_-$ of the simple Lie algebra $\mathfrak{g}(B_l)$ of type B_l. The remaining triangles are crystal graphs $\mathcal{B}_{B_l}(\theta)$ of the adjoint representation of $\mathfrak{g}(B_l)$ given over $L_{B_l}(\theta) \otimes t^{2i}$ with $i < 0$. Finally, the diagonals, which consist of $2l+1$ nodes, are crystal graphs $\mathcal{B}_{B_l}(\omega_1)$ of the vector representation $L_{B_l}(\omega_1)$ given over $\mathbb{C}^{2l+1} \otimes t^{2i+1}$ with $i < 0$. The weights of nodes in $\mathcal{B}^-_{D_{l+1}^{(2)}}$ are periodic with period 2δ. We place the Chevalley generator f_0 in the bottom row.

The main property of $\mathcal{B}^-_{D_{l+1}^{(2)}}$ is that the weights of the corresponding nodes on two adjacent diagonals differ by $-\alpha_i$ if there is an i-arrow between these two diagonals. The sequence of arrows between diagonals is determined by the sequence of arrows in $\mathcal{B}_{B_l}(\omega_1)$.

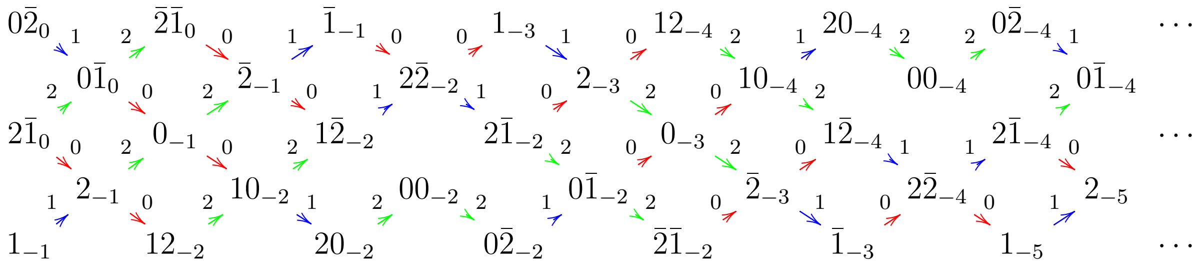

FIGURE 19. Arrangement of negative root vectors with elements b_i for $D_3^{(2)}$

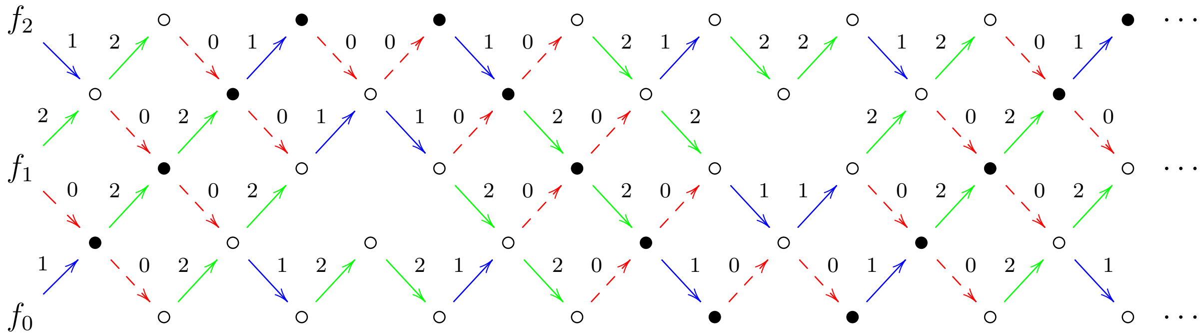

FIGURE 20. Arrangement of negative root vectors for $D_3^{(2)}$

Example 5.6. *The array $\mathcal{B}^-_{D_3^{(2)}}$ is given by Figure 19. The triangle with vertices $0\bar{2}_0$, $2\bar{1}_0$, $\bar{2}\bar{1}_0$ corresponds to the basis of the 4-dimensional Lie subalgebra $\mathfrak{n}_-$ of the simple Lie algebra of type B_2. The remaining triangles $\mathcal{B}_{B_2}(\theta)\otimes t^{2i}$, $i<0$, possess catheti consisting of two 2-arrows and one 1-arrow, which represent the action of f_2 and f_1, respectively. Finally, the diagonals $\mathcal{B}_{B_2}(\omega_1)\otimes t^{2i+1}$, $i<0$, consist of two 1-arrows, at the beginning and at the end, and two 2-arrows in the middle. The triangles and the diagonals are connected by the action of f_0 which is indicated by the dashed 0-arrows.*

As noted before, for our purposes it is not necessary to precisely specify the elements $b_i\in\mathcal{B}^-_{D_{l+1}^{(2)}}$; all the information needed is encoded in arrows and "positions of triangles $B(\theta)\otimes t^{2j}$ and lines $B(\omega_1)\otimes t^{2j+1}$" like in Figure 20.

5.3. **The array of negative root vectors for $A_{2l}^{(2)}$.** Since the crystal $\mathcal{B}_{B_l}(\lambda)$ parametrizes a weight basis of $\mathfrak{g}(B_l)$-module $L_{B_l}(\lambda)$, from (2.3) we get (a parametrization of) a weight basis of $\mathfrak{g}(A_{2l}^{(2)})$:

$$\{c,d\}\cup\mathcal{B}_{A_{2l}^{(2)}},\quad\text{where}\quad \mathcal{B}_{A_{2l}^{(2)}}=\bigcup_{j\in\mathbb{Z}}\mathcal{B}_{B_l}(\theta)\otimes t^{2j}\ \cup\ \bigcup_{j\in\mathbb{Z}}\mathcal{B}_{B_l}(2\omega_1)\otimes t^{2j+1}. \tag{5.3}$$

For $i\in\mathbb{Z}$, set $b_{2i}=b\otimes t^{2i}$ for $b\in\mathcal{B}_{B_l}(\theta)$ and $b_{2i+1}=b\otimes t^{2i+1}$ for $b\in\mathcal{B}_{B_l}(2\omega_1)$. Note that these are weight vectors in the affine Lie algebra $\mathfrak{g}(A_{2l}^{(2)})$ of weight

$$\mathrm{wt}(b\otimes t^j)=\mathrm{wt}_{\mathfrak{h}}(b)+j\delta.$$

We can glue the catheti of the small triangles $\mathcal{B}_{B_l}(\theta)\otimes t^{2i}$ to the catheti of big triangles $\mathcal{B}_{B_l}(2\omega_1)\otimes t^{2i\pm1}$ in $\mathcal{B}_{A_{2l}^{(2)}}$ along the congruent arrows (see the small and big triangles in Figures 8 and 10 for $l=2$ and $l=3$); for $l=2$, this is shown in Figure 21, and, in general, in Figure 22. The dashed 0-arrows represent the action of the Chevalley generator f_0 on the array of negative root vectors $\mathcal{B}_{A_{2l}^{(2)}}$, which changes weights of the corresponding vectors by

$$-\alpha_0 = 2\omega_1 - \delta = 2\epsilon_1 - \delta.$$

We shall usually write the array of negative root vectors $\mathcal{B}^-_{A_{2l}^{(2)}}$ rotated by $\pi/4$, starting with $f_0 = 0\bar{l} \otimes t^0$, $f_1 = l\overline{l-1} \otimes t^0$, $\ldots$, $f_{l-1} = 2\bar{1} \otimes t^0$ and $f_l = 11 \otimes t^{-1}$.

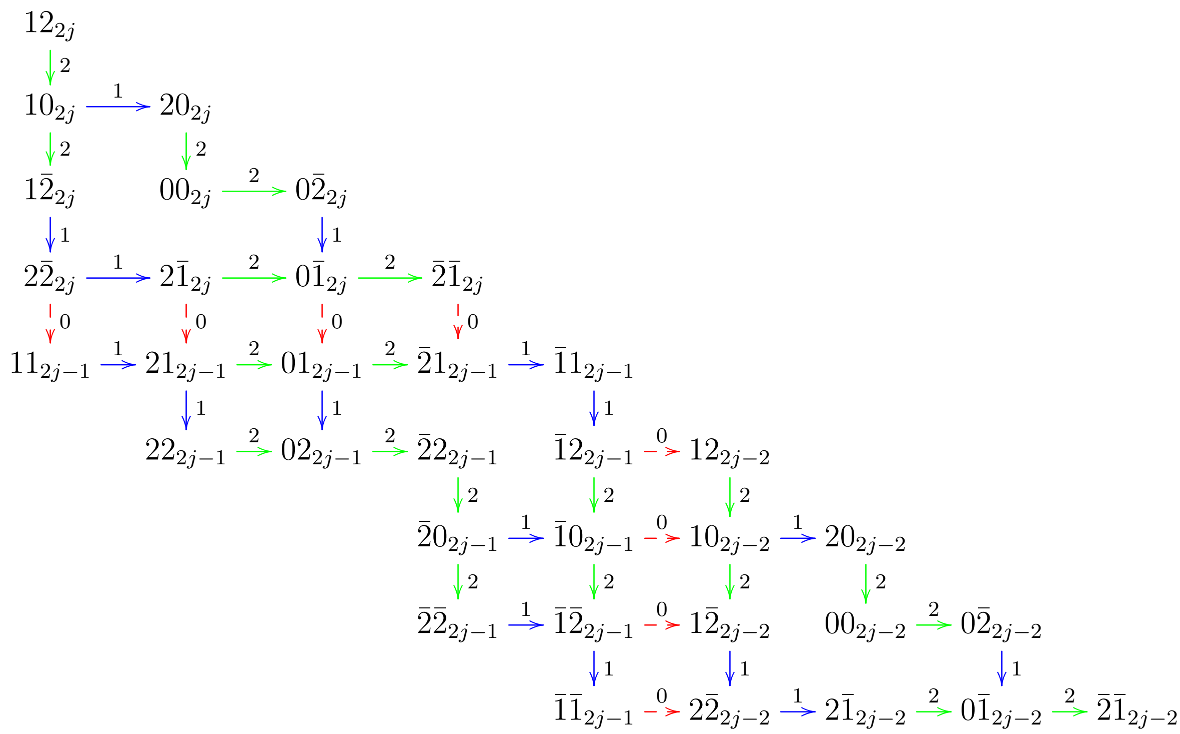

FIGURE 21. Gluing small and big triangles in $\mathcal{B}_{A_4^{(2)}}$

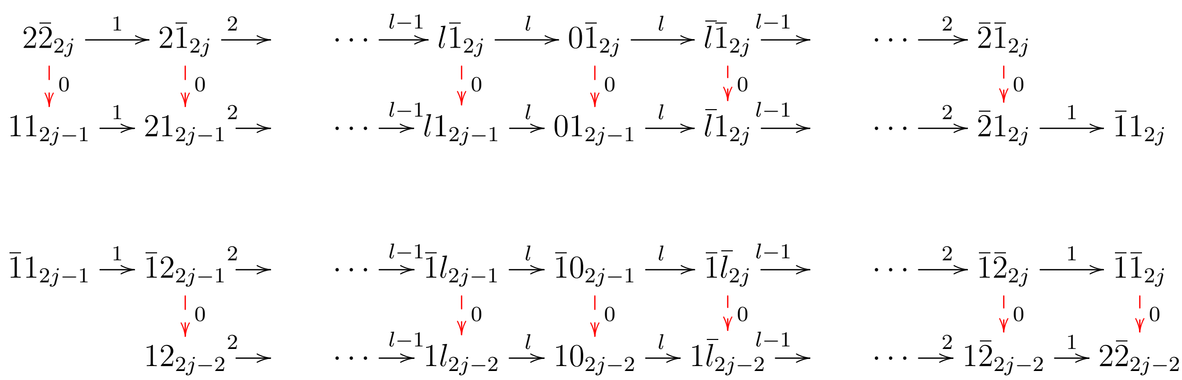

FIGURE 22. Gluing catheti of small and big triangles in $\mathcal{B}_{A_{2l}^{(2)}}$

Proposition 5.7. *The* array of negative root vectors for $A_{2l}^{(2)}$, *denoted by* $\mathcal{B}^-_{A_{2l}^{(2)}}$ *is a colored directed graph. Its nodes, which represent the basis vectors of* $\mathfrak{g}(A_{2l}^{(2)})^-$, *are organized into* $2l+1$ *rows and two sequences of diagonals with* $2l+1$ *(or fewer) nodes. Its arrows indicate the action of the Chevalley generators* $f_0, f_1, \ldots, f_l$ *on the negative root subspaces and they* *are colored by* $0, 1, \ldots, l$*, respectively. Removing the arrows of color* l*, which correspond to the action of* $f_l = 11 \otimes t^{-1}$*, the graph decomposes into an infinite union of connected subgraphs which we refer to as* triangles*. The first triangle, positioned in the upper left corner, corresponds to the root vector basis of the* l^2*-dimensional Lie subalgebra* $\mathfrak{n}_-$ *of the simple Lie algebra* $\mathfrak{g}(B_l)$ *of type* B_l*. The remaining triangles alternate between the crystals* $\mathcal{B}_{B_l}(2\omega_1)$ *of* $\mathfrak{g}(B_l)$ *of the representation given over* $L_{B_1}(2\omega_1) \otimes t^{2i+1}$ *with* $i < 0$*, and the crystals* $\mathcal{B}_{B_l}(\theta)$ *of the adjoint representation given over* $L_{B_1}(\theta) \otimes t^{2i}$ *with* $i < 0$*. We shall refer to these subgraphs as* big triangles *and* small triangles*. The hypotenuses of the big triangles miss the midpoints. The weights of nodes in* $\mathcal{B}^-_{A^{(2)}_{2l}}$ *are periodic with period* 2δ*. We place the hypotenuses of the big triangles on the bottom row; hence the Chevalley generator* f_0 *is in the top row.*

The main property of $\mathcal{B}^-_{A^{(2)}_{2l}}$ *is that the weights of the corresponding nodes on two adjacent diagonals differ by* $-\alpha_i$ *if there is an* i*-arrow between these two diagonals. The sequence of arrows between diagonals is determined by the sequence of arrows in* $\mathcal{B}_{B_l}(\omega_1)$*.*

Example 5.8. *The array* $\mathcal{B}^-_{A^{(2)}_{4}}$ *is given by Figure 23, and by Figure 24 without specifying vectors* b_i*.*

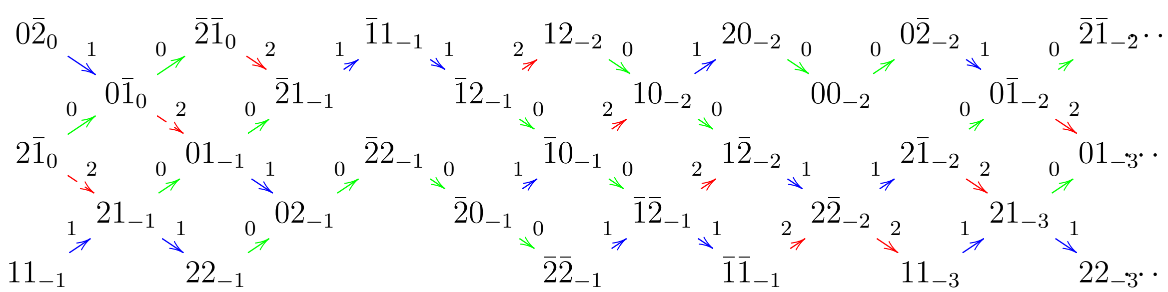

FIGURE 23. Arrangement of negative root vectors with elements b_i for $A^{(2)}_4$

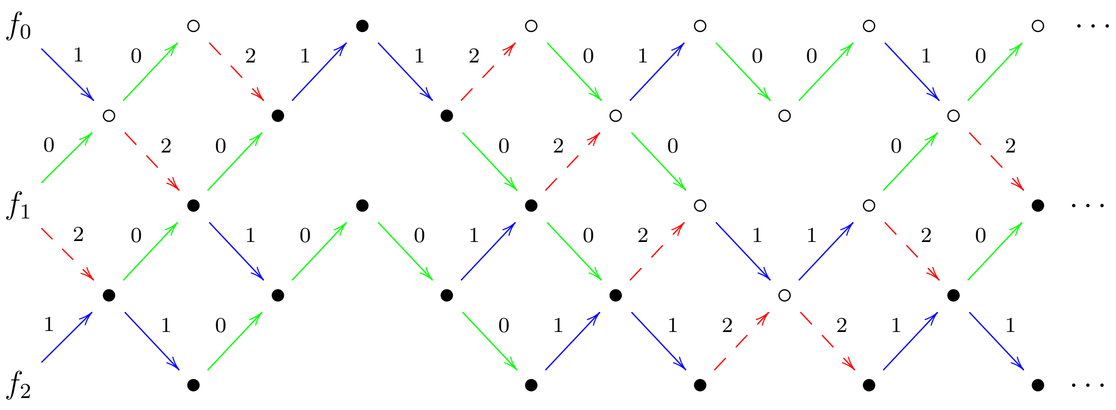

FIGURE 24. Arrangement of negative root vectors for $A^{(2)}_4$

6. Specialized arrays of negative root vectors

6.1. **Specialized arrays of negative root vectors for** $C^{(1)}_l$**.** Let $s_0, s_1, \ldots, s_l \in \mathbb{N}$ be fixed. By assigning degrees

$$\deg f_0 = s_0, \ \deg f_1 = s_1, \ \ldots, \ \deg f_l = s_l$$

to the arrows in the array $\mathcal{B}^-_{C_l^{(1)}}$ we obtain an array $\mathcal{N}^{(s_0,s_1,\dots,s_l)}_{C_l^{(1)}}$ of positive integers, which we call the $(s_0,s_1,\dots,s_l)$*-specialized weighted array for* $C_l^{(1)}$. The array $\mathcal{N}^{(s_0,s_1,\dots,s_l)}_{C_l^{(1)}}$ is periodic with period $s_0+2\sum_{i=1}^{l-1}s_i+s_l$.

For $s=(1,\dots,1)$ we have the so called *principal specialization*. In the principally specialized array of negative root vectors the labels of nodes increase by one when we move one place to the upper right or lower right. Hence we have the following lemma:

Lemma 6.1. *For $l\geq 1$, the array $\mathcal{N}^{(1,\dots,1)}_{C_l^{(1)}}$ consists of l copies of positive integers and one copy of odd positive integers. Equivalently,*

$$Q(1,\dots,1;C_l^{(1)})=\prod_{i\in\mathbb{N}}(1-q^i)^l\prod_{j\in\mathbb{N}}(1-q^{2j-1}).$$

Example 6.2. *Figure 25 represents the array $\mathcal{N}^{(1,1,1,1,1)}_{C_4^{(1)}}$. Note that the labels corresponding to the different adjoint triangles are written in italic and bold, so that they can be immediately detected in the image.*

$$\begin{array}{ccccccccccccccccc}
\mathit{1} & & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathit{9} & & \mathit{11} & & \mathit{13} & & \mathit{15} & & \\
 & \mathit{2} & & \mathit{4} & & \mathit{6} & & \mathbf{8} & & \mathit{10} & & \mathit{12} & & \mathit{14} & & \mathit{16} & \\
\mathit{1} & & \mathit{3} & & \mathit{5} & & \mathbf{7} & & \mathbf{9} & & \mathit{11} & & \mathit{13} & & \mathit{15} & & \\
 & \mathit{2} & & \mathit{4} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathit{12} & & \mathit{14} & & \mathit{16} & \\
\mathit{1} & & \mathit{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathit{13} & & \mathit{15} & & \dots \\
 & \mathit{2} & & \mathbf{4} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} & & \mathit{14} & & \mathit{16} & \\
\mathit{1} & & \mathbf{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathbf{13} & & \mathit{15} & & \\
 & \mathbf{2} & & \mathbf{4} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} & & \mathbf{14} & & \mathit{16} & \\
\mathbf{1} & & \mathbf{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathbf{13} & & \mathbf{15} & &
\end{array}$$

FIGURE 25. Array $\mathcal{N}^{(1,1,1,1,1)}_{C_4^{(1)}}$

Example 6.3. *The array $\mathcal{N}^{(2,1,1,1,2)}_{C_4^{(1)}}$ consists of three copies of positive integers, one copy of even positive integers and one copy of positive integers congruent to 5 modulo 10. We shall now demonstrate how such a conclusion can be easily obtained examining the array in the case $l=4$. First, note that the array $\mathcal{N}^{(2,1,1,1,2)}_{C_4^{(1)}}$ is periodic with period 10, so it is enough to consider the first two triangles corresponding to $\mathfrak{n}_-\otimes t^0$ and $\mathfrak{n}_+\otimes t^{-1}$ (see Figure 26). Then, as in Figure 25, we divide the array $\mathcal{N}^{(2,1,1,1,2)}_{C_4^{(1)}}$ into triangles; see Figure 26. Next, suppose that all triangles with italic labels slide one place down the catheti of the adjacent triangles with bold labels; see Figure 27. Finally, in Figure 27, we notice three copies of the positive integers (we indicate their positions in Figure 28 by the symbols ①, ②, ③), one copy of the even positive integers (indicated by Ⓔ in Figure 28) and one copy of the positive integers congruent to 5 modulo 10 (indicated by ⑤ in Figure 28).*

The previous argument can be generalized to the case of arbitrary $l\geq 3$:

Lemma 6.4. *The array $\mathcal{N}^{(2,1,\dots,1,2)}_{C_l^{(1)}}$ consists of $l-1$ copies of the positive integers, one copy of the even positive integers and one copy of the positive integers congruent to $l+1$ modulo $2(l+1)$. Equivalently,*

$$Q(2,1,\dots,1,2;C_l^{(1)})=\prod_{i\in\mathbb{N}}(1-q^i)^{l-1}\prod_{j\in\mathbb{N}}(1-q^{2j})\prod_{r\equiv(l+1)\mod 2(l+1)}(1-q^r).$$

$$\begin{array}{ccccccccc}
\mathit{2} & & \mathit{4} & & \mathit{6} & & \mathit{8} & & \\
 & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathbf{10} & \\
\mathit{1} & & \mathit{4} & & \mathit{6} & & \mathbf{9} & & \\
 & \mathit{2} & & \mathit{5} & & \mathbf{8} & & \mathbf{10} & \\
\mathit{1} & & \mathit{3} & & \mathbf{7} & & \mathbf{9} & & \cdots \\
 & \mathit{2} & & \mathbf{5} & & \mathbf{8} & & \mathbf{10} & \\
\mathit{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{9} & & \\
 & \mathbf{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{10} & \\
\mathbf{2} & & \mathbf{4} & & \mathbf{6} & & \mathbf{8} & &
\end{array}$$

FIGURE 26. Array $\mathcal{N}^{(2,1,1,1,2)}_{C_4^{(1)}}$

$$\begin{array}{cccccccccc}
\mathit{2} & & \mathit{4} & & \mathit{6} & & \mathit{8} & & \mathbf{10} & \\
 & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathbf{9} & & \\
\mathit{1} & & \mathit{4} & & \mathit{6} & & \mathbf{8} & & \mathbf{10} & \\
 & \mathit{2} & & \mathit{5} & & \mathbf{7} & & \mathbf{9} & & \\
\mathit{1} & & \mathit{3} & & \mathbf{5} & & \mathbf{8} & & \mathbf{10} & \cdots \\
 & \mathit{2} & & \mathbf{4} & & \mathbf{6} & & \mathbf{9} & & \\
\mathit{1} & & \mathbf{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{10} & \\
 & \mathbf{2} & & \mathbf{4} & & \mathbf{6} & & \mathbf{8} & &
\end{array}$$

FIGURE 27. Translated triangles in the array $\mathcal{N}^{(2,1,1,1,2)}_{C_4^{(1)}}$

$$\begin{array}{cccccccccc}
\textcircled{E} & & \textcircled{E} & & \textcircled{E} & & \textcircled{E} & & \textcircled{E} & \\
 & \textcircled{3} & & \textcircled{2} & & \textcircled{1} & & \textcircled{3} & & \\
\textcircled{1} & & \textcircled{3} & & \textcircled{2} & & \textcircled{1} & & \textcircled{3} & \\
 & \textcircled{1} & & \textcircled{3} & & \textcircled{2} & & \textcircled{1} & & \\
\textcircled{2} & & \textcircled{1} & & \textcircled{5} & & \textcircled{2} & & \textcircled{1} & \cdots \\
 & \textcircled{2} & & \textcircled{1} & & \textcircled{3} & & \textcircled{2} & & \\
\textcircled{3} & & \textcircled{2} & & \textcircled{1} & & \textcircled{3} & & \textcircled{2} & \\
 & \textcircled{3} & & \textcircled{2} & & \textcircled{1} & & \textcircled{3} & &
\end{array}$$

FIGURE 28. Three copies of the positive integers ($\textcircled{1}$, $\textcircled{2}$, $\textcircled{3}$), one copy of the even positive integers ($\textcircled{E}$) and one copy of the positive integers congruent to 5 modulo 10 ($\textcircled{5}$) in the array $\mathcal{N}^{(2,1,1,1,2)}_{C_4^{(1)}}$.

Lemma 6.5. *The array $\mathcal{N}^{(2,1,\ldots,1)}_{C_l^{(1)}}$ consists of l copies of the positive integers. Equivalently,*

$$Q(2,1,\ldots,1;C_l^{(1)}) = \prod_{i\in\mathbb{N}}(1-q^i)^l.$$

Proof. In each triangle the labels of nodes increase by one when we move one place to the upper right or lower right. Moreover, when we move one place from one triangle to another one, the labels increase by two. Figure 29 shows three adjacent triangles in the case $l = 4$. Now we can argue as before by sliding the upper triangles one place down the catheti of the adjacent lower triangles. □

Remark 6.6. *It is worth noting that the array $\mathcal{N}^{(1,1,\ldots,1,2)}_{C_l^{(1)}}$ is the transpose of the array $\mathcal{N}^{(2,1,\ldots,1,1)}_{C_l^{(1)}}$; see Figure 30 for $l = 4$.*

Example 6.7. *The labels of nodes in the specialized array $\mathcal{N}^{(4,3,2,3,4)}_{C_4^{(1)}}$ are the positive integers congruent to*

$$0,0,0,0,\pm2,\pm3,\pm3,\pm4,\pm4,\pm5,\pm5,\pm7,\pm7,\pm8,\pm9,\pm9,\pm10,\pm10,\pm12,\pm12$$

modulo 24; see Figure 31.

$$
\begin{array}{ccccccccccccccccc}
\mathit{1} & & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathit{11} & & \mathit{13} & & \mathit{15} & & \mathit{17} & & \\
& \mathit{2} & & \mathit{4} & & \mathit{6} & & \mathbf{9} & & \mathit{12} & & \mathit{14} & & \mathit{16} & & \mathit{18} & \\
\mathit{1} & & \mathit{3} & & \mathit{5} & & \mathbf{8} & & \mathbf{10} & & \mathit{13} & & \mathit{15} & & \mathit{17} & & \\
& \mathit{2} & & \mathit{4} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathit{14} & & \mathit{16} & & \mathit{18} & \\
\mathit{1} & & \mathit{3} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} & & \mathit{15} & & \mathit{17} & & \dots \\
& \mathit{2} & & \mathbf{5} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathbf{13} & & \mathit{16} & & \mathit{18} & \\
\mathit{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} & & \mathbf{14} & & \mathit{17} & & \\
& \mathbf{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathbf{13} & & \mathbf{15} & & \mathit{18} & \\
\mathbf{2} & & \mathbf{4} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} & & \mathbf{14} & & \mathbf{16} & &
\end{array}
$$

FIGURE 29. Array $\mathcal{N}_{C_4^{(1)}}^{(2,1,1,1,1)}$

$$
\begin{array}{ccccccccccccccccc}
\mathit{2} & & \mathit{4} & & \mathit{6} & & \mathit{8} & & \mathit{10} & & \mathit{12} & & \mathit{14} & & \mathit{16} & & \\
& \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathbf{9} & & \mathit{11} & & \mathit{13} & & \mathit{15} & & \mathit{18} & \\
\mathit{1} & & \mathit{4} & & \mathit{6} & & \mathbf{8} & & \mathbf{10} & & \mathit{12} & & \mathit{14} & & \mathit{17} & & \\
& \mathit{2} & & \mathit{5} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathit{13} & & \mathit{16} & & \mathit{18} & \\
\mathit{1} & & \mathit{3} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} & & \mathit{15} & & \mathit{17} & & \dots \\
& \mathit{2} & & \mathbf{4} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \mathbf{14} & & \mathit{16} & & \mathit{18} & \\
\mathit{1} & & \mathbf{3} & & \mathbf{5} & & \mathbf{8} & & \mathbf{10} & & \mathbf{13} & & \mathbf{15} & & \mathit{17} & & \\
& \mathbf{2} & & \mathbf{4} & & \mathbf{6} & & \mathbf{9} & & \mathbf{12} & & \mathbf{14} & & \mathbf{16} & & \mathit{18} & \\
\mathbf{1} & & \mathbf{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{11} & & \mathbf{13} & & \mathbf{15} & & \mathbf{17} & &
\end{array}
$$

FIGURE 30. Array $\mathcal{N}_{C_4^{(1)}}^{(1,1,1,1,2)}$

$$
\begin{array}{ccccccccccccccccc}
\mathit{4} & & \mathit{10} & & \mathit{14} & & \mathit{20} & & \mathit{28} & & \mathit{34} & & \mathit{38} & & \mathit{44} & & \\
& \mathit{7} & & \mathit{12} & & \mathit{17} & & \mathbf{24} & & \mathit{31} & & \mathit{36} & & \mathit{41} & & \mathit{48} & \\
\mathit{3} & & \mathit{9} & & \mathit{15} & & \mathbf{21} & & \mathbf{27} & & \mathit{33} & & \mathit{39} & & \mathit{45} & & \\
& \mathit{5} & & \mathit{12} & & \mathbf{19} & & \mathbf{24} & & \mathbf{29} & & \mathit{36} & & \mathit{43} & & \mathit{48} & \\
\mathit{2} & & \mathit{8} & & \mathbf{16} & & \mathbf{22} & & \mathbf{26} & & \mathbf{32} & & \mathit{40} & & \mathit{46} & & \dots \\
& \mathit{5} & & \mathbf{12} & & \mathbf{19} & & \mathbf{24} & & \mathbf{29} & & \mathbf{36} & & \mathit{43} & & \mathit{48} & \\
\mathit{3} & & \mathbf{9} & & \mathbf{15} & & \mathbf{21} & & \mathbf{27} & & \mathbf{33} & & \mathbf{39} & & \mathit{45} & & \\
& \mathbf{7} & & \mathbf{12} & & \mathbf{17} & & \mathbf{24} & & \mathbf{31} & & \mathbf{36} & & \mathbf{41} & & \mathit{48} & \\
\mathbf{4} & & \mathbf{10} & & \mathbf{14} & & \mathbf{20} & & \mathbf{28} & & \mathbf{34} & & \mathbf{38} & & \mathbf{44} & &
\end{array}
$$

FIGURE 31. Array $\mathcal{N}_{C_4^{(1)}}^{(4,3,2,3,4)}$

6.2. **Specialized arrays of negative roots for $D_{l+1}^{(2)}$.** As in the previous case, fix $s_0, s_1, \dots, s_l \in \mathbb{N}$ and then assign the degrees

$$\deg f_0 = s_0, \ \deg f_1 = s_1, \ \dots, \ \deg f_l = s_l$$

to arrows in the array $\mathcal{B}^-_{D_{l+1}^{(2)}}$. Thus, we obtain an array $\mathcal{N}_{D_{l+1}^{(2)}}^{(s_0,s_1,\dots,s_l)}$ of positive integers, which we call the $(s_0, s_1, \dots, s_l)$-*specialized weighted array for* $D_{l+1}^{(2)}$. The array $\mathcal{N}_{D_{l+1}^{(2)}}^{(s_0,s_1,\dots,s_l)}$ is periodic with period $2(s_0 + s_1 + \dots + s_l)$.

In the principally specialized array of negative root vectors the labels of nodes increase by one when we move one place to the upper right or lower right. Hence we have the following:

Lemma 6.8. *The array $\mathcal{N}_{D_{l+1}^{(2)}}^{(1,\ldots,1)}$ consists of l copies of the positive integers and one copy of the odd positive integers. Equivalently,*

$$Q(1,\ldots,1;D_{l+1}^{(2)})=\prod_{i\in\mathbb{N}}(1-q^i)^l\prod_{j\in\mathbb{N}}(1-q^{2j-1}).$$

Example 6.9. *In Figure 32 we have the array $\mathcal{N}_{D_5^{(2)}}^{(1,1,1,1,1)}$. Its triangles (resp. diagonals) are indicated by labels in italic (resp. bold).*

$$\begin{array}{cccccccccccccccccccccc}
\mathit{1} & & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathbf{9} & & \mathbf{11} & & \mathit{13} & & \mathit{15} & & \mathit{17} & & \mathit{19} & & & \\
 & \mathit{2} & & \mathit{4} & & \mathit{6} & & \mathbf{8} & & \mathit{10} & & \mathbf{12} & & \mathit{14} & & \mathit{16} & & \mathit{18} & & \mathit{20} & & \\
\mathit{1} & & \mathit{3} & & \mathit{5} & & \mathbf{7} & & \mathit{9} & & \mathit{11} & & \mathbf{13} & & \mathit{15} & & \mathit{17} & & \mathit{19} & & & \\
 & \mathit{2} & & \mathit{4} & & \mathbf{6} & & \mathit{8} & & \mathit{10} & & \mathit{12} & & \mathbf{14} & & \mathit{16} & & \mathit{18} & & \mathit{20} & & \\
\mathit{1} & & \mathit{3} & & \mathbf{5} & & \mathit{7} & & \mathit{9} & & \mathit{11} & & \mathit{13} & & \mathbf{15} & & \mathit{17} & & \mathit{19} & & & \ldots \\
 & \mathit{2} & & \mathbf{4} & & \mathit{6} & & \mathit{8} & & \mathit{10} & & \mathit{12} & & \mathit{14} & & \mathbf{16} & & \mathit{18} & & \mathit{20} & & \\
\mathit{1} & & \mathbf{3} & & \mathit{5} & & \mathit{7} & & \mathit{9} & & \mathit{11} & & \mathit{13} & & \mathit{15} & & \mathbf{17} & & \mathit{19} & & & \\
 & \mathbf{2} & & \mathit{4} & & \mathit{6} & & \mathit{8} & & \mathit{10} & & \mathit{12} & & \mathit{14} & & \mathit{16} & & \mathbf{18} & & \mathit{20} & & \\
\mathbf{1} & & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathit{9} & & \mathit{11} & & \mathit{13} & & \mathit{15} & & \mathit{17} & & \mathbf{19} & & &
\end{array}$$

FIGURE 32. Array $\mathcal{N}_{D_5^{(2)}}^{(1,1,1,1,1)}$

6.3. **Specialized arrays of negative root vectors for $A_{2l}^{(2)}$.** As before, for fixed integers $s_0,s_1,\ldots,s_l\in\mathbb{N}$ we assign the degrees

$$\deg f_0=s_0,\ \deg f_1=s_1,\ \ldots,\ \deg f_l=s_l$$

to the arrows in the array $\mathcal{B}_{A_{2l}^{(2)}}^-$, thus getting the array $\mathcal{N}_{A_{2l}^{(2)}}^{(s_0,s_1,\ldots,s_l)}$ of positive integers, which we call the $(s_0,s_1,\ldots,s_l)$*-specialized weighted array for* $A_{2l}^{(2)}$. It is periodic with period $2(s_l+2\sum_{i=0}^{l-1}s_i)$. By using the orientation-reversing automorphism of the Dynkin diagram of type $A_{2l}^{(2)}$ that sends the node i to $l-i$, we obtain a weighted crystal graph of type $\mathcal{B}_{A_{2l}^{(2)T}}^-$ and $(s_0,s_1,\ldots,s_l)$-specialized weighted crystal of negative roots for ${A_{2l}^{(2)}}^T$.

Lemma 6.10. *For $l\geq 1$, the array $\mathcal{N}_{A_{2l}^{(2)}}^{(1,\ldots,1)}$ consists of l copies of the positive integers and one copy of the odd positive integers which are not congruent to $2l+1$ modulo $2(2l+1)$. Equivalently,*

$$Q(1,\ldots,1;A_{2l}^{(2)})=\prod_{i\in\mathbb{N}}(1-q^i)^l\prod_{j\not\equiv l+1 \mod (2l+1)}(1-q^{2j-1}).$$

Example 6.11. *In Figure 33, we have the array $\mathcal{N}_{A_8^{(2)}}^{(1,1,1,1,1)}$. Its triangles are again indicated by italic and bold labels.*

1		*3*		*5*		*7*		**9**		*11*		*13*		*15*		*17*		
	2		*4*		*6*		**8**		**10**		*12*		*14*		*16*		*18*	
1		*3*		*5*		**7**		**9**		**11**		*13*		*15*		*17*		
	2		*4*		**6**		**8**		**10**		**12**		*14*		*16*		*18*	
1		*3*		**5**		**7**		**9**		**11**		**13**		*15*		*17*		...
	2		**4**		**6**		**8**		**10**		**12**		**14**		*16*		*18*	
1		**3**		**5**		**7**		**9**		**11**		**13**		**15**		*17*		
	2		**4**		**6**		**8**		**10**		**12**		**14**		**16**		*18*	
1		**3**		**5**		**7**				**11**		**13**		**15**		**17**		

FIGURE 33. Array $\mathcal{N}_{A_8^{(2)}}^{(1,1,1,1,1)}$

Example 6.12. *The labels of nodes in the specialized array* $\mathcal{N}_{A_8^{(2)}}^{(3,2,2,3,4)}$ *are the positive integers congruent to*

$$0,0,0,0,\pm 2,\pm 2,\pm 3,\pm 3,\pm 4,\pm 5,\pm 5,\pm 7,\pm 7,\pm 7,\pm 8,\pm 9,\pm 10,\pm 10,\pm 11,\pm 12$$

modulo 24 and the positive integers congruent to $\pm 4,\pm 10,\pm 14,\pm 18$ *modulo 48; see Figure 34.*

3		*8*		*12*		*17*		**24**		*31*		*36*		*40*		*45*		
	5		*10*		*15*		**21**		**27**		*33*		*38*		*43*		*48*	
2		*7*		*13*		**19**		**24**		**29**		*35*		*41*		*46*		
	4		*10*		**17**		**22**		**26**		**31**		*38*		*44*		*48*	
2		*7*		**14**		**20**		**24**		**28**		**34**		*41*		*46*		...
	5		**11**		**17**		**22**		**26**		**31**		**37**		*43*		*48*	
3		**9**		**14**		**19**		**24**		**29**		**34**		**39**		*45*		
	7		**12**		**16**		**21**		**27**		**32**		**36**		**41**		*48*	
4		**10**		**14**		**18**				**30**		**34**		**38**		**44**		

FIGURE 34. Array $\mathcal{N}_{A_8^{(2)}}^{(3,2,2,3,4)}$

Example 6.13. *The labels of nodes in the specialized array* $\mathcal{N}_{A_8^{(2)T}}^{(6,2,2,3,2)}$ *are the positive integers congruent to*

$$0,0,0,0,\pm 2,\pm 2,\pm 2,\pm 3,\pm 4,\pm 5,\pm 5,\pm 7,\pm 7,\pm 7,\pm 8,\pm 9,\pm 9,\pm 10,\pm 11,\pm 12$$

modulo 24 and the positive integers congruent to $\pm 6,\pm 10,\pm 14,\pm 20$ *modulo 48; see Figure 35.*

7. Explicit versions of Lepowsky's and Wakimoto's product formulas for $C_l^{(1)}$

7.1. **Lepowsky's formula of type** $(1,1,\ldots,1,1;\ C_l^{(1)})$**.** Let l be a nonnegative integer, and let $(s_0,s_1,\ldots,s_l)$ be $(l+1)$-tuple of positive integers. To write Lepowsky's product formula for $C_l^{(1)}$ in terms of generating function of $(s_0,s_1,\ldots,s_l)$-admissible partitions, the authors in [CMPP] introduced the *congruence triangle* as the multiset

$$\Delta(s_1,\ldots,s_l;D_{l+1}^{(2)}) = D(s_1,\ldots,s_l;D_{l+1}^{(2)}) \cup D(s_2,\ldots,s_l;D_{l+1}^{(2)}) \cup \cdots \cup D(s_l;D_{l+1}^{(2)}),$$

2		*7*		*12*		*16*		**24**		*32*		*36*		*41*		*46*		
	5		*9*		*14*		**22**		**26**		*34*		*39*		*43*		*48*	
3		*7*		*11*		**20**		**24**		**28**		*37*		*41*		*45*		
	5		*9*		**17**		**22**		**26**		**31**		*39*		*43*		*48*	
2		*7*		**15**		**19**		**24**		**29**		**33**		*41*		*46*		...
	4		**13**		**17**		**21**		**27**		**31**		**35**		*44*		*48*	
2		**10**		**15**		**19**		**24**		**29**		**33**		**38**		*46*		
	8		**12**		**17**		**22**		**26**		**31**		**36**		**40**		*48*	
6		**10**		**14**		**20**				**28**		**34**		**38**		**42**		

FIGURE 35. Array $\mathcal{N}^{(6,2,2,3,2)}_{A_8^{(2)T}}$

where for $0 \leq i \leq l$, $D(s_i, s_{i+1}, \ldots, s_l; D^{(2)}_{l+1})$ denotes the set of the $2(l-i)+1$ integers

$$\begin{gathered} s_i,\ s_i + s_{i+1},\ \ldots,\ s_i + s_{i+1} + \ldots + s_{l-2} + s_{l-1}, \\ s_i + s_{i+1} + \ldots + s_{l-1} + s_l,\ s_i + s_{i+1} + \ldots + s_{l-1} + 2s_l, \\ s_i + s_{i+1} + \ldots + 2s_{l-1} + 2s_l,\ \ldots,\ s_i + 2s_{i+1} + \ldots + 2s_{l-1} + 2s_l. \end{gathered}$$

The elements of the multiset $\Delta(s_1, \ldots, s_l; D^{(2)}_{l+1})$ correspond to the elements of the upper left right-angled triangle with vertices s_1, s_l and $s_1 + 2\sum_{i=2}^{l} s_i$ in the $(s_0, s_1, \ldots, s_l)$-specialized array $\mathcal{N}^{(s_0,s_1,\ldots,s_l)}_{D^{(2)}_{l+1}}$.

Example 7.1. *In Figure 36, the elements of the sets $D(s_i, \ldots, s_4; D^{(2)}_5)$ are denoted by ①, ②, ③, ④ for $i = 1, 2, 3, 4$, respectively. Hence, for example, the symbol ② represents the elements*

$$s_2,\ s_2 + s_3,\ s_2 + s_3 + s_4,\ s_2 + s_3 + 2s_4,\ s_2 + 2s_3 + 2s_4. \tag{7.1}$$

The remaining elements of the array which belong to a diagonal (resp. a triangle) are denoted by the symbol • (resp. ○).

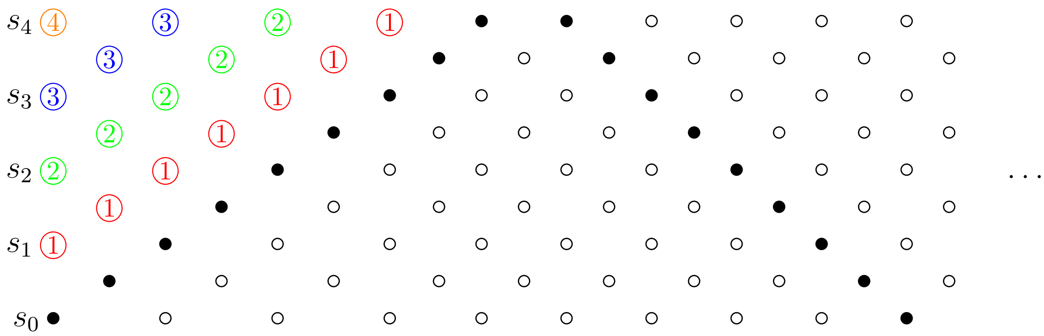

FIGURE 36. Congruence triangle $\Delta(s_1, s_2, s_3, s_4; D^{(2)}_5)$

In general, as we demonstrated in Subsection 6.2, the elements of the array $\mathcal{N}^{(s_0,\ldots,s_l)}_{D^{(2)}_{l+1}}$ can be organized into a disjoint union of triangles and diagonals. Suppose that all triangles, with an exception of the grey triangle $\Delta(s_1, \ldots, s_l; D^{(2)}_{l+1})$, are divided into two right-angled triangles; see Figure 37. Let us consider Figure 38, where the common catheti of these right-angled triangles are denoted by IJ. Then the grey triangles ABC are the

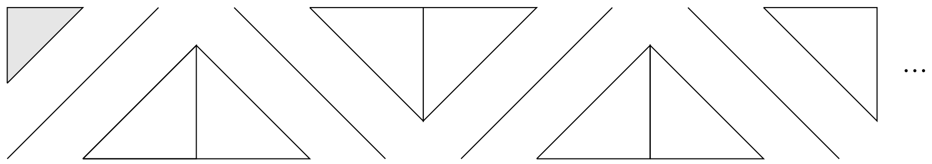

FIGURE 37. Array $\mathcal{N}_{D_{l+1}^{(2)}}^{(s_0,s_1,\dots,s_l)}$

specialization of $\mathfrak{n}_- \otimes t^{2j} \subset \mathcal{B}^-_{D_{l+1}^{(2)}}$, $j \leq 0$, the black triangles DEF are the specialization of $\mathfrak{n}_+ \otimes t^{2j}$, $j < 0$, and common catheti IJ are the specialization of $\mathfrak{h} \otimes t^{2j}$, $j < 0$; where $\mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ is the triangular decomposition of $\mathfrak{g}(B_l)$. On the other side, the lines HG are the specialization of $\mathcal{B}_{B_l}(\omega_1) \otimes t^{2j-1}$, $j \leq 0$. By closely examining the weights of $\mathcal{B}^-_{D_{l+1}^{(2)}}$, one observes that the elements of the congruence triangle $\Delta(s_1,\dots,s_l;D_{l+1}^{(2)})$ are congruent modulo $\pi = 2\sum_{i=0}^{l} s_i$ to the corresponding elements of other grey right-angled triangles ABC. The same observation holds true for the corresponding elements of the black triangles DEF, lines IJ and the diagonals GH, as indicated by the labels of their vertices. In particular, the elements of the common catheti IJ are multiples of 2π. Furthermore, the black triangles DEF consist of modular additive inverses modulo π of the corresponding elements of the grey triangles ABC.

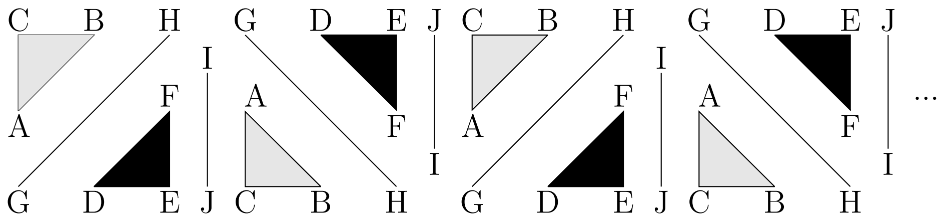

FIGURE 38. Positions of congruent elements in the array $\mathcal{N}_{D_{l+1}^{(2)}}^{(s_0,s_1,\dots,s_l)}$

Remark 7.2. *By a reasoning similar to above we can write formulas for $Q(s_0,s_1,\dots s_l;C_l^{(1)})$ and $Q(s_0,s_1,\dots s_l;A_{2l}^{(2)})$ needed for Theorems 7.7 and 7.11 below. Positions of congruent elements in the array $\mathcal{N}_{C_l^{(1)}}^{(s_0,s_1,\dots,s_l)}$ look like Figure 38 without lines GH.*

On the other side, the array $\mathcal{B}^-_{A_{2l}^{(2)}}$ consists of $\mathfrak{n}_- \otimes t^0$, small triangles $L_{B_l}(\theta) \otimes t^{2j}$ and big triangles $L_{B_l}(2\theta) \otimes t^{2j+1}$, $j < 0$. After specialization $\mathfrak{n}_- \otimes t^0$ becomes the gray triangle in the upper left corner of $\mathcal{N}_{A_{2l}^{(1)}}^{(s_0,s_1,\dots,s_1)}$, and the small triangle becomes the union of the black triangle DEF, the vertical line IJ and the gray triangle ABC with the hypotenuse in the top row. Finally, after specialization the big triangle has its hypotenuse in the bottom row, and the remaining part is the union of the black triangle DEF, the vertical line IJ and the gray triangle ABC. For $l = 2$ the last statement is obvious if we put the (rectified) small triangle in the lower left corner of Figure 8 and the big triangle without hypotenuse in the upper left corner: one is the transpose of the other.

Example 7.3. *We return to the setting of Example 7.1 to illustrate the preceding discussion. Consider Figure 39. It shows the array $\mathcal{N}_{D_5^{(2)}}^{(s_0,s_1,s_2,s_3,s_4)}$ modulo $\pi = 2\sum_{i=0}^{4} s_i$, where its triangle (resp. diagonal) elements are labeled by the upper case letters, lower case letters and zeros* $\boldsymbol{0}$ *(resp. bullets $\bullet_i$ for $i = 1,\dots,9$). Note that we use the upper* *case letters for the elements of the congruence triangle $\Delta(s_1, s_2, s_3, s_4; D_5^{(2)})$ and that the common catheti IJ from Figure 38 consists of zeros $\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}$.*

The triangle in Figure 39 with the vertices $\boldsymbol{0}$, d, D (which are underlined in the figure) and hypotenuse dD in the bottom row is the "$(s_0, s_1, s_2, s_3, s_4) \mod \pi$"-specialization of the triangle $L_{B_l}(\theta) \otimes t^2 \subset \mathcal{B}^-_{D_5^{(2)}}$. From Figures 8 and 10 we see that the positions of root vectors for roots α and $-\alpha$ are symmetric with respect to the "line corresponding to $\mathfrak{h}$" (i.e. IJ from Figure 38). Hence $A + a = B + b = \cdots = P + p = 0$.

$$
\begin{array}{ccccccccccccccccccccc}
A & & B & & C & & D & & \bullet_9 & & \bullet_1 & & d & & c & & b & & a & & \\
& E & & F & & G & & \bullet_8 & & \underline{\mathbf{0}} & & \bullet_2 & & g & & f & & e & & \mathbf{0} & \\
H & & I & & J & & \bullet_7 & & p & & P & & \bullet_3 & & j & & i & & h & & \\
& K & & L & & \bullet_6 & & o & & \mathbf{0} & & O & & \bullet_4 & & l & & k & & \mathbf{0} & \\
M & & N & & \bullet_5 & & n & & m & & M & & N & & \bullet_5 & & n & & m & & \dots \\
& O & & \bullet_4 & & l & & k & & \mathbf{0} & & K & & L & & \bullet_6 & & o & & \mathbf{0} & \\
P & & \bullet_3 & & j & & i & & h & & H & & I & & J & & \bullet_7 & & p & & \\
& \bullet_2 & & g & & f & & e & & \mathbf{0} & & E & & F & & G & & \bullet_8 & & \mathbf{0} & \\
\bullet_1 & & \underline{d} & & c & & b & & a & & A & & B & & C & & \underline{D} & & \bullet_9 & &
\end{array}
$$

FIGURE 39. Congruent elements in the array $\mathcal{N}^{(s_0,s_1,s_2,s_3,s_4)}_{D_5^{(2)}}$

By using the above analysis of the array $\mathcal{N}^{(s_0,s_1,\ldots,s_l)}_{D_{l+1}^{(2)}}$, one obtains Lepowsky's product formulas for the principally specialized character of a standard $C_l^{(1)}$-modules:

Theorem 7.4. *Lepowsky's product formula for the principally specialized character of the standard $C_l^{(1)}$-module $L_{C_l^{(1)}}(\Lambda)$ of highest weight $\Lambda = \sum_{i=0}^{l} k_i\Lambda_i$ can be written as*

$$
\begin{aligned}
&\mathrm{ch}^{(1,\ldots,1;\ C_l^{(1)})} L(k_0, k_1, \ldots, k_{l-1}, k_l) \\
&= \frac{\prod_{i \equiv a, \pm b \mod 2(k+l+1),\ a \in \{0\}^l \cup D(k_0+1, k_1+1, \ldots, k_l+1; D_{l+1}^{(2)}),\ b \in \Delta(k_1+1, k_2+1, \ldots, k_{l-1}+1, k_l+1; D_{l+1}^{(2)})} (1 - q^i)}{\prod_{j \in \mathbb{N}} (1 - q^j)^l \prod_{j \in \mathbb{N}} (1 - q^{2j-1})},
\end{aligned}
\tag{7.2}
$$

where $\{0\}^l$ denotes the multiset consisting of l copies of 0.

Proof. The principal specialization of the character of $L(\Lambda)$ is given by

$$
\mathrm{ch}^{(1,\ldots,1;\ C_l^{(1)})} L(k_0, k_1, \ldots, k_{l-1}, k_l) = \frac{Q(k_0 + 1, k_1 + 1, \ldots, k_l + 1; D_{l+1}^{(2)})}{Q(1, \ldots, 1; C_l^{(1)})}. \tag{7.3}
$$

Lemma 6.1 implies that the denominator $Q(1, \ldots, 1; C_l^{(1)})$ of (7.3) is equal to

$$
\prod_{j \in \mathbb{N}} (1 - q^j)^l \prod_{j \in \mathbb{N}} (1 - q^{2j-1}).
$$

The elements in the array $\mathcal{N}^{(k_0+1,k_1+1,\ldots,k_l+1)}_{D_{l+1}^{(2)}}$ are congruent modulo $2(k + l + 1)$. By using the direct calculations on the elements of the array $\mathcal{N}^{(k_0+1,k_1+1,\ldots,k_l+1)}_{D_{l+1}^{(2)}}$, follows that the numerator $Q(k_0 + 1, k_1 + 1, \ldots, k_l + 1; D_{l+1}^{(2)})$ of (7.3) can be written as

$$
\prod_{i \equiv a, \pm b \mod 2(k+l+1),\, a \in \{0\}^l \cup D(k_0+1, k_1+1, \ldots, k_l+1; D_{l+1}^{(2)}),\, b \in \Delta(k_1+1, k_2+1, \ldots, k_{l-1}+1, k_l+1; D_{l+1}^{(2)})} (1 - q^i). \qquad \square
$$

Remark 7.5. *Lepowsky's product formula is written in this way in* [CMPP]. *In the same vein we define three types of sets* $D(s_i, s_{i+1}, \ldots, s_l; X_l^{(r)})$ *and corresponding types of congruence triangles, where* $X_l^{(r)} = C_l^{(1)}$, $A_{2l}^{(2)}$ *or* $A_{2l}^{(2)^T}$.

7.2. **Wakimoto's formulas of type** $(2, 1, \ldots, 1, 2;\ C_l^{(1)})$**.** For $1 \le i \le l$, we define the set $D(s_i, s_{i+1}, \ldots, s_l; C_l^{(1)})$ of cardinality $2(l-i)+1$ which consists of the integers

$$s_i,\ s_i + s_{i+1},\ \ldots,\ s_i + s_{i+1} + \ldots + s_{l-2} + s_{l-1},\ s_i + s_{i+1} + \ldots + s_{l-1} + s_l,$$
$$s_i + s_{i+1} + \ldots + 2s_{l-1} + s_l,\ \ldots,\ 2s_i + 2s_{i+1} + \ldots + 2s_{l-1} + s_l.$$

Moreover, we introduce the *congruence triangle* $\Delta(s_1, \ldots, s_l; C_l^{(1)})$ as

$$\Delta(s_1, \ldots, s_l; C_l^{(1)}) = D(s_1, \ldots, s_l; C_l^{(1)}) \cup D(s_2, \ldots, s_l; C_l^{(1)}) \cup \cdots \cup D(s_l; C_l^{(1)}).$$

The elements of the multiset $\Delta(s_1, \ldots, s_l; C_l^{(1)})$ correspond to the elements of the upper left right-angled triangle with vertices s_1, s_l and $s_l + 2\sum_{i=1}^{l-1} s_i$ in the $(s_0, s_1, \ldots, s_l)$–specialized array of negative roots $\mathcal{N}_{C_l^{(1)}}^{(s_0, s_1, \ldots, s_l)}$; see Figure 41. In Figure 40, the elements of $D(s_i, \ldots, s_4; C_4^{(1)})$ are denoted by ①, ②, ③, ④, for $i = 1, 2, 3, 4$, respectively, e.g., the symbol ② represents the elements

$$s_2,\ s_2 + s_3,\ s_2 + s_3 + s_4,\ s_2 + 2s_3 + s_4,\ 2s_2 + 2s_3 + s_4.$$

The remaining elements of the array are denoted by the symbol $\circ$.

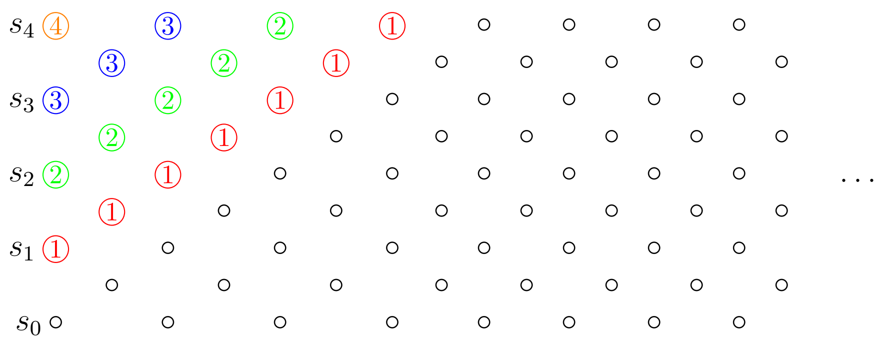

FIGURE 40. Congruence triangle $\Delta(s_1, s_2, s_3, s_4; C_4^{(1)})$

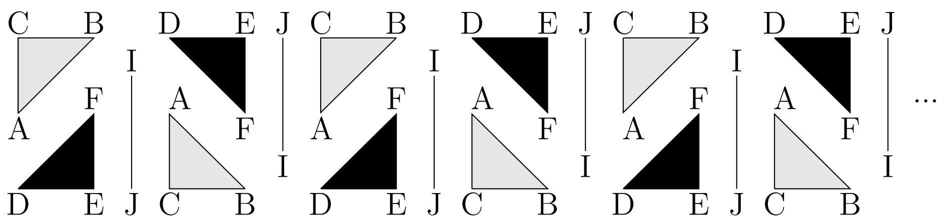

FIGURE 41. Positions of congruent elements in the array $\mathcal{N}_{C_l^{(1)}}^{(s_0, s_1, \ldots, s_l)}$

Example 7.6. *For $s=(4,3,2,3,4)$, we have*

$$\begin{aligned} D(3,2,3,4;C_4^{(1)}) &= \{3,5,8,12,15,17,20\},\\ D(2,3,4;C_4^{(1)}) &= \{2,5,9,12,14\},\\ D(3,4;C_4^{(1)}) &= \{3,7,10\},\\ D(4;C_4^{(1)}) &= \{4\}. \end{aligned}$$

The elements of these sets correspond to the italic nodes in the upper left triangle of the array $\mathcal{N}^{(4,3,2,3,4)}_{C_4^{(1)}}$ on Figure 31.

Theorem 7.7. *Wakimoto's product formula for the $(2,1,\ldots,1,2)$-specialized character of the standard module $L_{C_l^{(1)}}(\Lambda)$ of highest weight $\Lambda=\sum_{i=0}^{l} k_i\Lambda_i$ (3.4) can be written as*

$$\begin{aligned} &\mathrm{ch}^{(2,1,\ldots,1,2;\ C_l^{(1)})} L(k_0,k_1,\ldots,k_{l-1},k_l)\\ &=\frac{\prod_{i\equiv a,\pm b \mod 2(k+l+1),\ a\in\{0\}^l,\ b\in\Delta(k_1+1,k_2+1,\ldots,k_{l-1}+1,2(k_l+1);C_l^{(1)})}(1-q^i)}{\prod_{i\in\mathbb{N}}(1-q^i)^{l-1}\prod_{j\in\mathbb{N}}(1-q^{2j})\prod_{r\equiv(l+1) \mod 2(l+1)}(1-q^r)}. \end{aligned} \tag{7.4}$$

Remark 7.8. *We note that the congruence triangle does not depend on $s_0=2(k_0+1)$, but the product formula depends on k_0 via the modulus in the congruence condition* $\mathrm{mod}\ 2(k+l+1)$.

Example 7.9. *From Examples 7.6 and 6.3 follows the specialized character formula*

$$\begin{aligned} \mathrm{ch}^{(2,1,1,1,2;\ C_4^{(1)})} L(1,2,1,2,1) = &\prod_{j\equiv\pm1,\pm1,\pm2,\pm6,\pm6,\pm8,\pm11,\pm11 \mod 24}(1-q^j)^{-1}\\ &\times \prod_{\substack{i\in\mathbb{N}\\ i\not\equiv 0,12 \mod 24}}(1-q^i)^{-1}\prod_{\substack{s\in2\mathbb{N}\\ s\not\equiv 0,12 \mod 24}}(1-q^s)^{-1}\\ &\times\prod_{r\equiv 5 \mod 10}(1-q^r)^{-1}. \end{aligned}$$

7.3. **Wakimoto's formulas of types $(1,\ldots,1,2;\ C_l^{(1)})$ and $(2,1,\ldots,1;\ C_l^{(1)})$.** For $0\leq i\leq l-1$, we define the set $D(s_i,s_{i-1},\ldots,s_0;A_{2l}^{(2)})$ of cardinality $2i+1$ which consists of the integers

$$\begin{aligned} &s_i,\ s_i+s_{i-1},\ \ldots,\ s_i+s_{i-1}+\ldots+s_1+s_0,\\ &s_i+s_{i-1}+\ldots+s_1+2s_0,\ s_i+s_{i-1}+\ldots+2s_1+2s_0,\\ &\ldots,\ s_i+2s_{i-1}+\ldots+2s_1+2s_0. \end{aligned}$$

Moreover, we introduce the *congruence triangle*

$$\Delta(s_{l-1},\ldots,s_0;A_{2l}^{(2)})=D(s_{l-1},\ldots,s_0;A_{2l}^{(2)})\cup D(s_{l-2},\ldots,s_0;A_{2l}^{(2)})\cup\cdots\cup D(s_0;A_{2l}^{(2)}).$$

Its elements belong to the upper left right-angled triangle with vertices s_{l-1}, s_0 and $s_{l-1}+2\sum_{i=0}^{l-2}s_i$ of the $(s_0,s_1,\ldots,s_l)$-specialized array of negative roots $\mathcal{N}^{(s_0,s_1,\ldots,s_l)}_{A_{2l}^{(2)}}$. In Figure 42, the elements of $D(s_3,s_2,s_1,s_0;A_8^{(2)})$ are denoted by ③, ②, ①, ⓪, for $i=3,2,1,0$, respectively. For example, the symbol ② represents the elements

$$s_2,\ s_2+s_1,\ s_2+s_1+s_0,\ s_2+s_1+2s_0,\ s_2+2s_1+2s_0.$$

The remaining elements of the array are denoted by the symbols ∘ and •.

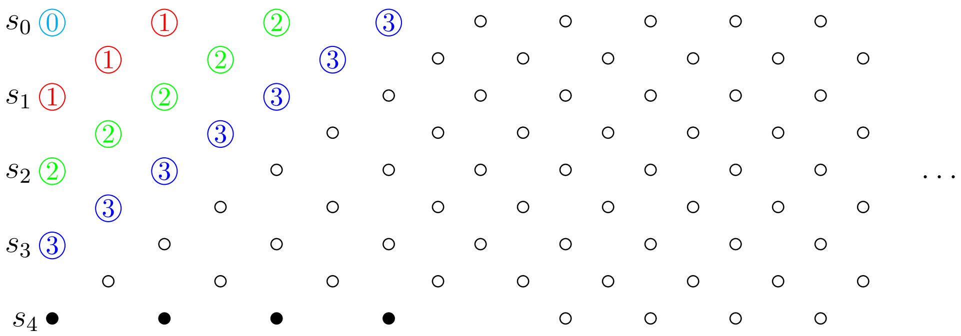

FIGURE 42. Congruence triangle $\Delta(s_3, s_2, s_1, s_0; A_8^{(2)})$

To write Wakimoto's formulas for the specialized character of type $(1, 1, \dots, 1, 2)$ (see (3.3)), we introduce the set

$$S(s_l, s_{l-1}, \dots, s_1; A_{2l}^{(2)}) = \{s_l, s_l + 2s_{l-1}, \dots, s_l + 2s_{l-1} + \dots + 2s_1\}$$

of cardinality l. In Figure 42, the elements of $S(s_4, s_3, s_2, s_1; A_8^{(2)})$, which are

$$s_4,\ s_4 + 2s_3,\ s_4 + 2s_3 + 2s_2,\ s_4 + 2s_3 + 2s_2 + 2s_1,$$

are indicated by the symbol $\bullet$.

Example 7.10. *For $s = (3, 2, 2, 3, 4)$, we have*

$$\begin{aligned} D(3, 2, 2, 3; A_8^{(2)}) &= \{3, 5, 7, 10, 13, 15, 17\}, \\ D(2, 2, 3; A_8^{(2)}) &= \{2, 4, 7, 10, 12\}, \\ D(2, 3; A_8^{(2)}) &= \{2, 5, 8\}, \\ D(3; A_8^{(2)}) &= \{3\}, \\ S(4, 3, 2, 2; A_8^{(2)}) &= \{4, 10, 14, 18\}. \end{aligned}$$

Theorem 7.11. *The Wakimoto product formula for the $(1, 1, \dots, 1, 2)$-specialized character of the standard module $L_{C_l^{(1)}}(\Lambda)$ of highest weight $\Lambda = \sum_{i=0}^{l} k_i \Lambda_i$ can be written as*

$$\begin{aligned} &\mathrm{ch}^{(1,1,\dots,1,2;\ C_l^{(1)})} L(k_0, k_1, \dots, k_{l-1}, k_l) \\ &= \prod_{i \in \mathbb{N}} (1 - q^i)^{-l} \prod_{i \equiv \pm a, b \mod 2(k+l+1),\ a \in \Delta(k_{l-1}+1, \dots, k_0+1; A_{2l}^{(2)}), b \in \{0\}^l} (1 - q^i) \\ &\quad \times \prod_{j \equiv \pm c \mod 4(k+l+1),\ c \in S(2(k_l+1), k_{l-1}+1, \dots, k_1+1; A_{2l}^{(2)})} (1 - q^j). \end{aligned} \tag{7.5}$$

Example 7.12. *From Example 7.10 follows the specialized character formula*

$$\begin{aligned} \mathrm{ch}^{(1,1,1,1,2;\ C_4^{(1)})} L(2, 1, 1, 2, 1) = &\prod_{\substack{i \in \mathbb{N} \\ i \not\equiv 0 \mod 24}} (1 - q^i)^{-1} \prod_{k \equiv \pm 6, \pm 20 \mod 48} (1 - q^k)^{-1} \\ &\times \prod_{j \equiv \pm 1, \pm 1, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 6, \pm 8, \pm 8, \pm 9, \pm 9, \pm 11, \pm 11, 12 \mod 24} (1 - q^j)^{-1}. \end{aligned}$$

For $1 \leq i \leq l$, define the *congruence triangles* as multisets

$$\Delta(s_1,\ldots,s_l;{A_{2l}^{(2)}}^T) = D(s_1,\ldots,s_l;{A_{2l}^{(2)}}^T) \cup D(s_2,\ldots,s_l;{A_{2l}^{(2)}}^T) \cup \cdots \cup D(s_l;{A_{2l}^{(2)}}^T),$$

where the sets $D(s_i,s_{i+1},\ldots,s_l;{A_{2l}^{(2)}}^T)$ consist of integers

$$\begin{aligned}&s_i,\ s_i+s_{i+1},\ \ldots,\ s_i+\ldots+s_{l-1}+s_l,\\ &s_i+s_{i+1}+\ldots+s_{l-1}+2s_l,\ \ldots,\ s_i+2s_{i+1}+\ldots+2s_{l-1}+2s_l.\end{aligned}$$

Furthermore, let

$$S(s_0,\ldots,s_{l-1};{A_{2l}^{(2)}}^T) = \{s_0, s_0+2s_1,\ldots,s_0+2s_1+\cdots+2s_{l-1}\}\,.$$

Example 7.13. *For $s=(2,3,2,2,6)$, we have*

$$\begin{aligned}D(2,2,3,2;{A_8^{(2)}}^T) &= \{2,4,7,9,11,14,16\},\\ D(2,3,2;{A_8^{(2)}}^T) &= \{2,5,7,9,12\},\\ D(3,2;{A_8^{(2)}}^T) &= \{3,5,7\},\\ D(2;{A_8^{(2)}}^T) &= \{2\},\\ S(6,2,2,3;{A_8^{(2)}}^T) &= \{6,10,14,20\}.\end{aligned}$$

Theorem 7.14. *The Wakimoto formula for the specialized character of type $(2,1,\ldots,1,1)$ of the standard module of highest weight $\Lambda=\sum_{i=0}^{l}k_i\Lambda_i$ can be written as*

$$\begin{aligned}\mathrm{ch}^{(2,1,\ldots,1,1;\ C_l^{(1)})}L(k_0,k_1,\ldots,k_{l-1},k_l) = &\prod_{i\in\mathbb{N}}(1-q^i)^{-l}\prod_{\substack{i\equiv\pm a,b \mod 2(k+l+1),\\ a\in\Delta(k_1+1,\ldots,k_l+1;{A_{2l}^{(2)}}^T),\, b\in\{0\}^l}}(1-q^i)\\ &\times\prod_{\substack{j\equiv\pm c \mod 4(k+l+1),\\ c\in S(2(k_0+1),k_1+1\ldots,k_{l-1}+1;{A_{2l}^{(2)}}^T)}}(1-q^j).\end{aligned}\tag{7.6}$$

Example 7.15. *Using Example 7.13 we obtain the specialized character formula*

$$\begin{aligned}\mathrm{ch}^{(2,1,1,1,1;\ C_4^{(1)})}L(2,1,1,2,1) = &\prod_{\substack{i\in\mathbb{N}\\ i\not\equiv 0 \mod 24}}(1-q^i)^{-1}\prod_{k\equiv\pm4,\pm18 \mod 48}(1-q^k)^{-1}\\ &\times\prod_{j\equiv\pm1,\pm1,\pm1,\pm3,\pm3,\pm4,\pm5,\pm6,\pm6,\pm8,\pm8,\pm9,\pm10,\pm11,\pm11,12 \mod 24}(1-q^j)^{-1}.\end{aligned}$$

Example 7.16. *The specialized characters*

$$\mathrm{ch}^{(2,1,1,1,1;\ C_4^{(1)})}L(1,2,1,2,1) \qquad \textit{and} \qquad \mathrm{ch}^{(1,1,1,1,2;\ C_4^{(1)})}L(1,2,1,2,1)$$

coincide and they are equal to

$$\prod_{\substack{i\in\mathbb{N}\\ i\not\equiv 0,12 \mod 24}}(1-q^i)^{-1}\prod_{j\equiv\pm1,\pm1,\pm1,\pm2,\pm3,\pm4,\pm4,\pm6,\pm6,\pm6,\pm8,\pm8,\pm9,\pm10,\pm11,\pm11,\pm11 \mod 24}(1-q^j)^{-1}.$$

7.4. **Wakimoto's formulas of type** $(s_0, s_1, \dots, s_l;\ C_l^{(1)})$**.** From (3.5)–(3.7), we have the following three sets of Wakimoto's formulas of type $(s_0, s_1, \dots, s_l;\ C_l^{(1)})$. Using notation from previous subsections we will write these formulas and give some examples.

Theorem 7.17. *The first set of Wakimoto's formulas of type* $(s_0, s_1, \dots, s_l;\ C_l^{(1)})$ *is*

$$\begin{aligned}
&\mathrm{ch}^{(s_0,s_1,\dots,s_l;\ C_l^{(1)})} L(2n-1, 2n-1, \dots, 2n-1, n-1) \\
&= \frac{\prod_{i \equiv \pm a, b \mod n(2\sum_{j=0}^{l} s_j + 2\sum_{j=1}^{l-1} s_j),\ a \in \Delta(2ns_1, \dots, 2ns_{l-1}, ns_l; {A_{2l}^{(2)}}^T),\, b \in \{0\}^l} (1-q^i)}{\prod_{i \equiv \pm a, b \mod (\sum_{j=0}^{l} s_j + \sum_{j=1}^{l-1} s_j),\ a \in \Delta(s_1, s_2, \dots, s_l; C_l^{(1)}),\, b \in \{0\}^l} (1-q^i)} \\
&\times \prod_{j \equiv \pm c \mod 2n(2\sum_{m=0}^{l} s_m + 2\sum_{m=1}^{l-1} s_m),\ c \in S(2ns_0, 2ns_1, \dots, 2ns_{l-1}; {A_{2l}^{(2)}}^T)} (1-q^j).
\end{aligned} \tag{7.7}$$

Example 7.18. *For* $l = 2$*,* $s = (3,1,1)$ *and* $n = 1$*, we have*

$$\mathrm{ch}^{(3,1,1;\ C_2^{(1)})} L(1,1,0) = \frac{\prod_{i \equiv \pm 5 \mod 12} (1+q^i)}{\prod_{j \in \mathbb{N}} (1-q^{2j-1}) \prod_{r \equiv \pm 6 \mod 24} (1-q^r)}.$$

For $l = 2$*,* $s = (3,1,1)$ *and* $n = 2$*, we have*

$$\mathrm{ch}^{(3,1,1;\ C_2^{(1)})} L(3,3,1) = \frac{\prod_{i \equiv \pm 10 \mod 24} (1+q^i)}{\prod_{j \in \mathbb{N}} (1-q^{2j-1})^2 \prod_{r \equiv \pm 6, 12 \mod 24} (1-q^r)}.$$

For $l = 2$*,* $s = (3,1,1)$ *and* $n = 3$*, we have*

$$\begin{aligned}
\mathrm{ch}^{(3,1,1;\ C_2^{(1)})} L(5,5,2) = &\prod_{i \in \mathbb{N}, i \not\equiv 0, \pm 30, 36 \mod 72} (1-q^i)^{-1} \\
&\times \prod_{r \equiv \pm 1, \pm 5, \pm 7, \pm 11, \pm 13, \pm 15, \pm 17 \mod 36} (1-q^r)^{-1}.
\end{aligned}$$

Theorem 7.19. *The second set of Wakimoto's formulas of type* $(s_0, s_1, \dots, s_l;\ C_l^{(1)})$ *is*

$$\begin{aligned}
&\mathrm{ch}^{(s_0,s_1,\dots,s_l;\ C_l^{(1)})} L(n-1, 2n-1, \dots, 2n-1, 2n-1) \\
&= \frac{\prod_{i \equiv \pm a, b \mod n(2\sum_{j=0}^{l} s_j + 2\sum_{j=1}^{l-1} s_j),\ a \in \Delta(2ns_{l-1}, \dots, 2ns_1, ns_0; A_{2l}^{(2)}),\, b \in \{0\}^l} (1-q^i)}{\prod_{i \equiv \pm a, b \mod (\sum_{j=0}^{l} s_j + \sum_{j=1}^{l-1} s_j),\ a \in \Delta(s_1, s_2, \dots, s_l; C_l^{(1)}),\, b \in \{0\}^l} (1-q^i)} \\
&\times \prod_{j \equiv \pm c \mod 2n(2\sum_{m=0}^{l} s_m + 2\sum_{m=1}^{l-1} s_m),\ c \in S(2ns_l, 2ns_{l-1}, \dots, 2ns_1; A_{2l}^{(2)})} (1-q^j).
\end{aligned} \tag{7.8}$$

Example 7.20. *For* $l = 2$*,* $s = (1,3,1)$ *and* $n = 1$*, we have*

$$\mathrm{ch}^{(1,3,1;\ C_2^{(1)})} L(0,1,1) = \frac{\prod_{i \in \mathbb{N}} (1+q^{2i-1})}{\prod_{j \equiv 4 \mod 16} (1-q^j)^2}.$$

For $l = 2$*,* $s = (1,3,1)$ *and* $n = 2$*, we have*

$$\mathrm{ch}^{(1,3,1;\ C_2^{(1)})} L(1,3,3) = \frac{\prod_{i \equiv \pm 1, \pm 7, \pm 9, \pm 15 \mod 32} (1+q^i)}{\prod_{j \in \mathbb{N}} (1-q^{2j-1}) \prod_{r \equiv \pm 4, \pm 8, \pm 8, \pm 12 \mod 32} (1-q^r)}.$$

Theorem 7.21. *The third set of Wakimoto's formulas of type $(s_0, s_1, \ldots, s_l;\ C_l^{(1)})$ is*

$$\mathrm{ch}^{(s_0,s_1,\ldots,s_l;\ C_l^{(1)})} L(n-1, 2n-1, \ldots, 2n-1, n-1) = \tag{7.9}$$

$$= \prod_{i\equiv \pm a, b \mod (\sum_{j=0}^{l} s_j + \sum_{j=1}^{l-1} s_j),\ a\in\Delta(s_1,s_2,\ldots,s_l;C_l^{(1)}),\, b\in\{0\}^l} (1-q^i)^{-1}$$

$$\times \prod_{\substack{i\equiv a, \pm b \mod 2n(\sum_{j=0}^{l} s_j + \sum_{j=1}^{l-1} s_j),\\ a\in\Delta(ns_0, 2ns_1,\ldots,2ns_{l-1}, ns_l; D_{l+1}^{(2)})\cup\{0\}^l,\, b\in\Delta(2ns_1,\ldots,2ns_{l-1},ns_l;D_{l+1}^{(2)})}} (1-q^i).$$

Example 7.22. *For $l=2$, $s=(3,1,1)$ and $n=1$, we have*

$$\mathrm{ch}^{(3,1,1;\ C_2^{(1)})} L(0,1,0) = \prod_{i\equiv\pm 1 \mod 6} (1-q^i)^{-1} \prod_{j\equiv 6 \mod 12} (1-q^j)^{-1}.$$

For $l=2$, $s=(3,1,1)$ and $n=2$, we have

$$\mathrm{ch}^{(3,1,1;\ C_2^{(1)})} L(1,3,1) = \prod_{i\in\mathbb{N}} (1-q^{2i-1})^{-2} \prod_{j\equiv 12 \mod 24} (1-q^j)^{-1}.$$

For $l=2$, $s=(3,1,1)$ and $n=3$, we have

$$\mathrm{ch}^{(3,1,1;\ C_2^{(1)})} L(2,5,2) = \prod_{\substack{i\in\mathbb{N}\\ i\not\equiv 0,\pm 9,\pm 27, 36 \mod 72}} (1-q^i)^{-1} \prod_{j\equiv\pm 1 \mod 6} (1-q^j)^{-1}.$$

8. Borcea's correspondence for $C_l^{(1)}$ and $A_{2l}^{(2)}$

Let $k\in\mathbb{N}$ be fixed. The number of level $k=\sum_{i=0}^{l} k_i$ standard $C_l^{(1)}$-modules of highest weight $\Lambda = \sum_{i=0}^{l} k_i\Lambda_i$ is $\binom{k+l}{l}$. This number is the same as the number of level $2k+1$ standard $A_{2l}^{(2)}$-modules of highest weight $(2k_0+1)\Lambda_0 + \sum_{i=1}^{l} k_i\Lambda_i$.

It follows directly from the Lepowsky–Wakimoto product formula for the specialized characters of the Weyl–Kac character formulas of types $(2,1,\ldots,1,1;C_l^{(1)})$ and $(1,1,\ldots,1;A_{2l}^{(2)})$ (see [W1, Theorem 1] and [W2, Corollary 2.2.8]) and Lemmas 6.5 and 6.10 that

Theorem 8.1. *Let $\mathrm{ch}^{(1,1,\ldots,1;A_{2l}^{(2)})} L(2k_0+1, k_1, \ldots, k_l)$ denote the principally specialized character of level $2k+1$ standard module $L(\Lambda)$ of the affine Lie algebra of type $A_{2l}^{(2)}$ with highest weight $\Lambda = (2k_0+1)\Lambda_0 + \sum_{i=1}^{l} k_i\Lambda_i$. We have*

$$\mathrm{ch}^{(2,1,\ldots,1;C_l^{(1)})} L(k_0, k_1 \ldots, k_l) = F(q)\cdot \mathrm{ch}^{(1,1,\ldots,1;A_{2l}^{(2)})} L(2k_0+1, k_1, \ldots, k_l), \tag{8.1}$$

where

$$F(q) = \prod_{i\equiv\pm 1,\pm 3,\ldots,\pm(2l-1) \mod 2(2l+1)} (1-q^i).$$

Proof. From [W1, Theorem 1] (see also [W2]) we have

$$\mathrm{ch}^{(2,1,\ldots,1;C_l^{(1)})} L(k_0, k_1 \ldots, k_l) = \frac{Q(k_l+1, \ldots, k_1+1, 2(k_0+1); A_{2l}^{(2)})}{Q(2,1,\ldots,1;C_l^{(1)})},$$

and

$$\mathrm{ch}^{(1,1,\ldots,1;A_{2l}^{(2)})} L(2k_0+1, k_1, \ldots, k_l) = \frac{Q(k_l+1, \ldots, k_1+1, 2(k_0+1); A_{2l}^{(2)})}{Q(1,1,\ldots,1;A_{2l}^{(2)})}.$$

The theorem now follows from Lemmas 6.5 and 6.10. □

In this work we are interested in affine Lie algebras of type $C_l^{(1)}$, $l \geq 2$, but most of our considerations hold as well for $l = 1$ when $C_1^{(1)} \cong A_1^{(1)}$. From that point of view the identity in (8.1) is a generalization of the identity obtained in [Bor] in the case when $l = 1$. In the case of $l = 1$ and $k = 1$, we have two identities (8.1)

$$\begin{aligned}\mathrm{ch}^{(2,1;A_1^{(1)})} L(1,0) &= \prod_{i\equiv\pm1 \mod 6} (1-q^i)\cdot \mathrm{ch}^{(1,1;A_2^{(2)})} L(3,0)\\ &= \frac{1}{\prod_{i\equiv\pm2,\pm3 \mod 12}(1-q^i)},\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}^{(2,1;A_1^{(1)})} L(0,1) &= \prod_{i\equiv\pm1 \mod 6} (1-q^i)\cdot \mathrm{ch}^{(1,1;A_2^{(2)})} L(1,1)\\ &= \frac{\prod_{i\equiv\pm2 \mod 12}(1-q^i)}{\prod_{i\in\mathbb{N}}(1-q^{2i-1})},\end{aligned}$$

and the corresponding products appear in [MP] as product sides of two combinatorial identities for level 1 standard $A_1^{(1)}$-modules and, on the other side, the same products appear in [C] as product sides of Capparelli's identities for level 3 standard $A_2^{(2)}$-modules. Moreover, by some sort of coincidence, the difference conditions for partitions coincide in both cases.

9. Combinatorial identities

Let $l \geq 2$. A downward path $\mathcal{Z}$ in the array $\mathcal{B}_{C_l^{(1)}}$ is a sequence (or a subset) with an element in the top row followed by an adjacent element in the second row, and so on all the way to an element in the bottom row. The vertices denoted by $\bullet$'s in Figure 43 gives an example of a downward path.

A monomial

$$\pi = \prod_{b(j)\in\mathcal{B}^-_{C_l^{(1)}}} b(j)^{m_{b(j)}} \in S(\mathfrak{g}(C_l^{(1)})) \tag{9.1}$$

can be interpreted as a colored partition π: for $m_{b(j)} > 0$ we say that $b(j)$ is a part of degree $|b(j)| = j$ and color $b \in \mathcal{B}_{C_l}(\theta)$ which appears in the partition $m_{b(j)}$ times. We define the degree, weight and length of a colored partition π as

$$|\pi| = \sum_{b(j)\in\bar{B}_\ell^{<0}} j\cdot m_{b(j)}, \quad \mathrm{wt}_{\mathfrak{h}}(\pi) = \sum_{b(j)\in\bar{B}_\ell^{<0}} \mathrm{wt}_{\mathfrak{h}}(b)\cdot m_{b(j)}, \quad \ell(\pi) = \sum_{b(j)\in\bar{B}_\ell^{<0}} m_{b(j)}. \tag{9.2}$$

We interpret 1 as an empty partition $\emptyset$ with no parts of degree 0 and length 0. A colored partition π can be also interpreted as a function

$$\pi\colon \mathcal{B}^-_{C_l^{(1)}} \to \mathbb{Z}_{\geq 0}, \qquad b(n) \mapsto m_{b(n)}$$

with finite support.

Let k be a positive integer. We say that a *colored partition π satisfies the level k-difference conditions* if

$$\sum_{b(j)\in\mathcal{Z}} m_{b(j)} \leq k \quad \text{for all downward paths } \mathcal{Z}\subset\mathcal{B}^-_{C_l^{(1)}}. \tag{9.3}$$

To formulate the *initial conditions* for π we need to extend $\mathcal{B}^-_{C_l^{(1)}}$ with Chevalley generators associated to simple roots and simple coroots in $\mathfrak{h}$, i.e.

$$\mathcal{B}^-_{C_l^{(1)}} \subset \mathcal{B}^{-e}_{C_l^{(1)}} = \{e_0, e_1, \dots, e_l\} \cup \{h_1, \dots, h_l\} \cup \mathcal{B}^-_{C_l^{(1)}} \subset \mathcal{B}_{C_l^{(1)}}.$$

For example, for $l = 2$ we have the extended array of root vectors on Figure 43 (with omitted arrows—compare with Figure 16). For nonnegative integers $k_0, k_1, \dots, k_l$, $k_0 +$

$$\begin{array}{ccccccccccc}
e_2 & & f_2 & & \circ & & \bullet & & \circ & & \circ \ \cdots \\
& h_2 & & \circ & & \bullet & & \circ & & \circ & \\
e_1 & & f_1 & & \circ & & \bullet & & \circ & & \circ \ \cdots \\
& h_1 & & \circ & & \circ & & \bullet & & \circ & \\
e_0 & & f_0 & & \circ & & \bullet & & \circ & & \circ \ \cdots
\end{array}$$

FIGURE 43. The extended arrangement of negative root vectors for $C_2^{(1)}$ and a downward path

$k_1 + \cdots + k_l = k$ and dominant integral $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_l\Lambda_l$ we say that π *is* Λ*-admissible if the monomial*

$$e_0^{k_0} e_1^{k_1} \dots e_l^{k_l} h_1^0 \dots h_l^0 \cdot \pi$$

satisfies the level k difference conditions on the extended array $\mathcal{B}^{-e}_{C_l^{(1)}}$. The generating function for Λ-admissible colored partitions is

$$\sum_{\pi \text{ is } \Lambda\text{-admissible}} e^{|\pi|\delta} e^{\mathrm{wt}_{\mathfrak{h}}(\pi)} \tag{9.4}$$

In [CMPP] it was (implicitly) conjectured:

Conjecture 9.1. *The generating function for* Λ*-admissible colored partitions is* $\mathrm{ch}\, e^{-\Lambda} L_{C_l^{(1)}}(\Lambda)$ *for* $l, k \geq 2$.

Remark 9.2. *For* $l = 1$ *the statement above is proved in* [MP, FKLMM] *for all* Λ *and in* [F] *for* $\Lambda = k\Lambda_0$, $k \geq 1$. *For* $l \geq 2$ *and* $k = 1$ *the statement above is proved in* [DK, R] *for all* Λ *and in* [PŠ1] *for* $\Lambda = \Lambda_0$.

The $(s_0, s_1, \dots, s_l)$-specialization defines a map

$$\mathcal{B}^-_{C_l^{(1)}} \to \mathcal{N}^{(s_0, s_1, \dots, s_l)}_{C_l^{(1)}}, \qquad b(j) \mapsto a_{b(j)} = \|b(j)\|,$$

which extends to a map

$$\pi = \prod_{b(j) \in \mathcal{B}^-_{C_l^{(1)}}} b(j)^{m_{b(j)}} \mapsto \|\pi\| = \sum_{a_{b(j)} \in \mathcal{N}^{(s_0, s_1, \dots, s_l)}_{C_l^{(1)}}} a_{b(j)} \cdot m_{b(j)}.$$

In other words, the $(s_0, s_1, \dots, s_l)$-specialization of a colored partition π gives a partition $\|\pi\|$ defined on the array $\mathcal{N}^{(s_0, s_1, \dots, s_l)}_{C_l^{(1)}}$, and Conjecture 9.1 implies that the generating function

$$\sum_{\pi \text{ is } \Lambda\text{-admissible}} q^{-\|\pi\|}$$

for Λ-admissible colored partitions $\|\pi\|$ is the specialized character $\mathrm{ch}^{(s_0,s_1,\ldots,s_l;C_l^{(1)})}L(\Lambda)$. If the specialized character can be written as an infinite periodic product, the conjecture gives a Rogers–Ramanujan-type combinatorial identity. By using Lepowsky's product formula for the principal specialization $(1,1,\ldots,1)$, the conjectured identities are formulated in [CMPP]. By using Wakimoto's product formulas we get other Rogers–Ramanujan-type combinatorial identities.

Example 9.3. *For* $l=2$ *and* $(2,1,1)$*-specialization we have the array* $\mathcal{N}_{C_2^{(1)}}^{(2,1,1)}$. *For* $\Lambda = k\Lambda_0$, $k\geq 2$, *we can apply Theorem 7.14 and get the conjectured identity:*

The generating function for partitions on the array

$$\begin{array}{cccccccccc}
 & & & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & \\
 & & & \mathit{5} & & \mathbf{8} & & \mathbf{10} & & \mathbf{12} \\
 & & \mathit{4} & & \mathit{6} & & \mathbf{9} & & \mathbf{11} & \\
 & \mathit{3} & & \mathit{5} & & \mathit{7} & & \mathbf{10} & & \mathit{13} \\
\mathit{2} & & \mathit{4} & & \mathit{6} & & \mathit{8} & & \mathit{12} &
\end{array}$$

satisfying level k *difference conditions is*

$$\prod_{i\in\mathbb{N}}(1-q^i)^{-2}\prod_{\substack{i\equiv\pm a,b \mod 2(k+3),\\ a\in\{1,1,2,3\},\, b\in\{0\}^2}}(1-q^i)\prod_{\substack{j\equiv\pm c \mod 4(k+3),\\ c\in\{2(k+1),2(k+2)\}}}(1-q^j).$$

For $k=1$ *this is a theorem due to results in* [PŠ1, DK, R], *and* $k>1$ *due to* [PT].

Remark 9.4. *The product formula in Example 7.9 is the conjectured generating function for* $(1,2,1,2,1)$*-admissible partitions*

$$n=\sum_{a\in\mathcal{N}} m_a\cdot a$$

on the array $\mathcal{N}=\mathcal{N}_{C_4^{(1)}}^{(2,1,1,1,2)}$ *that consists of* 3 *copies of the positive integers, one copy of the even positive integers and one copy of the positive integers congruent to* 5 *modulo* 10.

The product formulas in Examples 7.12 and 7.15 are the conjectured generating functions for $(2,1,1,2,1)$*-admissible partitions on two different arrays (see Figures 29 and 30) that consist of* 4 *copies of the positive integers. The product formula in Example 7.16 is another conjectured generating function for admissible partitions, also satisfying level* 7 *difference conditions.*

For specializations $(2,1,\ldots,1,2)$, $(2,1,\ldots,1,1)$ *and* $(1,1,\ldots,1,2)$, *it seems that we have the conjectured Rogers–Ramanujan-type partition identity for all* Λ*-admissible partitions. On the other side, Theorems 7.17, 7.19 and 7.21 correspond to Wakimoto's unspecialized product formulas, but only for specific highest weights* Λ. *The product formulas in Examples 7.18, 7.20 and 7.22 have combinatorial interpretations, and Conjecture 9.1 gives the corresponding Rogers–Ramanujan-type partition identities; it is the theorem in level* 1 *case due to* [DK], *and the conjectures in all the other cases. It seems that unspecialized Wakimoto's product formulas should provide many new (conjectured) Rogers–Ramanujan-type partition identities for different specializations.*

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